Superconformal Supergravity
superconformal tensor calculus approach

Taichiro Kugo
Yukawa Institute for Theoretical Physics, Kyoto University

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京都大学基礎物理学研究所

1　はじめに

Supersymmetry Q_α は最後の「時空対称性」：
如何なる粒子 $|\boldsymbol{p}\rangle$ もその singlet 表現（不変）ではあり得ず，必ず超対称相棒 (super-partner) $Q_\alpha|\boldsymbol{p}\rangle \neq 0$ が存在する．

$$[Q_\alpha,\, \bar{Q}_{\dot\beta}] = 2(\gamma_\mu)_{\alpha\dot\beta} P^\mu$$

$$Q_\alpha|\boldsymbol{p}\rangle = 0 \quad \rightarrow \quad P^\mu|\boldsymbol{p}\rangle = 0 \quad \rightarrow \quad |\boldsymbol{p}\rangle = |0\rangle$$

この世界には最終的にはSupersymmetryが存在する．重力も存在する．
$\rightarrow$　SUSY は SUGRA (Supergravity) でなければならない．

$$P^\mu : \text{local symm.} = \text{gravity} \quad \rightarrow \quad Q_\alpha : \text{local symm.} = \text{SUGRA}$$

しかし，SUGRA は不変Lagrangianを書くことも自明でない．Superconformal Tensor Calculus が，その最も実用的な方法を与える．この講義ではその基本構造の説明と実用的manualを与える．

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2.1 Conformal Group

massless理論では，エネルギー運動量テンソル$\Theta_{\mu\nu}$は対称，tracelessにとれる：

$$\partial^\nu \Theta_{\mu\nu} = 0, \quad \Theta_{\mu\nu} = \Theta_{\nu\mu}, \quad \Theta^\mu_\mu = 0. \tag{1}$$

この性質から次の多くの保存カレント・保存量の存在が導かれる：

$$\begin{aligned}
\Theta_{\mu\nu} : \quad &\to \quad P_\mu = \int d^3x \Theta_{\mu 0} \\
\mathcal{M}_{\mu\nu\rho} = x_\mu \Theta_{\nu\rho} - x_\nu \Theta_{\mu\rho} : \quad &\to \quad M_{\mu\nu} = \int d^3x \mathcal{M}_{\mu\nu 0} \\
\mathcal{D}_\mu = x^\rho \Theta_{\rho\mu} : \quad &\to \quad D = \int d^3x \mathcal{D}_0 \\
\mathcal{K}_{\mu\nu} = 2x_\mu x^\rho \Theta_{\rho\nu} - x^2 \Theta_{\mu\nu} : \quad &\to \quad K_\mu = \int d^3x \mathcal{K}_{\mu 0}.
\end{aligned}$$

Dの変換$D = ix^\rho \partial_\rho$の有限形はスケール変換:

$$D: \quad x^\mu \to x'^\mu = e^{-\rho} x^\mu, \tag{2}$$

$K_\mu = i(2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu)$ 変換の有限形は，反転空間 x^μ / x^2 における並進変換:

$$K: \quad x^\mu \to x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b\cdot x + b^2 x^2} \quad \Longleftrightarrow \quad \frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu. \tag{3}$$

微分演算子表現 $P_\mu = i\partial_\mu$, $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$ とを用いて，次の共形代数を満たすことが確かめられる:

$$\begin{aligned}
&[M_{\mu\nu},\, M_{\rho\sigma}] = -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\sigma} M_{\mu\rho}),\\
&[P_\rho,\, M_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu),\\
&[K_\rho,\, M_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu),\\
&[D,\, M_{\mu\nu}] = [P_\mu,\, P_\nu] = [K_\mu,\, K_\nu] = 0,\\
&[P_\mu,\, D] = iP_\mu, \quad [K_\mu,\, D] = -iK_\mu,\\
&[P_\mu,\, K_\nu] = 2i(\eta_{\mu\nu} D - M_{\mu\nu}).
\end{aligned} \tag{4}$$

$$
\begin{array}{lcl}
\text{Lorentz group} & \longleftrightarrow & \text{Conformal group} \\
SO(D-1,1) & & SO(\underset{+1}{D},\ \underset{+1}{2})
\end{array}
$$

$$
\eta_{\mu\nu} \qquad\qquad \eta_{MN} = \begin{array}{c} \\ 0\text{-}3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{ccc} 0\text{ - }3 & 4 & 5 \end{array} \\ \begin{pmatrix} \eta_{\mu\nu} & & \\ & -1 & \\ & & +1 \end{pmatrix} \end{array}
$$

$$[M_{MN},\ M_{RS}] = -i(\eta_{MR}M_{NS} - \eta_{NR}M_{MS} - \eta_{MS}M_{NR} + \eta_{NS}M_{MR})$$

$$M_{\mu 4} = \tfrac{1}{2}(P_\mu - K_\mu), \qquad M_{\mu 5} = \tfrac{1}{2}(P_\mu + K_\mu), \qquad M_{54} = D$$

$$\downarrow$$

$$[iD, P_\mu] = +1P_\mu, \quad [iD, (M_{\mu\nu}, D)] = 0, \quad [iD, K_\mu] = -1K_\mu$$

Conformal (Weyl) weight: で整理すると分かり易い.

$$
\begin{array}{lcl}
w = +1 & : & P_\mu \\
w = 0 & : & M_{\mu\nu}, \quad D \\
w = -1 & : & K_\mu
\end{array}
$$

Conformal Algebra $SO(D,\,2)$ á la Weyl weights:

$$
\begin{aligned}
w=+2 \ &: \qquad\qquad [P_\mu,\,P_\nu]=0 \\
w=+1 \ &: \qquad [P_\rho,\,M_{\mu\nu}]=i(\eta_{\rho\mu}P_\nu-\eta_{\rho\nu}P_\mu), \quad [P_\mu,D]=+iP_\mu \\
w=0 \ &: \ [M_{\mu\nu},\,M_{\rho\sigma}]=-i(\eta_{\mu\rho}M_{\nu\sigma}-\eta_{\nu\rho}M_{\mu\sigma}-\eta_{\mu\sigma}M_{\nu\rho}+\eta_{\nu\sigma}M_{\mu\rho}) \\
&\qquad [P_\mu,\,K_\nu]=2i(\eta_{\mu\nu}D-M_{\mu\nu}), \qquad [M_{\mu\nu},\,D]=0 \\
w=-1 \ &: \quad [K_\rho,\,M_{\mu\nu}]=i(\eta_{\rho\mu}K_\nu-\eta_{\rho\nu}K_\mu), \qquad [K_\mu,\,D]=-iK_\mu \\
w=-2 \ &: \qquad\qquad [K_\mu,\,K_\nu]=0\ .
\end{aligned}
\tag{5}
$$

以降 $D=4$.

SO(3,1)$\simeq$ SL(2;C) Weyl(chiral) spinor 上の 2×2 Special Linear 複素行列

SO(6) $\simeq$ SU(4) $\quad\rightarrow\quad$ SO(4, 2) $\simeq$ SU(2,2)

$SO(4,2)$ 群 chiral spinor(4 成分) 上の 4×4 γ_0-unitary 行列 SU(2,2): $\psi_1^\dagger\gamma_0\psi_2$ invariant.

$$
\begin{aligned}
M_{\mu\nu}&=\tfrac{1}{2}\sigma_{\mu\nu}=i\tfrac{1}{4}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)(=\tfrac{i}{2}\gamma_{\mu\nu}),\\
P_\mu&=\gamma_\mu\mathcal{P}_\mathrm{R}, \qquad (\mathcal{P}_\mathrm{R}=\tfrac{1}{2}(1+\gamma_5))\\
K_\mu&=\gamma_\mu\mathcal{P}_\mathrm{L}, \qquad (\mathcal{P}_\mathrm{L}=\tfrac{1}{2}(1-\gamma_5))\\
D&=\tfrac{1}{2}i\gamma_5.
\end{aligned}
$$

2.2 SuperConformal (SC) Group $SU(2,2|1)$

metric $\quad \alpha = \begin{pmatrix} \gamma_0 & \\ & 1 \end{pmatrix}$, $\qquad$ acting on $\Psi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix}$ $\begin{array}{l} \leftarrow \text{4-spinor (fermi)} \\ \leftarrow \text{bose (1成分)} \end{array}$

invariant $\qquad \Psi_1^\dagger \alpha \Psi_2 = \psi_1^\dagger \gamma_0 \psi_2 + \varphi_1^\dagger \varphi_2$

24 generators (supertraceless, α-hermitian)

$$M_{MN} = \frac{1}{2}\begin{pmatrix} \sigma_{MN} & \\ & 0 \end{pmatrix} \quad \leftarrow \text{15 generators of } SU(2,2)$$

$$\Sigma_{\hat\alpha} = 2\begin{pmatrix} 0_4 & 0 \\ \boxed{1\,0\,0\,0} & 0 \end{pmatrix} (\hat\alpha = 1),\ \begin{pmatrix} 0_4 & 0 \\ \boxed{0\,1\,0\,0} & 0 \end{pmatrix} (\hat\alpha = 2),$$
$$\begin{pmatrix} 0_4 & 0 \\ \boxed{0\,0\,1\,0} & 0 \end{pmatrix} (\hat\alpha = 3),\ \begin{pmatrix} 0_4 & 0 \\ \boxed{0\,0\,0\,1} & 0 \end{pmatrix} (\hat\alpha = 4),$$

$$\bar\Sigma^{\hat\alpha} = (\Sigma_{\hat\alpha})^{\mathrm{T}}\ (\hat\alpha = 1-4), \qquad A = -\frac{1}{4}\begin{pmatrix} 1_4 & \\ & 4 \end{pmatrix} \quad (\mathrm{str}\, A = 0)$$

Using

$$[\Sigma_{\hat\alpha},\, A] = \tfrac{3}{4}\Sigma_{\hat\alpha}, \quad [\Sigma_{\hat\alpha},\, \bar\Sigma^{\hat\beta}] = (\sigma^{MN})_{\hat\alpha}{}^{\hat\beta} M_{MN} - 4\delta_{\hat\alpha}{}^{\hat\beta} A$$

and defining

$$\Sigma_{\hat\alpha} = \begin{pmatrix} Q_\alpha \\ \bar S^{\dot\alpha} \end{pmatrix}, \quad \bar\Sigma^{\hat\alpha} = \begin{pmatrix} S^\alpha & \bar Q_{\dot\alpha} \end{pmatrix} = (\Sigma_{\hat\beta})^\dagger (\gamma_0)_{\hat\beta}{}^{\hat\alpha}, \quad (\alpha, \dot\alpha = 1, 2)$$

we find the SC algebra for the charges:

$$\begin{array}{lll} w = +1 \;: & P_\mu & \cdots n = 0 \\ w = 1/2 \;: & Q_{\hat\alpha} & \cdots n = \mp\tfrac{3}{2} \begin{pmatrix} Q_\alpha \\ \bar Q^{\dot\alpha} \end{pmatrix} \\ w = 0 \;: & M_{\mu\nu}, \quad D & \cdots n = 0 \\ w = -1/2 \;: & S_{\hat\alpha} & \cdots n = \pm\tfrac{3}{2} \\ w = -1 \;: & K_\mu & \cdots n = 0 \end{array}$$

$[iD,\, X_A] = w X_A$

S, K はWeyl weightを下げる演算.

chiral weight n

$[iA,\; X_A] = i\tfrac{1}{2} n X_A$

Superconformal $SU(2,2|1)$ algebra:

$$
\begin{aligned}
w = +2 \quad &: \quad [P_\mu,\, P_\nu] = 0 \\
w = +3/2 \quad &: \quad [P_\mu,\, Q] = 0 \\
w = +1 \quad &: \quad \{Q,\, \bar{Q}\} = 2\gamma^\mu P_\mu, \quad [P_\mu,\, A] = 0, \quad [P_\rho,\, M_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \\
w = +1/2 \quad &: \quad [Q,\, M_{\mu\nu}] = \tfrac{1}{2}\sigma_{\mu\nu}Q, \quad [Q,\, A] = \tfrac{3}{4}\gamma_5 Q, \quad [S,\, P_\mu] = \gamma_\mu Q, \\
w = 0 \quad &: \quad [M_{\mu\nu},\, A] = 0, \quad [M_{\mu\nu},\, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \cdots), \\
&\quad\ \{Q, \bar{S}\} = -2iD + \sigma^{\mu\nu}M_{\mu\nu} - 4\gamma_5 A, \quad [P_\mu,\, K_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}), \\
w = -1/2 \quad &: \quad [S,\, M_{\mu\nu}] = \tfrac{1}{2}\sigma_{\mu\nu}S, \quad [S,\, A] = -\tfrac{3}{4}\gamma_5 S, \quad [Q,\, K_\mu] = \gamma_\mu S, \\
w = -1 \quad &: \quad \{S,\, \bar{S}\} = 2\gamma^\mu K_\mu, \quad [K_\mu,\, A] = 0, \quad [K_\rho,\, M_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu), \\
w = -3/2 \quad &: \quad [K_\mu,\, S] = 0 \\
w = -2 \quad &: \quad [K_\mu,\, K_\nu] = 0\ .
\end{aligned}
\tag{6}
$$

using 4-component Q and S supercharges:

$$
Q_{\hat{\alpha}} = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{Q}^{\hat{\alpha}} = (Q^\dagger \gamma_0)^{\hat{\alpha}} = \begin{pmatrix} Q^\alpha & \bar{Q}_{\dot{\alpha}} \end{pmatrix}, \quad S_{\hat{\alpha}} = \begin{pmatrix} S_\alpha \\ \bar{S}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{S}^{\hat{\alpha}} = (S^\dagger \gamma_0)^{\hat{\alpha}}
$$

以降 $\mu, \nu, \cdots \ \to \ m, n, \cdots$ 内部空間 flat index に

2.3 Yang-Mills Theory of $SU(2,2|1)$

Anti-hermitian Generators of $SU(2,2|1)$: active operators, not repr. matrix

$$\begin{aligned} X_A &= i^{-1}(P_m,\, Q,\, M_{mn},\, D,\, A,\, S,\, K_m) \\ &\equiv \underset{w}{}(\underset{+1}{\boldsymbol{P}_m},\, \underset{+1/2}{\boldsymbol{Q}},\, \underset{0}{\boldsymbol{M}_{mn}},\, \underset{0}{\boldsymbol{D}},\, \underset{0}{\boldsymbol{A}},\, \underset{-1/2}{\boldsymbol{S}},\, \underset{-1}{\boldsymbol{K}_m})\,, \end{aligned}$$

and write the algebra

$$[X_A,\, X_B\} = f_{AB}{}^C X_C.$$

Define trf parameters ε^A and gauge fields h_μ^A:

$$\begin{aligned} \varepsilon^A X_A &= \xi^m \boldsymbol{P}_m + \bar{\varepsilon}\boldsymbol{Q} + \tfrac{1}{2}\lambda^{mn}\boldsymbol{M}_{mn} + \rho\boldsymbol{D} + \theta\boldsymbol{A} + \bar{\zeta}\boldsymbol{S} + \xi_K^m\boldsymbol{K}_m, \\ h_\mu^A X_A &= e^m{}_\mu \boldsymbol{P}_m + \bar{\psi}_\mu\boldsymbol{Q} + \tfrac{1}{2}\omega_\mu{}^{mn}\boldsymbol{M}_{mn} + b_\mu\boldsymbol{D} + A_\mu\boldsymbol{A} + \bar{\varphi}_\mu\boldsymbol{S} + f^m{}_\mu\boldsymbol{K}_m. \end{aligned}$$

Transformation of the field Φ by the active operator:

$$\delta(\varepsilon)\Phi = \varepsilon^A X_A \Phi \quad \Big(\equiv \sum_A \varepsilon^A (X_A\,\Phi)\Big)$$

Covariant derivative

$$D_\mu\Phi = (\partial_\mu - h_\mu^A X_A)\Phi = (\partial_\mu - \delta(h_\mu))\Phi$$

is defined by demanding the property

$$\delta(\varepsilon)(D_\mu\Phi) = \varepsilon^A D_\mu(X_A\Phi) \equiv D_\mu(\delta(\check{\varepsilon})\Phi)(\check{\varepsilon}\text{ receiving no differenciation})$$

from which follows ゲージ場変換則：

$$\delta(\varepsilon)h_\mu^A = \partial_\mu\varepsilon^A + \varepsilon^B h_\mu^C f_{CB}{}^A \equiv (D_\mu\varepsilon)^A \tag{7}$$

$\because$) Writing $h_\mu^A X_A \equiv h_\mu, \varepsilon^A X_A \equiv \varepsilon$,

$$\begin{aligned}
\delta(\varepsilon)(D_\mu\Phi) &= \partial_\mu(\delta(\varepsilon)\Phi) - (\delta(\varepsilon)h_\mu)\Phi - \delta(\varepsilon)(\check{h}_\mu\Phi) \\
&= \partial_\mu(\varepsilon\Phi) - (\varepsilon h_\mu)\Phi - \varepsilon\check{h}_\mu\Phi \\
\varepsilon^A D_\mu(X_A\Phi) &= D_\mu(\check{\varepsilon}\Phi) = \partial_\mu(\check{\varepsilon}\Phi) - h_\mu\varepsilon\Phi \\
\to (\varepsilon h_\mu)\Phi &= (\partial_\mu\varepsilon)\Phi + h_\mu\varepsilon\Phi - \varepsilon\check{h}_\mu\Phi \\
\to \delta(\varepsilon)h_\mu &= \partial_\mu\varepsilon + [\check{h}_\mu, \varepsilon] \\
\text{i.e.,}\delta(\varepsilon)h_\mu^A &= \partial_\mu\varepsilon^A + \varepsilon^B h_\mu^C f_{CB}{}^A
\end{aligned} \tag{8}$$

where $\check{O}$ means that O is neither transformed by ∂_μ nor X_A.

Curvature tensor (Field strength) $R^A_{\mu\nu}$ is defined by

$$
\begin{aligned}
&[D_\mu,\ D_\nu]\Phi \equiv R^A_{\mu\nu} X_A \Phi \quad \text{on} \ \ \forall\Phi \\
&\quad \rightarrow \quad R^A_{\mu\nu} = \partial_\nu h^A_\mu - \partial_\mu h^A_\nu - h^B_\nu h^C_\mu f_{CB}{}^A = \partial_\nu h^A_\mu - D_\mu h^A_\nu \\
&\text{or} \quad R_{\mu\nu} = \partial_\nu h_\mu - \partial_\mu h_\nu - [\check{h}_\mu, \check{h}_\nu]
\end{aligned}
$$

$\because$)

$$
\begin{aligned}
[D_\mu,\ D_\nu]\Phi &= \partial_\mu D_\nu \Phi - \delta(h_\mu) D_\nu \Phi - (\mu \leftrightarrow \nu) \\
&= \partial_\mu \partial_\nu \Phi - \partial_\mu (h_\nu \Phi) - D_\nu (\check{h}_\mu \Phi) - (\mu \leftrightarrow \nu) \\
&= \partial_\mu \partial_\nu \Phi - (\partial_\mu h_\nu)\Phi - \partial_\mu (\check{h}_\nu \Phi) - \partial_\nu (\check{h}_\mu \Phi) + h_\nu \check{h}_\mu \Phi - (\mu \leftrightarrow \nu) \\
&= -(\partial_\mu h_\nu)\Phi + h_\nu \check{h}_\mu \Phi - (\mu \leftrightarrow \nu) \\
&= (\partial_\nu h_\mu - \partial_\mu h_\nu)\Phi + [\check{h}_\nu, \check{h}_\mu]\Phi
\end{aligned} \tag{9}
$$

Curvature 変換則

$$
\begin{aligned}
\delta(\varepsilon) R_{\mu\nu} &= [\check{R}_{\mu\nu}, \varepsilon] \\
\delta(\varepsilon) R^A_{\mu\nu} &= \varepsilon^B R^C_{\mu\nu} f_{CB}{}^A
\end{aligned}
$$

Curvatures:

$$R_{\mu\nu}{}^{m}(P) = 2\partial_\nu e^m_{\ \mu} - 2\omega_\nu{}^{mn} e_{n\mu} + 2b_\nu e^m_{\ \mu} + 2i\bar{\psi}_\nu \gamma^m \psi_\mu$$

e.g., $f_{QQ}{}^{Pm} = -2i\gamma^m$

$$R_{\mu\nu}{}^{mn}(M) = 2\partial_\nu \omega_\mu{}^{mn} - 2\omega_\nu{}^{mc}\omega_{\mu c}{}^{n} + 4(f^m_{\ \nu} e^n_{\ \mu} - f^n_{\ \nu} e^m_{\ \mu}) + 4i\bar{\psi}_\nu \sigma^{mn}\varphi_\mu,$$

$$R_{\mu\nu}(D) = 2\partial_\nu b_\mu + 4f^n_{\ \nu} e_{n\mu} + 4\bar{\psi}_\nu \varphi_\mu$$

$$R_{\mu\nu}(A) = 2\partial_\nu A_\mu - 8i\bar{\psi}_\nu \gamma_5 \varphi_\mu$$

$$R_{\mu\nu}{}^{m}(K) = 2\partial_\nu f^m_{\ \mu} - 2\omega_\nu{}^{mn} f_{n\mu} - 2b_\nu f^m_{\ \mu} + 2i\bar{\varphi}_\nu \gamma^m \varphi_\mu,$$

$$\bar{R}_{\mu\nu}(Q) = 2D^\omega_\nu \bar{\psi}_\mu + b_\nu \bar{\psi}_\mu - \tfrac{3}{2} i A_\nu \bar{\psi}_\mu \gamma_5 + 2i\bar{\varphi}_\nu \gamma_m e^m_{\ \mu},$$

$$\bar{R}_{\mu\nu}(S) = 2D^\omega_\nu \bar{\varphi}_\mu - b_\nu \bar{\varphi}_\mu + \tfrac{3}{2} i A_\nu \bar{\varphi}_\mu \gamma_5 + 2i\bar{\psi}_\nu \gamma_m f^m_{\ \mu},$$

with

$$D^\omega_\nu \bar{\psi}_\mu \equiv \partial_\nu \bar{\psi}_\mu - \tfrac{i}{4}\omega_\nu{}^{mn} \bar{\psi}_\mu \sigma_{mn},$$

$$D^\omega_\nu \psi_\mu \equiv \partial_\nu \psi_\mu + \tfrac{i}{4}\omega_\nu{}^{mn} \sigma_{mn}\psi_\mu,$$

(and the same for φ_μ,) where antisymmetrization w.r.t. $\mu \leftrightarrow \nu$ implied like

$$\begin{aligned} R_{\mu\nu}(A) &= (2\partial_\nu A_\mu - 8i\bar{\psi}_\nu\gamma_5\varphi_\mu)_{\text{anti-symm}} \\ &= \partial_\nu A_\mu - \partial_\mu A_\nu - 4i(\bar{\psi}_\nu\gamma_5\varphi_\mu - \bar{\psi}_\mu\gamma_5\varphi_\nu) \end{aligned}$$

gauge場 変換則: （ $SU(2,2|1)$ group law と呼ぶ ）

$$
\begin{aligned}
\delta e^m{}_\mu &= \partial_\mu \xi^m + \lambda^{ml} e_{l\mu} - \omega_\mu{}^{mn}\xi_n - \rho e^m{}_\mu + b_\mu \xi^m - 2i\bar{\varepsilon}\gamma^m \psi_\mu, \\
\delta \omega_\mu{}^{mn} &= \partial_\mu \lambda^{mn} + 2\lambda^{ml}\omega_{\mu l}{}^n - 2(\xi_K^m e^n{}_\mu - \xi_K^n e^m{}_\mu) + 2(f^m{}_\mu \xi^n - f^n{}_\mu \xi^m) \\
&\qquad -2i\bar{\varepsilon}\sigma^{mn}\varphi_\mu - 2i\bar{\psi}_\mu \sigma^{mn}\zeta, \\
\delta b_\mu &= \partial_\mu \rho - 2\xi_K^n e_{n\mu} + 2f^n{}_\mu \xi_n - 2\bar{\varepsilon}\varphi_\mu + 2\bar{\psi}_\mu \zeta \\
\delta A_\mu &= \partial_\mu \theta + 4i\bar{\varepsilon}\gamma_5 \varphi_\mu - 4i\bar{\psi}_\mu \gamma_5 \zeta \\
\delta f_{\mu\nu}{}^m &= \partial_\mu \xi_K^m + \lambda^{\ mn} f_{n\mu} - \omega_\mu{}^{mn}\xi_{K\,n} + \rho f^m{}_\mu - b_\mu \xi_K^m - 2i\bar{\zeta}\gamma^m \varphi_\mu, \\
\delta \bar{\psi}_\mu &= D_\mu^\omega \bar{\varepsilon} + \tfrac{i}{4}\lambda^{mn}\bar{\psi}_\mu \sigma_{mn} - \tfrac{1}{2}\rho\bar{\psi}_\mu + \tfrac{1}{2}b_\mu \bar{\varepsilon} + \tfrac{3}{4}i\theta\bar{\psi}_\mu\gamma_5 - \tfrac{3}{4}iA_\mu \bar{\varepsilon}\gamma_5 \\
&\qquad -i\bar{\zeta}\gamma_m e^m{}_\mu + i\bar{\varphi}_\mu \gamma_m \xi^m\big), \\
\big(\delta \psi_\mu &= D_\mu^\omega \varepsilon - \tfrac{i}{4}\lambda^{mn}\sigma_{mn}\psi_\mu - \tfrac{1}{2}\rho\psi_\mu + \tfrac{1}{2}b_\mu \varepsilon + \tfrac{3}{4}i\theta\gamma_5\psi_\mu - \tfrac{3}{4}iA_\mu \gamma_5 \varepsilon \\
&\qquad +ie^m{}_\mu \gamma_m \zeta - i\xi^m \gamma_m \varphi_\mu\big), \\
\delta \bar{\varphi}_\mu &= D_\mu^\omega \bar{\zeta} + \tfrac{i}{4}\lambda^{mn}\bar{\varphi}_\mu \sigma_{mn} + \tfrac{1}{2}\rho\bar{\varphi}_\mu - \tfrac{1}{2}b_\mu \bar{\zeta} - \tfrac{3}{4}i\theta\bar{\varphi}_\mu\gamma_5 + \tfrac{3}{4}iA_\mu \bar{\zeta}\gamma_5 \\
&\qquad -i\bar{\varepsilon}\gamma_m f^m_\mu + i\bar{\psi}_\mu \gamma_m \xi_K^m, \\
\big(\delta \varphi_\mu &= D_\mu^\omega \zeta - \tfrac{i}{4}\lambda^{mn}\sigma_{mn}\varphi_\mu + \tfrac{1}{2}\rho\varphi_\mu - \tfrac{1}{2}b_\mu \zeta - \tfrac{3}{4}i\theta\varphi_\mu\gamma_5 + \tfrac{3}{4}iA_\mu \zeta\gamma_5 \\
&\qquad +if^m{}_\mu \gamma_m \varepsilon - i\xi_K^m \gamma_m \psi_\mu\big),
\end{aligned}
$$

For inverse vierbein,

$$\delta e_m{}^\mu = -e_n{}^\mu e_m{}^\nu (\delta e^n{}_\nu)$$
$$= -e_n{}^\mu \partial_m \xi^n - e_l{}^\mu \lambda^l{}_m + \omega_m{}^{\mu l}\xi_l + \rho e_m{}^\mu - b_m \xi^\mu + 2i\bar{\varepsilon}\gamma^\mu \psi_m.$$

Curvature の group変換則 $\delta R^A_{\mu\nu}$ は、上のgauge場の変換則 δh^A_μ で、$\partial_\mu \varepsilon^A$ を捨て、全ての h^B_μ を $R^B_{\mu\nu}$ に置き換えれば良い。

3 Local SC Algebra and Weyl Multiplet

上で与えたSC代数のYMゲージ理論は，非コンパクト群に基づき，負計量ゲージ場を含むので，物理的にはナンセンス．また，P_m は，内部空間の`並進’変換で，時空における並進と無関係．この理論を物理的に意味のある理論にし，P_m が時空における並進という意味を持たせるため，代数をdeform する．

これからやること：

$$[\delta_Q(\varepsilon_1),\, \delta_Q(\varepsilon_2)] = \delta_P(\xi^m) \quad \xi^m = -2i\bar{\varepsilon}_1\gamma^m\varepsilon_2$$
$$\to \delta_{\tilde{P}}(\xi^m) = \delta_{\mathrm{GC}}(\xi^\mu) - \sum_{A'\neq P} \delta_{A'}(\xi^\mu h^{A'}_\mu)$$

w	:	generator X_A	gauge field h_μ^A
$w=+1$	:	P_m	e_μ^m
$w=1/2$	:	Q	ψ_μ
$w=0$	:	M_{mn} D A	ω_μ^{mn} b_μ A_μ
$w=-1/2$	:	S	φ_μ
$w=-1$	:	K_m	f_μ^m

e_μ^m, ψ_μ, $\cdots$ の上で成立するよう順に要請.

$$\begin{array}{llll} \rightarrow \quad \text{curvatures} & R_{\mu\nu}(P) & R_{\mu\nu}(Q) & R_{\mu\nu}(M) \ \text{に constraint} \\ & \downarrow & \downarrow & \downarrow \\ & {\omega_\mu}^{mn} & \varphi_\mu & {f_\mu}^m \ \text{が dependent gauge 場} \end{array}$$

$\omega(e,\psi,b) \rightarrow Q$ 変換が，ω,φ,f に対し group law からずれる．
代数 $[P_m,\ Q],[P_m,\ P_n]$ だけ，$SU(2,2|1)$ algebra からずれる．
${f_{P_m Q}}^{A'}$, ${f_{P_m P_n}}^A$ が $\neq 0$ で現れ，field dependent structure functions を与える．

3.1 Deformation of $SU(2,2|1)$ algebra

GC transformation:

$$\begin{aligned}
\delta_{\rm GC}(\xi^\lambda)h_\mu^A &= \partial_\mu\xi^\lambda\!\cdot\! h_\lambda^A + \xi^\lambda\partial_\lambda h_\mu^A \\
&= D_\mu(\xi^\lambda\!\cdot\! h_\lambda^A) + \xi^\lambda(\partial_\lambda h_\mu^A - D_\mu h_\lambda^A) \\
&= [D_\mu(\xi\!\cdot\! h)]^A + \xi^\lambda R_{\mu\lambda}^A \\
&= \delta(\xi\!\cdot\! h)\, h_\mu^A + \xi^\lambda R_{\mu\lambda}^A\ ,
\end{aligned}$$

The last equality is because

$$\begin{aligned}
D_\mu h_\lambda^A &= \partial_\mu h_\lambda^A + h_\lambda^B h_\mu^C f_{CB}{}^A \\
\delta(\varepsilon)h_\mu^A &= (D_\mu\varepsilon)^A = \partial_\mu\varepsilon^A + \varepsilon^B h_\mu^C f_{CB}{}^A \\
\delta(\xi\!\cdot\! h)h_\mu^A &= \partial_\mu(\xi\!\cdot\! h^A) + (\xi\!\cdot\! h)^B h_\mu^C f_{CB}{}^A.
\end{aligned}$$

Note that

$$\delta(\xi\!\cdot\! h) = \xi^\lambda h_\lambda^A X_A = \delta_P(\xi^m) + \sum_{A'(\neq P)} \delta_{A'}(\xi\!\cdot\! h^{A'}),$$

$$\begin{aligned}
\xi^m &= \xi^\lambda e_\lambda^m, \\
\xi\!\cdot\! h^{A'} &= \xi^\lambda h_\lambda^{A'} = \xi^m h_m^{A'}
\end{aligned}$$

Therefore, we have *a key relation*:

$$\delta_P(\xi^m)h_\mu^A = \underbrace{\delta_{\rm GC}(\xi^\lambda)h_\mu^A - \sum_{B'}\delta_{B'}(\xi\cdot h^{B'})h_\mu^A}_{\stackrel{\rm def}{=}\ \delta_{\widetilde{P}}(\xi^m)h_\mu^A} - \xi^\lambda R_{\mu\lambda}^A.$$

Now, we deform the $SU(2,2|1)$ algebra so as to make a replacement

$$\delta_P(\xi^m)\ \rightarrow\ \delta_{\widetilde{P}}(\xi^m) = \delta_{\rm GC}(\xi^\lambda) - \sum_{B'}\delta_{B'}(\xi\cdot h^{B'}). \tag{10}$$

First we note that, among the commutators $[\delta_{A'},\,\delta_{B'}]$ for $A',\ B' \neq P$, the only one yielding δ_P in the RHS is $[\delta_Q(\varepsilon_2),\,\delta_Q(\varepsilon_1)] = \delta_P(-2i\bar{\varepsilon}_1\gamma^m\varepsilon_2)$. So we require first that

$$\boxed{[\delta_Q(\varepsilon_2),\,\delta_Q(\varepsilon_1)] = \delta_{\widetilde{P}}(\xi^m), \quad \text{with} \quad \xi^m \equiv -2i\bar{\varepsilon}_1\gamma^m\varepsilon_2}, \tag{11}$$

holds on *any independent gauge fields*, and find constraints necessary for that.

3.2 On $e^m_{\ \mu}$

On $e^m_{\ \mu}$, we originally have

$$[\delta_Q(\varepsilon_2),\ \delta_Q(\varepsilon_1)]e^m_{\ \mu} = \delta_P(\xi^m)e^m_{\ \mu},$$
$$= \delta_{\widetilde{P}}(\xi^m)e^m_{\ \mu} - \xi^\lambda R_{\mu\lambda}{}^m(P).$$

So it is necessary and sufficient to impose the constraint:

$$\boxed{0 = R_{\mu\nu}{}^m(P)} = 2\partial_\nu e^m_{\ \mu} - 2\omega_\nu{}^{mn}e_{n\mu} + 2b_\nu e^m_{\ \mu} + 2i\bar{\psi}_\nu\gamma^m\psi_\mu \tag{12}$$

This can be solved by the M gauge field $\omega_\mu{}^{mn}$ and yields

$$\omega_\mu{}^{mn} = \omega_\mu{}^{mn}(e, \psi, b), \tag{13}$$

so that $\omega_\mu{}^{mn}$ is no longer an *independent* gauge field. However, since the constraint $R_{\mu\nu}{}^m(P) = 0$ is invariant under M_{mn}, D, A, S, K_m, $\omega_\mu{}^{mn}$ still keeps the same transformation law as the original group transformation under M_{mn}, D, A, S, K_m transformations. On the other hand, the constraint $R_{\mu\nu}{}^m(P) = 0$ is *not* invariant under Q transformation, the Q transformation of $\omega_\mu{}^{mn}$ becomes different from the original group transformation law:

$$\delta_Q(\varepsilon)\omega_\mu{}^{mn}(e, \psi, b) = \delta_Q^{\text{group}}(\varepsilon)\omega_\mu{}^{mn} + \delta'_Q(\varepsilon)\omega_\mu{}^{mn}. \tag{14}$$

The difference can be easily found by noting that the constraint $R_{\mu\nu}{}^{m}(P) = 0$ is of course an identity and Q-invariant if $\omega_{\mu}{}^{mn}$ there is replaced by $\omega_{\mu}{}^{mn}(e,\psi,b)$, so that we have

$$\begin{aligned} 0 &= \delta_Q^{\mathrm{group}}(\varepsilon) R_{\mu\nu}{}^{m}(P) + \delta'_Q(\varepsilon)\omega_{\mu}{}^{m}{}_{\nu} - \delta'_Q(\varepsilon)\omega_{\nu}{}^{m}{}_{\mu} \\ &= -2i\bar{\varepsilon}\gamma^m R_{\mu\nu}(Q) + \delta'_Q(\varepsilon)\omega_{\mu}{}^{m}{}_{\nu} - \delta'_Q(\varepsilon)\omega_{\nu}{}^{m}{}_{\mu}. \end{aligned}$$

(Note that we are anticipating that $e^m{}_\mu$, ψ_μ, b_μ will remain to be independent gauge fields and receive no changes in the Q-transformation laws.) Solving this (in a similar way to solve Christoffel symbol in terms of $g_{\mu\nu}$), we find

$$\delta'_Q(\varepsilon)\omega_{\mu\,mn} = i\bar{\varepsilon}(\gamma_\mu R_{mn}(Q) + \gamma_m R_{\mu n}(Q) - \gamma_n R_{\mu m}(Q)) \equiv i\bar{\varepsilon}\mathcal{R}_{\mu mn}(Q). \quad (15)$$

3.3 On ψ_μ

Noting

$$\delta_Q(\varepsilon)\psi_\mu = (\partial_\mu + \tfrac{i}{4}\omega_{\mu}{}^{mn}\sigma_{mn} + \tfrac{1}{2}b_\mu - \tfrac{3}{4}i\gamma_5 A_\mu)\varepsilon \quad (16)$$

and that $\omega_{\mu}{}^{mn}$ now receives an extra Q transformation $\delta'_Q(\varepsilon)$ in addition to the original group transformation $\delta_Q^{\mathrm{group}}(\varepsilon)$, we find that the $[\delta_Q,\,\delta_Q]$ commutator on ψ_μ now reads

$$\begin{aligned}[\delta_Q(\varepsilon_2),\, \delta_Q(\varepsilon_1)]\psi_\mu &= [\delta_Q^{\text{group}}(\varepsilon_2),\, \delta_Q^{\text{group}}(\varepsilon_1)]\psi_\mu + \tfrac{i}{4}\big(\delta'_Q(\varepsilon_2)\omega_\mu\cdot\sigma\varepsilon_1 - (1\leftrightarrow 2)\big)\\ &= \delta_{\widetilde{P}}(\xi)\psi_\mu - \xi^m R_{\mu m}(Q) + \tfrac{i}{4}\big(\delta'_Q(\varepsilon_2)\omega_\mu\cdot\sigma\varepsilon_1 - (1\leftrightarrow 2)\big).\end{aligned}$$

So we see that the condition

$$\tfrac{i}{4}\big((i\bar{\varepsilon}_2\mathcal{R}_{\mu\, mn}(Q))\,\sigma^{mn}\varepsilon_1 - (1\leftrightarrow 2)\big) = -2i(\bar{\varepsilon}_1\gamma^m\varepsilon_2)R_{\mu m}(Q) \tag{17}$$

should hold. From this, after some calculation like Fierzing, we find a constraint

$$\boxed{\gamma^\rho R_{\mu\rho}(Q) = 0.} \tag{18}$$

is necessary and sufficient condition for the $[\delta_Q,\, \delta_Q]$ algebra Eq. (11) hold on ψ_μ. With this new constraint, the extra Q transformation for ${\omega_\mu}^{mn}$ can be simplified into

$$\boxed{\delta'_Q(\varepsilon)\omega_{\mu\, mn} = 2i\bar{\varepsilon}\gamma_\mu R_{mn}(Q)}\,(= -2i\bar{R}_{mn}(Q)\gamma_\mu\varepsilon). \tag{19}$$

The constraint (18), $\gamma^\mu R_{\mu\nu}(Q) = 0$, is solved by the S-gauge field φ_μ:

$$0 = \gamma^\mu R_{\mu\nu}(Q) = \gamma^\mu\big[(\partial_\nu + \tfrac{i}{4}\omega_\nu{\cdot}\sigma + \tfrac{1}{2}b_\nu - \tfrac{3}{4}i\gamma_5 A_\nu)\psi_\mu - (1\leftrightarrow 2)\big] - i\gamma^\mu(\gamma_\mu\varphi_\nu - \gamma_\nu\varphi_\mu)$$
$$\Rightarrow \quad \varphi_\mu = \varphi_\mu(e, \psi, b, A).$$

So φ_μ now become *dependent* gauge field. Since the constraint $\gamma^\mu R_{\mu\nu}(Q) = 0$ is $M_{mn},\ D,\ A,\ S,\ K_m$ invariant but not invariant under Q, the Q-transformation of φ_μ is modified: After much computation, we find the extra Q transformation $\delta'_Q(\varepsilon)\varphi_\mu$ to be given by

$$\begin{aligned}\delta'_Q(\varepsilon)\varphi_\mu = -\frac{i}{2}\big[&\gamma^m\varepsilon\,\big(-\tfrac{1}{12}e_{m\mu}R^{\text{cov.}}(M)^\rho_\rho + \tfrac{1}{2}R^{\text{cov.}}_{\mu m}(M) + \tfrac{1}{4}\widetilde{R}_{\mu m}(A)\big)\\ &+i\gamma^m\gamma_5\varepsilon\,\tfrac{1}{2}R_{\mu m}(A)\big]\end{aligned} \tag{20}$$

where

$$R^{\text{cov.}\,mn}_{\mu\nu}(M) \equiv R_{\mu\nu}{}^{mn}(M) + 2i\big(\bar\psi_\mu\gamma_\nu R^{mn}(Q) - \bar\psi_\nu\gamma_\mu R^{mn}(Q)\big)$$
$$R^{\text{cov.}}_{\mu m}(M) = R^{\text{cov.}\,np}_{\mu\nu}(M)e_n{}^\nu\eta_{pm}$$

3.4 On A_μ and b_μ

Noting

$$\delta_Q(\varepsilon)A_\mu = 4i\bar{\varepsilon}\gamma_5\varphi_\mu$$
$$\delta_Q(\varepsilon)b_\mu = -2\bar{\varepsilon}\varphi_\mu,$$

and that φ_μ now receives an extra Q transformation $\delta'_Q(\varepsilon)$ in addition to the original group transformation $\delta_Q^{\rm group}(\varepsilon)$, we find that the $[\delta_Q,\ \delta_Q]$ commutator on A_μ now reads

$$\begin{aligned}[\delta_Q(\varepsilon_2),\ \delta_Q(\varepsilon_1)]A_\mu &= [\delta_Q^{\rm group}(\varepsilon_2),\ \delta_Q^{\rm group}(\varepsilon_1)]A_\mu + 4i\big(\bar{\varepsilon}_1\gamma_5(\delta'_Q(\varepsilon_2)\varphi_\mu) - (1\leftrightarrow 2)\big) \\ &= \delta_{\widetilde{P}}(\xi)A_\mu - \xi^m R_{\mu m}(A) + 4i\big((-\tfrac{i}{2})(\bar{\varepsilon}_1\gamma_5 i\gamma^m\gamma_5\varepsilon_2)\,\tfrac{1}{2}R_{\mu m}(A) - (1\leftrightarrow 2) \\ &= \delta_{\widetilde{P}}(\xi)A_\mu - \xi^m R_{\mu m}(A) + 4i(-\tfrac{i}{2})\xi^m\tfrac{1}{2}R_{\mu m}(A) = \delta_{\widetilde{P}}(\xi)A_\mu\end{aligned}$$

This is automatic, so leading no new constraint!

On b_μ, on the other hand, the condition

$$[\delta_Q(\varepsilon_2),\ \delta_Q(\varepsilon_1)]b_\mu = \delta_{\widetilde{P}}(\xi)b_\mu \tag{21}$$

requires a new constraint

$$-\tfrac{1}{12}e_{m\mu}R^{\rm cov.}(M)_\rho^\rho + \tfrac{1}{2}R^{\rm cov.}_{\mu m}(M) + \tfrac{1}{4}\widetilde{R}_{\mu m}(A) = -R_{\mu m}(D). \tag{22}$$

This constraint can be shown to be rewritten into

$$\boxed{R^{\text{cov.}}_{\nu\mu}(M) + \tfrac{1}{2}\widetilde{R}_{\mu\nu}(A) = 0.} \tag{23}$$

This is the necessary and sufficient condition for the $[\delta_Q,\ \delta_Q]$ algebra Eq. (11) to hold on b_μ. Using this constraint, the extra Q transformation Eq. (20) of φ_μ is simplified into

$$\boxed{\delta'_Q(\varepsilon)\varphi_\mu = -\frac{i}{4}\gamma^m\big(\widetilde{R}_{\mu m}(A) + i\gamma_5 R_{\mu m}(A)\big)\varepsilon.}$$

The constraint Eq. (23) can be solved by the K_m gauge field $f^{\ m}_\mu$, which now becomes a dependent field:

$$f^{\ m}_\mu = f^{\ m}_\mu(e, \psi, b, A). \tag{24}$$

Since the constraint Eq. (23) is not Q-invariant and so $f^{\ m}_\mu$ also receives an extra Q-transformation, which can be found in a similar way as above:

$$\boxed{\delta'_Q(\varepsilon)f^{\ m}_\mu = -\frac{i}{2}\bar{\varepsilon}\big(\sigma^{m\nu}R^{\text{cov.}}_{\mu\nu}(S) + e^{m\nu}\widetilde{R}^{\text{cov.}}_{\mu\nu}(S)\big).} \tag{25}$$

3.5 Resultant Local SC algebra and Weyl multiplet: summary

Now that the M_{mn}, S and K_m gauge fields $\omega_\mu{}^{mn}$, φ_μ and $f^m{}_\mu$ have become dependent fields, there no longer remain other independent gauge fields. Thus the desired $[\delta_Q, \delta_Q]$ algebra (11)

$$[\delta_Q(\varepsilon_2), \delta_Q(\varepsilon_1)] = \delta_{\widetilde{P}}(\xi^m), \quad \text{with} \quad \xi^m \equiv -2i\bar{\varepsilon}_1\gamma^m\varepsilon_2 \tag{26}$$

already holds on all the independent gauge fields $e^m{}_\mu$, ψ_μ, A_μ and b_μ.

Now the resultant Local SuperConformal Algebra is as follows:

w	:	generator X_A	gauge field h^A_μ
$w=+1$	:	P_m	e^m_μ
$w=1/2$	:	Q	ψ_μ
$w=0$	:	M_{mn} D A	ω^{mn}_μ b_μ A_μ
$w=-1/2$	:	S	φ_μ
$w=-1$	:	K_m	f^m_μ

ω^{mn}_μ, φ_μ, f^m_μ become dependent gauge field by the constraints, respectively:

$$
\begin{aligned}
\hat{R}_{\mu\nu}{}^{m}(P) = 0 \ &\rightarrow\ \omega_\mu^{mn} = \omega_\mu^{mn}(e) + i(2\bar{\psi}_\mu\gamma^{[m}\psi^{n]} + \bar{\psi}^m\gamma_\mu\psi^n) - 2e_\mu^{[m}b^{n]}, \\
&\qquad\qquad \text{with}\ \ \omega_\mu^{mn}(e) \equiv -2e^{\nu[m}\partial_{[\mu}e_{\nu]}{}^{n]} + e^{\rho[m}e^{n]\sigma}e_\mu{}^c\partial_\rho e_{\sigma c}, \\
\gamma^\nu R_{\mu\nu}(Q) = 0 \ &\rightarrow\ \varphi_\mu^i = -\tfrac{i}{3}\gamma^m\hat{R}'_{\mu m}(Q) + \tfrac{i}{12}\gamma_{\mu mn}\hat{R}'^{mn}(Q), \\
R_{\nu\mu}^{\text{cov.}}(M) + \tfrac{1}{2}\widetilde{R}_{\mu\nu}(A) = 0 \ &\rightarrow\ f_\mu{}^m = \tfrac{1}{4}\hat{R}'_\mu{}^m(M) - \tfrac{1}{8}\widetilde{\hat{R}}_\mu{}^m(A) - \tfrac{1}{24}e_\mu{}^m\hat{R}'(M)\,.
\end{aligned}
$$

so that their trf laws are modified from the $SU(2,2|1)$ group law: $\delta_Q(\varepsilon) = \delta_Q^{\text{group}}(\varepsilon) + \delta'_Q(\varepsilon)$ with

$$
\begin{aligned}
\delta'_Q(\varepsilon)\varphi_\mu &= -\frac{i}{4}\gamma^m\big(\widetilde{R}_{\mu m}(A) + i\gamma_5 R_{\mu m}(A)\big)\varepsilon \\
\delta'_Q(\varepsilon)\omega_{\mu\, mn} &= 2i\bar{\varepsilon}\gamma_\mu R_{mn}(Q) \\
\delta'_Q(\varepsilon)f_\mu^m &= -\frac{i}{2}\bar{\varepsilon}\big(\sigma^{m\nu}R_{\mu\nu}^{\text{cov.}}(S) + e^{m\nu}\widetilde{R}_{\mu\nu}^{\text{cov.}}(S)\big)
\end{aligned}
$$

The P_m transformation is replaced by $\tilde{P}_m$ trf

$$
\boxed{\delta_P(\xi^m) = \delta_{\text{GC}}(\xi^m e_m^\mu) - \sum_{A\neq P}\delta_A(\xi^m h_m^A)} \tag{27}
$$

which defines SC covariant derivative $\hat{\mathcal{D}}_m (= D^{\rm c}_m)$:

$$\delta_P(\xi^m)\phi = \xi^m(\partial_\mu - \sum_{A\neq P} \delta_A(h^A_m))\phi = \xi^m \hat{\mathcal{D}}_m\phi. \tag{28}$$

The local SC algebra still takes the same form as the $SU(2,2|1)$ one

$$\boxed{[\delta_A(\varepsilon^A),\, \delta_B(\varepsilon^B)] = \sum_C \delta_C(\varepsilon^B\varepsilon^A f_{AB}{}^C)}$$

except for $[\delta_P,\, \delta_Q]$ and $[\delta_P,\, \delta_P]$, for which

$$\boxed{[\delta_P(\xi^m),\, \delta_Q(\varepsilon^Q)] = \sum_{B'=M,S,K} \delta_{B'}(\xi^m\delta'_Q(\varepsilon^Q)h^{B'}_m)} \tag{29}$$

$$\boxed{[\delta_P(\xi_1^m),\, \delta_{\tilde{P}}(\xi_2^n)] = \sum_{A\ \text{all}} \delta_A(\xi_1^m\xi_2^n R^{\text{cov. A}}_{mn})}.$$

$\xi^m\delta'_Q(\varepsilon^Q)h^{B'}_m = \varepsilon^Q\xi^m f_{P_m Q}{}^{B'}$ gives a NEW field-dependent structure function
$R^{\text{cov. A}}_{mn} = f_{P_n P_m}{}^A$ defines a NEW field-dependent structure function

3.6 Covariant derivative and curvatures

振り返って見るに，$\bar{A} \equiv \{P_m, A\}$

$$[\boldsymbol{X}_{\bar{A}}, \boldsymbol{X}_{\bar{B}}\} = f_{\bar{A}\bar{B}}{}^{\bar{C}} \boldsymbol{X}_{\bar{C}}. \tag{30}$$

(この $f_{\bar{A}\bar{B}}{}^{\bar{C}}$ は最終の構造関数). For any field ϕ carrying only flat indices, Define

$$\hat{\mathcal{D}}_\mu \phi = \partial_\mu \phi - h_\mu^A \boldsymbol{X}_A \phi$$
$$\boldsymbol{\nabla}_\mu \phi = \partial_\mu \phi - h_\mu^{\bar{A}} \boldsymbol{X}_{\bar{A}} \phi = \hat{\mathcal{D}}_\mu \phi - e_\mu^m P_m \phi$$

As usual covariance requirement $\boldsymbol{\nabla}_\mu$ for $\boldsymbol{\nabla}_\mu$,

$$\boldsymbol{X}_{\bar{A}}(\boldsymbol{\nabla}_\mu \phi) = \boldsymbol{\nabla}_\mu (\boldsymbol{X}_A \phi)$$

determines the transformation law of gauge fields as

$$\varepsilon^{\bar{B}} \boldsymbol{X}_{\bar{B}} h_\mu^{\bar{A}} \equiv \delta(\varepsilon) h_\mu^{\bar{A}} = \partial_\mu \varepsilon^{\bar{A}} + \varepsilon^{\bar{C}} h_\mu^{\bar{B}} f_{\bar{B}\bar{C}}{}^{\bar{A}}, \tag{31}$$

and the commutator of $\boldsymbol{\nabla}_\mu$ defines the curvature tensors $\boldsymbol{R}_{\mu\nu}{}^{\bar{A}}$ in the form

$$[\boldsymbol{\nabla}_\mu, \boldsymbol{\nabla}_\nu] = \boldsymbol{R}_{\mu\nu}{}^{\bar{A}} \boldsymbol{X}_{\bar{A}} \quad \rightarrow \quad \boldsymbol{R}_{\mu\nu}{}^{\bar{A}} \equiv 2\partial_{[\nu} h_{\mu]}^{\bar{A}} - h_\nu^{\bar{C}} h_\mu^{\bar{B}} f_{\bar{B}\bar{C}}{}^{\bar{A}}. \tag{32}$$

These are quite normal up to here. Now SUGRA の特殊な要求: for $\forall\phi$ carrying flat indices alone

$$\boldsymbol{\nabla}_\mu \phi = 0 \quad \Longrightarrow \quad \boldsymbol{R}_{\mu\nu}{}^{\bar{A}} = 0$$

Curvature in SUGRA, $\hat{R}_{ab}{}^{\bar{A}}$, is defined via the flat $\hat{\mathcal{D}}_a$ by

$$[\hat{\mathcal{D}}_a,\, \hat{\mathcal{D}}_b] = \hat{R}_{ab}{}^{\bar{A}} \boldsymbol{X}_{\bar{A}} \equiv \hat{R}_{ab}. \tag{33}$$

計算無しに

$$e_a^\mu \boldsymbol{\nabla}_\mu \phi = \hat{\mathcal{D}}_a \phi - \boldsymbol{P}_a \phi = 0 \quad \to \quad \boxed{\boldsymbol{P}_a = \hat{\mathcal{D}}_a \quad \text{on} \quad \forall\phi}$$

$$\to \quad \hat{R}_{ab}{}^{\bar{A}} = [\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b]^{\bar{A}} = [\boldsymbol{P}_a, \boldsymbol{P}_b]^{\bar{A}} = f_{ab}{}^{\bar{A}}$$

$$\text{よって}\ \hat{R}_{\mu\nu}{}^{\bar{A}} = e_\nu^b e_\mu^a f_{ab}{}^{\bar{A}} = 2\partial_{[\nu} h_{\mu]}^{\bar{A}} - h_\nu^{\bar{C}} h_\mu^{\bar{B}} f'_{\bar{B}\bar{C}}{}^{\bar{A}}, \tag{34}$$

ここで $f'_{\bar{B}\bar{C}}{}^{\bar{A}}$ は，$[\boldsymbol{P}_a,\, \boldsymbol{P}_b]$ 交換子の $f_{ab}{}^{\bar{A}}$ 部分を除く意味で，この等号は

$$\boldsymbol{R}_{\mu\nu}{}^{\bar{A}} \equiv 2\partial_{[\nu} h_{\mu]}^{\bar{A}} - h_\nu^{\bar{C}} h_\mu^{\bar{B}} f_{\bar{B}\bar{C}}{}^{\bar{A}} = 0 \tag{35}$$

より，$f_{ab}{}^{\bar{A}}$ 部分だけを分離すれば従う．

Curved index も持つゲージ場 $h_\mu^{\bar{A}}$ に対しても，

$$\delta_{\rm GC}(\xi^\nu) h_\mu^{\bar{A}} = (\xi^\nu h_\nu)^{\bar{B}} \boldsymbol{X}_{\bar{B}} h_\mu^{\bar{A}} - \xi^\nu \underbrace{\boldsymbol{R}_{\mu\nu}{}^{\bar{A}}}_{=0}$$

だから，$\bar{B}=\{P_a,B\}$ で P_a と B を分離すれば，

$$\xi^a \boldsymbol{P}_a h_\mu^{\bar{A}} = \delta_P(\xi) h_\mu^{\bar{A}} = \left[\delta_{\rm GC}(\xi^\nu = e_a^\nu \xi^a) - (\xi^a h_a^B)\boldsymbol{X}_B\right] h_\mu^{\bar{A}} \tag{36}$$

，すなわち，$\boldsymbol{P}_a$ 変換は，先に定義した $\tilde{P}_a$ 変換を与える．

$\hat{\mathcal{D}}_a\phi$ の変換公式：(Conformal d'Alembertian $\Box^c\phi \equiv \hat{\Box}\phi = \hat{\mathcal{D}}^a\hat{\mathcal{D}}_a\phi$ の計算に必要）$\forall\phi$ の上で，

$$[\boldsymbol{X}_A, \hat{\mathcal{D}}_a] = [\boldsymbol{X}_A, \boldsymbol{P}_a] = f_{Aa}{}^{\bar{B}} \boldsymbol{X}_{\bar{B}}$$

だから，$\boldsymbol{P}_a$ を含まない変換 $\delta(\varepsilon) \equiv \varepsilon^A \boldsymbol{X}_A$ に対し，

$$\underbrace{\delta(\varepsilon)}_{\varepsilon^A \boldsymbol{X}_A} \hat{\mathcal{D}}_a\phi = \varepsilon^A \hat{\mathcal{D}}_a(\boldsymbol{X}_A\phi) + \varepsilon^A f_{Aa}{}^{\bar{B}} \boldsymbol{X}_{\bar{B}}\phi. \tag{37}$$

$\varepsilon^A f_{Aa}{}^{\bar{B}}$ は，例えば．$\delta(\varepsilon)h_\mu^{\bar{B}}$ の $\varepsilon^A e_\mu^a$ 比例部分を見れば読み取れる．

次節以降はKugo-Uehara notation

Kugo-Uehara	Ours
$x_m = (x_k,\ x_4)$	$(x^k,\ it) \to$ write x^μ or $-x_\mu$
$\partial_m = (\partial_k,\ \partial_4)$	$(\partial_k,\ -i\partial_t) \to$ write ∂_μ or $-\partial^\mu$
$\delta_{mn}, \quad \partial_m\partial_m \equiv \Box$	$-\eta_{\mu\nu}, \quad -\partial_\mu\partial^\mu = -\Box$
$\gamma_m = (\gamma_k,\ \gamma_4)$	$(-i\gamma^k,\ \gamma^0) \to$ write $-i\gamma^\mu$ or $i\gamma_\mu$
$\gamma_m\partial_m \equiv \partial\!\!\!/$	$-i\gamma^\mu\partial_\mu = -i\partial\!\!\!/$
$\gamma_m B_m \equiv B\!\!\!/$	$-i\gamma^\mu(-V_\mu) = +iV\!\!\!/$
a_{mn}	$ia_{\mu\nu}$
γ_5	γ_5
$\sigma_{mn} \equiv (1/4)[\gamma_m,\ \gamma_n]$	$(1/4)[-i\gamma^\mu,\ -i\gamma^\nu] = (i/2)\sigma^{\mu\nu}$
$\mathcal{C},\ \mathcal{Z},\ \mathcal{H},\ \mathcal{K},\ \mathcal{B}_m,\ \Lambda,\ \mathcal{D}$	$C,\ \chi,\ N,\ -M,\ -V_\mu,\ \lambda,\ D$
$\mathcal{A},\ \mathcal{P}_{\mathrm{R}}\chi,\ \mathcal{F}$	$\varphi,\ \sqrt{2}\psi,\ \mathcal{F}$
$(\bar\psi_{\mathrm{R}}, \bar\psi_{\mathrm{L}}),\ \begin{pmatrix}\psi_{\mathrm{R}}\\ \psi_{\mathrm{L}}\end{pmatrix}$	$(\psi, \bar\psi),\ \begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}$
$P_m,\ A,\ D,\ Q$	$P_\mu/i,\ A/i,\ D/i,\ (1/2)Q/i$
$M_{mn},\ K_m,\ S$	$-M_{\mu\nu}/i,\ -K_\mu/i,\ -(1/2)S/i$ (**negative signs**!)
$x_m \ \to\ x^\mu$ or $-x_\mu$	の置き換えをするspace優先のルールでは
ε_{mnrs}	$-i\varepsilon_{\mu\nu\rho\sigma}$、$-i\varepsilon^{\mu\nu\rho\sigma}$ (どちらも $-i$)
super変換parameter ε	2α

4 Matter Multiplets and Invariant Action Formulas

4.1 Complex general multiplets

General complex multiplet in global SUSY:

$$\mathcal{V}(x,\theta) = \mathcal{C} + i\bar\theta\gamma_5\mathcal{Z} - \tfrac{1}{2}\bar\theta(\mathcal{H} - i\gamma_5\mathcal{K} - \gamma_5\gamma^\mu\mathcal{B}_\mu)\theta$$
$$-i(\bar\theta\theta)\bar\theta\gamma_5[\Lambda - \tfrac{i}{2}\gamma^\mu\partial_\mu\mathcal{Z}] + \tfrac{1}{4}(\bar\theta\theta)(\bar\theta\theta)[\mathcal{D} - \tfrac{1}{2}\hat\Box\mathcal{C}]$$

This can be generalized into local SC general multiplet

$$\mathcal{V}_A = [\underset{w}{\mathcal{C}_A},\ \underset{w+1/2}{\mathcal{Z}_{\alpha A}},\ \underset{w+1}{\mathcal{H}_A},\ \underset{w+1}{\mathcal{K}_A},\ \underset{w+1}{\mathcal{B}_{aA}},\ \underset{w+3/2}{\Lambda_{\alpha A}},\ \underset{w+2}{\mathcal{D}_A}] = [\![\mathcal{C}_A]\!] \tag{38}$$

with arbitrary external Lorentz spinor indices $A = (\alpha_1, ..., \alpha_m; \dot\beta_1, ..., \dot\beta_n)$.
Lowest Weyl weight component $\mathcal{C}_A$ の一般的な変換を仮定：

$$\begin{aligned}
\delta_Q(\varepsilon)\mathcal{C}_A &= \tfrac{1}{2}\bar\varepsilon i\gamma_5\mathcal{Z}_A \\
\delta_M(\lambda^{ab})\mathcal{C}_A &= \tfrac{1}{2}\lambda^{ab}(\Sigma_{ab})_A{}^B\mathcal{C}_B = \tfrac{1}{2}\lambda^{ab}(\Sigma_{ab}\mathcal{C})_A \\
\delta_D(\rho)\mathcal{C}_A &= w\rho\,\mathcal{C}_A \qquad (\leftarrow\ \text{有限形}\ \mathcal{C}'_A = e^{w\rho}\mathcal{C}_A) \\
\delta_A(\theta)\mathcal{C}_A &= \tfrac{i}{2}n\theta\,\mathcal{C}_A \qquad (\leftarrow\ \text{有限形}\ \mathcal{C}'_A = e^{\frac{i}{2}n\theta}\mathcal{C}_A) \\
\delta_S(\zeta)\mathcal{C}_A &= \delta_K(\xi_K^m)\mathcal{C}_A = 0
\end{aligned}$$

Lowest component $\mathcal{C}_A$ の Weyl weight と chiral weight $(w,\ n)$ で，multiplet $\mathcal{V}_A$ の Weyl weight, chiral weight と呼ぶ．S-, K_m-変換は Weyl weight$w$ を下げる演算なので，$\mathcal{C}_A$ は multiplet の中で w の一番低い成分場だという定義により，$\mathcal{C}_A$ の S-, K_m-変換不変性がでる．

Higher components の 変換則は,（成分場の位相や係数の定義の不定性を除いて）SC algebra から一意的に決まる．

1）$w \to w + 1/2$ の場 $\mathcal{Z}_{\alpha A}$:

$$\delta_Q \mathcal{C}_A \sim \mathcal{Z}_A \quad \text{defines} \quad \mathcal{Z}_A. \tag{39}$$

1-i) Q 以外の $X' \equiv (M_{mn},\ D,\ A,\ S,\ K_m)$ 変換は代数と $\mathcal{C}_A$ の変換則から unique.

$$\delta_{X'} \mathcal{Z}_A \ \sim\ \delta_{X'} \delta_Q \mathcal{C}_A \ \sim\ [\delta_{X'},\ \delta_Q] \mathcal{C}_A + \delta_Q(\delta_{X'} \mathcal{C}_A) \tag{40}$$

1-ii) Q 変換 $\delta_Q \mathcal{Z}_A$ は，$[Q,\ Q] = P_a$, i.e.,

$$[\delta_Q(\varepsilon_1),\ \delta_Q(\varepsilon_2)] = \xi^a \hat{\mathcal{D}}_a \qquad \xi^a = \tfrac{1}{2} \bar{\varepsilon}_2 \gamma^a \varepsilon_1 \tag{41}$$

が一つ前の成分場$\mathcal{C}_A$ の上で成り立つよう決める．

一般式

$$\delta_Q(\varepsilon)\mathcal{Z}_A = \frac{1}{2}\left(\mathcal{S}_A + i\gamma_5\mathcal{P}_A + \mathcal{V}_{mA}\gamma^m + \mathcal{A}_{mA}i\gamma^m\gamma_5 + \mathcal{T}_{mnA}\sigma^{mn}\right)\varepsilon \tag{42}$$

を書いて $[Q,\,Q]\mathcal{C}_A = P_a\mathcal{C}_A$ を要求すると，

$$\mathcal{A}_{mA} = \hat{\mathcal{D}}_m\mathcal{C}_A, \quad \mathcal{T}_{mnA} = 0 \tag{43}$$

不定に残る場 $\mathcal{S}_A, \mathcal{P}_A\,\mathcal{V}_{mA}$ は，次の $w+1$ の成分場

$$(\mathcal{S}_A,\ \mathcal{P}_A,\ \mathcal{V}_{mA}) \xrightarrow{\text{define}} (-\mathcal{K}_A,\ \mathcal{H}_A,\ \mathcal{B}_{mA}) \tag{44}$$

この手続きを続けて，各段階の Q 変換で不定に残る場が higher components 場を定義し，

$$\begin{aligned}
w+1/2:\ \delta_Q\mathcal{C}_A &\overset{\text{def}}{\Longrightarrow} \mathcal{Z}_A \\
w+1:\ \delta_Q\mathcal{Z}_A &\overset{\text{def}}{\Longrightarrow} \mathcal{H}_A,\ \mathcal{K}_A,\ \mathcal{B}_{mA} \\
w+3/2:\ \delta_Q\mathcal{H}_A &\overset{\text{def}}{\Longrightarrow} \Lambda_A \\
w+2:\ \delta_Q\Lambda_A &\overset{\text{def}}{\Longrightarrow} \mathcal{D}_A
\end{aligned}$$

$\delta_Q\mathcal{D}_A$ 変換の段階で，$\hat{\mathcal{D}}_m\Lambda_A$ だけで書けて不定項が残らず，$\mathcal{D}_A$ が highest の成分場であることがわかる．

最終的に次の変換則が得られる.

$$
\begin{aligned}
\delta_Q(\varepsilon)\mathcal{C}_A &= \tfrac{1}{2}\bar{\varepsilon}i\gamma_5\mathcal{Z}_A \\
\delta_Q(\varepsilon)\mathcal{Z}_A &= (-1)^A\tfrac{1}{2}\big(i\gamma_5\mathcal{H}_A - \mathcal{K}_A - \not{\mathcal{B}} + \hat{\not{\mathcal{D}}}\mathcal{C}_A i\gamma_5\big)\varepsilon \\
\delta_Q(\varepsilon)\mathcal{H}_A &= \tfrac{1}{2}\bar{\varepsilon}i\gamma_5(\hat{\not{\mathcal{D}}}\mathcal{Z}_A + \Lambda_A) \\
\delta_Q(\varepsilon)\mathcal{K}_A &= -\tfrac{1}{2}\bar{\varepsilon}(\hat{\not{\mathcal{D}}}\mathcal{Z}_A + \Lambda_A) \\
\delta_Q(\varepsilon)\mathcal{B}_{mA} &= -\tfrac{1}{2}\bar{\varepsilon}(\hat{\mathcal{D}}_m\mathcal{Z}_A + \gamma_m\Lambda_A) - \tfrac{1}{4}R_{ab}(Q)i\gamma_5\gamma_m\varepsilon(\Sigma^{ab}\mathcal{C})_A \\
\delta_Q(\varepsilon)\Lambda_A &= (-1)^A\tfrac{1}{2}(\sigma^{ab}\hat{\mathcal{F}}_{abA} + i\gamma_5\mathcal{D}_A)\varepsilon \\
&\quad +\tfrac{1}{8}\Big\{\gamma_m\varepsilon\big(R_{ab}(Q)\gamma_m(\Sigma^{ab}\mathcal{Z})_A\big) + \gamma_5\gamma_m\varepsilon\big(R_{ab}(Q)\gamma_5\gamma_m(\Sigma^{ab}\mathcal{Z})_A\big)\Big\} \\
\delta_Q(\varepsilon)\mathcal{D}_A &= \tfrac{1}{2}\bar{\varepsilon}i\gamma_5\hat{\not{\mathcal{D}}}\Lambda_A - \tfrac{1}{4}\bar{\varepsilon}\big(R_{ab}(A) + \gamma_5\tilde{R}_{ab}(A)\big)(\Sigma^{ab}\mathcal{Z})_A \\
&\quad +(-1)^A\tfrac{1}{4}\bar{\varepsilon}\Big\{i\gamma_5(\Sigma^{ab}\not{\mathcal{B}})_A - (\Sigma^{ab}\hat{\not{\mathcal{D}}}\mathcal{C})_A\Big\}\hat{R}_{ab}(Q)
\end{aligned}
$$

where $\hat{\mathcal{D}}_a$ is the conformal covariant derivative and $\hat{\mathcal{F}}_{abA}$ is a field-strength-like quantity given by

$$
\begin{aligned}
\hat{\mathcal{F}}_{abA} &= 2\hat{\mathcal{D}}_{[a}\mathcal{B}_{b]A} + \tfrac{1}{2}i\epsilon_{abcd}[\hat{\mathcal{D}}^c,\, \hat{\mathcal{D}}^d]\mathcal{C}_A \\
&= 2\hat{\mathcal{D}}_{[a}\mathcal{B}_{b]A} + \tfrac{i}{4}\varepsilon_{ab}{}^{cd}R_{cd}^{\mathrm{cov}.mn}(M)(\Sigma_{mn}\mathcal{C})_A \\
&\quad +\tfrac{1}{2}R_{ab}(Q)\mathcal{Z}_A + \tfrac{1}{2}w\mathcal{C}_A R_{ab}(A) - \tfrac{1}{2}n\mathcal{C}_A\tilde{R}_{ab}(A).
\end{aligned}
$$

Other SC transformation: with $\delta_{ASK}=\delta_S(\zeta)+\delta_K(\xi^a_K)+\delta_A(\theta)$,

$$
\begin{aligned}
\delta_{ASK}\mathcal{C}_A &= \tfrac{1}{2}in\theta\mathcal{C}_A,\\
\delta_{ASK}\mathcal{Z}_A &= \left(\tfrac{1}{2}in-\tfrac{3}{4}i\gamma_5\right)\mathcal{Z}_A\theta+i(n+w\gamma_5)\zeta\mathcal{C}_A+i\gamma_5\sigma_{ab}\zeta(\Sigma^{ab}\mathcal{C})_A,\\
\delta_{ASK}\mathcal{H}_A &= \left(\tfrac{1}{2}in\mathcal{H}_A+\tfrac{3}{2}\mathcal{K}_A\right)\theta+\tfrac{1}{2}\bar{\zeta}\left((w-2)i\gamma_5+in\right)\mathcal{Z}_A+\tfrac{1}{2}\bar{\zeta}i\gamma_5\sigma_{ab}(\Sigma^{ab}\mathcal{Z})_A,\\
\delta_{ASK}\mathcal{K}_A &= \left(\tfrac{1}{2}in\mathcal{K}_A-\tfrac{3}{2}\mathcal{H}_A\right)\theta+\tfrac{1}{2}\bar{\zeta}\left((w-2)+n\gamma_5\right)\mathcal{Z}_A+\tfrac{1}{2}\bar{\zeta}\sigma_{ab}(\Sigma^{ab}\mathcal{Z})_A,\\
\delta_{ASK}\mathcal{B}_{mA} &= \tfrac{1}{2}in\theta\mathcal{B}_{mA}+\tfrac{1}{2}\bar{\zeta}\left((w+1)+n\gamma_5\right)\gamma_m\mathcal{Z}_A+\tfrac{1}{2}\bar{\zeta}\sigma_{ab}\gamma_m(\Sigma^{ab}\mathcal{Z})_A\\
&\quad +2in\xi_{Km}\mathcal{C}_A+i\varepsilon_{mnab}(\Sigma^{ab}\mathcal{C})_A\xi^n_K,\\
\delta_{ASK}\Lambda_A &= (-1)^{A+1}\tfrac{1}{2}\left(i\gamma_5\mathcal{H}_A+\mathcal{K}_A+\not{\mathcal{B}}_A-\hat{\not{\mathcal{D}}}\mathcal{C}_Ai\gamma_5\right)(w+n\gamma_5)\zeta\\
&\quad +(-1)^A\tfrac{1}{2}\left(\Sigma^{ab}(i\gamma_5\mathcal{H}+\mathcal{K}+\not{\mathcal{B}}-\hat{\not{\mathcal{D}}}\mathcal{C}i\gamma_5)\right)_A\sigma_{ab}\zeta\\
&\quad +\left(\tfrac{1}{2}in+\tfrac{3}{4}i\gamma_5\right)\theta\Lambda_A+\left\{-(w+n\gamma_5)\gamma_m\mathcal{Z}_A+\sigma_{ab}\gamma_m(\Sigma^{ab}\mathcal{Z})_A\right\}\xi^m_K,\\
\delta_{ASK}\mathcal{D}_A &= \tfrac{1}{2}in\theta\mathcal{D}_A\\
&\quad +i\bar{\zeta}(w\gamma_5+n)\left(\Lambda_A+\tfrac{1}{2}\hat{\not{\mathcal{D}}}\mathcal{Z}_A\right)+\bar{\zeta}i\gamma_5\sigma_{ab}\left(\Sigma^{ab}(\Lambda+\tfrac{1}{2}\hat{\not{\mathcal{D}}}\mathcal{Z})\right)_A\\
&\quad -\left\{2w\hat{\mathcal{D}}_m\mathcal{C}_A+2in\mathcal{B}_{mA}+2(\Sigma_{mn}\hat{\mathcal{D}}^n\mathcal{C})_A-i\varepsilon_{mnab}(\Sigma^{ab}\mathcal{B}^n)_A\right\}\xi^m_K,
\end{aligned}
$$

注意）Λ_Aは，我々は$\delta_Q\mathcal{H}_A$の変換を使って定義したが，$\delta_Q\mathcal{K}_A$でも$\delta_Q\mathcal{B}_{mA}$

を使っても定義できた．したがって，上の計算では

$$\{Q,\, X'\}\ \text{algebra on}\ \ \mathcal{K}_A,\ \mathcal{B}_{mA},\ \mathcal{D}_A$$
$$\{Q,\, Q\}\ \text{algebra on}\ \ \mathcal{D}_A$$
$$\{X'_1,\, X'_2\}\ \text{algebra on all component fields}$$

を使わなかった．よってこれらを全てチェックし確認しなければならない．All are confirmed.

Superconformal multiplets の任意の set をひっくるめて $\boldsymbol{\phi}$ と記し，その成分場 $\boldsymbol{\phi} = \{\phi_1, \phi_2, \cdots, \phi_n\}$ の関数 $\mathcal{C}_A(\phi)$ を最初の成分とする SC multiplet (general multiplet)

$$\mathcal{V}_A(\boldsymbol{\phi}) = [\![\mathcal{C}_A(\phi)]\!] \tag{45}$$

が存在するか？

先ず，任意の成分場 ϕ_i の上で，SC algebra $[X_A,\, X_B\} = f_{AB}{}^C X_C$ が成り立つ．今 $f(\phi), g(\phi)$ の上で成り立つと，積 $f(\phi)\cdot g(\phi)$ や和 $f(\phi)+g(\phi)$ の上でも成り立つことが示せるので任意の関数の上で成り立つことがわかる．したがって，$\mathcal{C}_A(\phi)$ が，Lowest component の変換則を満たして

いれば，SC代数を用いる上の手続きで，次々とhigher components を決めることが出来るので，$\mathcal{V}_A(\boldsymbol{\phi})$の存在は保証される．

Lowest component の変換則で唯一Non-trivial な条件は，S-，K-不変性である．この条件はS-不変性

$$\boxed{\delta_S(\zeta)\mathcal{C}_A(\phi)=0} \tag{46}$$

を満たせば十分である．$\{S,\,S\}=K$ なのでK-不変性は自動的に成り立つ．

4.2 Covariant spinor derivative

$$D_\alpha=\frac{\partial}{\partial\theta^\alpha}+i(\sigma^m)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m \tag{47}$$

Local superconformal (SC) version

$$\mathcal{D}_\alpha\mathcal{V}_A=[\![\mathcal{Z}_{\alpha A}]\!]\qquad\text{exists or not?} \tag{48}$$

Note the S-trf law

$$\delta_S(\zeta)\mathcal{Z}_{\alpha A}=i(w+n)\zeta_\alpha\mathcal{C}_A-i(\sigma_{ab}\zeta)_\alpha(\Sigma^{ab}\mathcal{C})_A,$$

which leads for $A = \{\beta_1 \cdots \beta_m; \dot{\Gamma}\}$ to

$$\begin{aligned} \delta_S(\zeta)\mathcal{Z}_{(\alpha\beta_1\cdots\beta_m);\dot{\Gamma}} &= -i(w+n+m)\zeta_\alpha \mathcal{C}_{\beta_1\cdots\beta_m;\dot{\Gamma}} \\ \delta_S(\zeta)\mathcal{Z}^\alpha{}_{\alpha\beta_2\cdots\beta_m;\dot{\Gamma}} &== -i\big(w+n-(m+2)\big)\zeta^\alpha \mathcal{C}_{\alpha\beta_2\cdots\beta_m;\dot{\Gamma}} \end{aligned} \tag{49}$$

So,

$$\begin{aligned} \mathcal{D}_{(\alpha}\mathcal{V}_{\beta_1\cdots\beta_m);\dot{\Gamma}} &= \Big[\Big(\mathcal{Z}_{(\alpha\beta_1\cdots\beta_m);\dot{\Gamma}}\Big)\Big] \;:\; \text{SC-multiplet iff } \; w+n=-m \\ \mathcal{D}^\alpha\mathcal{V}_{\alpha\beta_2\cdots\beta_m;\dot{\Gamma}} &= \Big[\Big(\mathcal{Z}^\alpha{}_{\alpha\beta_2\cdots\beta_m;\dot{\Gamma}}\Big)\Big] \;:\; \text{SC-multiplet iff } \; w+n=m+2 \end{aligned} \tag{50}$$

$\mathcal{D}_\alpha$ carries Weyl and chiral weights $(w = 1/2, n = -3/2)$.
Similarly, for $\bar{\mathcal{D}}_{\dot{\alpha}}$ by $n \to -n$

$$\begin{aligned} \bar{\mathcal{D}}_{(\dot{\alpha}}\mathcal{V}_{\dot{\beta}_1\cdots\dot{\beta}_m);\Gamma} &= \Big[\Big(\mathcal{Z}_{(\dot{\alpha}\dot{\beta}_1\cdots\dot{\beta}_m);\Gamma}\Big)\Big] \;:\; \text{SC-multiplet iff } \; w-n=-m \\ \bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{V}_{\dot{\alpha}\dot{\beta}_2\cdots\dot{\beta}_m;\Gamma} &= \Big[\Big(\mathcal{Z}^{\dot{\alpha}}{}_{\dot{\alpha}\dot{\beta}_2\cdots\dot{\beta}_m;\Gamma}\Big)\Big] \;:\; \text{SC-multiplet iff } \; w-n=m+2 \end{aligned} \tag{51}$$

Note that these two conditions are not compatible, but that the second case is empty for $m = 0$.

4.3 chiral multiplet

The SC counterpart of the chiral multiplet condition $\bar{D}_{\dot\alpha}\phi = 0$ in the global SUSY case should look like

$$\bar{\mathcal{D}}_{\dot\alpha}\mathcal{V}_A = [\![\mathcal{P}_\mathrm{L}\mathcal{Z}]\!] = 0 \tag{52}$$

In order for this condition to be covariant, this should be a SC multiplet. From the condition , this is possible only when

$$\text{Weyl and chiral weights } (w,\, n) \text{ of } \mathcal{V}_A \text{ satisfy } w = n \tag{53}$$

and

$$\text{the index } A \text{ is purely undotted spinors; } A = (\alpha_1 \cdots \alpha_\ell) \tag{54}$$

Or, we can see this condition more directly from the S-transformation lawof this component $\mathcal{P}_\mathrm{L}\mathcal{Z}$

$$\delta_S(\zeta)\mathcal{P}_\mathrm{L}\mathcal{Z}_A = i(n-w)\mathcal{P}_\mathrm{L}\zeta\mathcal{C}_A - i\sigma_{ab}\mathcal{P}_\mathrm{L}\zeta(\Sigma^{ab}\mathcal{C})_A,$$

第２項は，$\sigma_{ab}\mathcal{P}_\mathrm{L}$が$[ab]$添え字についてselfdual，$(\Sigma^{ab}\mathcal{C})_A$は$A$がpurely undotted (right-handed) なら，anti-selfdualなので，a, bの縮約が消えるから．

この２条件を満たす場合 chiral multiplet

$$\mathbf{\Sigma}^{(w=n)}_{A=(\alpha_1\cdots\alpha_\ell)} = \left[z_A,\ \mathcal{P}_\mathrm{R}\chi_A,\ h_A\right] \tag{55}$$

が存在し，general multiplet に次の形で埋め込まれている．

$$\mathcal{V}(\boldsymbol{\Sigma}_A) = \left[z_A,\ -i\mathcal{P}_{\rm R}\chi_A,\ -h_A,\ ih_A,\ i\hat{\mathcal{D}}_m z_A,\ 0,\ 0\right] \tag{56}$$

変換則: $\delta_{QS} = \delta_Q(\varepsilon) + \delta_S(\zeta)$

$$\boxed{\begin{aligned} \delta_{QS} z_A &= \tfrac{1}{2}\bar{\varepsilon}\mathcal{P}_{\rm R}\chi_A, \\ \delta_{QS}(\mathcal{P}_{\rm R}\chi) &= (-1)^A\left(\hat{\not{\mathcal{D}}} z_A \mathcal{P}_{\rm L}\varepsilon + h_A\mathcal{P}_{\rm R}\varepsilon + \left\{2wz_A - (\Sigma^{ab}z)_A\sigma_{ab}\right\}\mathcal{P}_{\rm R}\zeta\right), \\ \delta_{QS} h_A &= \tfrac{1}{2}\bar{\varepsilon}\hat{\not{\mathcal{D}}}(\mathcal{P}_{\rm R}\chi) - \bar{\zeta}\left\{(w-1)\mathcal{P}_{\rm R}\chi_A + \tfrac{1}{2}\sigma_{ab}(\Sigma^{ab}\mathcal{P}_{\rm R}\chi)_A\right\}. \end{aligned}}$$

4.4 chiral projection $\Pi \sim \bar{D}^2$

The SC analogue to the rigid chiral projection $\bar{D}^2 V$ is

$$\Pi\mathcal{V}_A = [\![\tfrac{1}{2}(\mathcal{H}_A - i\mathcal{K}_A)]\!]$$

This can be a SC multiplet only when the original $\mathcal{V}_A$ satisfies

$$\text{i) } w = n+2, \ \text{ and ii) } A\text{: purely undotted spinor indices} \tag{57}$$

since, then, the first component is S-inert:

$$\delta_S(\zeta)(\mathcal{H}_A - i\mathcal{K}_A) = -i(w-2-n)\bar{\zeta}\mathcal{P}_{\rm L}\mathcal{Z}_A - i\bar{\zeta}\mathcal{P}_{\rm L}\sigma_{ab}(\Sigma^{ab}\mathcal{Z})_A \tag{58}$$

The chiral multiplet $\Pi\mathcal{V}_A$ carries Weyl and chiral weights $(w+1,\ n+3) = (w+1,\ w+1)$, since $\Pi \sim \bar{D}^2$ carries $(w=1, n=3)$ and $\bar{D}_{\dot{\alpha}}$ carries $(w=1/2, n=3/2)$. So, it indeed gives the chiral multiplet

$$\boxed{\Pi\mathcal{V}_A \ = \ \left[\tfrac{1}{2}(\mathcal{H}_A - i\mathcal{K}_A),\ i\mathcal{P}_{\mathrm{R}}(\hat{\not{\mathcal{D}}}\mathcal{Z}_A + \Lambda_A),\ -\tfrac{1}{2}(\mathcal{D}_A + \hat{\Box}\mathcal{C}_A + i\hat{\mathcal{D}}_m\mathcal{B}_{mA})\right]}$$

where $\hat{\Box}$ is the conformal d'Alembertian $\hat{\Box} \equiv \hat{\mathcal{D}}^a\hat{\mathcal{D}}_a$.

4.5 linear multiplet

The complex linear multiplet $\mathcal{L}_A$ is defined by the constraint

$$\Pi\mathcal{L}_A = 0 \tag{59}$$

so that it exists only for $w = n+2$ and purely undotted A case:

$$\mathcal{L}^{(w,n=w-2)}_{A=(\alpha_1\cdots\alpha_\ell)} = \left[\mathcal{C}_A,\ \mathcal{Z}_A,\ \mathcal{H}_A,\ -i\mathcal{H}_A,\ \mathcal{B}_{mA},\ \mathcal{P}_{\mathrm{L}}\Lambda_A + \hat{\not{\mathcal{D}}}\mathcal{P}_{\mathrm{R}}\mathcal{Z}_A,\ -\hat{\Box}\mathcal{C}_A - i\hat{\mathcal{D}}_m\mathcal{B}_{mA}\right]$$

Real linear multiplet $\boldsymbol{L}$, satisfying both conditions $\Pi\boldsymbol{L} = 0$ and $\bar{\Pi}\boldsymbol{L} = 0$, can exists only for $n = 0$ and hence for $w = 2$ and no external Lorentz index A:

$$\boldsymbol{L}^{w=2} = \left[C,\ Z,\ 0,\ 0,\ B_m,\ -\hat{\not{\mathcal{D}}}Z,\ -\hat{\Box}C\right] \tag{60}$$

with a constraint $\hat{\mathcal{D}}_m B^m = 0$, which can be solved in the form

$$B_m = e^{-1} e_{m\mu} \varepsilon^{\mu\nu\rho\sigma} \big(\partial_\nu a_{\rho\sigma} - \tfrac{1}{4} i \bar{\psi}_\nu \gamma_\rho \psi_\sigma \, C\big) - \bar{Z} \sigma_{mn} \psi^n \tag{61}$$

The anti-symmetric tensor gauge field $a_{\mu\nu}$ have ${}_4C_2 - {}_4C_1 + {}_4C_0 = 6 - 4 + 1 = 3$ off-shell degrees of freedom.

4.6 gauge field-strength multiplet: analog of $\bar{D}^2 D_\alpha V$

$$\begin{aligned}
W_\alpha^{(w=n=3/2)} &= \underset{(1,3)}{\Pi} \, \underset{(\frac{1}{2}, -\frac{3}{2})}{\mathcal{D}_\alpha} \, V^{(w=n=0)} \\
&= \big[\lambda_{\mathrm{R}\alpha}, \; -\mathcal{P}_{\mathrm{R}} (\sigma \cdot \hat{F} - iD)_{\alpha\beta}, \; (\mathcal{P}_{\mathrm{R}} \hat{\not{\mathcal{D}}} \lambda)_\alpha\big] \\
\hat{F}_{\mu\nu} &= \partial_\mu B'_\nu + \tfrac{1}{2} \bar{\psi}_\mu \gamma_\nu \lambda - (\mu \leftrightarrow \nu); \quad B'_\mu \equiv e_\mu^m B_m - \tfrac{1}{2} \bar{\psi}_\mu Z \\
\hat{\mathcal{D}}_\mu \lambda &= D_\mu^\omega \lambda - \tfrac{1}{2} (\sigma \cdot \hat{F} + i \gamma_5 D) \psi_\mu - \tfrac{3}{2} b_\mu \lambda - \frac{3}{4} i \gamma_5 A_\mu \lambda.
\end{aligned} \tag{62}$$

Note that the conditions $w + n = -m$ and $w = n + 2$ for $\mathcal{D}_\alpha$ and Π operations, respectively, are just met.

4.7 Function formula

Holomorphic function $g(\mathbf{\Sigma})$ of chiral multiplets $\mathbf{\Sigma}_{A_i} = [z_{A_i},\ \mathcal{P}_{\rm R}\chi_{A_i},\ h_{A_i}]$:

$$g(\mathbf{\Sigma}) = \left[g(z) \equiv g,\ \chi'_{\rm R}\, g,\ \left(h' - \tfrac{1}{4}\bar{\chi}'_{\rm R}\chi'_{\rm R}\right) g \,\right] \tag{63}$$

where

$$\chi'_{\rm R} \equiv \sum_i \mathcal{P}_{\rm R}\chi_{A_i}\frac{\partial}{\partial z_{A_i}}, \quad h' \equiv \sum_i h_{A_i}\frac{\partial}{\partial z_{A_i}}, \quad \text{etc}, \tag{64}$$

($g(z)$ に Q 変換して，chiral multiplet の変換表と比較して higher components を次々求める.)

General function $\Phi(\mathcal{V}) \equiv \mathbf{\Phi}$ of

general multiplets $\mathcal{V}_{A_i} = [\mathcal{C}_{A_i},\ \mathcal{Z}_{A_i},\ \mathcal{H}_{A_i},\ \mathcal{K}_{A_i},\ \mathcal{B}_{mA_i},\ \Lambda_{A_i},\ \mathcal{D}_{A_i}]$:

$$\mathcal{C}(\mathbf{\Phi}) = \Phi(\mathcal{C}) \equiv \Phi,$$

$$\mathcal{Z}(\mathbf{\Phi}) = \mathcal{Z}'\,\Phi,$$

$$\begin{pmatrix} \mathcal{H}(\mathbf{\Phi}) \\ \mathcal{K}(\mathbf{\Phi}) \\ \mathcal{B}_m(\mathbf{\Phi}) \end{pmatrix} = \left[\begin{pmatrix} \mathcal{H}' \\ \mathcal{K}' \\ \mathcal{B}'_m \end{pmatrix} + \tfrac{1}{4}\bar{\mathcal{Z}}' \begin{pmatrix} -1 \\ i\gamma_5 \\ i\gamma_5\gamma_m \end{pmatrix} \mathcal{Z}' \right] \Phi$$

$$\Lambda(\mathbf{\Phi}) = \cdots\ ,$$

$$\mathcal{D}(\mathbf{\Phi}) = \cdots\ .$$

with

$$\mathcal{Z}' = \sum_i \mathcal{Z}_{A_i} \frac{\partial}{\partial \mathcal{C}_{A_i}}, \quad \mathcal{H}' = \sum_i \mathcal{H}_{A_i} \frac{\partial}{\partial \mathcal{C}_{A_i}}, \text{etc}, \tag{65}$$

4.8 Invariant Action Formulas

Fully local superconformal invariant action formulas exist.

F-type invariant action formula applicable to scalar chiral multiplet $\Sigma = \left[\, z = \frac{1}{2}(A + iB),\, \mathcal{P}_{\rm R}\chi,\, h = \frac{1}{2}(F + iG) \,\right]$ with $w = n = 3$:

$$\boxed{\begin{aligned} \int d^4x \left[\, \Sigma^{(w=n=3)} \,\right]_F &= \int d^4x\, e \Big(h + \tfrac{1}{2}\bar{\psi}_{{\rm L}m}\gamma^m \chi_{\rm R} + \bar{\psi}_{{\rm L}m}\sigma^{mn}\psi_{{\rm L}n} z + \text{h.c.} \Big) \\ &= \int d^4x\, e \Big(F + \tfrac{1}{2}\bar{\psi}_a\gamma^a\chi + \tfrac{1}{2}\bar{\psi}_a\sigma^{ab}(A - i\gamma_5 B)\psi_b \Big). \qquad (66) \end{aligned}}$$

D-type invariant action formula applies to real vector multiplet $V = \left[C,\ Z,\ H,\ K,\ B_a,\ \lambda,\ D \right]$ with $w=2$ and $n=0$:

$$\boxed{\begin{aligned}&\int d^4x \left[V^{(w=2)} \right]_D = \int d^4x \left[-\Pi V^{(w=2)} \right]_F \\ &= \int d^4x\, e\Big(D + \hat{\Box} C - \tfrac{1}{2} i \bar{\psi}_a \gamma^a \gamma_5 (\gamma^b \hat{\mathcal{D}}_b Z + \lambda) - \tfrac{1}{2} \bar{\psi}_a \sigma^{ab} (H + i\gamma_5 K) \psi_b \Big) \\ &= \int d^4x\, e\Big(D - \tfrac{1}{2} \bar{\psi}_a \gamma^a i\gamma_5 \lambda - \bar{\varphi}_a \gamma^a i\gamma_5 Z + \tfrac{1}{3} C \big(\underbrace{R + e^{-1} \bar{\psi}_\mu \mathcal{R}^\mu}_{= \mathcal{L}_{\mathrm{SUGRA}} \times (-2)} \big) \\ &\qquad + \tfrac{i}{4} \varepsilon^{abcd} \bar{\psi}_a \gamma_b \psi_c \big(B_d - A_d C - \tfrac{1}{2} \bar{\psi}_d Z \big) \Big). \qquad (67)\end{aligned}}$$

$$R = R_{\mu\nu}{}^{mn}(M) e_m{}^\nu e_m{}^\mu \Big|_{f_\mu^m = \varphi = 0}, \quad \mathcal{R}^\mu = \varepsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu D^\omega_\rho \psi_\sigma \qquad (68)$$

5 Poincaré SUGRAs and Compensating Multiplets

There are three ways of SUGRA formulation, called

1) (old) minimal, 2) new minimal and 3) non-minimal SUGRAs.

They are different only in the non-dynamical auxiliary fields used and so in their tensor calculi.

All those Poincaré invariant SUGRAs can be unifiedly understood from the superconformal SUGRA approach. The former formulations result after fixing the extraneous gauge freedom $D, (A,)\ S, K_m$ from the latter. Those gauge fixing is done by using compensating multiplet, without which the SUGRA action cannot be written in a superconformal invariant way. The different choices of the compensating multiplet lead to different SUGRAs:

$$\begin{array}{rcc} \text{(old) minimal} & : & \text{chiral compensator } \Sigma_0 \\ \text{new minimal} & : & \text{real linear compensator } L_0 \\ \text{non-minimal} & : & \text{complex linear compensator } \mathcal{L}_0 \end{array} \tag{69}$$

5.1 Old Minimal SUGRA

We first explain the (old) minimal formulation of Poincaré supergravity, in the simplest pure SUGRA system. We take a chiral compensator with Weyl and chiral weight $w = n = 1$:

$$\Sigma_0 = \left[\, z_0,\ \chi_{\mathrm{R}0},\ h_0 \,\right] \tag{70}$$

Then, $\Sigma_0\bar{\Sigma}_0$ gives an real vector multiplet with $w = 2$ to which the D-term action formula can be applied. Pure (Poincaré) supergravity Lagrangian:

$$\mathcal{L}_{\text{pure SUGRA}} = -\tfrac{1}{2}\left[\, \Sigma_0\bar{\Sigma}_0 \,\right]_D \tag{71}$$

Let us fix the extraneous D, A, S, K_m gauges by the following conditions:

$$\begin{aligned} D:\quad \mathrm{Re}z_0 = \sqrt{3}, \qquad & A:\quad \mathrm{Im}z_0 = 0, \\ S:\quad \chi_{\mathrm{R}0} = 0, \qquad & K_m:\quad b_\mu = 0, \end{aligned} \tag{72}$$

where b_μ is the D gauge field. Then, rewriting $h_0 = \frac{1}{\sqrt{3}}(S - iP)$ and $A_\mu = -\frac{2}{3}A_\mu^{\mathrm{aux}}$,

$$\Sigma_0\bar{\Sigma}_0 = [3,\ 0,\ -2S,\ 2P,\ -2A_m^{\mathrm{aux}},\ 0,\ -\tfrac{1}{3}(S^2 + P^2 - A_m^{\mathrm{aux}\,2}) \times (-2)] \tag{73}$$

Substituting this into Eq. (71) and applying the D-term formula (67), we obtain

$$\mathcal{L}_{\text{pure SUGRA}} = e\Big[-\tfrac{1}{2}(R+e^{-1}\bar{\psi}_\mu\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D^\omega_\rho\psi_\sigma)-\tfrac{1}{3}(S^2+P^2-A_m^{\text{aux}\,2})\Big]. \quad (74)$$

S, P and A_μ^{aux} constitute the well-known minimal set of auxiliarly fields, hence the name of minimal Poincaré supergravity.

Q transformation in the Poincaré SUGRA

Note that the local SUSY transformation in the resultant Poincaré SUGRA is no longer the original Q trf in SC tensor calculus: The gauge fixing condition is not invariant under Q, but it is always possible to compensate. Recall

$$(\delta_Q(\varepsilon)+\delta_S(\zeta))\chi_{\text{R}0} = \hat{\not{\mathcal{D}}} z_0\mathcal{P}_{\text{L}}\varepsilon + h_0\mathcal{P}_{\text{R}}\varepsilon + 2wz_0\mathcal{P}_{\text{R}}\zeta$$

$$\hat{\mathcal{D}}_\mu z_0 = (\partial_\mu - wb_\mu - \tfrac{1}{2}iwA_\mu)z_0 - \tfrac{1}{2}\bar{\psi}_a\chi_{\text{R}0}$$

Putting the gauge-fixing conditions $z_0=\sqrt{3}$, $\chi_{\text{R}0}=0$ and $b_\mu=0$ on the RHS, and using $w=1$, $h_0=\frac{1}{\sqrt{3}}(S-iP)$ and $A_\mu=-\frac{2}{3}A_\mu^{\text{aux}}$, we have

$$(\delta_Q(\varepsilon)+\delta_S(\zeta))\chi_0 = \tfrac{1}{\sqrt{3}}(S-i\gamma_5 P+i\gamma_5\not{A}^{\text{aux}})\varepsilon + 2\sqrt{3}\zeta$$

Thus the Q transformation preserving the gauge-fixing condition is

$$\delta_Q^{\mathrm{P}}(\varepsilon) = \delta_Q(\varepsilon) + \delta_S(\tfrac{1}{2}\eta\varepsilon), \quad \eta = -\tfrac{1}{3}(S - i\gamma_5 P + i\gamma_5 \not{A}^{\mathrm{aux}})\varepsilon \tag{75}$$

identified with Q-transformation in old minimal Poincaré SUGRA.

5.2 New Minimal SUGRA

The compensator is a real linear multiplet $\boldsymbol{L}_0$ with Weyl weight $w = 2$. Obvious candidate action for the pure SUGRA would be

$$\mathcal{L}_{\text{pure SUGRA}} \overset{?}{=} -\tfrac{1}{2}\left[\,\boldsymbol{L}_0\,\right]_D \tag{76}$$

But this identically vanishes: just like

$$\int d^4x d^4\theta \boldsymbol{L}_0 = \int d^4x \left(\int d^2\theta\ \bar{D}^2 \boldsymbol{L}_0 + \text{h.c.} \right) = 0 \tag{77}$$

in global SUSY case, we have also in this local case

$$\int d^4x \left[\,\boldsymbol{L}_0\,\right]_D = \int d^4x \left[\,\Pi \boldsymbol{L}_0\,\right]_F = 0. \tag{78}$$

Correct Lagrangian is given by

$$\mathcal{L}_{\text{pure SUGRA}} = -\tfrac{1}{2}\left[\,\boldsymbol{L}_0 \ln\left(\frac{\boldsymbol{L}_0}{\Sigma_0 \bar{\Sigma}_0}\right)\,\right]_D \tag{79}$$

where we have added an extra chiral multiplet Σ_0 with weight $w = 1$. This addition actually do not introduce extra degrees of freedom; indeed, we have additional gauge symmetry under

$$\Sigma_0 \quad \to \quad e^{\Lambda}\Sigma_0, \qquad \bar{\Sigma}_0 \quad \to \quad e^{\bar{\Lambda}}\bar{\Sigma}_0, \tag{80}$$

with chiral trf parameter Λ as the change vanishes:

$$\int d^4x \left[\, \boldsymbol{L}_0(\Lambda + \bar{\Lambda}) \,\right]_D = \int d^4x \left[\, \Lambda\, \Pi \boldsymbol{L}_0 \,\right]_F + \text{h.c.} \tag{81}$$

Using this gauge invariance, we can fix $\Sigma_0 = 1$.

Using the component fields of $\boldsymbol{L}$

$$\boldsymbol{L}_0 = \left[\, C_0,\ Z_0,\ 0,\ 0,\ B_{0m},\ -\hat{\not{D}} Z_0,\ -\hat{\Box} C_0 \,\right] \tag{82}$$

the extraneous gauge $D,\ S,\ K_m$ gauges are fixed

$$\begin{aligned} D: &\quad C_0 = 3, &\quad \text{Additional gauge}: &\quad \Sigma_0 = \sqrt{3/e}, \\ S: &\quad Z_0 = 0, &\quad K_m: &\quad b_\mu = 0, \end{aligned} \tag{83}$$

Note that chiral A-gauge is not fixed since C_0 is real. So new minimal SUGRA retains the U(1) A gauge symmetry. Also, the B_{0m} gauge field is constrained and so is rewritten in terms of the anti-symmetric tensor gauge field $a_{\mu\nu}$.

$$B_{0m} = e^{-1} e_{m\mu} \varepsilon^{\mu\nu\rho\sigma} \left(\partial_\nu a_{\rho\sigma} - \frac{3}{4} i \bar{\psi}_\nu \gamma_\rho \psi_\sigma \right) \tag{84}$$

New minimal SUGRA also has this gauge symmetry.

6 Improved SC-Gauge Fixing: general YM-matter coupled SUGRA

6.1 Fixing the extraneous gauges D, S, K

Consider the general matter coupled SUGRA system in the old minimal formulation:

$$\mathcal{L} = -\tfrac{1}{2} \Big[\Sigma_c \bar{\Sigma}_c \, \tilde{\Phi}(S, \bar{S} e^{2V_G}) \Big]_D + \Big[\Sigma_c^3 \, g(S) \Big]_F + \Big[f_{ab}(S) W_\alpha^a \varepsilon^{\alpha\beta} W_\beta^b \Big]_F \tag{85}$$

Here a, b are YM group index, S denotes a set of chiral matter multiplets $\{ S_i \}$. We here assume (without loss of generality) that only the compensator Σ_c carries Weyl weight $w = 1$ while all matter S_i's carry $w = 0$. Now we can explain another virtue of our superconformal tensor calculus. To explain the point, we omit the YM interaction terms for a while.

First, for the system with non-vanishing superpotential $g(S)$, we eliminate $g(S)$ by redefining the compensator as $g^{1/3}(S)\Sigma_c \equiv \Sigma_0$, and rewrite the Lagrangian into the following form using $\Phi \equiv \tilde{\Phi}/|g|^{2/3}$:

$$\mathcal{L} = -\tfrac{1}{2}\Big[\,\Sigma_0\bar{\Sigma}_0\,\Phi(S,\bar{S})\,\Big]_D + \Big[\,\Sigma_0^3\,\Big]_F\,, \tag{86}$$

It is clear that the system depends only on the function

$$\mathcal{G} \equiv 3\ln(\Phi/3) = 3\ln(\tilde{\Phi}/3) - \ln|g|^2$$

In this matter coupled system, the multiplet $\Sigma_0\bar{\Sigma}_0\,\Phi(\phi,\bar{\phi}) \equiv V$ in the D-term has the following first two components:

$$\begin{aligned} C(V) &= |z_0|^2\,\Phi(z,z^*) \\ \tfrac{1}{2}Z(V) &= i\,|z_0|^2\,(\Phi_i\chi_{\mathrm{L}}^i - \Phi^i\chi_{\mathrm{R}i}) + i\Phi\,(z_0\chi_{\mathrm{L}0} - z_0^*\chi_{\mathrm{R}0})\,, \end{aligned}$$

with notation $\Phi \equiv \Phi(z,z^*)$, $\Phi^i \equiv \partial\Phi(z,z^*)/\partial z_i$, $\Phi_i \equiv \partial\Phi(z,z^*)/\partial z^{*\,i}$. Recalling here that the invariant action formula took the form

$$\Big[\,V^{(w=2)}\,\Big]_D = e\Big(D + \cdots - \bar{\varphi}_a\gamma^a i\gamma_5 Z + \tfrac{1}{3}C\underbrace{\big(R + e^{-1}\bar{\psi}_\mu\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho^\omega\psi_\sigma\big)}_{=\mathcal{L}_{\mathrm{SUGRA}}\times(-2)} + \cdots\Big)$$

we see that, to obtain the canonical form of Einstein-Hilbert as well as Rarita-Schwinger action it would be best to take the gauge conditions realizing

$$C(V) = 3, \quad Z(V) = 0. \tag{87}$$

Therefore we take the following gauge conditions for the extraneous gauges D, A, S, K_m on the compensator $\Sigma_0 = [z_0, \chi_{R0}, h_0]$ as [Kugo-Uehara1982]

$$\begin{aligned} D: \quad \mathrm{Re} z_0 &= \sqrt{3}\Phi^{-1/2}, \qquad A: \quad \mathrm{Im} z_0 = 0, \\ S: \quad \chi_{R0} &= -z_0 \Phi^{-1}\Phi^i \chi_{Ri}, \qquad K_m: \quad b_\mu = 0. \end{aligned} \tag{88}$$

This gauge, called improved, or, KU gauge, is a very clever gauge. In fact, Cremmer-Ferrara-Girardello-VanProeyen calculated by using the same gauge condition as for the pure SUGRA's case; then they have to do the painful field-dependent field re-definitions corresponding to the Weyl rescaling (D-), A- and S-gauge transformations. This laborious computation is actually double: the same redefinition has to be done also in deriving the SUSY trf law. All those calculation can be bypassed in this improved KU gauge!

6.2 Old Minimal SUGRA Action

$$
\begin{aligned}
\tfrac{1}{2}e^{-1}\big[\Phi\big]_D = {}& \mathcal{G}^i_j \nabla_m z_i \nabla_m z^{*j} + \big\{ \mathcal{G}^i_j \bar{\chi}^j_{\mathrm{L}} \not{\nabla} \chi_{\mathrm{R}i} + \text{ h.c. } \big\} + \mathcal{L}_{\mathrm{SG}} \\
& + \big\{ (\mathcal{G}^{ik}_j + \tfrac{1}{2}\mathcal{G}^i_j \mathcal{G}^k)(\bar{\chi}^j_{\mathrm{L}} \not{\nabla} z_k \chi_{\mathrm{R}i}) - \mathcal{G}^i_j (\bar{\psi}_{m\mathrm{L}} \not{\nabla} z_i \gamma_m \chi^j_{\mathrm{L}}) + \text{ h.c. } \big\} \\
& - \big\{ \mathcal{G}^{ij}_{kl} + \tfrac{1}{2}\mathcal{G}^i_k \mathcal{G}^j_l - \mathcal{G}^{ij}_m (\mathcal{G}^{-1})^m_n \mathcal{G}^n_{kl} \big\} (\bar{\chi}_{\mathrm{R}i}\chi_{\mathrm{R}j})(\bar{\chi}^k_{\mathrm{L}}\chi^l_{\mathrm{L}}) \\
& - \tfrac{1}{2}\mathcal{G}^i_j (\bar{\chi}^j_{\mathrm{L}} \gamma_m \chi_{\mathrm{R}i})(\bar{\psi}_{n\mathrm{R}} \gamma_m \psi_{n\mathrm{L}}) \\
& + \tfrac{1}{8}\varepsilon^{mnk\ell}(\bar{\psi}_m \gamma_n \psi_k)\big\{ (\mathcal{G}^i \nabla_\ell z_i - \mathcal{G}_i \nabla_\ell z^{*i}) + 2\mathcal{G}^i_j (\bar{\chi}^j_{\mathrm{L}} \gamma_\ell \chi_{\mathrm{R}i}) \big\} \\
& + i\tilde{g}\big\{ 2\mathcal{G}^i_j T^{ak}_i z_k (\bar{\chi}^j_{\mathrm{L}} \lambda^a_{\mathrm{L}}) + \tfrac{1}{2}\mathcal{G}^i T^{aj}_i z_j (\bar{\psi}_{\mathrm{L}} \cdot \gamma \lambda^a_{\mathrm{R}}) - \text{ h.c. } \big\} \\
& + e^{-1}\mathcal{L}^{[\Phi]}_{\text{aux.}}
\end{aligned}
\tag{89}
$$

where ∇_μ is covariant only under YM and LL gauge trf, i.e.,

$$
\begin{aligned}
\nabla_\mu z_i &= \partial_\mu z_i - i\tilde{g} B^a_\mu T^{aj}_i z_j, \\
\nabla_\mu \chi_{\mathrm{R}i} &= (\partial_\mu + \tfrac{1}{2}\omega^{mn}_\mu(e,\psi)\sigma_{mn})\chi_{\mathrm{R}i} - i\tilde{g} B^a_\mu T^{aj}_i \chi_{\mathrm{R}j},
\end{aligned}
\tag{90}
$$

and notations are used

$$
\mathcal{G}^i \equiv \frac{\partial \mathcal{G}(z,z^*)}{\partial z_i}, \quad \mathcal{G}^i_j \equiv \frac{\partial^2 \mathcal{G}(z,z^*)}{\partial z_i \partial z^{*j}}, \quad \text{etc} \tag{91}
$$

and

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SG}} &= -\tfrac{1}{2}\Big(R + e^{-1}\bar{\psi}_\mu \varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D^\omega_\rho \psi_\sigma\Big), \\
e^{-1}\mathcal{L}^{[\Phi]}_{\mathrm{aux.}} &= -\tilde{g}\mathcal{G}^i T^{aj}_i z_j D^a - \mathcal{G}^i_j \tilde{h}_i \tilde{h}^{*j} \\
&\quad - 3\Big|\tilde{h}_0 z_0^{-1} - \tfrac{1}{3}\big(\mathcal{G}_{ij} - \tfrac{1}{3}\mathcal{G}_i\mathcal{G}_j\big)(\bar{\chi}^i_{\mathrm{L}}\chi^j_{\mathrm{L}})\Big|^2 \\
&\quad + \tfrac{3}{4}\Big[A_m + \tfrac{1}{3}i\big(\mathcal{G}^i\nabla_m z_i - \mathcal{G}_i\nabla_m z^{*i} + \mathcal{G}^i_j\bar{\chi}^i_{\mathrm{L}}\gamma_m\chi_{\mathrm{R}i}\big)\Big]^2, \\
\tilde{h}_i &\equiv h_i - (\mathcal{G}^{-1})^j_i \mathcal{G}^{kl}_j \bar{\chi}_{\mathrm{R}k}\chi_{\mathrm{R}l}
\end{aligned} \tag{92}
$$

Next, the superpotential term:

$$
e^{-1}\big[\Sigma^3_0\big]_F = \Big\{3z_0^2\tilde{h}_0 - z_0^3\mathcal{G}^i h_i - 6\bar{\chi}_{\mathrm{R}0}\chi_{\mathrm{R}0}z_0 + 3\bar{\psi}_{\mathrm{L}}\cdot\gamma\chi_{\mathrm{R}0}z_0^2 + \bar{\psi}_{m\mathrm{L}}\sigma^{mn}\psi_{n\mathrm{L}}z_0^3 + \text{ h.c.}\Big\}
$$

Finally the YM kinetic term:

$$
\begin{aligned}
&-\tfrac{1}{4}e^{-1}\left[fW^2\right]_F \\
&= \mathrm{Re}f_{ab}\big(-\tfrac{1}{4}F^a_{\mu\nu}F^b_{\mu\nu} - \tfrac{1}{2}\bar{\lambda}^a \not{\nabla}\lambda^b\big) \\
&\ + \tfrac{1}{4}\mathrm{Im}f_{ab}\big(iF^a_{\mu\nu}\tilde{F}^b_{\mu\nu} - e^{-1}\nabla_\mu(e\bar{\lambda}^a i\gamma_5\gamma^\mu\lambda^b)\big) \\
&\ + \Big\{ \tfrac{1}{4}\mathrm{Re}f_{ab}(\bar{\psi}_m\sigma^{\mu\nu}\gamma^m\lambda^a)(F^b_{\mu\nu} + \tfrac{1}{2}\bar{\psi}_\mu\gamma_\nu\lambda^b) \\
&\qquad - \tfrac{1}{2}f^i_{ab}(\bar{\chi}_{\mathrm{R}i}\sigma^{\mu\nu}\lambda^a_{\mathrm{R}})(F^b_{\mu\nu} + \tfrac{1}{2}\bar{\psi}_{\mu\mathrm{R}}\gamma_\nu\lambda^b_{\mathrm{L}}) + \tfrac{1}{8}f_{ab}(\bar{\psi}_{\mathrm{L}}\cdot\gamma\chi_{\mathrm{R}i})\bar{\lambda}^a_{\mathrm{R}}\lambda^b_{\mathrm{R}} \\
&\qquad + \big(\tfrac{1}{4}f^{ij}_{ab} - \tfrac{1}{16}f^i_{ac}(\mathrm{Re}f)^{-1}_{cd}f^j_{db}\big)(\bar{\chi}_{\mathrm{R}i}\chi_{\mathrm{R}j})(\bar{\lambda}^a_{\mathrm{R}}\lambda^b_{\mathrm{R}}) \\
&\qquad - \tfrac{1}{16}f^i_{ac}(\mathrm{Re}f)^{-1}_{cd}f^j_{db}(\bar{\chi}_{\mathrm{R}i}\sigma_{mn}\chi_{\mathrm{R}j})(\bar{\lambda}^a_{\mathrm{R}}\sigma_{mn}\lambda^b_{\mathrm{R}}) \\
&\qquad + \tfrac{1}{16}f^i_{ac}(\mathrm{Re}f)^{-1}_{cd}f^*_{db\,j}(\bar{\chi}_{\mathrm{L}j}\gamma_m\chi_{\mathrm{R}i})(\bar{\lambda}^a_{\mathrm{R}}\gamma_m\lambda^b_{\mathrm{L}}) + \text{h.c.} \Big\} \\
&\ + e^{-1}\mathcal{L}^{[fW^2]}_{\mathrm{aux.}}
\end{aligned}
\tag{93}
$$

where

$$
\begin{aligned}
e^{-1}\mathcal{L}^{[fW^2]}_{\mathrm{aux.}} &= \tfrac{1}{2}\mathrm{Re}f_{ab}\tilde{D}^a\tilde{D}^b - \tfrac{3}{4}i\mathrm{Re}f_{ab}A_m\bar{\lambda}^a_{\mathrm{R}}\gamma_m\lambda^b_{\mathrm{L}} \\
&\quad - \tfrac{1}{4}\big\{h_i f^i_{ab}(\bar{\lambda}^a_{\mathrm{R}}\lambda^b_{\mathrm{R}}) + \text{h.c.}\big\} \\
\tilde{D}^a &\equiv D^a + \tfrac{1}{2}(\mathrm{Re}f)^{-1}_{ab}\big(if^i_{bc}\bar{\chi}_{\mathrm{R}i}\lambda^c_{\mathrm{R}} + \text{h.c.}\big).
\end{aligned}
\tag{94}
$$

If we eliminate the auxiliary fields by plugging the solution of e.o.m.,

$$\begin{aligned} z_0^{-1}\tilde{h}_0 &= \tfrac{1}{3}\big(\mathcal{G}_{ij} - \tfrac{1}{3}\mathcal{G}_i\mathcal{G}_j\big)\bar{\chi}^i_{\mathrm{L}}\chi^j_{\mathrm{L}} + e^{\mathcal{G}/2} \\ h_i &= (\mathcal{G}^{-1})^j_i\left[\mathcal{G}^{kl}_j\bar{\chi}_{\mathrm{R}k}\chi_{\mathrm{R}l} - \tfrac{1}{4}f^*_{abj}\bar{\lambda}^a_{\mathrm{L}}\lambda^b_{\mathrm{L}} - e^{-\mathcal{G}/2}\mathcal{G}_j\right] \\ A_m &= -\tfrac{i}{3}\left[\mathcal{G}^i\nabla_m z_i - \mathcal{G}_i\nabla_m z^{*i} + \mathcal{G}^i_j\bar{\chi}^j_{\mathrm{L}}\gamma_m\chi_{\mathrm{R}i} - \tfrac{3}{2}\mathrm{Re}f_{ab}\bar{\lambda}^a_{\mathrm{R}}\gamma_m\lambda^b_{\mathrm{L}}\right] \\ D^a &= \tfrac{1}{2}(\mathrm{Re}f)^{-1}_{ab}\left[\tilde{g}\mathcal{G}^i T^{b\,j}_i z_j - i f^i_{bc}\bar{\chi}_{\mathrm{R}i}\lambda^c_{\mathrm{R}} + \text{h.c.}\right] \end{aligned} \tag{95}$$

the auxiliary terms yields

$$
\begin{aligned}
& e^{-1}\left\{\mathcal{L}^{[\Phi]}_{\text{aux.}} + \left[\Sigma^3_0\right]_F + \mathcal{L}^{[fW^2]}_{\text{aux.}}\right\} \\
& \Rightarrow e^{-\mathcal{G}}\left[3 + \mathcal{G}_i(\mathcal{G}^{-1})^i_j\mathcal{G}^j\right] - \tfrac{1}{2}\tilde{g}^2(\mathrm{Re}f)^{-1}_{ab}(\mathcal{G}^i T^{a\,j}_i z_j)(\mathcal{G}^k T^{b\,l}_k z_l) \\
& \quad + e^{-\mathcal{G}/2}\Big\{\bar{\psi}_{m\mathrm{L}}\sigma^{mn}\psi_{n\mathrm{L}} - \mathcal{G}^i\bar{\psi}_{\mathrm{L}}\cdot\gamma\chi_{\mathrm{R}i} + \tfrac{1}{4}\mathcal{G}^i(\mathcal{G}^{-1})^j_i f^*_{ab\,j}\bar{\lambda}^a_{\mathrm{L}}\lambda^b_{\mathrm{L}} \\
& \qquad\qquad + \left(\mathcal{G}^{ij} - \mathcal{G}^i\mathcal{G}^j - \mathcal{G}^k(\mathcal{G}^{-1})^l_k\mathcal{G}^{ij}_l\right)\bar{\chi}_{\mathrm{R}i}\chi_{\mathrm{R}j} + \text{h.c.}\Big\} \\
& \quad + \Big\{\tfrac{i}{2}\tilde{g}\mathcal{G}^i T^{a\,j}_i z_j(\mathrm{Re}f)^{-1}_a b f^k_{bc}\bar{\chi}_{\mathrm{R}k}\lambda^c_{\mathrm{R}} \\
& \qquad - \tfrac{1}{4}\left(\mathcal{G}^i\nabla_m z_i + \tfrac{1}{2}\mathcal{G}^i_j\bar{\chi}^j_{\mathrm{L}}\gamma_m\chi_{\mathrm{R}i}\right)\mathrm{Re}f_{ab}\bar{\lambda}^a_{\mathrm{R}}\gamma_m\lambda^b_{\mathrm{L}} \\
& \qquad + \tfrac{1}{32}f^i_{ab}(\mathcal{G}^{-1})^j_i f^*_{cd\,j}(\bar{\lambda}^a_{\mathrm{R}}\lambda^b_{\mathrm{R}})(\bar{\lambda}^c_{\mathrm{L}}\lambda^d_{\mathrm{L}}) + \tfrac{3}{32}\left(\mathrm{Re}f_{ab}\bar{\lambda}^a_{\mathrm{R}}\gamma_m\lambda^b_{\mathrm{L}}\right)^2 \\
& \qquad - \tfrac{1}{4}\mathcal{G}^{ij}_k(\mathcal{G}^{-1})^k_l f^l_{ab}(\bar{\chi}_{\mathrm{R}i}\chi_{\mathrm{R}j})(\bar{\lambda}^a_{\mathrm{R}}\lambda^b_{\mathrm{R}}) + \text{h.c.}\Big\}
\end{aligned}
\tag{96}
$$

6.3 remarks

The minimal system posseing the canonical kinetic terms should have

$$
\begin{aligned}
\mathcal{G}^j_i &= -\delta^j_i, \qquad f_{ab} = \delta_{ab} \\
\Rightarrow \qquad & \mathcal{G} = -z_i z^{*i} - \ln|g(z)|^2
\end{aligned}
\tag{97}
$$

Recall

$$\mathcal{G} = 3\ln\frac{\tilde{\Phi}(z,z^*)}{3} - \ln|g(z)|^2 \tag{98}$$

So, the minimal sysytem has the function $\tilde{\Phi}$

$$3\ln(\tfrac{\tilde{\Phi}(z,z^*)}{3}) = -\sum_i z_i z^{*i} \equiv -K(z,z^*) \tag{99}$$

If we call this Kähler potential, then the D-term action looks like

$$\mathcal{L} = -\tfrac{1}{2}\Big[\,\Sigma_c\bar{\Sigma}_c\, 3e^{-K(S,\bar{S}e^{2V_G})/3}\,\Big]_D + \Big[\,\Sigma_c^3\, g(S)\,\Big]_F + \Big[W_\alpha^a\varepsilon^{\alpha\beta}W_\beta^a\Big]_F \tag{100}$$

That is, Kähler potential appears in the exponent in the old minimal SUGRA.

Now the scalar potential of the system is given by

$$V = e^{-\mathcal{G}}\left(-\mathcal{G}_i(\mathcal{G}^{-1})^i_j\mathcal{G}^j - 3\right) \tag{101}$$

For the above minimal system,

$$e^{-\mathcal{G}} = e^{\sum_i z_i z^{*i}}\,|g(z)|^2, \qquad \mathcal{G}^i = -z_i - \frac{g_i^*}{g^*}$$

Substituting this we obtain

$$V = e^{\sum_i |z_i|^2}\left(\left|g^i + z^{*i} g\right|^2 - 3\,|g|^2\right)$$
$$= e^{\sum_i \frac{z_i z^{*i}}{M^2}}\left(\left|g^i(z) + \frac{z^{*i}}{M^2} g(z)\right|^2 - \frac{3}{M^2}\,|g(z)|^2\right) \tag{102}$$

In the last line we recovered the Planck mass M (Note $\dim g(z) = 3$). Note that negative contribution comes from the compensator auxiliary field h_0.

7 Relation between different compensator SUGRAs

See S. Ferrara, L. Girardello, T. Kugo and A. Van Proeyen, "Relation Between Different Auxiliary Field Formulations of $N = 1$ Supergravity Coupled to Matter," Nucl. Phys. B **223** (1983), 191-217

General system in new minimal SUGRA:

$$\mathcal{L}_{\text{new minimal}} = \left[L \left\{ \ln\left(\frac{f(L, S_i, \bar{S}^i e^{m_i g_R V_R} e^{2\tilde{g}V})}{S_0 \bar{S}_0} \right) + g_R V_R \right\} \right]_D$$

8 Covariant spinor derivative and vector derivatives in Poincaré SUGRA: u-associated derivatives

See the Section 4 of Ref.[2].

9 Gravity Multiplets

There are Superconformal multiplets called

$$\begin{aligned}
&\text{scalar curvature multiplet } R \text{ (chiral)}:\\
&\qquad R^{(1,1)} = \Sigma_0^{-1}\Pi\bar{\Sigma}_0\\
&\text{Ricci tensor multiplet } E_a \text{ (general)}:\\
&\qquad E_a^{(1,0)} = [\![A_a]\!]^{\mathrm{P}} = \left[\!\!\left[\left(A_a - \bar{\psi}_a i\gamma_5\lambda_S + \frac{3}{4}\bar{\lambda}_S i\gamma_5\gamma_a\lambda_S\right)\right]\!\!\right]\\
&\text{Weyl tensor multiplet } W_{\alpha\beta\gamma}^{(3/2,3/2)} \text{ (chiral)}:\\
&\qquad W_{\alpha\beta\gamma} = [\![(\ (\sigma^{ab})_{\alpha\beta}R_{ab}(Q)_\gamma\ )]\!]
\end{aligned} \tag{103}$$

containing scalar curvature R, Ricci tensor $R_{\mu\nu}$ and Weyl tensor $C_{\mu\nu\rho\sigma}$, respevtively, (in the F-component, vector B_{ma} component, and $\chi_{\delta\alpha\beta\gamma}$ component).

See P. K. Townsend and P. van Nieuwenhuizen, "Anomalies, Topological Invariants and the Gauss-Bonnet Theorem in Supergravity," Phys. Rev. D **19** (1979), 3592.

10 SUGRA version of $f(R)$ gravity

See S. Cecotti, "HIGHER DERIVATIVE SUPERGRAVITY IS EQUIVALENT TO STANDARD SUPERGRAVITY COUPLED TO MATTER. 1.," Phys. Lett. B **190** (1987), 86-92.

11 Relation to superspace approach

Superspace formulation of Superconformal SUGRA was first given rather recently by Butter in Ref.[7]. The relation between the Superconformal tensor calculus approach and superspace approach was clarified in detail in Refs.[5] and [6].

my homepage URL: http://www2.yukawa.kyoto-u.ac.jp/~taichiro.kugo/

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