Entanglement of Purification in AdS/CFT

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Based on

[1] T. Takayanagi, KU, to appear in Nature Physics [arXiv:1708.09393].

[2] A. Bhattacharyya, T. Takayanagi, KU, [arXIv:1802.09545].

[2] -> Arpan's talk!

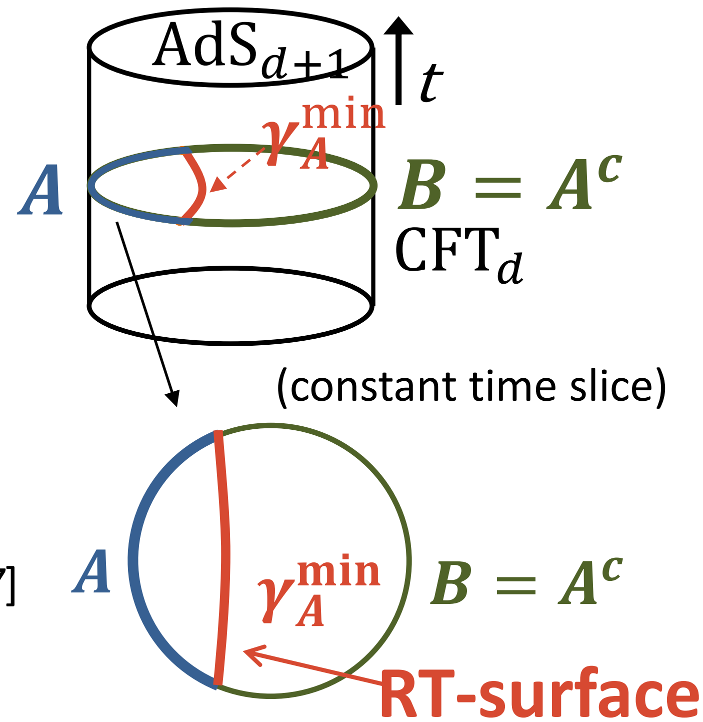
AdS/CFT and Quantum Information

Holographic entanglement entropy

$$S_A = \min_{\gamma_A} \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

(CFT) (AdS)

[*Ryu-Takayanagi '06*][*Hubeny-Rangamani-Takayanagi '07*]



Entanglement entropy

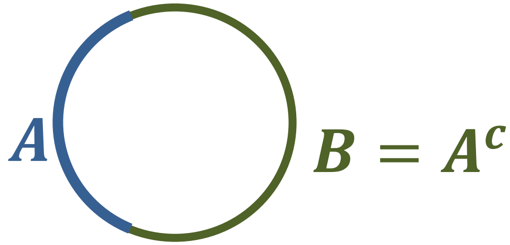
$$S_A \equiv S(\rho_A) = -\mathrm{Tr}\rho_A \log \rho_A. \quad (\rho_A := \mathrm{Tr}_B[|\Psi\rangle\langle\Psi|_{AB}])$$

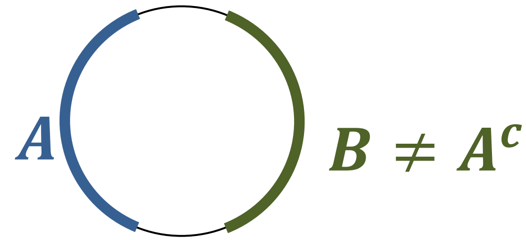
It is a **unique** measure of entanglement **for pure states $|\Psi\rangle_{AB}$**.

[*Donald-Horodecki-Rudolph '02*]

For mixed states ρ_{AB}

$|\Psi\rangle_{AB}$ **pure states** ➡ $\rho_{AB} = \Sigma_n p_n |\Psi_n\rangle\langle\Psi_n|_{AB}$ **mixed**





We can **not** use S_A as a measure of correlation.

e.g.) No correlation in $\rho_{AB} = \rho_A \otimes \rho_B$, but in general $S_A > 0$.

There are many good **correlation measures for mixed states**:

- Relative entropy of entanglement E_R ,
- Squashed entanglement E_{sq},
- Entanglement of Formation E_F,
- … etc.

⬇

Holographic counterparts of them?

Motivation

We want to investigate a new dual relationship between **correlation measures for mixed states** and **spacetime geometry** in AdS/CFT.

It will recover an information-theoretic understanding of holography for mixed states.

Entanglement Wedge
Cross Section

What is a geometrical counterpart of “correlation”?

Information of ρ_{AB} is included in **entanglement wedge M_{AB}** in AdS.

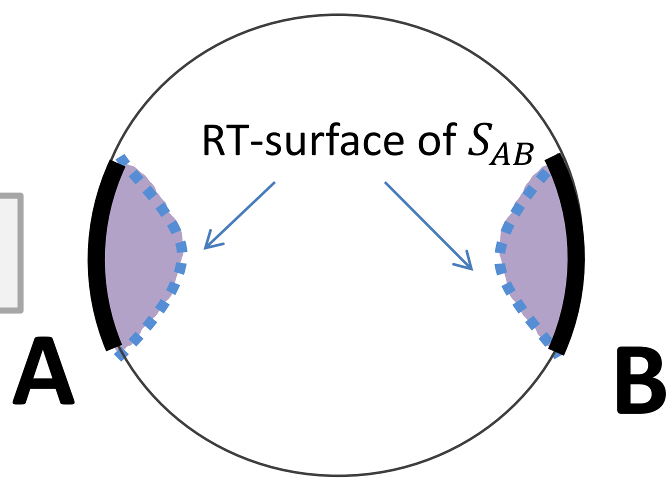
CFT

No correlation

$$\rho_{AB} = \rho_A \otimes \rho_B$$

$$\Leftrightarrow S_{AB} = S_A + S_B$$

AdS

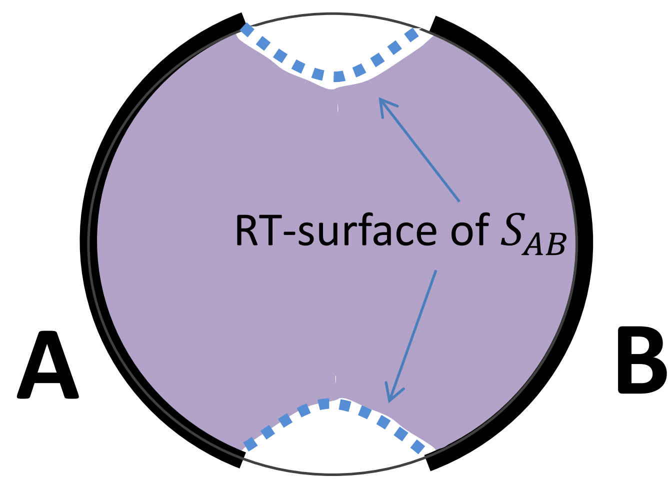


$$\therefore \boldsymbol{M_{AB} = M_A \cup M_B}$$

Correlated

$$\rho_{AB} \neq \rho_A \otimes \rho_B$$

$$\Leftrightarrow S_{AB} < S_A + S_B$$



$$\therefore \boldsymbol{M_{AB} \supsetneq M_A \cup M_B}$$

What is a geometrical counterpart of "correlation"?

Information of ρ_{AB} is included in **entanglement wedge** $\boldsymbol{M_{AB}}$ in AdS.

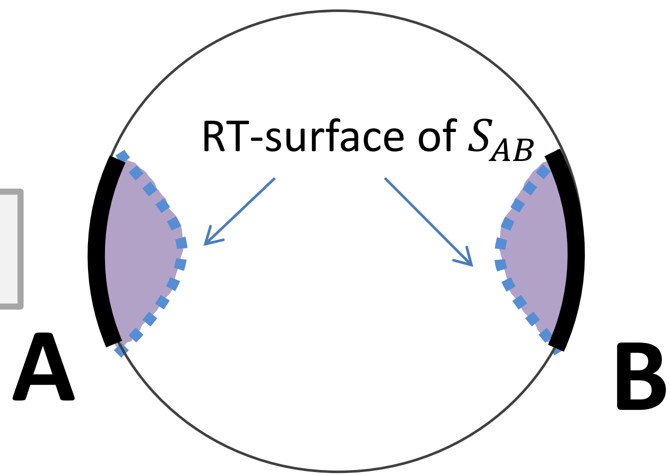
CFT

No correlation

$\rho_{AB} = \rho_A \otimes \rho_B$

$\Leftrightarrow S_{AB} = S_A + S_B$

AdS



$\therefore \boldsymbol{M_{AB} = M_A \cup M_B}$

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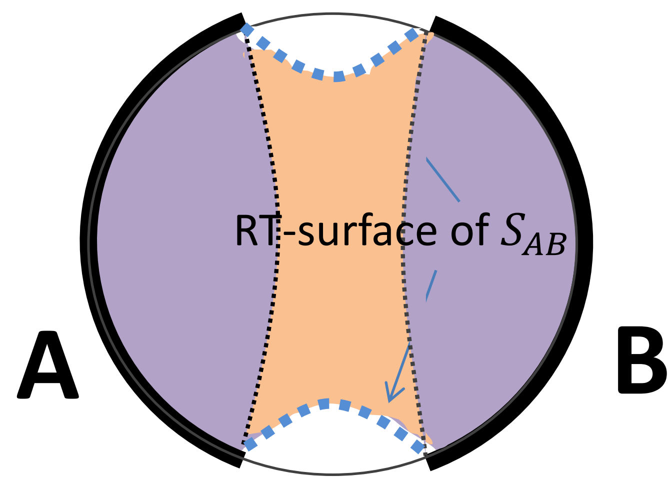
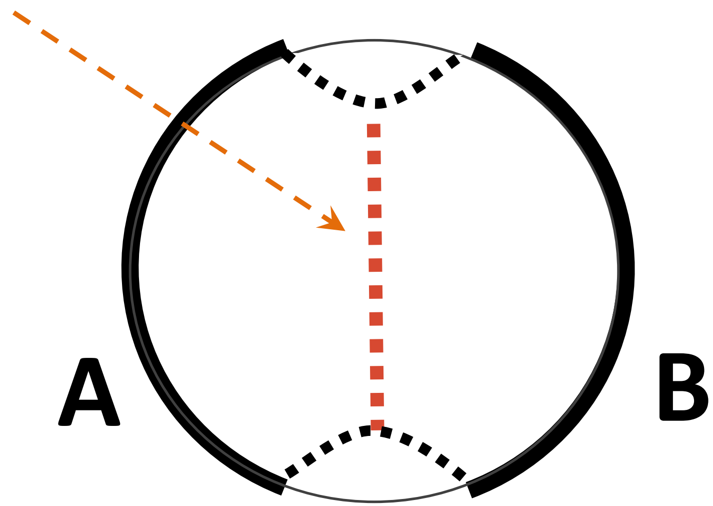


$\therefore \boldsymbol{M_{AB} \supsetneq M_A \cup M_B}$

Emerges from correlations

We define its minimal area by

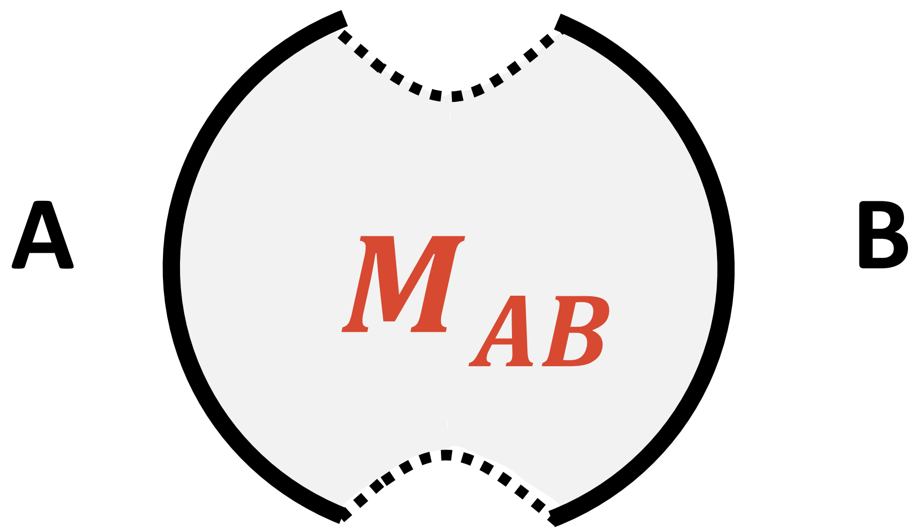
entanglement wedge cross section



as **a measure of correlation** between A and B in AdS side.

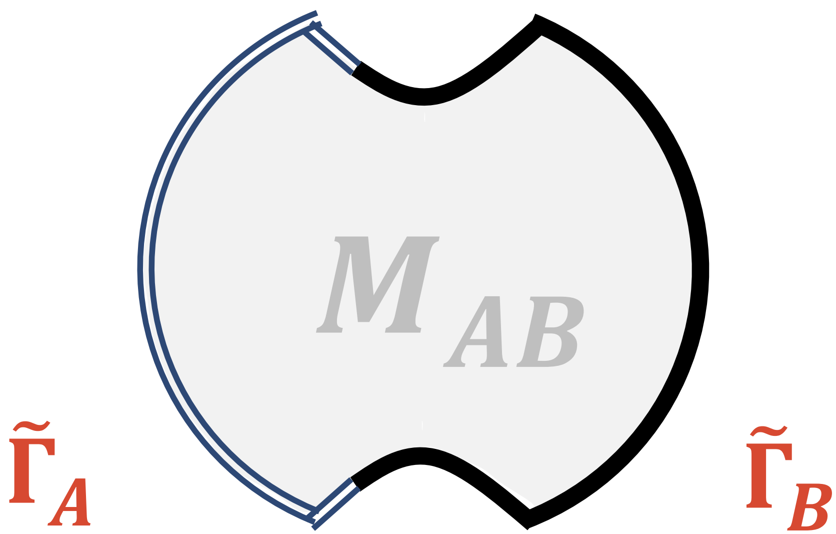
Definition of entanglement wedge cross section

Step 1. **Entanglement wedge M_{AB}** dual to ρ_{AB}
(and forget all the other regions)



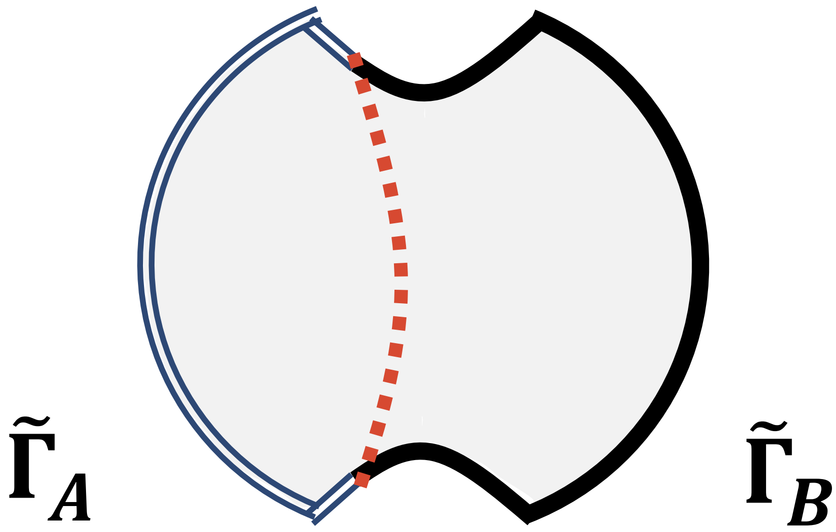
Definition of entanglement wedge cross section

Step 2. Regard ∂M_{AB} **as a new boundary** and divide it into two subsets $\tilde{\Gamma}_A$ and $\tilde{\Gamma}_B$ so that $A, B \subset \tilde{\Gamma}_{A,B}$, respectively.



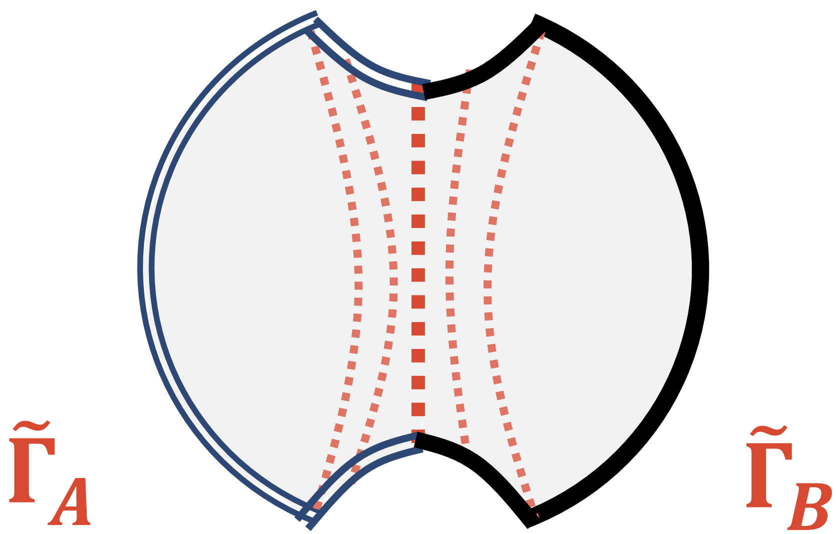
Definition of entanglement wedge cross section

Step 3. Find the **RT-surface** of $\tilde{\Gamma}_A$ (or equivalently $\tilde{\Gamma}_B$)



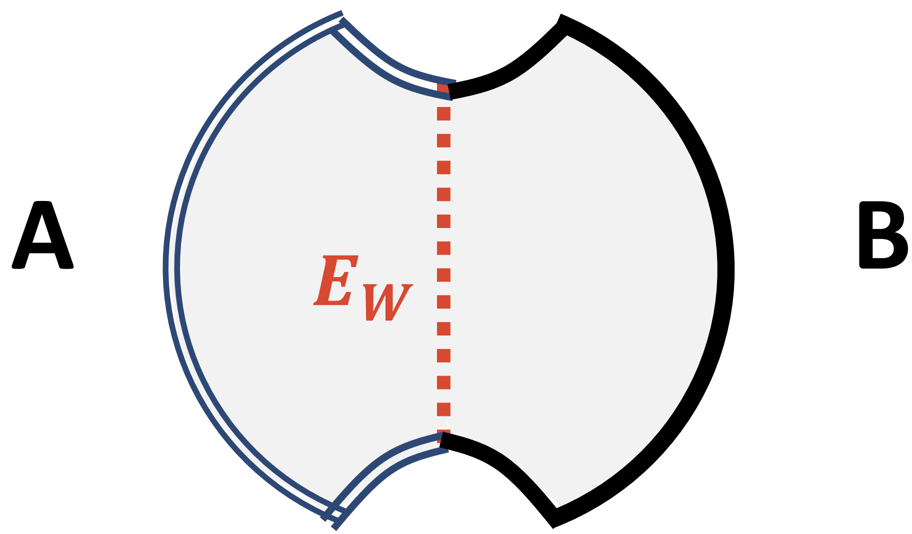
Definition of entanglement wedge cross section

Step 4. **Minimize** the area of the RT-surfaces over **all possible divisions** of ∂M_{AB}



Definition of entanglement wedge cross section

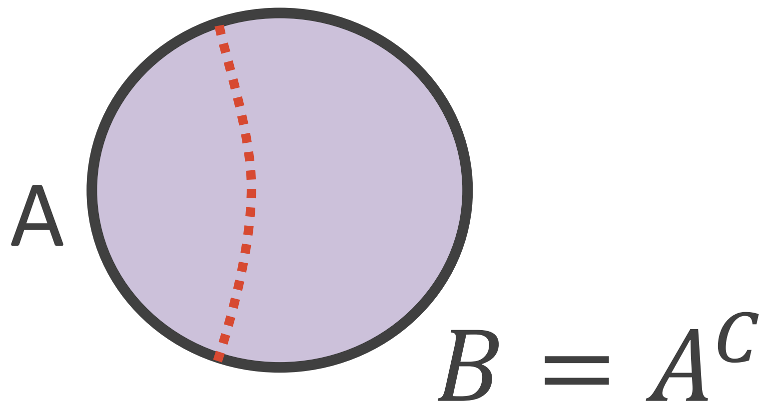
Step 5. This **minimal area** (divided by $4G_N$) is defined as the **entanglement wedge cross section** of ρ_{AB}.



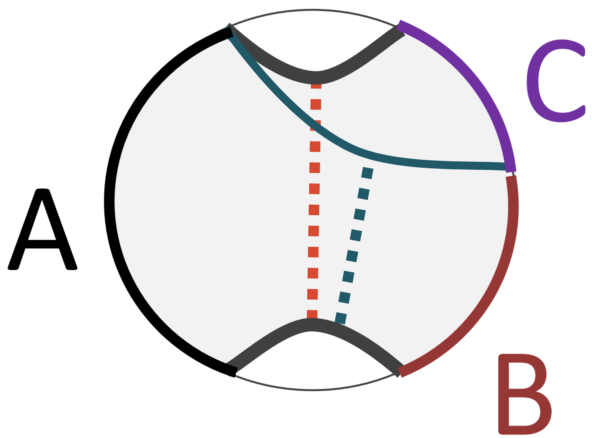
$$E_W(A:B) := \min_{\widetilde{\Gamma}_A} \frac{\mathrm{Area}(\Sigma_{AB}^{min})}{4G_N}$$

We prove the **geometric properties** of E_W.

e.g. ▪ $E_W(A\!:\!B) = S_A$ for pure states.



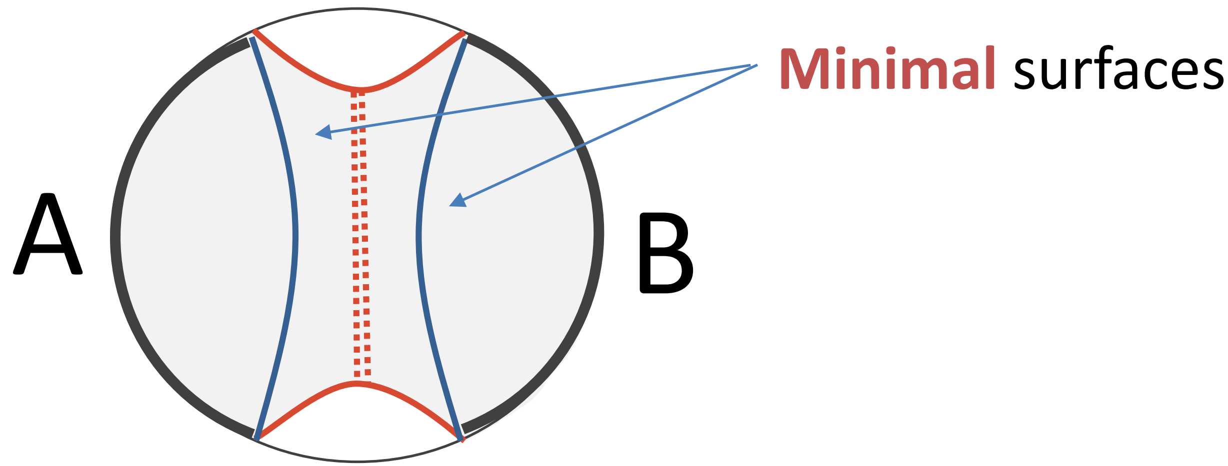
▪ $E_W(A\!:\!BC) \geq E_W(A\!:\!B)$.



・Larger than a half of mutual information.

$$E_W(A:B) \geq \frac{I(A:B)}{2}.$$

Holographic Proof



$$2E_W(\rho_{AB}) + S_{AB} \geq S_A + S_B.$$

$$\Leftrightarrow \quad E_W(A:B) \geq \frac{S_A + S_B - S_{AB}}{2} = \frac{I(A:B)}{2}. \quad \blacksquare$$

Properties

E_W

(1) $E(A\!:\!B) = 0$ iff $\rho_{AB} = \rho_A \otimes \rho_B$. ☑

(2) $E(A\!:\!B) = S_A$ for pure states $|\Psi\rangle_{AB}$. ☑

(3) $E(A\!:\!B) \leq \min[S_A, S_B]$. ☑

(4) Never increasing upon discarding ancilla: ☑

$$E(A\!:\!BC) \geq E(A\!:\!B).$$

(5) Larger than a half of mutual information: ☑

$$E(A\!:\!B) \geq I(A\!:\!B)/2.$$

(6) $E(A\!:\!BC) \geq (I(A\!:\!B) + I(A\!:\!C))/2$. ☑

(7) Additivity (with a condition): ☑

$$E(\rho_{AB} \otimes \sigma_{\tilde{A}\tilde{B}}) = E(\rho_{AB}) + E(\sigma_{\tilde{A}\tilde{B}})$$

Properties

	E_W	E_P
(1) $E(A{:}B) = 0$ iff $\rho_{AB} = \rho_A \otimes \rho_B$.	☑	☑
(2) $E(A{:}B) = S_A$ for pure states $\|\Psi\rangle_{AB}$.	☑	☑
(3) $E(A{:}B) \leq \min[S_A, S_B]$.	☑	☑
(4) Never increasing upon discarding ancilla: $E(A{:}BC) \geq E(A{:}B)$.	☑	☑
(5) Larger than a half of mutual information: $E(A{:}B) \geq I(A{:}B)/2$.	☑	☑
(6) $E(A{:}BC) \geq (I(A{:}B) + I(A{:}C))/2$.	☑	☑
(7) Additivity (with a condition): $E(\rho_{AB} \otimes \sigma_{\tilde{A}\tilde{B}}) = E(\rho_{AB}) + E(\sigma_{\tilde{A}\tilde{B}})$	☑	☑

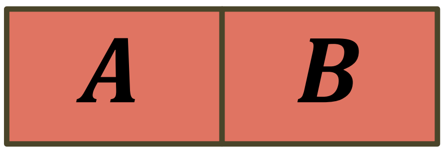
Entanglement of Purification

Entanglement of Purification

Entanglement of purification

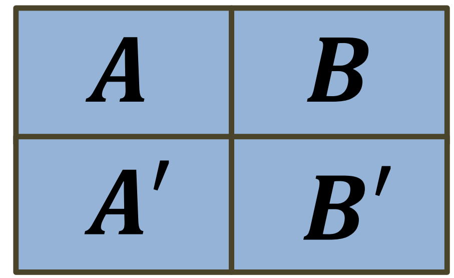
[*Terhal-Horodecki-Leung-DiVincenzo '02*]

ρ_{AB}



"Purify" →

$|\Psi\rangle_{AA'BB'}$



s.t. $\mathrm{Tr}_{A'B'}[|\Psi\rangle\langle\Psi|_{AA'BB'}] = \rho_{AB}$.

Quantum entanglement between AA' and BB' :

$$S_{AA'} = -\mathrm{Tr}\rho_{AA'}\log\rho_{AA'}$$

$$(\rho_{AA'} = \mathrm{Tr}_{BB'}[|\Psi\rangle\langle\Psi|_{AA'BB'}])$$

Definition

$$E_P\,(A:B) := \min_{\text{all purifications } |\Psi\rangle_{AA'BB'}} S_{AA'}$$

Example

$$\rho_{AB} = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow|_{AB} + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|_{AB})$$

Make a purification

$$\left|\boldsymbol{\Psi}^{(1)}\right\rangle_{AA'BB'} = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\uparrow\rangle_{AA'BB'} + |\downarrow\downarrow\downarrow\downarrow\rangle_{AA'BB'})$$

EE

$$S_{AA'}^{(1)} = \log 2$$

An upper bound

$$E_P(A{:}B) = \min S_{AA'} \leq S_{AA'}^{(1)} = \log 2.$$

To find E_P, we have to consider ***all* purifications**.

- E_P is a generalization of **entanglement entropy** for mixed states and measures both **quantum correlation** and **classical correlation** between A and B.

- Numerical results:
  - a pair of ½-spin system [*Terhal-Horodecki-Leung-DiVincenzo '02*]
  - Ising model with MPS ansatz [*Nguyen-Devakul-Halbasch-Zaletel-Swingle '17*]
  - Free scalar fields with Gaussian ansatz [*Bhattacharyya-Takayanagi-KU '18*]

- The regularized EoP $E_{LOq}(\rho_{AB}) = \lim_{n\to\infty} E_P(\rho_{AB}^{\otimes n})/n$ is **the minimal number of EPR pairs** needed to create ρ_{AB} by local operations and almost zero communication (LOq).

[*Terhal-Horodecki-Leung-DiVincenzo '02*]

Our conjecture

We propose a holographic dual relationship:

[*Takayanagi-KU '17*]

[*Nguyen-Devakul-Halbasch-Zaletel-Swingle '17*]

$$E_W(A:B) = E_P(A:B)$$

(AdS) (CFT)

- E_W**: Entanglement wedge cross section.**

- E_P **: Entanglement of purification.**

[*Terhal-Horodecki-Leung-DiVincenzo '02*]

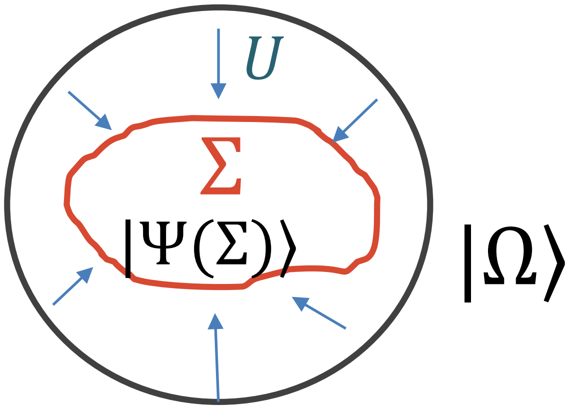
Properties

	E_W	E_P
(1) $E(A:B) = 0$ iff $\rho_{AB} = \rho_A \otimes \rho_B$.	☑	☑
(2) $E(A:B) = S_A$ for pure states $\|\Psi\rangle_{AB}$.	☑	☑
(3) $E(A:B) \leq \min[S_A, S_B]$.	☑	☑
(4) Never increasing upon discarding ancilla: $E(A:BC) \geq E(A:B)$.	☑	☑
(5) Larger than a half of mutual information: $E(A:B) \geq I(A:B)/2$.	☑	☑
(6) $E(A:BC) \geq (I(A:B) + I(A:C))/2$.	☑	☑
(7) Additivity (with a condition): $E(\rho_{AB} \otimes \sigma_{\tilde{A}\tilde{B}}) = E(\rho_{AB}) + E(\sigma_{\tilde{A}\tilde{B}})$	☑	☑

A heuristic derivation of $E_W = E_P$

The surface/state correspondence [*Miyaji-Takayanagi '15*]

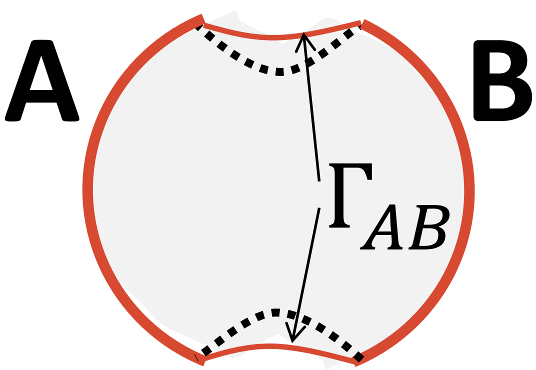
in tensor network description of AdS/CFT. [*Swingle '09*]



On any closed convex surfaces Σ, we can define a state $|\Psi(\Sigma)\rangle$.

$$|\Psi(\Sigma)\rangle \equiv U(\Sigma)|\Omega\rangle_{total},$$

Unitary operator: $U^\dagger U = I$



$$\Sigma \equiv A \cup B \cup \Gamma_{AB},$$

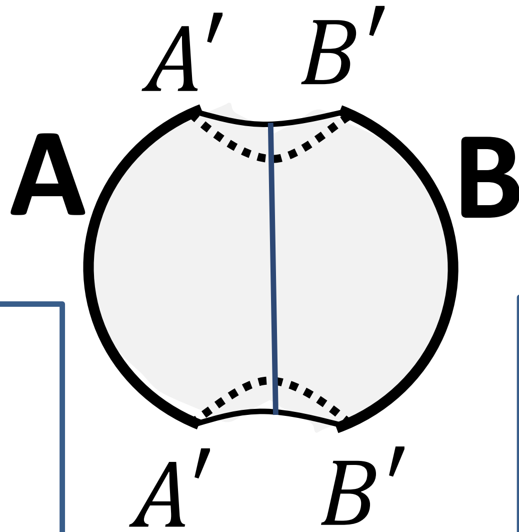
$$\mathrm{Tr}_{\Gamma_{AB}}[|\Psi(\Sigma)\rangle\langle\Psi(\Sigma)|]$$

$$= \mathrm{Tr}_{(AB)^c}[|\Omega\rangle\langle\Omega)|] = \rho_{AB}.$$

$\therefore$ $|\Psi(\Sigma)\rangle_{AB\Gamma_{AB}}$ is a purification of ρ_{AB}

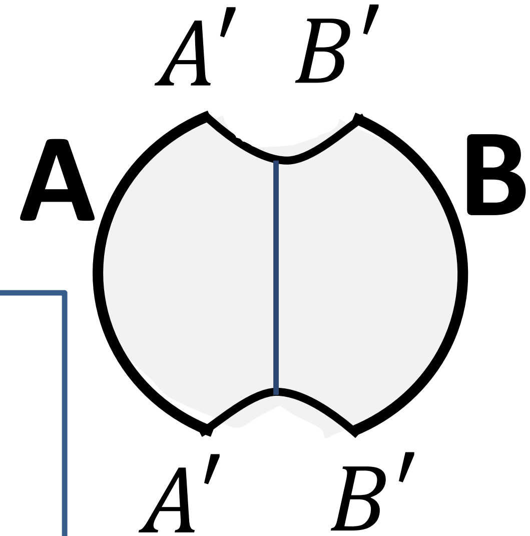
Once we assume “purified geometry”, then

$E_P(A{:}B) :=$



Devide Γ_{AB}
&
Calculate $S_{AA'}$

Minimize it
over all
$|\Psi\rangle_{\Gamma'}$ & A'



$= E_W(A{:}B).$

$\therefore$ One can derive $\boldsymbol{E_P = E_W}$.

Remark: We also assumed that an optimal purification has a classical gravity dual.

Summary and Future works

We proposed a dual relation between entanglement wedge cross section and entanglement of purification:

$$E_W = E_P,$$

Based on

(1) coincidence of the properties

(2) a heuristic derivation by tensor network.

- Purifications in QFTs/holography
- Calculate E_P in holographic CFTs and compare to E_W
- Holographic LOCC/LOq and EPR pairs
- Holographic mixed/multipartite entanglement

A generalization of this relation is given in [Bao-Halpern '17].

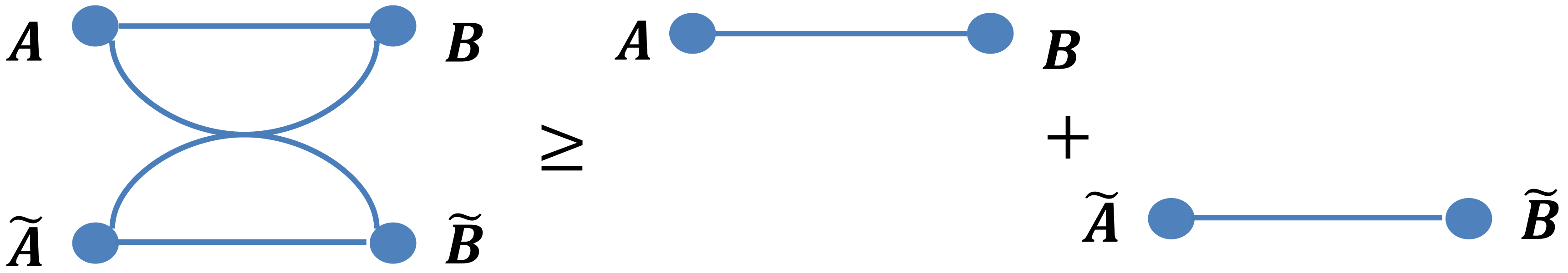
Thank you for your attention.

Appendices

Strong superadditivity

Definition:

$$E_{\#}\big(\rho_{(A\tilde{A})(B\tilde{B})}\big) \geq E_{\#}(\rho_{AB}) + E_{\#}(\rho_{\tilde{A}\tilde{B}}).$$



It is a feature of measures of **quantum entanglement**.

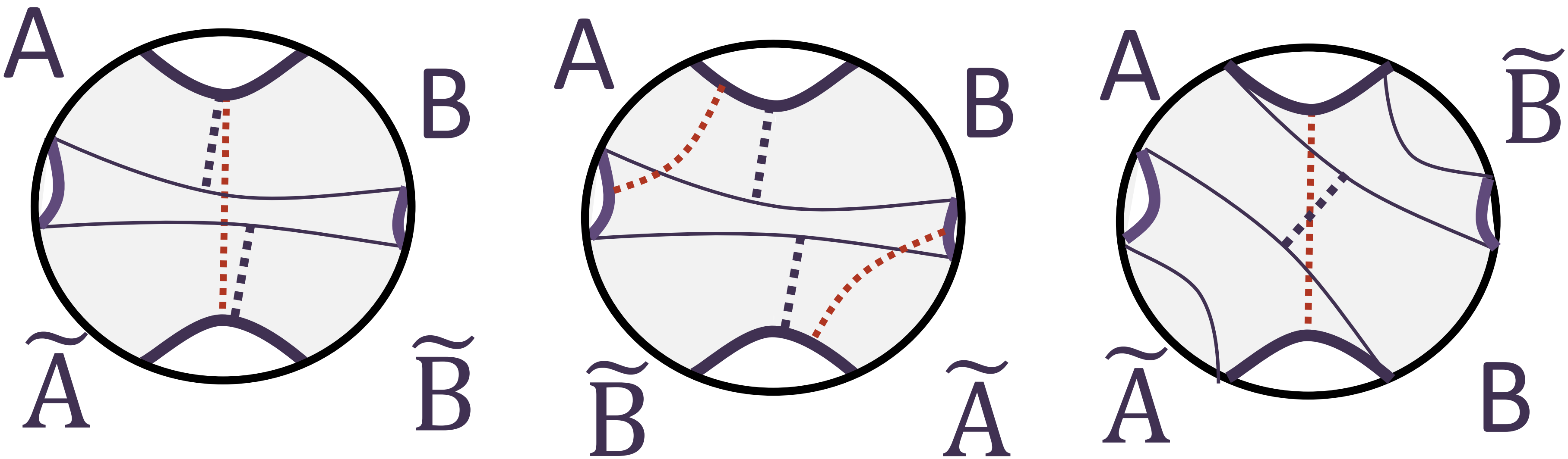
cf. distillable entanglement, squashed entanglement

E_W satisfies the strong superadditivity.

[*Takayanagi-KU '17*]

$$E_W\left(\rho_{(A\tilde{A})(B\tilde{B})}\right) \geq E_W(\rho_{AB}) + E_W(\rho_{\tilde{A}\tilde{B}}).$$

Examples



- E_P does ***not*** always satisfies the strong superadditivity.

Is it a contradiction?

- ***No!*** This give us **a constraint** (or a necessary condition) on the class of **holographic states**:

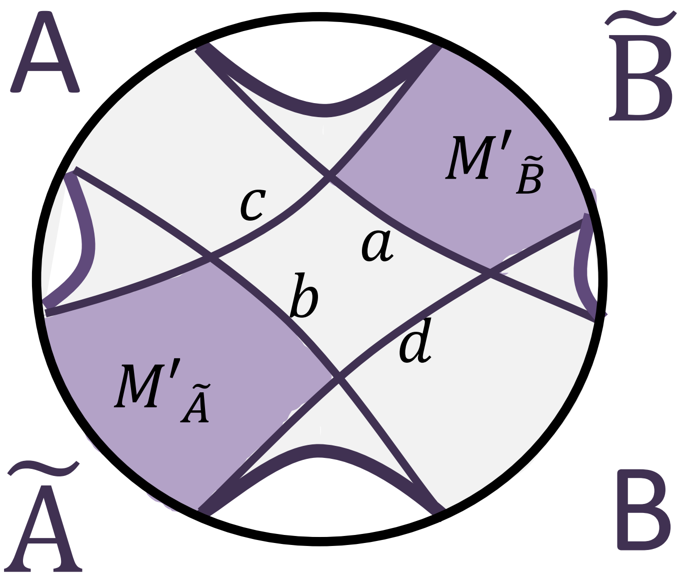
Holographic states is the ones for which

E_P **satisfies the strong superadditivity**.

- Cf. Other similar constraints:
  - Monogamy of mutual information [*Hayden-Headrick-Maloney '11*]
  - Entropy cones [*Bao-Nezami-Ooguri-Stoica-Sully-Walter '15*]

No crossing bridge

If M_{AB} is connected, then $M_{\tilde{A}\tilde{B}}$ is disconnected (and vice versa).



Proof:

If M_{AB} is connected,
$a + b < c + d$ should hold.
Then, for $M_{\tilde{A}\tilde{B}}$ at least
the disconnected wedge
$M'_{\tilde{A}} \cup M'_{\tilde{B}}$ is preferred. ■

Origin of the monogamy of M.I.

- "Squashed entanglement":

$$E_{sq}(\rho_{AB}) \coloneqq \frac{1}{2} \min_{\mathrm{Tr}_C \rho_{ABC} = \rho_{AB}} I(A{:}B|C)$$
$$= \frac{1}{2} \min_{\mathrm{Tr}_C \rho_{ABC} = \rho_{AB}} [S_{AC} + S_{BC} - S_{ABC} - S_C].$$

- E_{sq} is the most promising measure of entanglement for mixed states, and known to be **always monogamous**.
- In our picture $\boldsymbol{E_{sq} = \frac{I}{2}}$ in holography.
- This is discussed in [Hayden-Headrick-Maloney '11].

"Regularized" E_P

"The minimal number of EPR pairs
which is **needed to produce** $\boldsymbol{\rho_{AB}}$ using only
local operations and **vanishing communications**."

$$E_{LOq}(\rho_{AB}) :=$$
$$\inf_r \left\{ r \middle| \lim_{n\to\infty} \left[\inf_{\Lambda\in\mathrm{LOq}} D_{tr}\left(\rho_{AB}^{\otimes n}, \Lambda\left(\Phi_{2^{rn}}^{+}\right)\right)\right] = 0 \right\}.$$

Thm. $E_{LOq}(\rho_{AB}) = \lim_{n\to\infty} \frac{E_P\left(\rho_{AB}^{\otimes n}\right)}{n}$.

[Terhal-Horodecki-Leung-DiVincenzo '02]

∴ When it's additive, $E_{LOq} = E_P$.

Time-dependent case

- Replacing the "minimal surface $\Sigma_{AB}^{\min}$"
  → "**extremal surface $\boldsymbol{\Sigma_{AB}^{ext}}$** "
  following HRT formula.

[Hubeny-Rangamani-Takayanagi '07]

- All properties are proven by using of the **"maximin surfaces" prescription** discussed by A.Wall in [Class. Quant. Grav. 31 (2014) no.22, 225007]

Relative entropy of entanglement

$$E_R(\rho_{AB}) \coloneqq \min_{\sigma_{AB} \in \text{Seprable states}} R(\rho_{AB}||\sigma_{AB}).$$

where $R(\rho_{AB}||\sigma_{AB})$ is relative entropy.

- However… It must be less than $I(A{:}B)$:

$$E_R(\rho_{AB}) \leq I(A{:}B).$$