“Holography, Quantum Entanglement and Higher Spin Gravity II” 2018/03/15 @YITP, Kyoto

Optimization of Path-Integrals in CFTs and Complexity

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Next Talk!

- Phys.Rev.Lett. 119 (2017) no.7, 071602
- JHEP 1711 (2017) 097

w/ Pawel Caputa, Nilay Kundu, Masamichi Miyaji and Tadashi Takayanagi (YITP)

AdS/CFT & Quantum Information

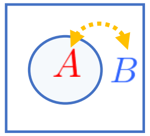
Modern Perspective of AdS/CFT ⟶ A "Geometrization" of Quantum States

Quantum Entanglement ⟷ Emergent Geometry "Gravity" "Spacetime"

AdS/CFT
Tensor Network

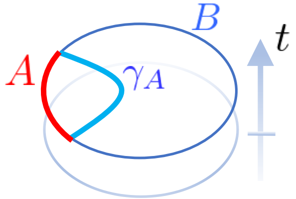
Ex: Entanglement Entropy (EE) ⟷ Min. Area of codim.-2 Surface

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A$$

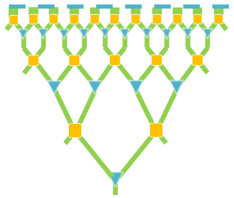


$$\min_{\gamma_A}\left[\frac{Area(\gamma_A)}{4G_N}\right]$$

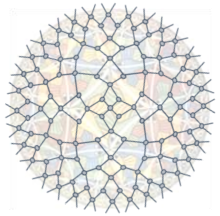
[Ryu-Takayanagi 06]



MERA network
(for CFT vacuum)
[Vidal 05,06]



⟶ Hyperbolic geometry (a time slice of AdS)
[Swingle 09]



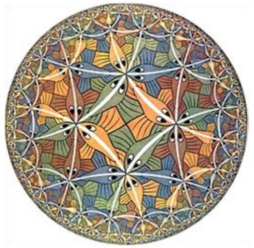
⟶ Active research for QFT and dual gravity using tools and concepts developed in QI & CM!!

Motivation & Proposal (1)

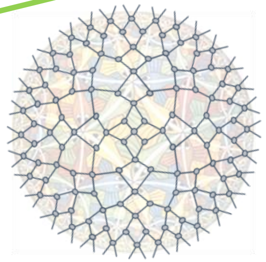
AdS/CFT → 20 years old! but still mysterious…

Direct or Systematic Way to Get Information about Dual Geometries?

Quantum State ←AdS/CFT→ Geometry (a time slice of AdS)



- MERA (Tensor Network) —AdS/TN→ Geometry
- Euclidean Path-Integral → MERA (Tensor Network)
- Euclidean Path-Integral → Geometry: Direct way ?



A toy model of AdS/CFT

Our Proposal [Caputa-Kundu-Miyaji-Takayanagi-KW'17]

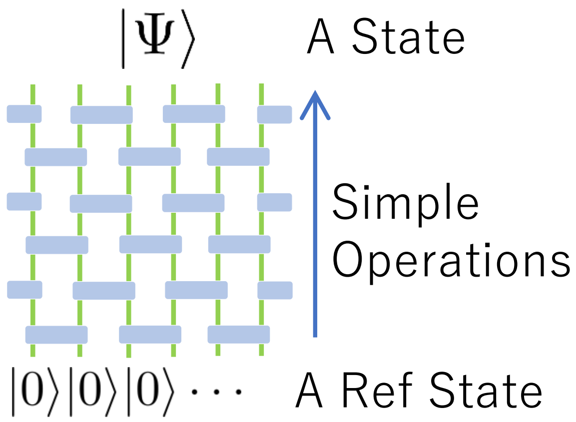
"Optimization" of Euclidean Path-Integral for Wave Functional in CFTs

In 2d CFT → "Minimization" of the Liouville Action of the Back-ground Metric

Motivation & Proposal (2)

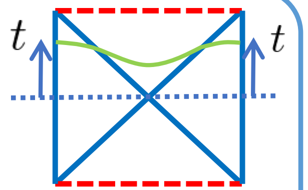
CFT Analogue of "Complexity of Quantum State" ?

$C_\Psi =$ min[#(Operations)]



- Holographic Complexity → A new probe for dual spacetime beyond HEE

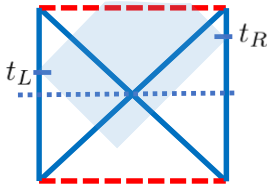
Complexity ⟺ "Volume"



[Susskind, +Stanford 14]

(Max. codim-1 surface)

or "Action"



[Brown-Roberts-Susskind-Swingle-Zhao 15]

(Action on WDW patch("codim-0"))

- CFT analogue? → No Definition of the complexity in CFTs so far···

Only a few attempts··· (including ours)

[Susskind, + collaborators···] [Chapman-Heller-Marrochio-Pastawski 17] [Jefferson-Myers 17] [Yang, + collaborators 17]

Motivation & Proposal (2)

CFT Analogue of "Complexity of Quantum State" ?

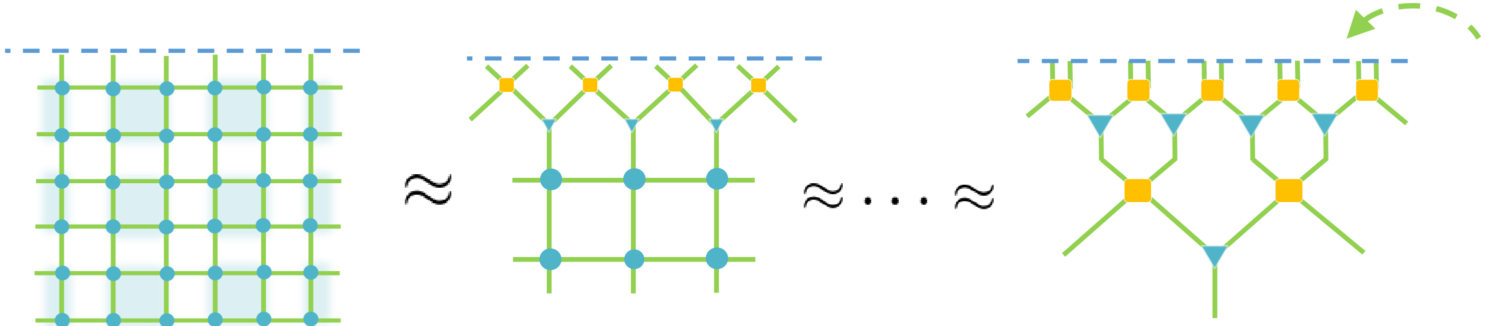
Our Proposal [Caputa-Kundu-Miyaji-Takayanagi-KW'17]

Complexity of States in CFTs ⟺ "Optimized Action"

In 2d CFT, "Liouvlille Action"

Hint Tensor Network Renormalization (TNR) Procedure → "Optimization"

$\Psi[\tilde{\varphi}(x)] \approx \cdots \approx \cdots \approx \cdots \approx$



(Refining & Coarse-graining tensors)

"Optimized" TN

Ex: MERA (CFT vacuum)

→ Complexity $\sim$ min#[tensors]

Optimization of Euclidean Path-Integrals & Complexity in CFTs

[Caputa-Kundu-Miyaji-Takayanagi-KW'17]

More detail···

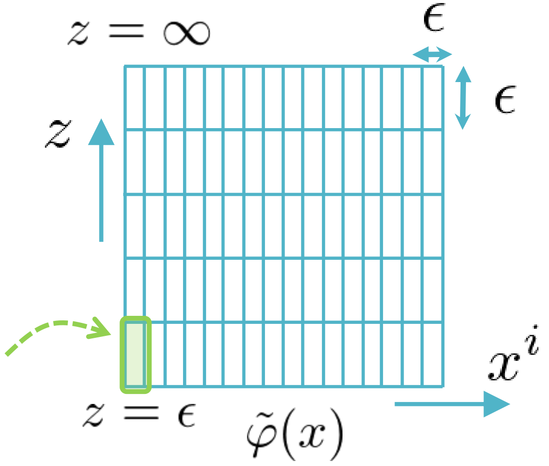
Basic Rules for Our "Optimization" Procedure

(CFT Analogue of TNR Procedure)

- Discretization of Euclidean path-integral
  (Regularization)

$$\Psi[\tilde{\varphi}(x)] = \int \left(\prod_{x} \prod_{\epsilon<z<\infty} D\varphi(z,x) \right) e^{-S_{CFT}(\varphi)} \cdot \prod_{x} \delta(\varphi(\epsilon,x) - \tilde{\varphi}(x))$$

→ Metric (one cell = unit area) $ds^2 = \epsilon^{-2} \cdot (dz^2 + dx^i dx^i)$

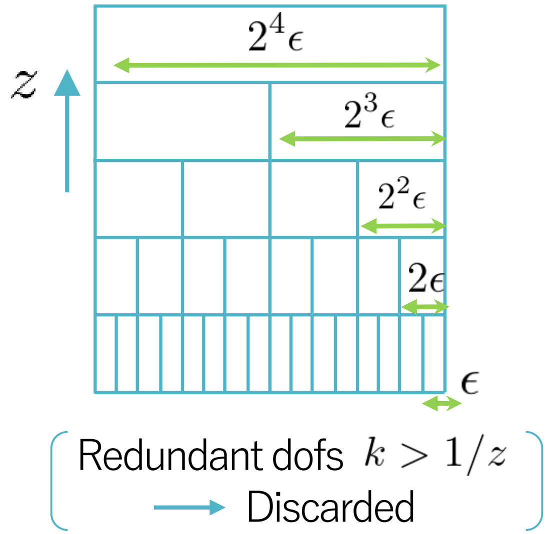


- "Optimization" of the path-integral

→ Changing the geometry of the lattice regularization

→ Modifying the back-ground metric for the path-integral with fixing the UV bdy condition

$$g_{zz}(z=\epsilon,x) = \epsilon^{-2}, \quad g_{ij}(z=\epsilon,x) = \delta_{ij} \cdot \epsilon^{-2} \qquad g_{iz}(z=,x) = 0$$



Basic Rules for Our "Optimization" Procedure

(CFT Analogue of TNR Procedure)

- After optimization, reproduce the correct wave functional up to a normalization

(Ansatz)

$$\longrightarrow \quad \Psi_{g_{ab}}[\tilde{\varphi}(x)] = N_{(g,\delta)} \cdot \Psi_{\delta_{ab}}[\tilde{\varphi}(x)]$$

Estimate redundant dofs → Minimize this!!

- "Optimization" of the path-integral

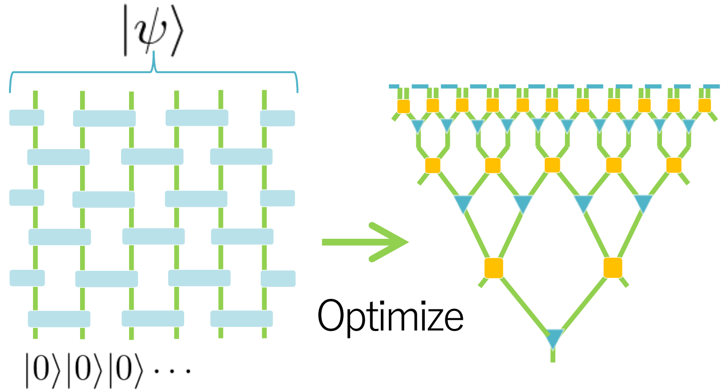
→ Minimizing "# of lattice points"

(= "# of tensors in a TN")

→ Our Conjecture for Complexity in CFT

$$N_{opt} \approx \exp(\text{Complexity})$$

→ Min[#(Operations)]



For CFT vacuum, MERA network

2d CFT case

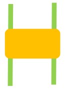
- In 2d CFT, we can diagonalize the general back-ground metric

Weyl Scaling $$ds^2 = e^{2\phi(z,x)}(dz^2 + dx^2) \quad \text{with} \quad e^{2\phi(z=\epsilon,x)} = \frac{1}{\epsilon^2}$$

- The change of the measure is characterized by the Liouville action

$$[D\varphi]_{g_{ab}=e^{2\phi}\delta_{ab}} = e^{S_L[\phi]-S_L[0]} \cdot [D\varphi]_{g_{ab}=\delta_{ab}}$$

$$S_L[\phi] = \frac{c}{24\pi}\int_{-\infty}^{\infty}dx\int_{\epsilon}^{\infty}dz\left[(\partial_x\phi)^2+(\partial_z\phi)^2+\mu e^{2\phi}+R_0\phi\right]$$

UV regularization ---> # of unitaries 

Conformal Anomaly ---> # of isometries 

[Czech 17]

$$\longrightarrow \quad \Psi_{g_{ab}=e^{2\phi}\delta_{ab}}[\tilde{\varphi}(x)] = e^{S_L[\phi]-S_L[0]} \cdot \Psi_{g_{ab}=\delta_{ab}}[\tilde{\varphi}(x)]$$

Minimize this !!

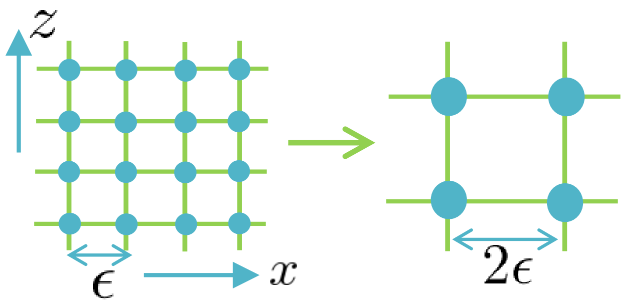
$$\min_{\phi}[S_L(\phi) - S_L(0)] \approx (\text{Complexity})$$

Naïve Estimation: Liouville Action From TNR

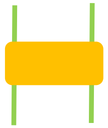
[Czech 17] [Caputa-Kundu-Miyaji-Takayanagi-KW17]

- Suppose each tensor has unit area in the original square lattice

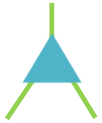
$$\frac{dxdz}{\epsilon^2} = e^{2\phi(z,x)} dxdz \xrightarrow{\text{Coarse-graining}} \frac{dxdz}{(2\epsilon)^2}$$



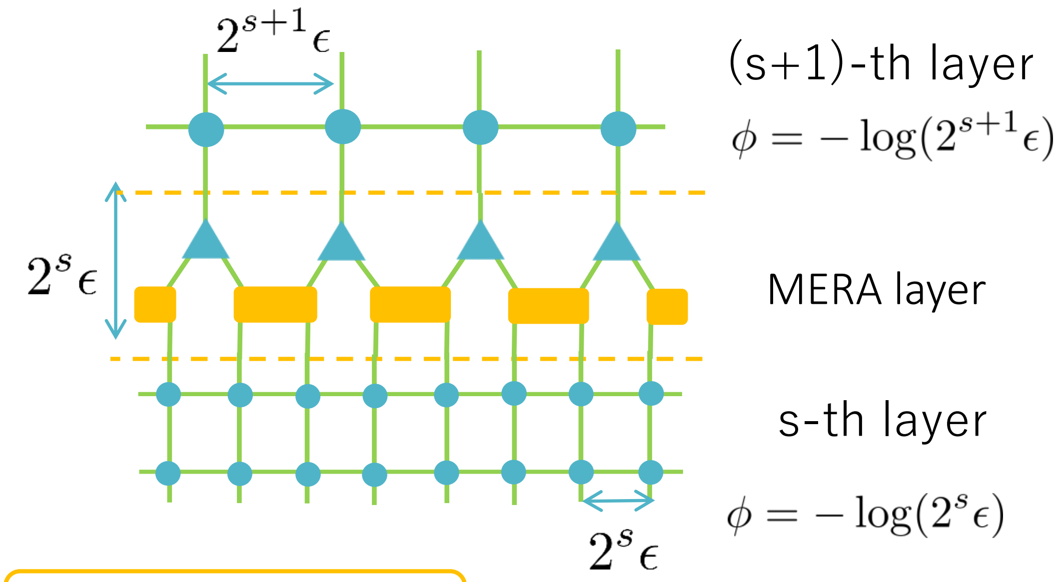
- For the s-th layer of MERA network, per unit cell,



$$\frac{dxdz}{(2^s\epsilon)^2} \longrightarrow e^{2\phi}$$



$$\frac{dxdz}{(2^{s-1}\epsilon)(2^s\epsilon)} \longrightarrow \partial\phi$$

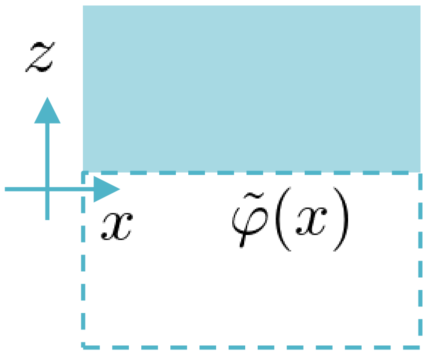


- Total # of tensors in the optimal network

$$\longrightarrow S_L[\phi] = \frac{c}{24\pi}\int_{-\infty}^{\infty} dx \int_{\epsilon}^{\infty} dz[(\partial\phi)^2 + \mu e^{2\phi}] \sim \boxed{\text{Complexity !!}}$$

Vacuum on Plane (Poincare AdS_3)

$$w = z + ix$$



EOM : $\partial_w \partial_{\bar{w}} \phi = \frac{\mu}{4} e^{2\phi} \longrightarrow e^{2\phi} = \frac{4}{\mu} \cdot \frac{A'(w)B'(\bar{w})}{(1 - A(w)B(\bar{w}))^2}$

Especially, $A(w) = w, \quad B(\bar{w}) = -1/\bar{w}$

a time slice of Poincare AdS_3

$$\longrightarrow e^{2\phi} = \frac{4}{\mu} \cdot \frac{1}{(w+\bar{w})^2} = \frac{1}{\mu} \cdot \frac{1}{z^2} \longrightarrow ds^2 = \frac{dx^2 + dz^2}{z^2} \quad : H_2$$

This solution clearly minimizes the Liouville action : $e^{2\phi(z=\infty, x)} = 0 \qquad L = \int dx$

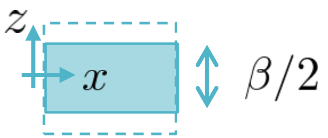
$$S_L = \frac{c}{24\pi} \int dx dz \left[(\partial_x \phi)^2 + (\partial_z \phi + \sqrt{\mu} e^{\phi})^2 \right] - \frac{c}{12\pi} \int dx [\sqrt{\mu} e^{\phi}]_{z=\epsilon}^{z=\infty} \geq \frac{c\sqrt{\mu} L}{12\pi\epsilon}$$

Volume divergence!!

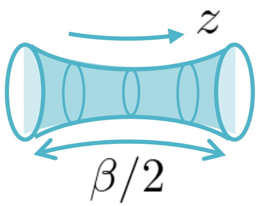
Other Examples

Work well for simples examples!!

- Finite T or TFD state

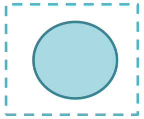


→ Wormhole



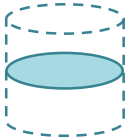
(A time slice of BTZ BH)

- Vacuum on disk



$|w| < 1$

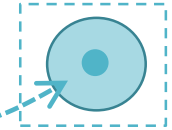
→ Hyperbolic disk H_2



(A slice of global AdS3)

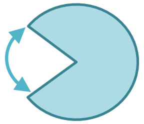
- Primary State

$O(w, \bar{w}) \propto e^{-2h\phi}$



→ Conical Singular Geometry (Match to AdS_3/CFT_2 for $h \ll c$)

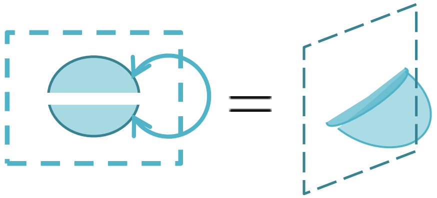
Deficit angle



Similarly ,

- $\rho_A = \mathrm{Tr}_{A^c} |\Psi\rangle\langle\Psi|$

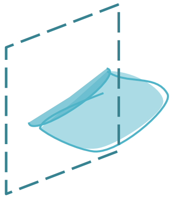
→ Entanglement Wedge



- $S_A = -\mathrm{Tr} \rho_A \log \rho_A$

$n \to 1$

→ Holographic EE $S_A \propto$ Min[Area]



Examples : Our "Complexity" in CFTs

2d CFT

$$\min_{\phi}[S_L(\phi) - S_L(0)] \approx (\text{Complexity})$$

- Vacuum on plane (Poincare AdS3) $S_L = \frac{cL}{12\pi\epsilon}$ ← Volume divergence !! $L = \int dx$
- Vacuum on disk (Global AdS3) $S_L = \frac{c}{6} \cdot \left(\frac{1}{\epsilon} - 1\right)$
- TFD (BTZ BH) $S_L = \frac{c}{3} \cdot \left(\frac{1}{\epsilon} - \frac{\pi^2}{2\beta}\right)$

With a naïve extension…

3d CFT

- (Global AdS4) $N = \frac{1}{8\pi G_N}$

$$S_L = 4\pi N \cdot \left(\frac{1}{\epsilon^2} + \frac{1}{2} + \log\left(\frac{2}{\epsilon}\right)\right)$$

4d CFT

- (Global AdS5) $N = \frac{3}{16\pi G_N}$

$$S_L = 2\pi^2 N \cdot \left(\frac{2}{3\epsilon^3} + \frac{1}{\epsilon} - \frac{5}{12}\right)$$

→
- [The volume law leading divergence & The divergence structure] agree with the holographic complexities !!

[Chapman-Marrochio-Myers 16]
[Lehner-Poisson-Myers-Sorkin 16]
[Carmi-Myers-Rath 16] [Reynolds-Ross16]

- The relative coefficients are different in general…

Summary [Caputa-Kundu-Miyaji-Takayanagi-KW'17]

- A proposal to define complexity of states in CFTs
  using "Optimization" of Euclidean Path-Integrals (→ Metric on a time slice of dual spacetime)
  → Some checks for some states in 2d CFTs, ρ_A (→ EW), S_A (→ HEE)
- "Complexity" = "Optimal Action"
  = "the Liouville Action" (2d) → $\min_{\phi}[S_L(\phi) - S_L(0)] \approx (\text{Complexity})$
- (Qualitative) Matching to the Holographic Conjectures!!

Future Works

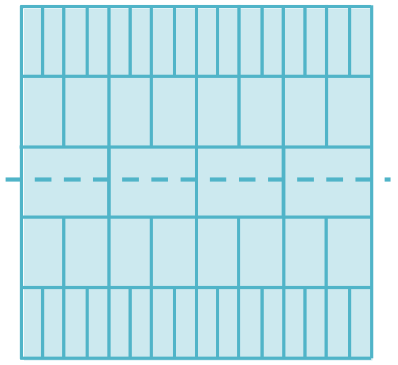
- Higher dims, Time depend., Relation to Holographic Complexity (skipped)
- Non-CFT case → Pawel's Talk !
- Phase transition of the surfaces?
- Monotonicity under RG flows? ⋯ etc

Thanks!!

Back-up Slides

Finite T or TFD state (BTZ BH)

CFT1 $\tilde{\varphi}_1(x)$



$z = -\frac{\beta}{4}$

$z = 0$

$z = +\frac{\beta}{4}$

CFT2 $\tilde{\varphi}_2(x)$

$$|\Psi_{TFD}\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2}E_n} |E_n\rangle_1 |E_n\rangle_2$$: TFD state

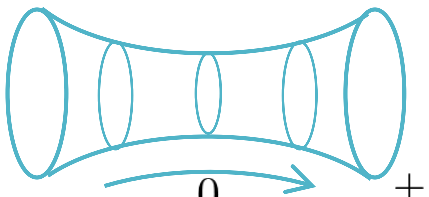
$$\Psi[\tilde{\varphi}_1(x), \tilde{\varphi}_2(x)] = \int \left(\prod_x \prod_{-\frac{\beta}{4}<z<\frac{\beta}{4}} D\varphi(z,x) \right) e^{-S_{CFT}(\varphi)}$$
$$\times \prod_{-\infty<x<\infty} \delta(\varphi(z_1,x) - \tilde{\varphi}_1(x))\, \delta(\varphi(z_2,x) - \tilde{\varphi}_2(x))$$

$$A(w) = e^{\frac{2\pi i w}{\beta}}, \quad B(\bar{w}) = -e^{\frac{2\pi i \bar{w}}{\beta}}$$

$$\longrightarrow \quad e^{2\phi} = \frac{16\pi^2}{\mu\beta^2} \frac{e^{\frac{2\pi i}{\beta}(w+\bar{w})}}{\left(1 + e^{\frac{2\pi i}{\beta}(w+\bar{w})}\right)^2} = \frac{4\pi^2}{\mu\beta^2} \sec^2\left(\frac{2\pi z}{\beta}\right)$$

a time slice of BTZ BH

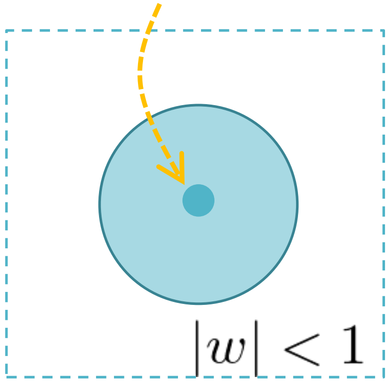
Wormhole



$-\frac{\beta}{4}$ $\quad 0 \quad$ z $\quad +\frac{\beta}{4}$

Primary op Excitation (Conical Singularity in AdS_3)

$$O(w,\bar{w}) \propto e^{-2h\phi} \longrightarrow \Psi_{g_{ab}=e^{2\phi}\delta_{ab}} \simeq e^{S_L[\phi]-S[0]} \cdot e^{-2h\phi(0)} \cdot \Psi_{g_{ab}=\delta_{ab}}$$



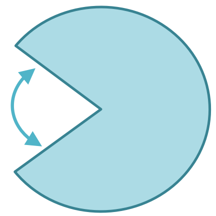
$$\partial_w \partial_{\bar{w}} \phi - \frac{\mu}{4} e^{2\phi} + \frac{\pi}{2}(1-a)\delta^2(w) = 0$$

$$a = 1 - \frac{12h}{c}$$

$$\longrightarrow \quad e^{2\phi} = \frac{4}{\mu} \cdot \frac{a^2}{|w|^{2(1-a)}(1-|w|^{2a})^2}$$

$$\zeta = w^a$$

Deficit angle $2\pi(1-a)$



On the other hand, in AdS_3/CFT_2 ,

$$a = \sqrt{1 - \frac{24h}{c}}$$

$\longrightarrow$ Only for $h \ll c$, they are consistent (very small back-reaction)

$\longrightarrow$ Correction from Quantum Liouville theory compensates it

Note : Correction from Quantum Liouville theory

However, in the quantum Liouville theory, we find a perfect matching

Introduce a parameter γ $\longrightarrow$ $c = 1 + 3Q^2 \qquad Q = \frac{2}{\gamma} + \gamma$

$$e^{\frac{2\beta}{\gamma}\phi} \longrightarrow h = \frac{\beta(Q-\beta)}{2} \longrightarrow a \simeq 1 - \beta\gamma \simeq \sqrt{1 - \frac{24h}{c}} \qquad \begin{matrix} \gamma \ll 1 \\ \beta \ll 1 \end{matrix}$$

This agreement suggests that, in “Quantum ver.” of our optimization procedure, we optimize :

$$\Psi_{\mathrm{opt}}[\tilde{\varphi}] = \left[\int D\phi(x,z) e^{-S_L[\phi]} \left(\Psi_{g_{ab}=\delta_{ab}}[\tilde{\varphi}]\right)^{-1} \right]^{-1}$$

which reproduces our classical opt. in the semi classical approx.

Higher dim case

[Caputa-Kundu-Miyaji-Takayanagi-KW17]

For simplicity, focus on $ds^2 = e^{2\phi(z,x^i)} \cdot (dz^2 + dx^i dx^i)$

(Conformal gauge)

A candidate of the action we minimize (extremalize) :

$$S_d = N \int dz dx^{d-1} \left[e^{d\phi} + e^{(d-2)\phi} \left((\partial_z \phi)^2 + (\partial_i \phi)^2 + \frac{R_0}{(d-1)(d-2)} \right) \right]$$

$$+2N \int_{bdy} dx^{d-1} \left[\frac{e^{(d-2)\phi} \cdot K_0}{(d-1)(d-2)} + \frac{\mu_B \cdot e^{(d-2)\phi}}{d-1} \right]$$

Reproduce

- Time slices of global and Poincare AdS for CFT vacua
- HEEs and entanglement wedges for round ball subsystems

by setting $N = \frac{(d-1) R_{AdS}^{d-1}}{16 \pi G_N}$

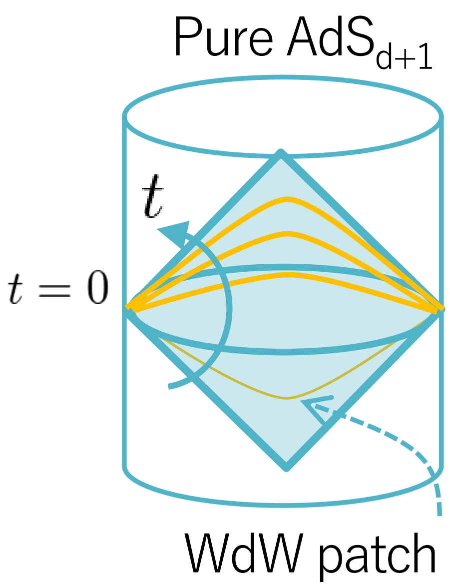
Relation with Complexity = Action Conjecture

- Consider a pure AdS_{d+1} and pick up the following patch

$$ds^2 = -dt^2 + \cos^2 t \cdot e^{2\phi(x)} h_{ij} dx^i dx^j \qquad -\pi/2 \le t \le \pi/2$$

→ It covers the WdW (Wheeler-deWitt) patch

d-dim Hyperbolic space



- Gravity Action in the WdW patch

[Complexity = Action Conjecture
[Brown-Roberts-Susskind-Swingle-Zhao 15]]

$$S_d^{WdW} = \frac{1}{16\pi G_N}\int \sqrt{|g|}(R-2\Lambda) + \frac{1}{8\pi G_N}\int_\partial \sqrt{|h|}K$$

$$= (d-2)\cdot n_d \cdot S_d(\phi) + (\text{IR Surface Term})$$

$$\left[\, n_d = \frac{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} \,\right]$$

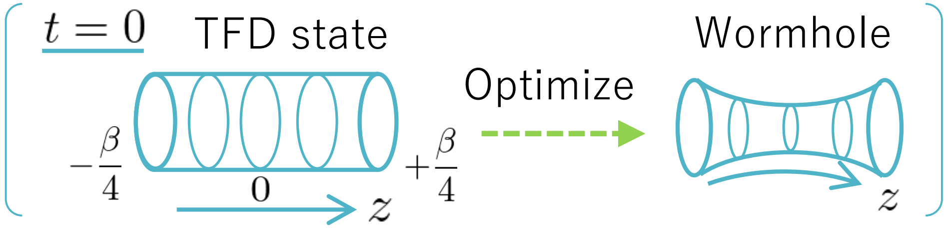
Our Complexity Function !!

For d=2 ,

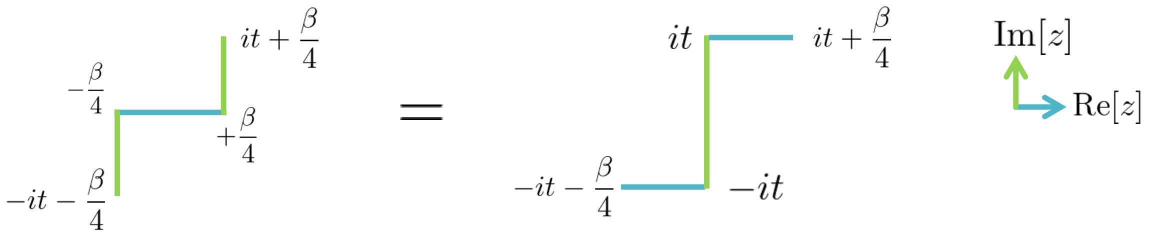
$$\lim_{d\to 2}\frac{S_d^{WdW}(\phi) - S_d^{WdW}(0)}{d-2} = \lim_{d\to 2}(S_d(\phi) - S_d(0)) = \begin{cases} 0 & \text{Poincare AdS3} \\ \dfrac{\pi}{4G_N} = \dfrac{\pi c}{6} & \text{Global AdS3} \end{cases}$$

[Agree with [Reynolds-Ross16]]

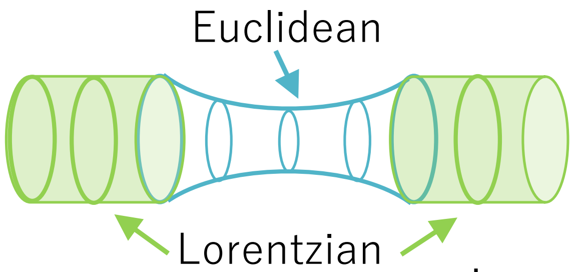
Hint for Time-dependent Case?

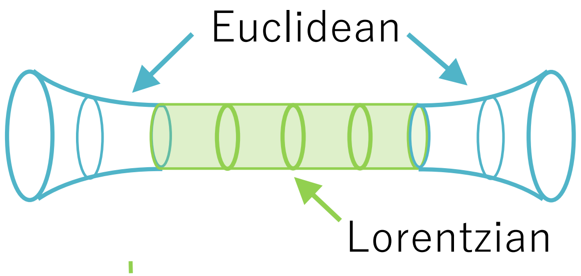


- Time Evolution → Two contour choices?



Optimize





Larger Volume ↔ Smaller Volume!

Favored!

$$\frac{dS_L}{dt} = \frac{c}{6}\left(\frac{2\pi}{\beta}\right)^2 = 2E_{CFT}$$

Match to the holographic conjectures $\frac{dC_{V,A}}{dt_L} = \frac{2M_{BH}}{\pi\hbar}$

The Lorentzian time evolution is sensitive to the initial state → No optimization may occur