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Supersymmetry

九後「ゲージ場の量子論」の convention に従う。

Table I. Wess-Bagger との換算表

Ours	Wess-Bagger
$p^\mu,\ x^\mu,\ \partial_\mu$	same $p^\mu,\ x^\mu,\ \partial_\mu$
$\eta^{\mu\nu}=\eta_{\mu\nu}=\mathrm{diag}(+1,-1,-1,-1)$	$-\eta^{\mu\nu},\ -\eta_{\mu\nu}$
$\varepsilon^{\mu\nu\rho\sigma}\quad \varepsilon^{0123}=+1$	same $\varepsilon^{\mu\nu\rho\sigma}$
$\psi_\alpha,\ \bar\psi^{\dot\beta}$	same $\psi_\alpha,\ \bar\psi^{\dot\beta}$
$\varepsilon^{\alpha\beta}=\varepsilon^{\dot\alpha\dot\beta}=i\sigma_2$	same $\varepsilon^{\alpha\beta},\ \varepsilon^{\dot\alpha\dot\beta}$
$\varepsilon_{\alpha\beta}=\varepsilon_{\dot\alpha\dot\beta}=i\sigma_2$	$-\varepsilon_{\alpha\beta},\ -\varepsilon_{\dot\alpha\dot\beta}$
$\sigma_\mu,\ \gamma_\mu$	same $\sigma_\mu,\ \gamma_\mu$
$\sigma^\mu,\ \gamma^\mu$	$-\sigma^\mu,\ -\gamma^\mu$
γ_5	$i\gamma_5$
$\sigma_{\mu\nu},\ \bar\sigma_{\mu\nu}$	$2i\sigma_{\mu\nu},\ 2i\bar\sigma_{\mu\nu}$
thus, $\Box,\ \sigma^\mu\partial_\mu,\bar\sigma^\mu\partial_\mu$	$-\Box,\ -\sigma^\mu\partial_\mu,-\bar\sigma^\mu\partial_\mu$

Standard spinor index position:

4-component Majorana Ψ

$$\Psi=\begin{pmatrix}\psi_\alpha\\ \bar\psi^{\dot\alpha}=\varepsilon^{\dot\alpha\dot\beta}(\psi_\beta)^*\end{pmatrix},\qquad \bar\Psi\equiv\Psi^\dagger\gamma_0=(\,\psi^\alpha\quad\bar\psi_{\dot\alpha}\,)=\Psi^{\mathrm{T}}C. \tag{1}$$

2-component Weyl spinors

$$\begin{aligned}&\text{at right-end:}\qquad \psi_\alpha,\quad \bar\psi^{\dot\alpha},\\ &\text{at left-end:}\qquad \psi^\alpha,\quad \bar\psi_{\dot\alpha},\end{aligned} \tag{2}$$

so that

$$\begin{aligned}&\psi\phi\equiv\psi^\alpha\phi_\alpha,\qquad \bar\psi\bar\phi\equiv\bar\psi_{\dot\alpha}\bar\phi^{\dot\alpha},\\ &\psi\sigma_\mu\bar\phi\equiv\psi^\alpha(\sigma_\mu)_{\alpha\dot\beta}\bar\phi^{\dot\beta},\qquad \bar\psi\bar\sigma_\mu\phi\equiv\bar\psi_{\dot\alpha}(\bar\sigma_\mu)^{\dot\alpha\beta}\phi_\beta,\\ &\psi\sigma_{\mu\nu}\phi\equiv\psi^\alpha(\sigma_{\mu\nu})_\alpha{}^\beta\phi_\beta,\qquad \bar\psi\bar\sigma_{\mu\nu}\bar\phi\equiv\bar\psi_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar\phi^{\dot\beta}.\end{aligned} \tag{3}$$

Formulas:

$$(\sigma_\mu)^{\alpha\dot\beta}\equiv\varepsilon^{\alpha\gamma}\varepsilon^{\dot\beta\dot\delta}(\sigma_\mu)_{\gamma\dot\delta}=(\varepsilon\sigma_\mu\varepsilon^T)^{\alpha\dot\beta}=(\bar\sigma_\mu)^{\dot\beta\alpha}. \tag{4}$$

$$\psi_\alpha\phi_\beta = -\frac{1}{2}\varepsilon_{\alpha\beta}(\psi\phi) + \psi_{(\alpha}\phi_{\beta)}, \qquad \bar\psi^{\dot\alpha}\bar\phi^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}(\bar\psi\bar\phi) + \bar\psi^{(\dot\alpha}\bar\phi^{\dot\beta)},$$
$$\psi^\alpha\phi^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}(\psi\phi) + \psi^{(\alpha}\phi^{\beta)}, \qquad \bar\psi_{\dot\alpha}\bar\phi_{\dot\beta} = \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}(\bar\psi\bar\phi) + \bar\psi_{(\dot\alpha}\bar\phi_{\dot\beta)}. \tag{5}$$

Ordering change

$$\begin{aligned}
&\psi\phi = \phi\psi, \qquad \bar\psi\bar\phi = \bar\phi\bar\psi,\\
&\psi\sigma_\mu\bar\phi = -\bar\phi\bar\sigma_\mu\psi,\\
&\psi\sigma_\mu\bar\sigma_\nu\phi = \phi\sigma_\nu\bar\sigma_\mu\psi, \qquad \bar\psi\bar\sigma_\mu\sigma_\nu\bar\phi = \bar\phi\bar\sigma_\nu\sigma_\mu\bar\psi,\\
&\psi\sigma_{\mu\nu}\phi = -\phi\sigma_{\mu\nu}\psi, \qquad \bar\psi\bar\sigma_{\mu\nu}\bar\phi = -\bar\phi\bar\sigma_{\mu\nu}\bar\psi.
\end{aligned} \tag{6}$$

hermitian conjugation

$$\begin{aligned}
&(\psi\phi)^\dagger = \bar\phi\bar\psi,\\
&(\psi\sigma_\mu\bar\phi)^\dagger = \phi\sigma_\mu\bar\psi, \qquad (\bar\psi\bar\sigma_\mu\phi)^\dagger = \bar\phi\bar\sigma_\mu\psi,\\
&(\psi\sigma_\mu\bar\sigma_\nu\phi)^\dagger = \bar\phi\bar\sigma_\nu\sigma_\mu\bar\psi,\\
&(\psi\sigma_{\mu\nu}\phi)^\dagger = \bar\phi\bar\sigma_{\mu\nu}\bar\psi.
\end{aligned} \tag{7}$$

Fierz (4-components)

$$\begin{aligned}
(\bar\Psi_1\Psi_2)(\bar\Psi_3\Psi_4) = -\frac{1}{4}\Big[& (\bar\Psi_1 1\Psi_4)(\bar\Psi_3 1\Psi_2) + (\bar\Psi_1\gamma_5\Psi_4)(\bar\Psi_3\gamma_5\Psi_2)\\
&+(\bar\Psi_1\gamma^\mu\Psi_4)(\bar\Psi_3\gamma_\mu\Psi_2) - (\bar\Psi_1\gamma^\mu\gamma_5\Psi_4)(\bar\Psi_3\gamma_\mu\gamma_5\Psi_2)\\
&+\frac{1}{2}(\bar\Psi_1\sigma^{\mu\nu}\Psi_4)(\bar\Psi_3\sigma_{\mu\nu}\Psi_2) \Big].
\end{aligned} \tag{8}$$

This yields 2-component Fierz formulas:

$$\begin{aligned}
(\bar\Psi_1\mathcal{P}_{\substack{R\\L}}\Psi_2)(\bar\Psi_3\mathcal{P}_{\substack{R\\L}}\Psi_4) = -\frac{1}{2}\Big[& (\bar\Psi_1\mathcal{P}_{\substack{R\\L}}\Psi_4)(\bar\Psi_3\mathcal{P}_{\substack{R\\L}}\Psi_2)\\
&+\frac{1}{4}(\bar\Psi_1\mathcal{P}_{\substack{R\\L}}\sigma^{\mu\nu}\Psi_4)(\bar\Psi_3\mathcal{P}_{\substack{R\\L}}\sigma_{\mu\nu}\Psi_2) \Big].
\end{aligned} \tag{9}$$

$$(\bar\Psi_1\mathcal{P}_{\substack{R\\L}}\Psi_2)(\bar\Psi_3\mathcal{P}_{\substack{L\\R}}\Psi_4) = -\frac{1}{2}\Big[(\bar\Psi_1\mathcal{P}_{\substack{R\\L}}\gamma^\mu\Psi_4)(\bar\Psi_3\mathcal{P}_{\substack{L\\R}}\gamma_\mu\Psi_2) \Big]. \tag{10}$$

e.g.,

$$\begin{aligned}
&(\theta\psi)(\theta\phi) = (\theta\psi)(\phi\theta) = -\frac{1}{2}\Big[(\theta\theta)(\phi\psi) + \frac{1}{4}(\theta\sigma^{\mu\nu}\theta)(\phi\sigma_{\mu\nu}\psi)\Big] = -\frac{1}{2}(\theta\theta)(\phi\psi),\\
&(\theta\sigma_\rho\bar\psi)(\theta\sigma_\lambda\bar\phi) = -(\theta\sigma_\rho\bar\psi)(\bar\phi\bar\sigma_\lambda\theta) = +\frac{1}{2}(\theta\theta)(\bar\phi\bar\sigma_\lambda\sigma_\rho\bar\psi) = \frac{1}{2}(\theta\theta)\big(\eta_{\lambda\rho}\bar\phi\bar\psi - i\bar\phi\bar\sigma_{\lambda\rho}\bar\psi\big),\\
&(\theta\psi)(\bar\theta\bar\phi) = -\frac{1}{2}(\theta\sigma^\mu\bar\theta)(\bar\phi\bar\sigma_\mu\psi) = +\frac{1}{2}(\theta\sigma^\mu\bar\theta)(\psi\sigma_\mu\bar\phi)
\end{aligned} \tag{11}$$

Vector multiplet field:

$$\begin{aligned}
V &= [C,\ \chi,\ M,\ N,\ V_\mu,\ \lambda,\ D] \\
&= C + i\bar\theta\gamma_5\chi - \tfrac{1}{2}\bar\theta(N - i\gamma_5 M - \gamma_5\gamma^\mu V_\mu)\theta \\
&\quad -i(\bar\theta\theta)\bar\theta\gamma_5[\lambda - \tfrac{i}{2}\gamma^\mu\partial_\mu\chi] + \tfrac{1}{4}(\bar\theta\theta)(\bar\theta\theta)[D - \tfrac{1}{2}\Box C] \\
&= C + i\theta\chi - i\bar\theta\bar\chi - \tfrac{1}{2}(\theta\theta)\mathcal{H} - \tfrac{1}{2}(\bar\theta\bar\theta)\bar{\mathcal{H}} + (\theta\sigma^\mu\bar\theta)V_\mu \\
&\quad +i(\theta\theta)\bar\theta[\bar\lambda - \tfrac{i}{2}\bar\sigma^\mu\partial_\mu\chi] - i(\bar\theta\bar\theta)\theta[\lambda - \tfrac{i}{2}\sigma^\mu\partial_\mu\bar\chi] + \tfrac{1}{2}(\theta\theta)(\bar\theta\bar\theta)[D - \tfrac{1}{2}\Box C],
\end{aligned} \tag{12}$$

where

$$\mathcal{H} \equiv N - iM, \qquad \bar{\mathcal{H}} \equiv N + iM. \tag{13}$$

Chiral multiplet field ϕ: constrained by $\bar D_{\dot\alpha}\phi = 0$

$$\begin{aligned}
\phi &= \exp(-i\theta\partial\!\!\!/\bar\theta)[\varphi + \theta\sqrt{2}\psi + \theta\theta\mathcal{F}] \\
&= \varphi + \theta\sqrt{2}\psi + \theta\theta\mathcal{F} + (\theta\sigma^\mu\bar\theta)(-i\partial_\mu\varphi) \\
&\quad +i(\theta\theta)\bar\theta[-\tfrac{1}{2}\bar\sigma^\mu\partial_\mu\sqrt{2}\psi] + \tfrac{1}{2}(\theta\theta)(\bar\theta\bar\theta)[-\tfrac{1}{2}\Box\varphi] \\
&= [\varphi,\ -i\sqrt{2}\mathcal{P}_{\rm R}\psi,\ -i\mathcal{F},\ -\mathcal{F},\ -i\partial_\mu\varphi,\ 0,\ 0]
\end{aligned} \tag{14}$$

Anti-chiral multiplet field $\bar\phi$:

$$\bar\phi = [\varphi^*,\ i\sqrt{2}\mathcal{P}_{\rm L}\psi,\ i\mathcal{F}^*,\ -\mathcal{F}^*,\ i\partial_\mu\varphi^*,\ 0,\ 0] \tag{15}$$

(real) Linear multiplet field:

$$\begin{aligned}
L :&\quad \text{constrained by}\ \ DD\,L = \bar D\bar D\,L = 0 \\
&= C + i\theta\chi - i\bar\theta\bar\chi + (\theta\sigma^\mu\theta)V_\mu \\
&\quad +i(\theta\theta)\bar\theta[\tfrac{i}{2}\bar\sigma^\mu\partial_\mu\chi] - i(\bar\theta\bar\theta)\theta[\tfrac{i}{2}\sigma^\mu\partial_\mu\bar\chi] + \tfrac{1}{2}(\theta\theta)(\bar\theta\bar\theta)[\tfrac{1}{2}\Box\varphi] \\
&= [C,\ \chi,\ 0,\ 0,\ V_\mu, i\gamma^\mu\partial_\mu\chi,\ \Box C]
\end{aligned} \tag{16}$$

with V_μ constrained by $\partial V = 0$, which is solved as

$$\begin{aligned}
V^\mu &= \varepsilon^{\mu\nu\rho\sigma}\partial_\nu a_{\rho\sigma}, \\
\delta a_{\mu\nu} &= -\frac{1}{2}\bar\alpha\sigma_{\mu\nu}\gamma_5\chi
\end{aligned} \tag{17}$$

Supersymmetry transformation law:

Vector multiplet: in 4-component notation

$$\begin{aligned}
\delta C &= i\bar{\alpha}\gamma_5\chi, \\
\delta\chi &= (-i\gamma^\mu V_\mu - \partial_\mu C\gamma_5\gamma^\mu + M + i\gamma_5 N)\alpha, \\
\delta M &= \bar{\alpha}\lambda - i\bar{\alpha}\gamma^\mu\partial_\mu\chi, \\
\delta N &= i\bar{\alpha}\gamma_5\lambda + \bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\chi, \\
\delta V_\mu &= i\bar{\alpha}\gamma_\mu\lambda + \bar{\alpha}\partial_\mu\chi, \\
\delta\lambda &= -\frac{i}{2}\sigma^{\mu\nu}\alpha(\partial_\mu V_\nu - \partial_\nu V_\mu) + i\gamma_5 D\alpha, \\
\delta D &= \bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\lambda.
\end{aligned} \tag{18}$$

This reads, in 2-component notation,

$$\begin{aligned}
\delta C &= i\alpha\chi - i\bar{\alpha}\bar{\chi}, \\
\delta\begin{pmatrix}\chi \\ \bar{\chi}\end{pmatrix} &= \begin{pmatrix}-i\sigma^\mu(V_\mu - i\partial_\mu C)\bar{\alpha} + i\mathcal{H}\alpha \\ -i\bar{\sigma}^\mu(V_\mu + i\partial_\mu C)\alpha - i\bar{\mathcal{H}}\bar{\alpha}\end{pmatrix}, \\
\delta\mathcal{H} &= -2i\bar{\alpha}\bar{\lambda} - 2\bar{\alpha}\bar{\sigma}^\mu\partial_\mu\chi, \\
\delta\bar{\mathcal{H}} &= 2i\alpha\lambda + 2\alpha\sigma^\mu\partial_\mu\bar{\chi}, \\
\delta V_\mu &= i(\alpha\sigma_\mu\bar{\lambda} + \bar{\alpha}\bar{\sigma}_\mu\lambda) + (\alpha\partial_\mu\chi + \bar{\alpha}\partial_\mu\bar{\chi}), \\
\delta\begin{pmatrix}\lambda \\ \bar{\lambda}\end{pmatrix} &= \begin{pmatrix}-\frac{i}{2}\sigma^{\mu\nu}\alpha(\partial_\mu V_\nu - \partial_\nu V_\mu) + iD\alpha \\ -\frac{i}{2}\sigma^{\mu\nu}\bar{\alpha}(\partial_\mu V_\nu - \partial_\nu V_\mu) - iD\bar{\alpha}\end{pmatrix}, \\
\delta D &= \alpha\sigma^\mu\partial_\mu\bar{\lambda} - \bar{\alpha}\bar{\sigma}^\mu\partial_\mu\lambda.
\end{aligned} \tag{19}$$

Noting the embedding formula of chiral and antichiral into vector:

$$\begin{aligned}
\text{chiral} \qquad & \text{anti-chiral} \\
C = \varphi, \qquad & C = \varphi^*, \\
\chi = -i\sqrt{2}\psi, \qquad & \bar{\chi} = +i\sqrt{2}\bar{\psi}, \\
\mathcal{H} = -2\mathcal{F}, \qquad & \bar{\mathcal{H}} = -2\mathcal{F}^*, \\
V_\mu = -i\partial_\mu\varphi, \qquad & V_\mu = +i\partial_\mu\varphi^*, \\
\text{other compts.} = 0 \qquad & \text{other compts.} = 0,
\end{aligned} \tag{20}$$

the vector susy transformation law leads to

$$\begin{cases}\delta\varphi = \sqrt{2}\alpha\psi, \\ \delta\psi = -i\sqrt{2}\partial\!\!\!/\varphi\bar{\alpha} + \sqrt{2}\mathcal{F}\alpha, \\ \delta\mathcal{F} = -i\sqrt{2}\bar{\alpha}\bar{\partial\!\!\!/}\psi,\end{cases} \qquad \begin{cases}\delta\varphi^* = \sqrt{2}\bar{\alpha}\bar{\psi}, \\ \delta\bar{\psi} = -i\sqrt{2}\bar{\partial\!\!\!/}\varphi^*\alpha + \sqrt{2}\mathcal{F}^*\bar{\alpha}, . \\ \delta\mathcal{F}^* = -i\sqrt{2}\alpha\partial\!\!\!/\bar{\psi},\end{cases} \tag{21}$$

Multiplication law:

Function of vector multiplet fields $\Phi(V_i)$

In 4-component notation,

$$
\begin{aligned}
C_\Phi &= \Phi(C_i), \\
\chi_\Phi &= \chi' \Phi(C_i), \\
\begin{pmatrix} M_\Phi \\ N_\Phi \\ V_{\Phi\mu} \end{pmatrix} &= \left[\begin{pmatrix} M' \\ N' \\ V'_\mu \end{pmatrix} - \frac{1}{4}\bar{\chi}' \begin{pmatrix} i\gamma_5 \\ 1 \\ \gamma_\mu\gamma_5 \end{pmatrix} \chi' \right] \Phi(C_i), \\
\lambda_\Phi &= \left[\lambda' + \tfrac{1}{2}(N' + i\gamma_5 M' - \gamma_5\gamma^\mu V'_\mu + i\partial\!\!\!/ C')\chi' - \tfrac{1}{4}\chi'(\bar{\chi}'\chi') \right] \Phi(C_i) \\
D_\Phi &= \Big[D' + \tfrac{1}{2}(N'N' + M'M' + V'^\mu V'_\mu + \partial^\mu C' \partial_\mu C') - \bar{\lambda}'\chi' + \tfrac{1}{2}\bar{\chi}' i\partial\!\!\!/ \chi' \\
&\qquad - \tfrac{1}{4}\bar{\chi}'(N' + i\gamma_5 M' - \gamma_5\gamma^\mu V'_\mu)\chi' + \tfrac{1}{16}(\bar{\chi}'\chi')(\bar{\chi}'\chi') \Big] \Phi(C_i)
\end{aligned}
\tag{22}
$$

In 2-component notation, this reads

$$
\begin{aligned}
C_\Phi &= \Phi(C), \\
\begin{pmatrix} \chi_\Phi \\ \bar{\chi}_\Phi \end{pmatrix} &= \begin{pmatrix} \chi_i \\ \bar{\chi}_i \end{pmatrix} \Phi_i, \\
\mathcal{H}_\Phi &= \mathcal{H}_i \Phi_i - \frac{1}{2}\chi_i\chi_j \Phi_{ij} \\
\bar{\mathcal{H}}_\Phi &= \bar{\mathcal{H}}_i \Phi_i - \frac{1}{2}\bar{\chi}_i\bar{\chi}_j \Phi_{ij} \\
V_{\Phi\mu} &= V_{i\mu}\Phi_i + \frac{1}{2}\chi_i\sigma_\mu\bar{\chi}_j \Phi_{ij} \\
\begin{pmatrix} \lambda_\Phi \\ \bar{\lambda}_\Phi \end{pmatrix} &= \begin{pmatrix} \lambda_i \\ \bar{\lambda}_i \end{pmatrix} \Phi_i + \frac{1}{2} \begin{pmatrix} \bar{\mathcal{H}}_i\chi_j - (V_\mu - i\partial_\mu C)_i \sigma^\mu \bar{\chi}_j \\ \mathcal{H}_i\bar{\chi}_j + (V_\mu + i\partial_\mu C)_i \bar{\sigma}^\mu \chi_j \end{pmatrix} \Phi_{ij} \\
&\quad - \frac{1}{4} \begin{pmatrix} \chi_i(\bar{\chi}_j\bar{\chi}_k) \\ \bar{\chi}_i(\chi_j\chi_k) \end{pmatrix} \Phi_{ijk} \\
D_\Phi &= D_i\Phi_i + \Big[\tfrac{1}{2}\big(\mathcal{H}_i\bar{\mathcal{H}}_j + (V^\mu - i\partial^\mu C)_i (V_\mu + i\partial_\mu C)_j\big) \\
&\qquad - (\lambda_i\chi_j + \bar{\lambda}_i\bar{\chi}_j) + \tfrac{1}{2}(\chi_i i\partial\!\!\!/ \bar{\chi}_j + \bar{\chi}_i i\bar{\partial\!\!\!/} \chi_j) \Big] \Phi_{ij} \\
&\quad + \Big[-\tfrac{1}{4}(\chi_i\chi_j)\bar{\mathcal{H}}_k - \tfrac{1}{4}(\bar{\chi}_i\bar{\chi}_j)\mathcal{H}_k + \tfrac{1}{2}(\chi_i\sigma^\mu\bar{\chi}_j)V_{k\mu} \Big] \Phi_{ijk} \\
&\quad + \frac{1}{8}(\chi_i\chi_j)(\bar{\chi}_k\bar{\chi}_l)\Phi_{ijkl}
\end{aligned}
\tag{23}
$$

Function of chiral multiplet fields $W(\phi_i)$

$$\begin{aligned}
\varphi_W &= W(\varphi_i), \\
\psi_W &= \psi' W(\varphi_i) = \psi_i W^i, \\
\mathcal{F}_W &= \left[\mathcal{F}' - \frac{1}{2}\psi'\psi'\right] W(\varphi_i) = \mathcal{F}_i W^i - \frac{1}{2}\psi_i\psi_j W^{ij}(\varphi_i).
\end{aligned} \tag{24}$$

By the help of chiral-to-vector embedding formulas, the general function formula $\Phi(V_i)$ for vector multiplets leads to the following formula for $K(\phi_i,\ \bar{\phi}^j)$:

$$\begin{aligned}
C_K &= K(\varphi, \varphi^*) \equiv K, \\
\chi_K &= -i\sqrt{2}\psi_i K^i, \qquad \bar{\chi}_K = +i\sqrt{2}\bar{\psi}^i K_i, \\
\begin{pmatrix} \mathcal{H}_K \\ \bar{\mathcal{H}}_K \\ V_{K\mu} \end{pmatrix} &= \begin{pmatrix} -2\mathcal{F}_i K^i + \psi_i\psi_j K^{ij} \\ -2\mathcal{F}^{*i} K_i + \bar{\psi}^i\bar{\psi}^j K_{ij} \\ -i\partial_\mu\varphi_i K^i + i\partial_\mu\varphi^{*i} K_i + \psi_i\sigma_\mu\bar{\psi}^j K^i_j \end{pmatrix}, \\
\begin{pmatrix} \lambda_K \\ \bar{\lambda}_K \end{pmatrix} &= \begin{pmatrix} i\sqrt{2}\left[(\psi_i\mathcal{F}^{*j} + i\partial\!\!\!/\varphi_i\bar{\psi}^j)K^i_j - \frac{1}{2}(\bar{\psi}^j\bar{\psi}^k)\psi_i K^i_{jk}\right] \\ -i\sqrt{2}\left[(\bar{\psi}^i\mathcal{F}_j + i\bar{\partial\!\!\!/}\varphi^{*i}\psi_j)K^j_i - \frac{1}{2}(\psi_j\psi_k)\bar{\psi}^i K^{jk}_i\right] \end{pmatrix} \\
\frac{1}{2}D_K &= \left(\partial^\mu\varphi_i\partial_\mu\varphi^{*j} + \frac{1}{2}(\psi_i i\sigma^\mu\partial_\mu\bar{\psi}^j + \bar{\psi}^j i\bar{\sigma}^\mu\partial_\mu\psi_i) + \mathcal{F}_i\mathcal{F}^{*j}\right) K^i_j \\
&\quad + \frac{1}{2}\left(i(\bar{\psi}^k\bar{\sigma}^\mu\psi_i)\partial_\mu\varphi_j - (\psi_i\psi_j)\mathcal{F}^{*k}\right) K^{ij}_k \\
&\quad + \frac{1}{2}\left(i(\psi_k\sigma^\mu\bar{\psi}^i)\partial_\mu\varphi^{*j} - (\bar{\psi}^i\bar{\psi}^j)\mathcal{F}_k\right) K^k_{ij} \\
&\quad + \frac{1}{4}(\psi_i\psi_j)(\bar{\psi}^k\bar{\psi}^l) K^{ij}_{kl}
\end{aligned} \tag{25}$$

Super Yang-Mills

gauge transformation:

$$\phi \to \phi' = e^{i\Lambda}\phi, \qquad \bar{\phi} \to \bar{\phi}' = \bar{\phi}e^{-i\Lambda^\dagger},$$
$$e^{-2V} \to e^{-2V'} = e^{+i\Lambda^\dagger}e^{-2V}e^{-i\Lambda}, \quad e^{2V} \to e^{2V'} = e^{+i\Lambda}e^{2V}e^{-i\Lambda^\dagger}. \tag{26}$$

In Wess-Zumino gauge:

$$V = (\theta\sigma^\mu\bar{\theta})V_\mu + i(\theta\theta)\bar{\theta}\bar{\lambda} - i(\bar{\theta}\bar{\theta})\theta\lambda + \tfrac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D. \tag{27}$$

$$\begin{aligned} W_\alpha &\equiv \frac{1}{8}\bar{D}\bar{D}(e^{2V}D_\alpha e^{-2V}) \\ &= e^{-i\theta\bar{\partial\!\!\!/}\bar{\theta}}\big[i\lambda_\alpha - \theta_\alpha D + \tfrac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + (\theta\theta)(\sigma^\mu D_\mu\bar{\lambda})_\alpha\big], \\ W^\alpha W_\alpha\big|_{\theta\theta} &= e^{-i\theta\bar{\partial\!\!\!/}\bar{\theta}}2(\theta\theta)\Big[\lambda i\bar{D\!\!\!\!/}\bar{\lambda} - \tfrac{1}{4}(F_{\mu\nu}F^{\mu\nu} + iF_{\mu\nu}\tilde{F}^{\mu\nu}) + \tfrac{1}{2}D^2\Big] \end{aligned} \tag{28}$$

where

$$\begin{aligned} D_\mu\bar{\lambda} &= \partial_\mu\bar{\lambda} - i[V_\mu,\, \bar{\lambda}], \\ F_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu]. \end{aligned} \tag{29}$$

$$\begin{aligned} \bar{\phi}e^{-2V}\phi\big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} =& (\theta\theta)(\bar{\theta}\bar{\theta})\big[D^\mu\varphi^* D_\mu\varphi + \bar{\psi}i\bar{\sigma}^\mu D_\mu\psi + \mathcal{F}^*\mathcal{F} \\ &+ \sqrt{2}i[(\bar{\psi}\bar{\lambda})\varphi - \varphi^*(\lambda\psi)] + \text{tot.der.}\big] \end{aligned} \tag{30}$$

where

$$\begin{aligned} &D_\mu\varphi = \partial_\mu\varphi - iV_\mu\varphi, \quad D_\mu\varphi^* = \partial_\mu\varphi^* + i\varphi^* V_\mu, \\ &D_\mu\psi = \partial_\mu\psi - iV_\mu\psi. \end{aligned} \tag{31}$$

Formula

$$\begin{aligned} &(\sigma^\mu\bar{\theta})_\alpha(\theta\sigma^\nu\bar{\theta}) = +\tfrac{1}{2}(\bar{\theta}\bar{\theta})(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha \\ &(\sigma^\mu\bar{\theta})_\alpha(\bar{\theta}\bar{\lambda}) = -\tfrac{1}{2}(\bar{\theta}\bar{\theta})(\sigma^\mu\bar{\lambda})_\alpha \\ &\theta_\alpha(\bar{\theta}\bar{\lambda})(\theta\sigma^\mu\bar{\theta}) = +\tfrac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})(\sigma^\mu\bar{\lambda})_\alpha. \end{aligned} \tag{32}$$

Table II. Kugo-Uehara との換算表

Kugo-Uehara	Ours
$x_m = (x_k,\ x_4)$	$(x^k,\ it) \to$ write x^μ or $-x_\mu$
$\partial_m = (\partial_k,\ \partial_4)$	$(\partial_k,\ -i\partial_t) \to$ write ∂_μ or $-\partial^\mu$
$\partial_m\partial_m \equiv \Box$	$-\partial_\mu\partial^\mu = -\Box$
δ_{mn}	$-\eta_{\mu\nu}$
$\gamma_m = (\gamma_k,\ \gamma_4)$	$(-i\gamma^k,\ \gamma^0) \to$ write $-i\gamma^\mu$ or $i\gamma_\mu$
$\gamma_m\partial_m \equiv \partial\!\!\!/$	$-i\gamma^\mu\partial_\mu = -i\partial\!\!\!/$
$\gamma_m B_m \equiv B\!\!\!/$	$-i\gamma^\mu(-V_\mu) = +iV\!\!\!/$
a_{mn}	$ia_{\mu\nu}$
γ_5	γ_5
$\sigma_{mn} \equiv (1/4)[\gamma_m,\ \gamma_n]$	$(1/4)[-i\gamma^\mu,\ -i\gamma^\nu] = (i/2)\sigma^{\mu\nu}$
$\mathcal{C},\ \mathcal{Z},\ \mathcal{H},\ \mathcal{K},\ \mathcal{B}_m,\ \mathit{\Lambda},\ \mathcal{D}$	$C,\ \chi,\ N,\ -M,\ -V_\mu,\ \lambda,\ D$
$\mathcal{A},\ \mathcal{P}_{\rm R}\chi,\ \mathcal{F}$	$\varphi,\ \sqrt{2}\psi,\ \mathcal{F}$
$(\bar\psi_{\rm R}, \bar\psi_{\rm L}),\ \begin{pmatrix}\psi_{\rm R}\\ \psi_{\rm L}\end{pmatrix}$	$(\psi, \bar\psi),\ \begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}$
$P_m,\ A,\ D,\ Q$	$P_\mu/i,\ A/i,\ D/i,\ (1/2)Q/i$
$M_{mn},\ K_m,\ S$	$-M_{\mu\nu}/i,\ -K_\mu/i,\ -(1/2)S/i$ (**negative signs**!)
$x_m\ \to\ x^\mu$ or $-x_\mu$	の置き換えをする space 優先のルールでは
ε_{mnrs}	$-i\varepsilon_{\mu\nu\rho\sigma}$、$-i\varepsilon^{\mu\nu\rho\sigma}$ (どちらも $-i$)
super 変換 parameter ε	2α

§1. Superconformal group $SU(2,2|1)$

4D Conformal Group: $SO(4,2) \cong SU(2,2)$

If the system has only massless particles, the energy momentum tensor $\Theta_{\mu\nu}$ can be chosen to be symmetric and traceless:

$$\partial^\nu \Theta_{\mu\nu} = 0, \quad \Theta_{\mu\nu} = \Theta_{\nu\mu}, \quad \Theta^\mu_\mu = 0. \tag{33}$$

Then we can have the following conserved currents and charges:

$$\Theta_{\mu\nu}: \quad \to \quad P_\mu = \int d^3x \Theta_{\mu 0}$$

$$\mathcal{M}_{\mu\nu\rho} = x_\mu\Theta_{\nu\rho} - x_\nu\Theta_{\mu\rho}: \quad \to \quad M_{\mu\nu} = \int d^3x \mathcal{M}_{\mu\nu 0}$$

$$\mathcal{D}_\mu = x^\nu \Theta_{\nu\mu} : \quad \to \quad D = \int d^3x \mathcal{D}_0$$

$$\mathcal{K}_{\mu\nu} = 2x_\mu x^\rho \Theta_{\rho\nu} - x^2 \Theta_{\mu\nu} : \quad \to \quad K_\mu = \int d^3x \mathcal{K}_{\mu 0}. \tag{34}$$

(Note that this K_μ has opposite sign to that of van Nieuwenhuizen.) Or, in terms of the differential operators, they can be expressed (by replacing $\Theta_{\mu 0} \to i\partial_\mu$) as

$$\begin{aligned} &P_\mu = i\partial_\mu, \qquad M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \\ &D = ix^\rho \partial_\rho, \qquad K_\mu = i(2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu). \end{aligned} \tag{35}$$

Identifying

$$\delta x^\mu = (ia^\rho P_\rho + i\frac{1}{2}\varepsilon^{\rho\sigma} M_{\rho\sigma} x^\mu + i\rho D + ib^\rho K_\rho) x^\mu, \tag{36}$$

we find

$$\delta x^\mu = -a^\mu + \varepsilon^\mu{}_\rho x^\rho - \rho x^\mu + (-2b^\rho x_\rho x^\mu + b^\mu x^2) \tag{37}$$

The finite K_μ transformation is the translation in the inverted space:

$$x^\mu \to x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2} \quad \leftrightarrow \quad \frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu. \tag{38}$$

They satisfy the following *4D conformal algebra*:

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\sigma} M_{\mu\rho}), \\ &[P_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu), \\ &[K_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu), \\ &[D, M_{\mu\nu}] = [P_\mu, P_\nu] = [K_\mu, K_\nu] = 0, \\ &[P_\mu, D] = iP_\mu, \quad [K_\mu, D] = -iK_\mu, \\ &[P_\mu, K_\nu] = 2i(\eta_{\mu\nu} D - M_{\mu\nu}). \end{aligned} \tag{39}$$

This 4D conformal group is in fact identical with the extended Lorentz group $SO(4,2)$ in 6 dimensions with metric

$$\eta_{ab} = \begin{pmatrix} \eta_{\mu\nu} & & \\ & -1 & \\ & & +1 \end{pmatrix}, \tag{40}$$

for which the generators $M_{ab} = -M_{ba}$ $(a, b = 0, 1, \cdots, 5)$ satisfy

$$[M_{ab}, M_{cd}] = -i(\eta_{ac} M_{bd} - \eta_{bc} M_{ad} - \eta_{ad} M_{bc} + \eta_{bd} M_{ac}). \tag{41}$$

This is easily seen if we identify the Lorentz generators in the extra dimensions labeled by 4 and 5 as

$$M_{\mu 4} \equiv \frac{1}{2}(P_\mu - K_\mu), \quad M_{\mu 5} \equiv \frac{1}{2}(P_\mu + K_\mu), \quad M_{54} \equiv D. \tag{42}$$

By considering the spinor representation, this algebra is also seen to be isomorphic with $SU(2,2)$. The generators Γ^a of the Clifford algebra for $SO(4,2)$, satisfying

$$\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\eta^{ab}, \tag{43}$$

can be represented, for instance, by the following 8×8 matrices:

$$\begin{aligned}
\Gamma^\mu &= \gamma^\mu \otimes \sigma^1 = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}, \\
\Gamma^4 &= i\gamma_5 \otimes \sigma^1 = \begin{pmatrix} 0 & i\gamma_5 \\ i\gamma_5 & 0 \end{pmatrix}, \\
\Gamma^5 &= 1_4 \otimes (-\sigma^2) = \begin{pmatrix} 0 & i1_4 \\ -i1_4 & 0 \end{pmatrix}.
\end{aligned} \tag{44}$$

The "chirality" matrix (analogous to γ_5 in 4 dimension) is given by

$$\Gamma_7 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5 = 1_4 \otimes \sigma_3 = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}. \tag{45}$$

The Lorentz generators M_{ab} of $SO(4,2)$ are then represented by

$$\begin{aligned}
& M_{ab} = \frac{i}{4}[\Gamma_a,\, \Gamma_b] = \frac{1}{2}\begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar\sigma_{ab} \end{pmatrix}, \\
& \sigma_{a=\mu,b=\nu} = \sigma_{\mu\nu}, \quad \sigma_{\mu 4} = \gamma_\mu\gamma_5, \quad \sigma_{\mu 5} = \gamma_\mu, \quad \sigma_{54} = i\gamma_5 \\
& \bar\sigma_{a=\mu,b=\nu} = \sigma_{\mu\nu}, \quad \bar\sigma_{\mu 4} = \gamma_\mu\gamma_5, \quad \bar\sigma_{\mu 5} = -\gamma_\mu, \quad \bar\sigma_{54} = -i\gamma_5.
\end{aligned} \tag{46}$$

For the 8 component Dirac spinor

$$\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}, \tag{47}$$

the invariant spinor inner-product is given by $\bar\Psi\Phi \equiv \Psi^\dagger A \Phi$ with a metric matrix

$$A = i\Gamma^0\Gamma^5\Gamma_7 = \gamma_0 \otimes 1_2 = \begin{pmatrix} \gamma^0 & 0 \\ 0 & \gamma^0 \end{pmatrix}, \qquad A^\dagger = A = A^{-1}. \tag{48}$$

Indeed, Γ_a are hermitian under this metric, $\Gamma_a^\dagger A = A\Gamma_a$, so are the $SO(4,2)$ generators M_{ab}:

$$A^{-1}M_{ab}^\dagger A = M_{ab} \quad i.e., \quad \gamma^0\sigma_{ab}^\dagger\gamma^0 = \sigma_{ab}. \tag{49}$$

Clearly the 4-component Weyl spinor ψ (or ϕ) gives an irreducible representation of the Lorentz group $SO(4,2)$, for which the Lorentz group element $\Lambda = \exp(\frac{i}{2}\varepsilon^{ab}M_{ab})$ is represented by

$$\exp(\frac{i}{4}\varepsilon^{ab}\sigma_{ab}). \tag{50}$$

These 4×4 matrices belong to $SU(2,2)$ since σ_{ab} are traceless and hermitian under the metric $a \equiv i\gamma^0\gamma_5$ (which has two $+1$ and two -1 eigenvalues). Moreover, Since $6\times 5/2 = 15$ σ_{ab} exist and give a complete set for such traceless and hermitian 4×4 matrices, any $SU(2,2)$ matrix is expressed in the form Eq. (50) (at least in the neighborhood of the identity) and so we have the isomorphism of the algebra $SO(4,2)\simeq SU(2,2)$.

With this isomorphism $SO(4,2)\simeq SU(2,2)$, we thus find a simple 4 dimensional representation for the 4D conformal group:

$$\begin{aligned} M_{\mu\nu} &= \tfrac{1}{2}\sigma_{\mu\nu}, \\ P_\mu &= \gamma_\mu \mathcal{P}_{\mathrm{R}}, \qquad (\mathcal{P}_{\mathrm{R}} = \tfrac{1}{2}(1+\gamma_5)) \\ K_\mu &= \gamma_\mu \mathcal{P}_{\mathrm{L}}, \qquad (\mathcal{P}_{\mathrm{L}} = \tfrac{1}{2}(1-\gamma_5)) \\ D &= \tfrac{1}{2} i\gamma_5. \end{aligned} \tag{51}$$

We have seen that the 4D conformal algebra $SO(4,2)\simeq SU(2,2)$ can be represented by traceless 4×4 matrices $M_{ab} = \frac{1}{2}\sigma_{ab}$ acting on a 4-component spinor ψ. Then it is clear that it can be extended to the superconformal algebra $SU(2,2|1)$ acting on a (4+1)-component super-spinor (ψ,φ) by adding another single component φ (which should have opposite statistics to the original component ψ). $SU(2,2|1)$ is defined to be a supergroup consisting of 5×5 matrices (of unimodular superdeterminant) which leave the innerproduct

$$\psi_1^\dagger \gamma_0 \psi_2 + \varphi_1^\dagger \varphi_2 \tag{52}$$

invariant. Clearly, there are 24 independent generators as a whole, which we can take, for instance,

$$\begin{aligned} M_{ab} &= \frac{1}{2}\begin{pmatrix} \sigma_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = -\frac{1}{4}\begin{pmatrix} 1_4 & 0 \\ 0 & 4 \end{pmatrix}, \\ \Sigma_\alpha &= 2\begin{pmatrix} 0_4 & 0 \\ \delta_\alpha^j & 0 \end{pmatrix}, \qquad \bar{\Sigma}^\alpha = 2\begin{pmatrix} 0_4 & \delta_i^\alpha \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{53}$$

Note that a diagonal (supertraceless) matrix A appears here. This gives the defining representation of $SU(2,2|1)$ algebra. From this we can easily find the following albebra written in 6 dimensional notation:

$$\begin{aligned} &[\Sigma,\, M_{ab}] = \tfrac{1}{2}\sigma_{ab}\Sigma, \qquad [\bar{\Sigma},\, M_{ab}] = -\tfrac{1}{2}\bar{\Sigma}\sigma^{ab} \\ &[\Sigma,\, A] = +\tfrac{3}{4}\Sigma, \qquad [\bar{\Sigma},\, A] = -\tfrac{3}{4}\bar{\Sigma}, \qquad [M_{ab},\, A] = 0, \\ &\{\Sigma,\, \Sigma\} = \{\bar{\Sigma},\, \bar{\Sigma}\} = 0, \qquad \{\Sigma,\, \bar{\Sigma}\} = \sigma^{ab}M_{ab} - 4A. \end{aligned} \tag{54}$$

(The $U(1)$ charge A is defined to coincide with van Nieuwenhuizen and $A = \dfrac{3}{4}R$ for Sohnius's charge R.) where, in confirming the last relation $\{\Sigma,\, \bar{\Sigma}\} = \sigma^{ab}M_{ab} - 4A$, we need the completeness relation

$$\tfrac{1}{4}\big[\tfrac{1}{2}(\sigma^{ab})_i{}^j(\sigma_{ab})_k{}^l + \delta_i{}^j\delta_k{}^l\big] = \delta_i{}^l\delta_k{}^j. \tag{55}$$

This shows that $\varSigma$ is an $SU(2,2) \simeq SO(4,2)$ spinor generator and $\bar{\varSigma}$ charge is its conjugate, so that they can be decomposed into two 2-component Weyl spinors in 4-dimension as follows:

$$\varSigma = \begin{pmatrix} Q_\alpha \\ \bar{S}^{\dot{\alpha}} \end{pmatrix}, \qquad \bar{\varSigma} = \varSigma^\dagger\gamma_0 = (S^\alpha,\ \bar{Q}_{\dot{\alpha}}). \tag{56}$$

Clearly, these 15+4+4+1 = 24 matrices again span a complete set of 5×5 (supertraceless) matrices and give the whole generators of $SU(2,2|1)$ superconformal algebra. The $SU(2,2|1)$ group acts on the 5 component super spinor as

$$\exp i\big(\tfrac{1}{2}\theta^{ab}M_{ab} + \theta A + \bar{\varepsilon}\varSigma + \bar{\varSigma}\varepsilon\big)\begin{pmatrix} \psi \\ \varphi \end{pmatrix} \tag{57}$$

(where ψ is an $SU(2,2)$ spinor field and φ a single component field and they should be fermion and boson (or vice versa), respectively, since the spinor transformation parameter ε is Grassmann odd) with $\bar{\varepsilon} = \varepsilon^\dagger\gamma^0$, which leaves invariant the innerproduct with metric

$$\alpha \equiv \begin{pmatrix} \gamma^0 & \\ & 1 \end{pmatrix}. \tag{58}$$

Indeed, the generators in the exponent

$$\bar{\varepsilon}\varSigma + \bar{\varSigma}\varepsilon = 2\begin{pmatrix} 0 & \varepsilon \\ \bar{\varepsilon} & 0 \end{pmatrix} \tag{59}$$

satisfies the hermitisity under this metric α:

$$(\bar{\varepsilon}\varSigma + \bar{\varSigma}\varepsilon)^\dagger\alpha = \alpha(\bar{\varepsilon}\varSigma + \bar{\varSigma}\varepsilon). \tag{60}$$

Rewriting Eq. (54) into 4 dimensional notation, we find the following algebra in addition to the $SO(4,2) \simeq SU(2,2)$ subalgebra:

$$\begin{aligned}
&[\begin{pmatrix} Q \\ S \end{pmatrix},\, M_{\mu\nu}] = \tfrac{1}{2}\sigma_{\mu\nu}\begin{pmatrix} Q \\ S \end{pmatrix}, \qquad [\begin{pmatrix} Q \\ S \end{pmatrix},\, A] = \tfrac{3}{4}\gamma_5\begin{pmatrix} Q \\ -S \end{pmatrix}, \\
&\begin{matrix} [Q,\,P_\mu] = 0, & [Q,\,K_\mu] = \gamma_\mu S, \\ [S,\,P_\mu] = \gamma_\mu Q, & [S,\,K_\mu] = 0, \end{matrix}\ , \qquad [\begin{pmatrix} Q \\ S \end{pmatrix},\, D] = i\tfrac{1}{2}\begin{pmatrix} Q \\ -S \end{pmatrix}, \\
&[A,\,M_{\mu\nu}] = [A,\,P_\mu] = [A,\,K_\mu] = [A,\,D] = 0, \\
&\{Q,\,\bar{Q}\} = 2\gamma^\mu P_\mu, \qquad \{S,\,\bar{S}\} = 2\gamma^\mu K_\mu, \\
&\{S,\,\bar{Q}\} = 2iD + \sigma^{\mu\nu}M_{\mu\nu} + 4\gamma_5 A, \quad (\{Q,\,\bar{S}\} = -2iD + \sigma^{\mu\nu}M_{\mu\nu} - 4\gamma_5 A).
\end{aligned} \tag{61}$$

§2. Yang-Mills theory of superalgebra

Consider a superalgebra whose generators (devided by i), $X_A = T_A/i$ satisfying

$$[X_A,\, X_B\} = f_{AB}{}^C X_C. \tag{62}$$

For definiteness, we here mean by X_A always an *operator* acting on any fields $\varPhi$, for which the infinitesimal transformation is given by

$$\delta(\varepsilon)\varPhi = \varepsilon\varPhi, \qquad \varepsilon \equiv \varepsilon^A X_A, \tag{63}$$

where ε^A are the transformation parameters. Introduce the gauge field by

$$h_\mu = h_\mu^A X_A. \tag{64}$$

The covariant derivative

$$D_\mu\varPhi \equiv (\partial_\mu - h_\mu)\varPhi = \partial_\mu\varPhi - \delta(h_\mu)\varPhi \tag{65}$$

is defined by a property

$$\delta(\varepsilon)(D_\mu\varPhi) = \varepsilon^A\, D_\mu(X_A\varPhi) = D_\mu(\delta(\breve{\varepsilon})\varPhi), \tag{66}$$

($\breve{\varepsilon}$ は、微分 ∂_μ を受けないとする意。) from which follows the transformation law of gauge field:

$$\delta(\varepsilon)h_\mu = \partial_\mu\varepsilon + [\check{h}_\mu, \varepsilon], \quad \to \quad \delta(\varepsilon)h_\mu^A = \partial_\mu\varepsilon^A + \varepsilon^B h_\mu^C f_{CB}{}^A. \tag{67}$$

This is because

$$\begin{aligned}
\delta(\varepsilon)(D_\mu\varPhi) &= \partial_\mu(\delta(\varepsilon)\varPhi) - (\delta(\varepsilon)h_\mu)\varPhi - \delta(\varepsilon)(\check{h}_\mu\varPhi) \\
&= \partial_\mu(\varepsilon\varPhi) - (\varepsilon h_\mu)\varPhi - \varepsilon\check{h}_\mu\varPhi \\
\varepsilon^A D_\mu(X_A\varPhi) &= D_\mu(\breve{\varepsilon}\varPhi) = \partial_\mu(\breve{\varepsilon}\varPhi) - h_\mu\varepsilon\varPhi \\
\to (\varepsilon h_\mu)\varPhi &= (\partial_\mu\varepsilon)\varPhi + h_\mu\varepsilon\varPhi - \varepsilon\check{h}_\mu\varPhi \\
\to \delta(\varepsilon)h_\mu &= \partial_\mu\varepsilon + [\check{h}_\mu, \varepsilon] \\
\text{i.e.,}\delta(\varepsilon)h_\mu^A &= \partial_\mu\varepsilon^A + \varepsilon^B h_\mu^C f_{CB}{}^A
\end{aligned} \tag{68}$$

where $\check{O}$ means that O is neither transformed by ∂_μ nor X_A.

The curvature tensor (field strength) is defined by

$$
\begin{aligned}
&[D_\mu,\, D_\nu]\Phi \equiv R^A_{\mu\nu} X_A \Phi \quad \text{on} \quad \forall\Phi \\
&\rightarrow \quad R^A_{\mu\nu} = \partial_\nu h^A_\mu - \partial_\mu h^A_\nu - h^B_\nu h^C_\mu f_{CB}{}^A = \partial_\nu h^A_\mu - D_\mu h^A_\nu \\
&\text{or} \quad R_{\mu\nu} = \partial_\nu h_\mu - \partial_\mu h_\nu - [\check{h}_\mu, \check{h}_\nu] \\
\because)\quad &[D_\mu,\, D_\nu]\Phi = \partial_\mu D_\nu \Phi - \delta(h_\mu) D_\nu \Phi - (\mu \leftrightarrow \nu) \\
&\qquad = \partial_\mu \partial_\nu \Phi - \partial_\mu (h_\nu \Phi) - D_\nu (\check{h}_\mu \Phi) - (\mu \leftrightarrow \nu) \\
&\qquad = \partial_\mu \partial_\nu \Phi - (\partial_\mu h_\nu)\Phi - \partial_\mu(\check{h}_\nu \Phi) - \partial_\nu(\check{h}_\mu \Phi) + h_\nu \check{h}_\mu \Phi - (\mu \leftrightarrow \nu) \\
&\qquad = -(\partial_\mu h_\nu)\Phi + h_\nu \check{h}_\mu \Phi - (\mu \leftrightarrow \nu) \\
&\qquad = (\partial_\nu h_\mu - \partial_\mu h_\nu)\Phi + [\check{h}_\nu, \check{h}_\mu]\Phi
\end{aligned}
\tag{69}
$$

The curvature tensor is covariant as usual:

$$
\delta(\varepsilon) R_{\mu\nu} = [\check{R}_{\mu\nu}, \varepsilon] \quad \rightarrow \quad \delta(\varepsilon) R^A_{\mu\nu} = \varepsilon^B R^C_{\mu\nu} f_{CB}{}^A
$$

Note) When the transformation by X_A is linearly realized on the fields Φ^i, then it is possible to use the representation matrix t_A instead of the *active operators* X_A as adopted above. We should, however, note that the operator X_A is represented by a *transposed matrix* or by *right multiplication*:

$$
X_A \Phi^i = (t^{\mathrm{T}}_A)^i{}_j \Phi^j = \Phi^j (t_A)_j{}^i \tag{70}
$$

in order for the same structure constant appears as above:

$$
[X_A, X_B\} = f_{AB}{}^C X_C \;\leftrightarrow\; [t_A, t_B\} = f_{AB}{}^C t_C \tag{71}
$$

This is because

$$
\begin{aligned}
X_A X_B \Phi^i &= X_A (t^{\mathrm{T}}_B)^i{}_j \Phi^j = (t^{\mathrm{T}}_B)^i{}_j X_A \Phi^j = (t^{\mathrm{T}}_B)^i{}_j (t^{\mathrm{T}}_A)^j{}_k \Phi^k = (t^{\mathrm{T}}_B t^{\mathrm{T}}_A)^i{}_k \Phi^k \\
[X_A, X_B\}\Phi^i &= (X_A X_B - (-1)^{AB} X_B X_A)\Phi^i = (t^{\mathrm{T}}_B t^{\mathrm{T}}_A - (-1)^{AB} t^{\mathrm{T}}_A t^{\mathrm{T}}_B)^i_j \Phi^j = \big([t^{\mathrm{T}}_B, t^{\mathrm{T}}_A\}\big)^i{}_j \Phi^j \\
&= \big([t_A, t_B\}^{\mathrm{T}}\big)^i{}_j \Phi^j = \Phi^j \big([t_A, t_B\}\big)_j{}^i \\
&= \Phi^j f_{AB}{}^C (t_C)_j{}^i = f_{AB}{}^C (t^{\mathrm{T}}_C)^i{}_j \Phi^j = f_{AB}{}^C X_C \Phi^i
\end{aligned}
\tag{72}
$$

Or, this is simply owing to the fact that the *right multiplication* of matrix preserves the multiplication order of the operator actions:

$$
X_A X_B \Phi^i = X_A \Phi^k (t_B)_k{}^i = \Phi^j (t_A)_j{}^k (t_B)_k{}^i = \Phi^j (t_A t_B)_j{}^i \tag{73}
$$

Superconformal Algebra:

$$
\begin{aligned}
&[P_\rho,\, M_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu),\\
&[P_\mu,\, D] = iP_\mu, \qquad \{Q,\, \bar{Q}\} = 2\gamma^\mu P_\mu,\\
&[M_{\mu\nu},\, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho}),\\
&[P_\mu,\, K_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}),\\
&\{S,\, \bar{Q}\} = 2iD + \sigma^{\mu\nu}M_{\mu\nu} + 4\gamma_5 A, \quad \big(\{Q,\, \bar{S}\} = -2iD + \sigma^{\mu\nu}M_{\mu\nu} - 4\gamma_5 A\big).\\
&[K_\rho,\, M_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu),\\
&[K_\mu,\, D] = -iK_\mu, \qquad \{S,\, \bar{S}\} = 2\gamma^\mu K_\mu,\\
&[Q,\, M_{\mu\nu}] = \tfrac{1}{2}\sigma_{\mu\nu}Q, \qquad \big([\bar{Q},\, M_{\mu\nu}] = -\tfrac{1}{2}\bar{Q}\sigma_{\mu\nu}\big),\\
&[Q,\, D] = i\frac{1}{2}Q, \qquad [Q,\, A] = \frac{3}{4}\gamma_5 Q, \qquad [S,\, P_\mu] = \gamma_\mu Q,\\
&\big([\bar{Q},\, D] = i\frac{1}{2}\bar{Q}, \qquad [\bar{Q},\, A] = \frac{3}{4}\bar{Q}\gamma_5, \qquad [\bar{S},\, P_\mu] = -\bar{Q}\gamma_\mu\big),\\
&[S,\, M_{\mu\nu}] = \frac{1}{2}\sigma_{\mu\nu}S, \qquad \big([\bar{S},\, M_{\mu\nu}] = -\tfrac{1}{2}\bar{S}\sigma_{\mu\nu}\big),\\
&[S,\, D] = -i\frac{1}{2}S, \qquad [S,\, A] = -\frac{3}{4}\gamma_5 S, \qquad [Q,\, K_\mu] = \gamma_\mu S,\\
&\big([\bar{S},\, D] = -i\frac{1}{2}\bar{S}, \qquad [\bar{S},\, A] = -\frac{3}{4}\bar{S}\gamma_5, \qquad [\bar{Q},\, K_\mu] = -\bar{S}\gamma_\mu\big).
\end{aligned} \tag{74}
$$

Superconformal (anti-hermitian) generators and trf parameters and gauge fields:

$$
\begin{aligned}
X_A &= i^{-1}(P_m,\, Q,\, M_{mn},\, D,\, A,\, S,\, K_m)\\
&\equiv (\boldsymbol{P}_m,\, \boldsymbol{Q},\, \boldsymbol{M}_{mn},\, \boldsymbol{D},\, \boldsymbol{A},\, \boldsymbol{S},\, \boldsymbol{K}_m),\\
\varepsilon^A X_A &= \xi^m \boldsymbol{P}_m + \bar{\varepsilon}\boldsymbol{Q} + \tfrac{1}{2}\lambda^{mn}\boldsymbol{M}_{mn} + \rho\boldsymbol{D} + \theta\boldsymbol{A} + \bar{\zeta}\boldsymbol{S} + \xi_K^m\boldsymbol{K}_m,\\
h_\mu^A X_A &= e_\mu^{\ m}\boldsymbol{P}_m + \bar{\psi}_\mu\boldsymbol{Q} + \tfrac{1}{2}\omega_\mu^{\ mn}\boldsymbol{M}_{mn} + b_\mu\boldsymbol{D} + A_\mu\boldsymbol{A} + \bar{\varphi}_\mu\boldsymbol{S} + f_\mu^{\ m}\boldsymbol{K}_m.
\end{aligned} \tag{75}
$$

From the SC algebra table (74), let us calculate

$$
[\check{h}_\nu,\, \check{h}_\mu] \equiv [\check{h}_\nu^B X_B,\, \check{h}_\mu^C X_C] = h_\mu^C h_\nu^B f_{BC}{}^A X_A. \tag{76}
$$

Note that

$$
\begin{aligned}
[\delta(\varepsilon_1),\, \delta(\varepsilon_2)] &= [\varepsilon_1^B X_B,\, \varepsilon_2^C X_C] = \varepsilon_2^C\varepsilon_1^B[X_B,\, X_C\}\\
&= \varepsilon_2^C\varepsilon_1^B f_{BC}{}^A X_A = \delta(\varepsilon_3^A = \varepsilon_2^C\varepsilon_1^B f_{BC}{}^A).
\end{aligned} \tag{77}
$$

In the following, all the gauge fields are understood not transformed by X_A, though the check symbol $\check{O}$ is omitted:

$$
[e\cdot\boldsymbol{P},\, \tfrac{1}{2}\omega\cdot\boldsymbol{M} + b\boldsymbol{D}] = (e_n\omega^{nm} + e^m b)\boldsymbol{P}_m,
$$

$$
\begin{aligned}
&[\bar\psi_1\boldsymbol{Q},\,\bar{\boldsymbol{Q}}\psi_2]=(-2i\bar\psi_1\gamma^m\psi_2)\boldsymbol{P}_m,\\
&[\tfrac{1}{2}\omega_1\cdot\boldsymbol{M},\,\tfrac{1}{2}\omega_2\cdot\boldsymbol{M}]=\tfrac{1}{2}(\omega_1^{ml}\omega_{2l}{}^n-\omega_1^{nl}\omega_{2l}{}^m)\boldsymbol{M}_{mn},\\
&[e\cdot\boldsymbol{P},\,f\cdot\boldsymbol{K}]=2e\cdot f\boldsymbol{D}-\tfrac{1}{2}[2(e^mf^n-e^nf^m)]\boldsymbol{M}_{mn},\\
&[\bar\psi\boldsymbol{Q},\,\bar{\boldsymbol{S}}\varphi]=-2\bar\psi\varphi\boldsymbol{D}+\tfrac{1}{2}(-2i\bar\psi\sigma^{mn}\varphi)\boldsymbol{M}_{mn}+4i\bar\psi\gamma_5\varphi\boldsymbol{A},\\
&\big([\bar\varphi\boldsymbol{S},\,\bar{\boldsymbol{Q}}\psi]=+2\bar\varphi\psi\boldsymbol{D}+\tfrac{1}{2}(-2i\bar\varphi\sigma^{mn}\psi)\boldsymbol{M}_{mn}-4i\bar\varphi\gamma_5\psi\boldsymbol{A}\big),\\
&[f\cdot\boldsymbol{K},\,\tfrac{1}{2}\omega\cdot\boldsymbol{M}+b\boldsymbol{D}]=(f_n\omega^{nm}-f^mb)\boldsymbol{K}_m,\\
&[\bar\varphi_1\boldsymbol{S},\,\bar{\boldsymbol{S}}\varphi_2]=(-2i\bar\varphi_1\gamma^m\varphi_2)\boldsymbol{K}_m,\\
&[\bar\psi\boldsymbol{Q},\,\tfrac{1}{2}\omega\cdot\boldsymbol{M}+b\boldsymbol{D}+A\boldsymbol{A}]=\bar\psi(-\tfrac{i}{4}\sigma\cdot\omega+\tfrac{1}{2}b-\tfrac{3}{4}i\gamma_5A)\boldsymbol{Q},\\
&\big([\tfrac{1}{2}\omega\cdot\boldsymbol{M}+b\boldsymbol{D}+A\boldsymbol{A},\,\bar{\boldsymbol{Q}}\psi]=\bar{\boldsymbol{Q}}(-\tfrac{i}{4}\sigma\cdot\omega-\tfrac{1}{2}b+\tfrac{3}{4}i\gamma_5A)\psi\big),\\
&[\bar\varphi\boldsymbol{S},\,e\cdot\boldsymbol{P}]=-i\bar\varphi\gamma_me^m\boldsymbol{Q},\quad\big([e\cdot\boldsymbol{P},\,\bar{\boldsymbol{S}}\varphi]=-i\bar{\boldsymbol{Q}}\gamma_me^m\varphi\big),\\
&[\bar\varphi\boldsymbol{S},\,\tfrac{1}{2}\omega\cdot\boldsymbol{M}+b\boldsymbol{D}+A\boldsymbol{A}]=\bar\varphi(-\tfrac{i}{4}\sigma\cdot\omega-\tfrac{1}{2}b+\tfrac{3}{4}i\gamma_5A)\boldsymbol{S},\\
&\big([\tfrac{1}{2}\omega\cdot\boldsymbol{M}+b\boldsymbol{D}+A\boldsymbol{A},\,\bar{\boldsymbol{S}}\varphi]=\bar{\boldsymbol{S}}(-\tfrac{i}{4}\sigma\cdot\omega+\tfrac{1}{2}b-\tfrac{3}{4}i\gamma_5A)\varphi\big),\\
&[\bar\psi\boldsymbol{Q},\,f\cdot\boldsymbol{K}]=-i\bar\psi\gamma_mf^m\boldsymbol{S},\quad\big([f\cdot\boldsymbol{K},\,\bar{\boldsymbol{Q}}\psi]=-i\bar{\boldsymbol{S}}\gamma_mf^m\psi\big),
\end{aligned}
\tag{78}
$$

Curvatures: $R^A_{\mu\nu}=2\partial_\nu h^A_\mu+h^B_\mu h^C_\nu f_{CB}{}^A$ (上の結果から読み取ると、以下の表式は$h^B_\mu h^C_\nu f_{CB}{}^A$部分が皆符号が逆のようだ。これで、Kugo-Ueharaとは合っているのだが、要conventionチェック。)

$$
\begin{aligned}
&R_{\mu\nu}{}^m(P)=2\partial_\nu e^m{}_\mu-2\omega_\nu{}^{mn}e_{n\mu}+2b_\nu e^m{}_\mu+2i\bar\psi_\nu\gamma^m\psi_\mu,\\
&R_{\mu\nu}{}^{mn}(M)=2\partial_\nu\omega_\mu{}^{mn}-2\omega_\nu{}^{mc}\omega_{\mu c}{}^n+4(f^m{}_\nu e^n{}_\mu-f^n{}_\nu e^m{}_\mu)+4i\bar\psi_\nu\sigma^{mn}\varphi_\mu,\\
&R_{\mu\nu}(D)=2\partial_\nu b_\mu+4f^n{}_\nu e_{n\mu}+4\bar\psi_\nu\varphi_\mu\\
&R_{\mu\nu}(A)=2\partial_\nu A_\mu-8i\bar\psi_\nu\gamma_5\varphi_\mu\\
&R_{\mu\nu}{}^m(K)=2\partial_\nu f^m_\mu-2\omega_\nu{}^{mn}f_{n\mu}-2b_\nu f^m{}_\mu+2i\bar\varphi_\nu\gamma^m\varphi_\mu,\\
&\bar R_{\mu\nu}(Q)=2D^\omega_\nu\bar\psi_\mu+b_\nu\bar\psi_\mu-\tfrac{3}{2}iA_\nu\bar\psi_\mu\gamma_5+2i\bar\varphi_\nu\gamma_me^m{}_\mu,\\
&\bar R_{\mu\nu}(S)=2D^\omega_\nu\bar\varphi_\mu-b_\nu\bar\varphi_\mu+\tfrac{3}{2}iA_\nu\bar\varphi_\mu\gamma_5+2i\bar\psi_\nu\gamma_mf^m{}_\mu,
\end{aligned}
\tag{79}
$$

with

$$
\begin{aligned}
&D^\omega_\nu\bar\psi_\mu\equiv\partial_\nu\bar\psi_\mu-\tfrac{i}{4}\omega_\nu{}^{mn}\bar\psi_\mu\sigma_{mn},\\
&D^\omega_\nu\psi_\mu\equiv\partial_\nu\psi_\mu+\tfrac{i}{4}\omega_\nu{}^{mn}\sigma_{mn}\psi_\mu,
\end{aligned}
\tag{80}
$$

(and the same for φ_μ,) where antisymmetrization w.r.t. $\mu\leftrightarrow\nu$ like

$$R_{\mu\nu}(A)=(2\partial_\nu A_\mu-8i\bar\psi_\nu\gamma_5\varphi_\mu)_{\text{anti-symm}}$$

$$= \partial_\nu A_\mu - \partial_\mu A_\nu - 4i(\bar\psi_\nu\gamma_5\varphi_\mu - \bar\psi_\mu\gamma_5\varphi_\nu) \tag{81}$$

変換則: $\delta(\varepsilon)h_\mu^A = \partial_\mu\varepsilon^A + \varepsilon^B h_\mu^C f_{CB}{}^A$

$$\begin{aligned}
&\delta e^m_{\ \mu} = \partial_\mu\xi^m + \lambda^{ml}e_{l\mu} - \omega_\mu^{\ mn}\xi_n - \rho e^m_{\ \mu} + b_\mu\xi^m - 2i\bar\varepsilon\gamma^m\psi_\mu, \\
&\delta\omega_\mu^{\ mn} = \partial_\mu\lambda^{mn} + 2\lambda^{ml}\omega_{\mu l}{}^n - 2(\xi_K^m e^n_{\ \mu} - \xi_K^n e^m_{\ \mu}) + 2(f^m_{\ \mu}\xi^n - f^n_{\ \mu}\xi^m) \\
&\qquad\qquad\qquad -2i\bar\varepsilon\sigma^{mn}\varphi_\mu - 2i\bar\psi_\mu\sigma^{mn}\zeta, \\
&\delta b_\mu = \partial_\mu\rho - 2\xi_K^n e_{n\mu} + 2f^n_{\ \mu}\xi_n - 2\bar\varepsilon\varphi_\mu + 2\bar\psi_\mu\zeta \\
&\delta A_\mu = \partial_\mu\theta + 4i\bar\varepsilon\gamma_5\varphi_\mu - 4i\bar\psi_\mu\gamma_5\zeta \\
&\delta f_{\mu\nu}{}^{m} = \partial_\mu\xi_K^m + \lambda^{\ mn}f_{n\mu} - \omega_\mu^{\ mn}\xi_{K\,n} + \rho f^m_{\ \mu} - b_\mu\xi_K^m - 2i\bar\zeta\gamma^m\varphi_\mu, \\
&\delta\bar\psi_\mu = D^\omega_\mu\bar\varepsilon + \tfrac{i}{4}\lambda^{mn}\bar\psi_\mu\sigma_{mn} - \tfrac12\rho\bar\psi_\mu + \tfrac12 b_\mu\bar\varepsilon + \tfrac34 i\theta\bar\psi_\mu\gamma_5 - \tfrac34 iA_\mu\bar\varepsilon\gamma_5 \\
&\qquad\qquad\qquad -i\bar\zeta\gamma_m e^m_{\ \mu} + i\bar\varphi_\mu\gamma_m\xi^m\big), \\
&\big(\delta\psi_\mu = D^\omega_\mu\varepsilon - \tfrac{i}{4}\lambda^{mn}\sigma_{mn}\psi_\mu - \tfrac12\rho\psi_\mu + \tfrac12 b_\mu\varepsilon + \tfrac34 i\theta\gamma_5\psi_\mu - \tfrac34 iA_\mu\gamma_5\varepsilon \\
&\qquad\qquad\qquad +ie^m_{\ \mu}\gamma_m\zeta - i\xi^m\gamma_m\varphi_\mu\big), \\
&\delta\bar\varphi_\mu = D^\omega_\mu\bar\zeta + \tfrac{i}{4}\lambda^{mn}\bar\varphi_\mu\sigma_{mn} + \tfrac12\rho\bar\varphi_\mu - \tfrac12 b_\mu\bar\zeta - \tfrac34 i\theta\bar\varphi_\mu\gamma_5 + \tfrac34 iA_\mu\bar\zeta\gamma_5 \\
&\qquad\qquad\qquad -i\bar\varepsilon\gamma_m f^m_{\ \mu} + i\bar\psi_\mu\gamma_m\xi_K^m, \\
&\big(\delta\varphi_\mu = D^\omega_\mu\zeta - \tfrac{i}{4}\lambda^{mn}\sigma_{mn}\varphi_\mu + \tfrac12\rho\varphi_\mu - \tfrac12 b_\mu\zeta - \tfrac34 i\theta\varphi_\mu\gamma_5 + \tfrac34 iA_\mu\zeta\gamma_5 \\
&\qquad\qquad\qquad +if^m_{\ \mu}\gamma_m\varepsilon - i\xi_K^m\gamma_m\psi_\mu\big),
\end{aligned} \tag{82}$$

For inverse vierbein,

$$\begin{aligned}
\delta e_m^{\ \mu} &= -e_n^{\ \mu}e_m^{\ \nu}(\delta e^n_{\ \nu}) \\
&= -e_n^{\ \mu}\partial_m\xi^n - e_l^{\ \mu}\lambda^l_{\ m} + \omega_m^{\ \mu l}\xi_l + \rho e_m^{\ \mu} - b_m\xi^\mu + 2i\bar\varepsilon\gamma^\mu\psi_m.
\end{aligned} \tag{83}$$

Curvature の group 変換則 $\delta R^A_{\mu\nu}$ は、上の gauge 場の変換則 δh^A_μ で、$\partial_\mu\varepsilon^A$ を捨て、全ての h^B_μ を $R^B_{\mu\nu}$ に置き換えれば良い。

§3. Deformation of the $SU(2,2|1)$ algebra

$$\begin{aligned}
\delta_{\rm GC}(\xi^\lambda)h^A_\mu &= \partial_\mu\xi^\lambda\cdot h^A_\lambda + \xi^\lambda\partial_\lambda h^A_\mu \\
&= D_\mu(\xi^\lambda\cdot h^A_\lambda) + \xi^\lambda(\partial_\lambda h^A_\mu - D_\mu h^A_\lambda) \\
&= [D_\mu(\xi\cdot h)]^A + \xi^\lambda R^A_{\mu\nu} \\
&= \delta(\xi\cdot h)\,h^A_\mu + \xi^\lambda R^A_{\mu\nu},
\end{aligned} \tag{84}$$

The last equality is because

$$\begin{aligned}\delta(\varepsilon)h_\mu^A &= (D_\mu\varepsilon)^A\\ \delta(\xi\cdot h)h_\mu^A &= \partial_\mu(\xi\cdot h^A) + (\xi\cdot h)^B h_\mu^C f_{BC}{}^A.\end{aligned} \tag{85}$$

Note that

$$\begin{aligned}\delta(\xi\cdot h) &= \delta_P(\xi^m) + \sum_{A'(\neq P)} \delta_{A'}(\xi\cdot h^{A'}),\\ \xi^m &= \xi^\lambda e^m{}_\lambda,\\ \xi\cdot h^{A'} &= \xi^\lambda h_\lambda^{A'} = \xi^m h_m^{A'}\end{aligned} \tag{86}$$

Therefore, we have *a key relation*:

$$\delta_P(\xi^m)h_\mu^A = \underbrace{\delta_{\mathrm{GC}}(\xi^\lambda)h_\mu^A - \sum_{B'}\delta_{B'}(\xi\cdot h^{B'})h_\mu^A}_{\equiv\, \delta_{\widetilde{P}}(\xi^m)h_\mu^A} - \xi^\lambda R_{\mu\lambda}^A. \tag{87}$$

Now, we deform the $SU(2,2|1)$ algebra by making a replacement

$$\delta_P(\xi^m) \ \to\ \delta_{\widetilde{P}}(\xi^m) = \delta_{\mathrm{GC}}(\xi^\lambda) - \sum_{B'}\delta_{B'}(\xi\cdot h^{B'}). \tag{88}$$

First we note that, among the commutators $[\delta_{A'},\, \delta_{B'}]$ for $A',\ B' \neq P$, the only one yielding δ_P in the RHS is $[\delta_Q(\varepsilon_2),\, \delta_Q(\varepsilon_1)] = \delta_P(-2i\bar{\varepsilon}_1\gamma^m\varepsilon_2)$. So we require first that

$$[\delta_Q(\varepsilon_2),\, \delta_Q(\varepsilon_1)] = \delta_{\widetilde{P}}(\xi^m), \quad \text{with} \quad \xi^m \equiv -2i\bar{\varepsilon}_1\gamma^m\varepsilon_2, \tag{89}$$

holds on *any independent gauge fields*, and find constraints necessary for that.

3.1. *On* $e^m{}_\mu$

On $e^m{}_\mu$, we originary have

$$\begin{aligned}[][\delta_Q(\varepsilon_2),\, \delta_Q(\varepsilon_1)]e^m{}_\mu &= \delta_P(\xi^m)e^m{}_\mu,\\ &= \delta_{\widetilde{P}}(\xi^m)e^m{}_\mu - \xi^\lambda R_{\mu\lambda}{}^m(P).\end{aligned} \tag{90}$$

So it is necessary and sufficient to impose the constraint:

$$\boxed{0 = R_{\mu\nu}{}^m(P)} = 2\partial_\nu e^m{}_\mu - 2\omega_\nu{}^{mn}e_{n\mu} + 2b_\nu e^m{}_\mu + 2i\bar{\psi}_\nu\gamma^m\psi_\mu \tag{91}$$

This can be solved by the M gauge field $\omega_\mu{}^{mn}$ and yields

$$\omega_\mu{}^{mn} = \omega_\mu{}^{mn}(e,\psi,b), \tag{92}$$

so that $\omega_\mu{}^{mn}$ is no longer an *independent* gauge field. However, since the constraint $R_{\mu\nu}{}^m(P) = 0$ is invariant under M_{mn}, D, A, S, K_m, $\omega_\mu{}^{mn}$ still keeps the same transformation law as the original group transformation under M_{mn}, D, A, S, K_m transformations. On the other hand, the constraint $R_{\mu\nu}{}^m(P) = 0$ is *not* invariant under Q transformation, the Q transformation of $\omega_\mu{}^{mn}$ becomes different from the original group transformation law:

$$\delta_Q(\varepsilon)\omega_\mu{}^{mn}(e.\psi, b) = \delta_Q^{\rm group}(\varepsilon)\omega_\mu{}^{mn} + \delta'_Q(\varepsilon)\omega_\mu{}^{mn}. \tag{93}$$

The difference can be easily found by noting that the constraint $R_{\mu\nu}{}^m(P) = 0$ is of course an identity and Q-invariant if $\omega_\mu{}^{mn}$ there is replaced by $\omega_\mu{}^{mn}(e, \psi, b)$, so that we have

$$\begin{aligned} 0 &= \delta_Q^{\rm group}(\varepsilon)R_{\mu\nu}{}^m(P) + \delta'_Q(\varepsilon)\omega_\mu{}^m{}_\nu - \delta'_Q(\varepsilon)\omega_\nu{}^m{}_\mu \\ &= -2i\bar\varepsilon\gamma^m R_{\mu\nu}(Q) + \delta'_Q(\varepsilon)\omega_\mu{}^m{}_\nu - \delta'_Q(\varepsilon)\omega_\nu{}^m{}_\mu. \end{aligned} \tag{94}$$

(Note that we are anticipating that e^m_μ, ψ_μ, b_μ will remain to be independent gauge fields and receive no changes in the Q-transformation laws.) Solving this (in a similar way to solve Christoffel symbol in terms of $g_{\mu\nu}$), we find

$$\delta'_Q(\varepsilon)\omega_{\mu\, mn} = i\bar\varepsilon(\gamma_\mu R_{mn}(Q) + \gamma_m R_{\mu n}(Q) - \gamma_n R_{\mu m}(Q)) \equiv i\bar\varepsilon\mathcal{R}_{\mu mn}(Q). \tag{95}$$

3.2. *On ψ_μ*

Noting

$$\delta_Q(\varepsilon)\psi_\mu = (\partial_\mu + \tfrac{i}{4}\omega_\mu{}^{mn}\sigma_{mn} + \tfrac{1}{2}b_\mu - \tfrac{3}{4}i\gamma_5 A_\mu)\varepsilon \tag{96}$$

and that $\omega_\mu{}^{mn}$ now receives an extra Q transformation $\delta'_Q(\varepsilon)$ in addition to the original group transformation $\delta_Q^{\rm group}(\varepsilon)$, we find that the $[\delta_Q,\, \delta_Q]$ commutator on ψ_μ now reads

$$\begin{aligned} [\delta_Q(\varepsilon_2),\, \delta_Q(\varepsilon_1)]\psi_\mu &= [\delta_Q^{\rm group}(\varepsilon_2),\, \delta_Q^{\rm group}(\varepsilon_1)]\psi_\mu + \tfrac{i}{4}\big(\delta'_Q(\varepsilon_2)\omega_\mu{\cdot}\sigma\varepsilon_1 - (1\leftrightarrow 2)\big) \\ &= \delta_{\widetilde P}(\xi)\psi_\mu - \xi^m R_{\mu m}(Q) + \tfrac{i}{4}\big(\delta'_Q(\varepsilon_2)\omega_\mu{\cdot}\sigma\varepsilon_1 - (1\leftrightarrow 2)\big). \end{aligned} \tag{97}$$

So we see that the condition

$$\tfrac{i}{4}\big((i\bar\varepsilon_2\mathcal{R}_{\mu\, mn}(Q))\,\sigma^{mn}\varepsilon_1 - (1\leftrightarrow 2)\big) = -2i(\bar\varepsilon_1\gamma^m\varepsilon_2)R_{\mu m}(Q) \tag{98}$$

is necessary and sufficient for the $[\delta_Q,\, \delta_Q]$ algebra Eq. (89) hold on ψ_μ. Applying Fierz, and noting then that only $(\bar\varepsilon_1\gamma^\rho\varepsilon_2)$ and $(\bar\varepsilon_1\sigma^{\rho\sigma}\varepsilon_2)$ terms appear by the antisymmetry under $1 \leftrightarrow 2$, we find the LHS to be

$$-\frac{2}{4\cdot 4}\big[(\bar\varepsilon_1\gamma^\rho\varepsilon_2)\sigma^{mn}\gamma_\rho\mathcal{R}_{\mu\, mn}(Q) + \tfrac{1}{2}(\bar\varepsilon_1\sigma^{\rho\sigma}\varepsilon_2)\sigma^{mn}\sigma_{\rho\sigma}\mathcal{R}_{\mu\, mn}(Q)\big] \tag{99}$$

so that the condition is rewritten into

$$
\begin{aligned}
&\sigma^{mn}\gamma_\rho \mathcal{R}_{\mu\, mn}(Q) = 16iR_{\mu\rho}(Q) \\
&\sigma^{mn}\sigma_{\rho\sigma}\mathcal{R}_{\mu\, mn}(Q) = 0.
\end{aligned} \tag{100}
$$

Multiplying the first equation by γ^ρ and using $\gamma^\rho\sigma^{mn}\gamma_\rho = 0$, we immediately find a constraint

$$
\boxed{\gamma^\rho R_{\mu\rho}(Q) = 0.} \tag{101}
$$

Once this holds, various identities for $R_{\mu\nu}(Q)$ follows:

$$
1. \qquad \varepsilon^{mnrs}\gamma_n R_{rs}(Q) = 0 \tag{102}
$$

$$
2. \qquad \mathcal{R}_{\mu mn}(Q) = 2\gamma_\mu R_{mn}(Q) \tag{103}
$$

$$
3. \qquad \sigma^{mn}\gamma_\rho\gamma_\mu R_{mn}(Q) = 8iR_{\mu\rho}(Q) \tag{104}
$$

$$
4. \qquad \sigma^{mn}\sigma_{\rho\sigma}\gamma_\mu R_{mn}(Q) = 0 \tag{105}
$$

Proof)

$$
\begin{aligned}
1.\quad & \varepsilon^{mnrs}\gamma_n \propto \gamma_5\gamma^{mrs} = \gamma_5[\gamma^m\gamma^{rs} - (\eta^{mr}\gamma^s - \eta^{ms}\gamma^r)] \\
2.\quad & \mathcal{R}_{\mu mn}(Q) = \gamma_\mu R_{mn}(Q) + \gamma_m R_{\mu n}(Q) - \gamma_n R_{\mu m}(Q) \\
& \qquad = 2\gamma_\mu R_{mn}(Q) = 2\gamma_m R_{\mu n}(Q) \qquad \text{(by identity 1.)} \\
3.\quad & \sigma^{mn}\gamma_\rho\gamma_\mu R_{mn}(Q) = (-i\gamma^n\gamma^m)\gamma_\rho\cdot\gamma_m R_{n\mu}(Q) = -i\gamma^n(-2\gamma_\rho)R_{n\mu}(Q) \\
& \qquad = +2i\gamma^n(\gamma_n)R_{\mu\rho}(Q) = +8iR_{\mu\rho}(Q) \\
4.\quad & \sigma_{\rho\sigma}\gamma_\mu = \text{linear combi. of } \gamma_\tau\ (\tau = \rho,\sigma,\mu) \text{ and } \gamma_5\gamma_\tau \\
& \qquad \text{while } \sigma^{mn}\gamma_\cdot R_{mn}(Q) = 0
\end{aligned} \tag{106}
$$

Now by the identities 2. – 4., both the conditions in Eq. (100) are seen to be satisfied, so that the constraint $\gamma^\mu R_{\mu\nu}(Q) = 0$ in Eq. (101) alone is necessary and sufficient condition for the $[\delta_Q,\, \delta_Q]$ algebra Eq. (89) hold on ψ_μ. By the identity 2., the extra Q transformation for ${\omega_\mu}^{mn}$ now takes a simple form:

$$
\boxed{\delta'_Q(\varepsilon)\omega_{\mu\, mn} = 2i\bar{\varepsilon}\gamma_\mu R_{mn}(Q)}\,(= -2i\bar{R}_{mn}(Q)\gamma_\mu\varepsilon). \tag{107}
$$

The constraint (101), $\gamma^\mu R_{\mu\nu}(Q) = 0$, is solved by the S-gauge field φ_μ:

$$
\begin{aligned}
& 0 = \gamma^\mu R_{\mu\nu}(Q) = \gamma^\mu\big[(\partial_\nu + \tfrac{i}{4}\omega_\nu\cdot\sigma + \tfrac{1}{2}b_\nu - \tfrac{3}{4}i\gamma_5 A_\nu)\psi_\mu - (1\leftrightarrow 2)\big] - i\gamma^\mu(\gamma_\mu\varphi_\nu - \gamma_\nu\varphi_\mu) \\
& \quad \Rightarrow \quad \varphi_\mu = \varphi_\mu(e,\psi,b,A).
\end{aligned} \tag{108}
$$

So φ_μ now become *dependent* gauge field. Since the constraint $\gamma^\mu R_{\mu\nu}(Q) = 0$ is M_{mn}, D, A, S, K_m invariant but not invariant under Q, the Q-transformation of φ_μ is modified:

$$\begin{aligned} 0 = \delta_Q(\varepsilon)(\gamma^\mu R_{\mu\nu}(Q)) &= \gamma^\mu \delta_Q^{\text{group}}(\varepsilon) R_{\mu\nu}(Q) + (\delta_Q(\varepsilon) e_m{}^\mu)\gamma^m R_{\mu\nu}(Q) \\ &+ \tfrac{i}{4}\gamma^\mu \big[(\delta'_Q(\varepsilon)\omega_\nu{}^{mn})\sigma_{mn}\psi_\mu - (\mu \leftrightarrow \nu)\big] - i(4\delta_\nu^\mu - \gamma^\mu\gamma_\nu)\delta'_Q(\varepsilon)\varphi_\mu \end{aligned} \tag{109}$$

where

$$\begin{aligned} \delta_Q(\varepsilon) e_m{}^\mu &= 2i\bar\varepsilon\gamma^\mu\psi_m \\ \delta_Q^{\text{group}}(\varepsilon) R_{\mu\nu}(Q) &= \big(\tfrac{i}{4}R_{\mu\nu}(M)\cdot\sigma + \tfrac{1}{2}R_{\mu\nu}(D) - \tfrac{3}{4}i\gamma_5 R_{\mu\nu}(A)\big)\varepsilon \end{aligned} \tag{110}$$

Using Eq. (107), we find $(\mu \leftrightarrow \nu)$ term $(\delta'_Q(\varepsilon)\omega_\mu{}^{mn})\gamma^\mu\sigma_{mn}\psi_\nu$ vanishes by the identity:

$$\gamma^\mu\sigma^{mn}\forall\eta(\forall\bar\varepsilon\gamma_\mu R_{mn}(Q)) = 0. \tag{111}$$

Indeed,

$$(\bar\varepsilon\gamma_\mu R_{\ mn}(Q))(\bar\phi\gamma^\mu\sigma^{mn}\psi_\nu) = -\frac{1}{4}\Big[\sum_A (\bar\varepsilon\varGamma^A\psi_\nu)(\bar\phi\gamma^\mu\sigma^{mn}\varGamma_A\gamma_\mu R_{mn}(Q))\Big] \tag{112}$$

whereas $\varGamma_A = 1$, γ_5 terms vanish by $\gamma^\mu\sigma^{mn}\gamma_\mu = 0$, $\varGamma_A = \gamma_\rho$, $\gamma_\rho\gamma_5$ terms vanish by the identity 3. in Eq. (105) since $\gamma^\mu\sigma^{mn}\gamma_\rho\gamma_\mu R_{mn}(Q) = \gamma^\mu \cdot 8iR_{\mu\rho}(Q) = 0$, and $\varGamma_A = \sigma_{\rho\sigma}$ term vanish by the identity 4.

By Fierzing similarly, we find

$$\begin{aligned} &\frac{i}{4}(\bar\phi\gamma^\mu\sigma_{mn}\psi_\mu)(\delta'_Q(\varepsilon)\omega_\nu{}^{mn}) = \frac{i}{4}(\bar\phi\gamma^\mu\sigma_{mn}\psi_\mu)(i\bar\varepsilon\mathcal{R}_{\nu\, mn}) \\ &= -\frac{1}{4}(\bar\psi_\mu\sigma^{mn}\gamma^\mu\phi)(\bar\varepsilon\mathcal{R}_{\nu\, mn}) \\ &= \frac{1}{16}\Big[\sum_A(\bar\psi_\mu\sigma^{mn}\varGamma_A\mathcal{R}_{\nu\, mn})(\bar\varepsilon\varGamma^A\gamma^\mu\phi)\Big] \\ &= \frac{1}{16}\Big[(\bar\psi_\mu\ \underbrace{\sigma^{mn}\gamma_\rho\mathcal{R}_{\nu\, mn}}_{=16iR_{\nu\rho}(Q)})(\bar\varepsilon\gamma^\rho\gamma^\mu\phi) + (\bar\psi_\mu\ \underbrace{\sigma^{mn}\gamma_\rho\gamma_5\mathcal{R}_{\nu\, mn}}_{-16i\gamma_5 R_{\nu\rho}(Q)})(\bar\varepsilon\gamma_5\gamma^\rho\gamma^\mu\phi)\Big] \\ &= i\big[(\bar\phi\gamma^\mu\gamma^\rho\varepsilon)(\bar\psi_\mu R_{\nu\rho}(Q)) - (\bar\phi\gamma^\mu\gamma^\rho\gamma_5\varepsilon)(\bar\psi_\mu\gamma_5 R_{\nu\rho}(Q))\big] \end{aligned} \tag{113}$$

$$\begin{aligned} &(\bar\phi\gamma^m R_{\mu\nu}(Q))(\delta_Q(\varepsilon)e_m{}^\mu) = (\bar\phi\gamma^m R_{\mu\nu}(Q))(2i\bar\varepsilon\gamma^\mu\psi_m) = -2i(\bar\phi\gamma^m R_{\mu\nu}(Q))(\bar\psi_m\gamma^\mu\varepsilon) \\ &= -2i(-\frac{1}{4})\Big[\sum_A(\bar\phi\gamma^m\varGamma^A\varepsilon)(\bar\psi_m\gamma^\mu\varGamma_A R_{\mu\nu}(Q))\Big] \\ &= \frac{i}{2}\Big[(\bar\phi\gamma^m\gamma^\rho\varepsilon)(\bar\psi_m\ \underbrace{\gamma^\mu\gamma_\rho}_{\to[\gamma^\mu,\gamma_\rho]=2\delta^\mu_\rho}\ R_{\mu\nu}(Q)) + (\bar\phi\gamma^m\gamma_5\gamma^\rho\varepsilon)(\bar\psi_m\ \underbrace{\gamma^\mu\gamma_\rho}_{\to 2\delta^\mu_\rho}\ \gamma_5 R_{\mu\nu}(Q)) \end{aligned}$$

$$
\begin{aligned}
&\qquad +\tfrac{1}{2}(\bar{\phi}\gamma^m\sigma^{ab}\varepsilon)(\bar{\psi}_m \underbrace{\gamma^\mu\sigma_{ab}} R_{\mu\nu}(Q))] \\
&\qquad\qquad\qquad [\gamma^\mu,\, \sigma_{ab}] = 2i(\delta^\mu_a\gamma_b - \delta^\mu_b\gamma_a) \\
&= i\big[(\bar{\phi}\gamma^m\gamma^\rho\varepsilon)(\bar{\psi}_m R_{\rho\nu}(Q)) + (\bar{\phi}\gamma^m\gamma_5\gamma^\rho\varepsilon)(\bar{\psi}_m\gamma_5 R_{\rho\nu}(Q)) \\
&\qquad +\tfrac{i}{2}(\bar{\phi}\gamma^m\sigma^{ab}\varepsilon)\bar{\psi}_m\underbrace{(\gamma_b R_{a\nu}(Q) - \gamma_a R_{b\nu}(Q))}_{\gamma_\nu R_{ab}(Q)})\big] \\
&= -i\big[(\bar{\phi}\gamma^m\gamma^\rho\varepsilon)(\bar{\psi}_m R_{\nu\rho}(Q)) - (\bar{\phi}\gamma^m\gamma^\rho\gamma_5\varepsilon)(\bar{\psi}_m\gamma_5 R_{\nu\rho}(Q))\big] \\
&\qquad -\frac{1}{2}(\bar{\phi}\gamma^m\sigma^{ab}\varepsilon)(\bar{\psi}_m\gamma_\nu R_{ab}(Q))
\end{aligned}
\tag{114}
$$

The first two terms just cancel the $\frac{i}{4}(\bar{\phi}\gamma^\mu\sigma_{mn}\psi_\mu)(\delta'_Q(\varepsilon)\omega_\nu{}^{mn})$ in Eq. (113), we thus find Eq. (109) leads to

$$
\begin{aligned}
-i(4\delta^\mu_\nu - \gamma^\mu\gamma_\nu)\delta'_Q(\varepsilon)\varphi_\mu &= \tfrac{i}{4}\gamma^\mu\sigma^{mn}\varepsilon R_{\mu\nu\, mn}(M) + \tfrac{1}{2}\gamma^\mu\varepsilon R_{\mu\nu}(D) - \tfrac{3}{4}i\gamma^\mu\gamma_5\varepsilon R_{\mu\nu}(A) \\
&\quad -\frac{1}{2}\gamma^\mu\sigma^{mn}\varepsilon(\bar{\psi}_\mu\gamma_\nu R_{mn}(Q))
\end{aligned}
\tag{115}
$$

In the last term, the factor $\bar{\psi}_\mu\gamma_\nu R_{mn}(Q)$ can be replaced by $\bar{\psi}_\mu\gamma_\nu R_{mn}(Q) - \bar{\psi}_\nu\gamma_\mu R_{mn}(Q)$ owing to the identity Eq. (111), $\gamma^\mu\sigma^{mn}\forall\eta(\forall\bar{\varepsilon}\gamma_\mu R_{mn}(Q)) = 0$. Then, defining covariantized $R_{\mu\nu}{}^{mn}(M)$ by

$$
R^{\mathrm{cov.}\,mn}_{\mu\nu}(M) \equiv R_{\mu\nu}{}^{mn}(M) + 2i\big(\bar{\psi}_\mu\gamma_\nu R^{mn}(Q) - \bar{\psi}_\nu\gamma_\mu R^{mn}(Q)\big),
\tag{116}
$$

the last term can be absorbed into the $R_{\mu\nu\, mn}(M)$ term and we obtain:

$$
\begin{aligned}
\delta'_Q(\varepsilon)\varphi_\mu &= -\frac{i}{2}(\delta^\nu_\mu - \tfrac{1}{6}\gamma_\mu\gamma^\nu)\mathcal{R}_\nu\varepsilon = -\frac{i}{2}(\mathcal{R}_\mu - \tfrac{1}{6}\gamma_\mu\gamma\cdot\mathcal{R})\varepsilon \\
\mathcal{R}_\mu &\equiv \tfrac{i}{4}\gamma^\mu\sigma^{mn}\varepsilon R^{\mathrm{cov.}}_{\mu\nu\, mn}(M) + \tfrac{1}{2}\gamma^\mu\varepsilon R_{\mu\nu}(D) - \tfrac{3}{4}i\gamma^\mu\gamma_5\varepsilon R_{\mu\nu}(A).
\end{aligned}
\tag{117}
$$

This quantity $(\mathcal{R}_\mu - \frac{1}{6}\gamma_\mu\gamma\cdot\mathcal{R})\varepsilon$ can be much simplified if we use the Bianchi identity.

The Bianchi identity is

$$
\begin{aligned}
0 = \varepsilon^{\mu\nu\rho\sigma}[D_\nu,\, R_{\rho\sigma}] &= \varepsilon^{\mu\nu\rho\sigma}\big(\partial_\nu R_{\rho\sigma} - [h_\mu,\, R_{\rho\sigma}]\big). \\
&\to \varepsilon^{\mu\nu\rho\sigma}\big(\partial_\nu R^A_{\rho\sigma} - h^B_\mu R^C_{\rho\sigma} f_{BC}{}^A\big) = 0.
\end{aligned}
\tag{118}
$$

This reads for $A = P_m$

$$
\begin{aligned}
0 = \varepsilon^{\mu\nu\rho\sigma}\big(&\partial_\nu R_{\rho\sigma}{}^m(P) - \omega_\nu{}^m{}_n R_{\rho\sigma}{}^n(P) + e_{n\nu} R_{\rho\sigma}{}^{mn}(M) \\
&+ b_\nu R_{\rho\sigma}{}^m(P) - e^m_{\ \nu} R_{\rho\sigma}(D) + 2i\bar{\psi}_\nu\gamma^m R_{\rho\sigma}(Q)\big).
\end{aligned}
\tag{119}
$$

Again, the last term can be absorbed into the $R^{\mathrm{cov.}\,mn}_{\rho\sigma}(M)$ term, and using the constraint $R_{\rho\sigma}{}^m(P) = 0$, we have an identity

$$
\varepsilon^{\mu\nu\rho\sigma}\big(R^{\mathrm{cov.}\,m}_{\rho\sigma\,\nu}(M) + e^m_{\ \nu} R_{\rho\sigma}(D)\big) = 0,
\tag{120}
$$

or, equivalently,

$$\varepsilon^{\mu\nu\rho\sigma}R^{\mathrm{cov.}\,m}_{\rho\sigma\,\nu}(M) = -2\widetilde{R}^{\mu\alpha}(D). \tag{121}$$

where tilde is generally defined by

$$\widetilde{R}^{\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}R_{\rho\sigma}, \qquad \text{(note that } \widetilde{\widetilde{R}}{}^{\mu\nu} = -R^{\mu\nu}\text{)} \tag{122}$$

Writing Eq. (120) into the form

$$R^{\mathrm{cov.}}_{rs\,nm}(M) + (\text{cyclic in } rsn) + R_{rs}(D)\eta_{mn} + (\text{cyclic in } rsn) = 0, \tag{123}$$

and adding the same form term with the indices s, m interchanged, we have

$$R^{\mathrm{cov.}}_{[rs,nm]}(M) + R^{\mathrm{cov.}}_{[rm,ns]}(M) = -(R(D)\eta)_{rs\,mn+sn\,mr+rm\,sn+mn\,sr+2nr\,ms} \tag{124}$$

with

$$R^{\mathrm{cov.}}_{[rs,nm]}(M) \equiv R^{\mathrm{cov.}}_{rs\,nm}(M) - R^{\mathrm{cov.}}_{nm\,rs}(M) \tag{125}$$

Then adding to Eq. (124) the same identity with indices r, s, n replaced by n, r, s and subtracting the one replaced by by s, n, r, we find

$$R^{\mathrm{cov.}}_{[rs,mn]}(M) = \eta_{rm}R_{sn}(D) - \eta_{sm}R_{rn}(D) - \eta_{rn}R_{sm}(D) + \eta_{sn}R_{rm}(D) \tag{126}$$

Contracting by ε^{arsm} and using the identity Eq. (121), we obtain

$$\varepsilon_m{}^{abc}R^{\mathrm{cov.}}_{na\,bc}(M) = -2\widetilde{R}_{mn}(D). \tag{127}$$

Contraction with η^{rn} gives yet another identity

$$R^{\mathrm{cov.}}_{\mu\nu}(M)|_{\text{antisymm. part}} \equiv \tfrac{1}{2}(R^{\mathrm{cov.}}_{\mu\nu}(M) - R^{\mathrm{cov.}}_{\nu\mu}(M)) = -R_{\mu\nu}(D), \tag{128}$$

where

$$R^{\mathrm{cov.}}_{\mu\nu}(M) \equiv R^{\mathrm{cov.}\,mn}_{\mu\rho}(M)e_m{}^{\rho}e_{n\nu}. \tag{129}$$

Using these identities Eqs. (121), (127) and (128), as well as

$$\gamma^{\mu}\sigma_{mn} = i(\delta^{\mu}_{m}\gamma_n - \delta^{\mu}_{n}\gamma_m) - \varepsilon^{\mu}{}_{mnr}\gamma^r\gamma_5, \tag{130}$$

we can compute $(\mathcal{R}_\mu - \frac{1}{6}\gamma_\mu\gamma{\cdot}\mathcal{R})\varepsilon$ into the following form

$$\begin{aligned}(\mathcal{R}_\mu - \tfrac{1}{6}\gamma_\mu\gamma{\cdot}\mathcal{R})\varepsilon = \gamma^m\varepsilon\big(&-\tfrac{1}{12}e_{m\mu}R^{\mathrm{cov.}}(M)^{\rho}_{\rho} + \tfrac{1}{2}R^{\mathrm{cov.}}_{\mu m}(M) + \tfrac{1}{4}\widetilde{R}_{\mu m}(A)\big)\\ &+i\gamma^m\gamma_5\varepsilon\,\tfrac{1}{2}R_{\mu m}(A)\end{aligned} \tag{131}$$

Thus the extra Q transformation $\delta'_Q(\varepsilon)\varphi_\mu$ is given by

$$\begin{aligned}\delta'_Q(\varepsilon)\varphi_\mu &= -\frac{i}{2}(\mathcal{R}_\mu - \tfrac{1}{6}\gamma_\mu\gamma{\cdot}\mathcal{R})\varepsilon\\ &= -\frac{i}{2}\big[\gamma^m\varepsilon\big(-\tfrac{1}{12}e_{m\mu}R^{\mathrm{cov.}}(M)^{\rho}_{\rho} + \tfrac{1}{2}R^{\mathrm{cov.}}_{\mu m}(M) + \tfrac{1}{4}\widetilde{R}_{\mu m}(A)\big)\\ &\qquad +i\gamma^m\gamma_5\varepsilon\,\tfrac{1}{2}R_{\mu m}(A)\big]\end{aligned} \tag{132}$$

3.3. *On A_μ and b_μ*

Noting

$$
\begin{aligned}
\delta_Q(\varepsilon)A_\mu &= 4i\bar{\varepsilon}\gamma_5\varphi_\mu \\
\delta_Q(\varepsilon)b_\mu &= -2\bar{\varepsilon}\varphi_\mu,
\end{aligned} \tag{133}
$$

and that φ_μ now receives an extra Q transformation $\delta'_Q(\varepsilon)$ in addition to the original group transformation $\delta_Q^{\text{group}}(\varepsilon)$, we find that the $[\delta_Q,\,\delta_Q]$ commutator on A_μ and b_μ now reads

$$
\begin{aligned}
[\delta_Q(\varepsilon_2),\,\delta_Q(\varepsilon_1)]A_\mu &= [\delta_Q^{\text{group}}(\varepsilon_2),\,\delta_Q^{\text{group}}(\varepsilon_1)]A_\mu + 4i\big(\bar{\varepsilon}_1\gamma_5(\delta'_Q(\varepsilon_2)\varphi_\mu) - (1\leftrightarrow 2)\big) \\
&= \delta_{\widetilde{P}}(\xi)A_\mu - \xi^m R_{\mu m}(A) + 4i\big((-\tfrac{i}{2})(\bar{\varepsilon}_1\gamma_5 i\gamma^m\gamma_5\varepsilon_2)\,\tfrac{1}{2}R_{\mu m}(A) - (1\leftrightarrow 2)\big) \\
&= \delta_{\widetilde{P}}(\xi)A_\mu - \xi^m R_{\mu m}(A) + 4i(-\tfrac{i}{2})\xi^m\tfrac{1}{2}R_{\mu m}(A) = \delta_{\widetilde{P}}(\xi)A_\mu \quad \text{OK!} \\
[\delta_Q(\varepsilon_2),\,\delta_Q(\varepsilon_1)]b_\mu &= [\delta_Q^{\text{group}}(\varepsilon_2),\,\delta_Q^{\text{group}}(\varepsilon_1)]b_\mu - 2\big(\bar{\varepsilon}_1(\delta'_Q(\varepsilon_2)\varphi_\mu) - (1\leftrightarrow 2)\big) \\
&= \delta_{\widetilde{P}}(\xi)b_\mu - \xi^m R_{\mu m}(D) \\
&\quad -2(-\tfrac{i}{2})\times 2(\bar{\varepsilon}_1\gamma^m\varepsilon_2)\big(-\tfrac{1}{12}e_{m\mu}R^{\text{cov.}}(M)^\rho_\rho + \tfrac{1}{2}R^{\text{cov.}}_{\mu m}(M) + \tfrac{1}{4}\widetilde{R}_{\mu m}(A)\big) \\
&= \delta_{\widetilde{P}}(\xi)b_\mu - \xi^m R_{\mu m}(D) \\
&\quad -\xi^m\big(-\tfrac{1}{12}e_{m\mu}R^{\text{cov.}}(M)^\rho_\rho + \tfrac{1}{2}R^{\text{cov.}}_{\mu m}(M) + \tfrac{1}{4}\widetilde{R}_{\mu m}(A)\big)
\end{aligned} \tag{134}
$$

Thus, the $[\delta_Q,\,\delta_Q]$ commutator on A_μ requires no constraint but that on b_μ requires a condition

$$
-\tfrac{1}{12}e_{m\mu}R^{\text{cov.}}(M)^\rho_\rho + \tfrac{1}{2}R^{\text{cov.}}_{\mu m}(M) + \tfrac{1}{4}\widetilde{R}_{\mu m}(A) = -R_{\mu m}(D) \tag{135}
$$

which leads, by separating the symmetric and antisymmetric parts and using Eq. (128), to

$$
\begin{aligned}
&R^{\text{cov.}}_{\mu m}(M)|_{\text{symm. part}} = 0, \\
&-\tfrac{1}{2}R_{\mu m}(D) + \tfrac{1}{4}\widetilde{R}_{\mu m}(A) = -R_{\mu m}(D).
\end{aligned} \tag{136}
$$

The latter condition is rewritten into

$$
R_{\mu m}(D) = -\tfrac{1}{2}\widetilde{R}_{\mu m}(A) \quad \text{or } \rightarrow \quad \widetilde{R}_{\mu m}(D) = +\tfrac{1}{2}R_{\mu m}(A). \tag{137}
$$

If Eq. (128) is used, these two conditions can be rewritten into a constraint

$$
R^{\text{cov.}}_{\nu\mu}(M) + \tfrac{1}{2}\widetilde{R}_{\mu\nu}(A) = 0. \tag{138}
$$

This is the necessary and sufficient condition for the $[\delta_Q,\,\delta_Q]$ algebra Eq. (89) to hold on b_μ. Then the extra Q transformation Eq. (132) of φ_μ is simplified into

$$
\begin{aligned}
\delta'_Q(\varepsilon)\varphi_\mu &= -\frac{i}{2}\big[\gamma^m\varepsilon\,\tfrac{1}{2}\widetilde{R}_{\mu m}(A) + i\gamma^m\gamma_5\varepsilon\,\tfrac{1}{2}R_{\mu m}(A)\big] \\
&\boxed{\delta'_Q(\varepsilon)\varphi_\mu = -\frac{i}{4}\gamma^m\big(\widetilde{R}_{\mu m}(A) + i\gamma_5 R_{\mu m}(A)\big)\varepsilon.}
\end{aligned} \tag{139}
$$

The constraint Eq. (138) can be solved by the K_m gauge field $f^m_{\ \mu}$, which now becomes a dependent field:

$$f^m_{\ \mu} = f^m_{\ \mu}(e, \psi, b, A). \tag{140}$$

Since the constraint Eq. (138) is not Q-invariant and so $f^m_{\ \mu}$ also receives an extra Q-transformation, which can be derived in the same way as above:

$$\boxed{\delta'_Q(\varepsilon) f^m_{\ \mu} = -\frac{i}{2}\bar{\varepsilon}\big(\sigma^{m\nu} R^{\text{cov.}}_{\mu\nu}(S) + e^{m\nu}\widetilde{R}^{\text{cov.}}_{\mu\nu}(S)\big).} \tag{141}$$

3.4. *Resultant modified $SU(2,2|1)$ algebra*

Now that the M_{mn}, S and K_m gauge fields $\omega_\mu^{\ mn}$, φ_μ and $f^m_{\ \mu}$ have become dependent fields, there no longer remain other independent gauge fields. Thus the desired $[\delta_Q, \delta_Q]$ algebra (89)

$$[\delta_Q(\varepsilon_2),\, \delta_Q(\varepsilon_1)] = \delta_{\widetilde{P}}(\xi^m), \quad \text{with} \quad \xi^m \equiv -2i\bar{\varepsilon}_1\gamma^m\varepsilon_2 \tag{142}$$

already holds on all the independent gauge fields $e^m_{\ \mu}$, ψ_μ, A_μ and b_μ.

This implies that

Proposition: *For all the transformations other than $\widetilde{P}_m$ transformation, (which we denote by primed index X' henceforth, $X' \in \{Q,\, M_{mn},\, D,\, A,\, S,\, K_m\}$), the commutators*

$$\boxed{[\delta_{Y'}(\varepsilon^{Y'}),\, \delta_{X'}(\varepsilon^{X'})] = \sum_C \delta_C(\varepsilon^{X'}\varepsilon^{Y'} f_{X'Y'}{}^C)} \tag{143}$$

of the same form as the original $SU(2,2|1)$ algebra, still hold. Note that when P_m appears in the C sum, it is always understood to stand for $\widetilde{P}_m$.

Proof) For the case $X' = Q$ and $Y' = Q$, we have already seen that $[\delta_Q(\varepsilon_2),\, \delta_Q(\varepsilon_1)] = \delta_{\widetilde{P}}(\xi^m)$ holds and so the Proposition holds. So it is enough to prove it for the other cases in which either X' or Y' is not equal to Q. We assume $X' \neq Q$ without loss of generality.

Since the M_{mn}, D, A, S, K_m transformations are the same as the original group transforamtions even for the dependent gauge fields, we clearly have

$$[\delta_{Y'}(\varepsilon^{Y'}),\, \delta_{X'}(\varepsilon^{X'})]h^G_\mu = \sum_C \delta^{\text{group}}_C(\varepsilon^{X'}\varepsilon^{Y'} f_{X'Y'}{}^C)h^G_\mu + \delta^Q_{Y'} \sum_{Z'=M,S,K} \varepsilon^{X'}\delta'_Q(\varepsilon^{Y'})h^{Z'}_\mu f_{X'Z'}{}^G \tag{144}$$

on any independent gauge fields h^G_μ. The second extra Q transformation terms may exist only when $Y' = Q$ and only for the dependent gauge fields $h^{Z'}_\mu$ with $Z' \in \{M,\, S,\, K\}$. However we show that this term is in fact absent. Consider the Weyl weights of the generators. The generator G for the independent gauge fields is one of P, Q, D, A carrying weights 1, 1/2, 0, 0, respectively. The sum of the weights of the generators X' and Z', $w(X') + w(Z')$, should be $w(G)$ in order for the structure constant $f_{X'Z'}{}^G$ to be non-zero. But, since

$w(G) \geq 0$, $w(X') \leq 0$ (recall $X' \neq Q$) and $w(Z') \leq 0$, the only possibility satisfying this condition is $w(G) = w(X') = w(Z') = 0$, which corresponds to the cases $G \in \{D, A\}$, $X' \in \{M, D, A\}$ and $Z' = M$. But, the commutators $[X', Z']$ for such cases can yield only M since $[\{M, D, A\}, M] \propto M$, so that $f_{X'Z'}{}^G = 0$ for $G \in \{D, A\}$. Therefore the group law holds. q.e.d.

Proposition:

$$\boxed{[\delta_{\tilde{P}}(\xi^m), \delta_{A'}(\varepsilon^{A'})] = \sum_{B \text{ all}} \delta_B(\varepsilon^{A'}\xi^m f_{A'P_m}{}^B) + \delta^Q_{A'} \sum_{B'=M,S,K} \delta_{B'}(\xi^m \delta'_Q(\varepsilon^{A'}) h^{B'}_m)} \tag{145}$$

Proof) Straightforward calculation using

$$\delta_{\tilde{P}}(\xi^m) = \delta_{\rm GC}(\xi^\lambda = \xi^m e_m{}^\lambda) - \sum_{B'} \delta_{B'}(\xi\cdot h^{B'}) \tag{146}$$

leads to the above result. First, using

$$\delta_{\rm GC}(\xi^\lambda) h^A_\mu = \partial_\mu \xi^\lambda \cdot h^A_\lambda + \xi^\lambda \partial_\lambda h^A_\mu, \tag{147}$$

we derive, for field-independent ξ^λ case,

$$[\delta_{\rm GC}(\xi^\lambda), \delta_{A'}(\varepsilon^{A'})] = -\delta_{A'}(\xi^\lambda \partial_\lambda \varepsilon^{A'}). \tag{148}$$

For the field-dependent case, we derive

$$[\delta_{\rm GC}(\xi^\lambda = \xi^m e_m{}^\lambda), \delta_{A'}(\varepsilon^{A'})] = -\delta_{A'}(\xi^\lambda \partial_\lambda \varepsilon^{A'}) - \delta_{\rm GC}\big(\xi^m \delta_{A'}(\varepsilon^{A'}) e_m{}^\lambda\big) \tag{149}$$

where

$$\delta_{A'}(\varepsilon^{A'}) e_m{}^\lambda = -e_m{}^\nu e_n{}^\mu \big(\delta_{A'}(\varepsilon^{A'}) e^n{}_\nu\big) = -\varepsilon^{A'} h^C_m f_{A'C}{}^{P_n} e_n{}^\mu. \tag{150}$$

Next we have

$$[-\delta_{B'}(\xi^m h^{B'}_m), \delta_{A'}(\varepsilon^{A'})] = -\sum_C \delta_C(\varepsilon^{A'} \xi\cdot h^{B'} f_{A'B'}{}^C) + \delta_{B'}(\xi^m \delta_{A'}(\varepsilon^{A'}) h^{B'}_m) \tag{151}$$

where

$$\begin{aligned}\delta_{A'}(\varepsilon^{A'}) h^{B'}_m &= \delta_{A'}(\varepsilon^{A'})(e_m{}^\mu h^{B'}_\mu) \\ &= -\varepsilon^{A'} h^C_m f_{A'C}{}^{P_n} h^{B'}_n + \delta^{B'}_{A'} \partial_m \varepsilon^{A'} + \varepsilon^{A'} h^C_m f_{A'C}{}^{B'} + \delta^Q_{A'} \delta'_Q(\varepsilon^{A'}) h^{B'}_m.\end{aligned} \tag{152}$$

Using these we can show the Proposition. q.e.d.

Proposition:

$$\boxed{[\delta_{\tilde{P}}(\xi_1), \delta_{\tilde{P}}(\xi_2)] = \sum_A \delta_A(\xi_1^m \xi_2^n R^A_{mn}) + \sum_{B'=M,S,K} \delta_{B'}\big(\delta'_Q(\xi_1\cdot\psi)\xi_2\cdot h^{B'} - \delta'_Q(\xi_2\cdot\psi)\xi_1\cdot h^{B'}\big)} \tag{153}$$

or, equivalently,

$$\boxed{[\delta_{\widetilde{P}}(\xi_1),\, \delta_{\widetilde{P}}(\xi_2)] = \sum_A \delta_A(\xi_1^m \xi_2^n R_{mn}^{\text{cov.}\,A})} \tag{154}$$

Proof) Straightforward calculation. First show, for field-independent ξ^λ case,

$$[\delta_{\text{GC}}(\xi_2^\rho),\, \delta_{\text{GC}}(\xi_1^\lambda)] = \delta_{\text{GC}}((\xi_1\!\cdot\!\partial)\xi_2^\lambda - (\xi_2\!\cdot\!\partial)\xi_1^\lambda) \tag{155}$$

$$[\delta_{B'}(\xi_2^m h_m^{B'}),\, \delta_{\text{GC}}(\xi_1^\lambda)] = \delta_{B'}\big((\xi_1\!\cdot\!\partial\, \xi_2^m)h_m^{B'}\big) \tag{156}$$

Next, for the field-dependent case, we derive

$$[\delta_{\text{GC}}(\xi_2^n e_n{}^\rho),\, \delta_{\text{GC}}(\xi_1^m e_m{}^\lambda)] = \delta_{\text{GC}}\big(\xi_1^m \xi_2^n (\partial_n e_m{}^\lambda - \partial_m e_n{}^\lambda)\big) \tag{157}$$

$$[\delta_{B'}(\xi_2^m h_m^{B'}),\, \delta_{\text{GC}}(\xi_1\!\cdot\! e)] = \delta_{B'}\big((\xi_1\!\cdot\!\partial\, \xi_2^m)h_m^{B'}\big) - \sum_C \delta_{\text{GC}}\big((\xi_2\!\cdot\! h^{B'})(\xi_1\!\cdot\! h^C) f_{B'C}{}^{P_n} e_n{}^\lambda\big) \tag{158}$$

Using these, we derive

$$\begin{aligned}
[\delta_{\widetilde{P}}(\xi_2),\, \delta_{\text{GC}}(\xi_1\!\cdot\! e)] &= \delta_{\text{GC}}\big(\xi_1^m \xi_2^n (\partial_n e_m{}^\lambda - \partial_m e_n{}^\lambda)\big) - \sum_{B'} \delta_{B'}\big((\xi_1\!\cdot\!\partial\, \xi_2^m)h_m^{B'}\big) \\
&\quad + \sum_{B',C} \delta_{\text{GC}}\big((\xi_2\!\cdot\! h^{B'})(\xi_1\!\cdot\! h^C) f_{B'C}{}^{P_n} e_n{}^\lambda\big)
\end{aligned} \tag{159}$$

Using the previous Proposition, we have

$$\begin{aligned}
[\delta_{\widetilde{P}}(\xi_2),\, \sum_{B'} \delta_{B'}(\xi_1\!\cdot\! h^{B'})] &= \sum_{B',C(all)} \delta_C\big((\xi_1\!\cdot\! h^{B'})\xi_2^m f_{B'P_m}{}^C\big) + \sum_{B'=M,S,K} \delta_{B'}(\xi_2^m \delta'_Q(\xi_1\!\cdot\!\psi) h_m^{B'}) \\
&\quad + \sum_{B'} \delta_{B'}(\xi_1^m \delta_{\widetilde{P}}(\xi_2) h_m^{B'})
\end{aligned} \tag{160}$$

where

$$\begin{aligned}
\xi_1^m \delta_{\widetilde{P}}(\xi_2) h_m^{B'} &= \xi_1^m (\xi_2\!\cdot\!\partial) h_m^{B'} - \xi_1^m \sum_{A'} \delta_{A'}(\xi_2\!\cdot\! h^{A'}) h_m^{B'} \\
&= \xi_1^m (\xi_2\!\cdot\!\partial) h_m^{B'} + \sum_{A',C} \big\{(\xi_2\!\cdot\! h^{A'})(\xi_1\!\cdot\! h^C) f_{A'C}{}^{P_n} h_n^{B'} - \delta_{A'}^{B'} (\xi_1\!\cdot\!\partial)(\xi_2\!\cdot\! h^{A'}) \\
&\qquad - (\xi_2\!\cdot\! h^{A'})(\xi_1\!\cdot\! h^C) f_{A'C}{}^{B'} - \delta_{A'}^Q \delta'_Q(\xi_2\!\cdot\! h^{A'}) h_m^{B'}\big\}.
\end{aligned} \tag{161}$$

Using these we can show the Proposition. q.e.d.

Also note

$$R_{mn}^{\text{cov.}\,A} = R_{mn}^A - \big(\delta'_Q(\psi_n) h_m^A - \delta'_Q(\psi_m) h_n^A\big) \tag{162}$$

§4. $N = 1$ Superconformal Tensor Calculus

4.1. *Matter multiplets*

The general, or so-called vector, (complex, unconstrained) superconformal multiplet $\mathcal{V}$ corresponding to the superfield

$$
\begin{aligned}
V(x,\theta) = &\mathcal{C} + i\bar{\theta}\gamma_5\mathcal{Z} - \tfrac{1}{2}\bar{\theta}(\mathcal{N} - i\gamma_5\mathcal{M} - \gamma_5\gamma^\mu\mathcal{V}_\mu)\theta \\
&-i(\bar{\theta}\theta)\bar{\theta}\gamma_5[\Lambda - \tfrac{i}{2}\gamma^\mu\partial_\mu\mathcal{Z}] + \tfrac{1}{4}(\bar{\theta}\theta)(\bar{\theta}\theta)[\mathcal{D} - \tfrac{1}{2}\Box\mathcal{C}]
\end{aligned} \tag{163}
$$

in the rigid supersymmetry case, is now denoted by $\mathcal{V} = [\mathcal{C}, \mathcal{Z}, \mathcal{N}, \mathcal{M}, \mathcal{V}_m, \Lambda, \mathcal{D}]$. (Real vector multiplet is denoted as $V = [C, Z, N, M, V_m, \lambda, D]$, by using the corresponding roman letters.) The basic quantum numbers of the superconformal matter multiplet are Weyl weight w and chiral weight n, which are defined through the transformation law of the first component $\mathcal{C}$:

$$
[\delta_D(\rho) + \delta_A(\theta)]\,\mathcal{C}(x) = (w\rho + \tfrac{1}{2}in\theta)\,\mathcal{C}(x). \tag{164}
$$

This vector multiplet $\mathcal{V}$ exists for any Weyl and chiral weights $w,\ n$ (and even $\mathcal{V}_A$ with arbitrary external Lorentz index $A = (\alpha_1, \cdots, \alpha_n; \dot{\beta}_1, \cdots, \dot{\beta}_m)$. On the contrary, the constrained type multiplets can exist only for particular values of $(w,\ n)$ (and for particular external Lorentz indices A). For instance, the chiral multiplets exist only when they carry the same values of Weyl and chiral weights, $w = n$ (and only with purely undotted spinor indices $A = (\alpha_1, \cdots, \alpha_n)$).

Here we do not give the transformation laws for the vector multiplet $\mathcal{V}$, but give those for the chiral multiplet $\phi = [z, \chi, h]$ possessing no external Lorentz index, which is embedded into the vector multiplet as follows:

$$
\mathcal{V}(\phi) = [\ z,\ -i\chi_{\mathrm{R}},\ -h,\ ih,\ iD^c_m z,\ 0,\ 0\ ]. \tag{165}
$$

The chiral multiplet transforms under Q, S, D and A as

$$
\begin{aligned}
\delta_{QSDA}\, z &\equiv (\delta_Q(\varepsilon) + \delta_S(\zeta) + \delta_D(\rho) + \delta_A(\theta))\, z = \tfrac{1}{2}\bar{\varepsilon}_{\mathrm{R}}\chi_{\mathrm{R}} + (w\rho + \tfrac{1}{2}iw\theta)z \\
\delta_{QSDA}\, \chi_{\mathrm{R}} &= \not{D}^c z \cdot \varepsilon_{\mathrm{L}} + h\varepsilon_{\mathrm{R}} + 2wz\zeta_{\mathrm{R}} + [(w + \tfrac{1}{2})\rho + i(\tfrac{1}{2}w - \tfrac{3}{4})\theta]\chi_{\mathrm{R}} \\
\delta_{QSDA}\, h &= \tfrac{1}{2}\bar{\varepsilon}_{\mathrm{L}}\not{D}^c\chi_{\mathrm{R}} + (1-w)\bar{\zeta}_{\mathrm{R}}\chi_{\mathrm{R}} + [(w+1)\rho + i(\tfrac{1}{2}w - \tfrac{3}{2})\theta]h,
\end{aligned} \tag{166}
$$

and inert under K_m, where D^c_m denotes conformal covariant derivative:

$$
\begin{aligned}
D^c_m z &= (\partial_m - wb_m - \tfrac{1}{2}iwA_m)z - \tfrac{1}{2}\bar{\psi}_{\mathrm{R}}\chi_{\mathrm{R}} \\
D^c_m \chi_{\mathrm{R}} &= \big(D^\omega_m - (w + \tfrac{1}{2})b_m - i(\tfrac{1}{2}w - \tfrac{3}{4})A_m\big)\chi_{\mathrm{R}} \\
&\quad -(\not{D}^c z \cdot \psi_{\mathrm{L}m} + h\psi_{\mathrm{R}m}) - 2wz\varphi_{\mathrm{R}m}
\end{aligned} \tag{167}
$$

with local Lorentz covariant derivative D^ω_m.

4.2. *Invariant action formula*

F-term formula: applicable to chiral multiplet with weight $w = n = 3$, $\phi_{w=n=3} = [\,z{=}\frac{1}{2}(A{+}iB),\ \chi_{\rm R},\ h{=}\frac{1}{2}(F{+}iG)\,]$

$$
\begin{aligned}
I_F &= \int d^4x\ \big[\phi_{(w=n=3)}\big]_F = \int d^4x\, e\,\Big[h + \tfrac{1}{2}\bar\psi_{{\rm L}m}\gamma^m\chi_{\rm R} + \bar\psi_{{\rm L}m}\sigma^{mn}z\,\psi_{{\rm L}n} + \text{h.c.}\Big] \\
&= \int d^4x\, e\,\Big[F + \tfrac{1}{2}\bar\psi_m\gamma^m\chi + \tfrac{1}{2}\bar\psi_m\sigma^{mn}(A - i\gamma_5 B)\psi_n\Big]
\end{aligned}
\tag{168}
$$

The full superconformal invariance of this action can be confirmed by checking only the S-invariance , since the GC(P) and LL(M) invariance together with D, A and K invariances are manifest; the non-trivial Q-invariance automatically follows from (the commutator of) GC and S invariances.

The next action formula can be derived from this. Since the chiral projection (analogue of $\bar D\bar DV$) of real vector multiplet V with Weyl weight $w = 2$ gives a chiral multiplet $\varPi V$ with weight $w = n = 3$:

$$
\varPi V = [\,\tfrac{1}{2}(H - iK),\ i\not{D}^c Z_{\rm L} + \varLambda_{\rm R},\ -\tfrac{1}{2}(D + \Box^c C + iD^c_m B^m)\,]
\tag{169}
$$

We can apply the above F-term formula to this chiral multiplet $\varPi V$ and obtain
D-term formula: applicable to real vector multiplet $V = [\,C,\ Z,\ H,\ K,\ B_m,\ \lambda,\ D\,]$ with weight $w = 2$ $n = 0$:

$$
\begin{aligned}
I_D &= \int d^4x\ \big[V_{(w=2,\,n=0)}\big]_D = \int d^4x\ [-\varPi V]_F \\
&= \int d^4x\, e\big[D - \tfrac{1}{2}\bar\psi_m\gamma^m i\gamma_5\lambda - \bar\varphi_m\gamma^m i\gamma_5 Z + \tfrac{1}{3}C\left(R + e^{-1}\bar\psi_\mu R^\mu\right) \\
&\qquad\qquad + \tfrac{1}{4}i\varepsilon^{mnkl}\bar\psi_m\gamma_n\psi_k\left(B_l - A_l C - \tfrac{1}{2}\bar\psi_l Z\right)\big]
\end{aligned}
\tag{170}
$$

where

$$
R = R_{\mu\nu}{}^{mn}(M)e_m{}^\nu e_n{}^\mu, \qquad R^\mu = \varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D^\omega_\rho\psi_\sigma\,.
\tag{171}
$$

4.3. *$N = 1$ SUGRA Lagrangian*

One may have wondered in the above why we consider such a superconformal framework possessing rather large local symmetry while we want supergravity which has only local Poincaré invariance. We can now answer to this question. All the possible theories of Poincaré supergravity can be obtained from our superconformal framework simply fixing the gauges for the extraneous gauge symmetries, dilatation D, chiral A, conformal supersymmetry S and special conformal K_m symmetries. Then, we need special matter multiplet(s) called *compensator*, whose component fields are used to fix those extraneous gauges. Choosing different type of multiplet as the compensator yields a different formulation of Poincaré supergravity: namely, chiral multiplet compensator leads to (old) minimal formulation, (real) linear multiplet compensator to new-minimal formulation and complex linear multiplet compensator to Breitenlohner formulation. One of the virtue of the superconformal framework is that all those different formulations of Poincaré supergravity can be dreived in a unified way from this unique framework. There is another and more important advantage in the superconformal tensor calculus actually, which we explain shortly.

We explain only the (old) minimal formulation of Poincaré supergravity. Pure (Poincaré) supergravity Lagrangian is given by

$$\mathcal{L}_{\text{pure SUGRA}} = \big[\Sigma\bar{\Sigma}\big]_D \tag{172}$$

where Σ is a chiral multiplet with weight $w = n = 1$, the compensator of the (old) minimal formulation. Denoting the components of this compensator as $\Sigma = [\mathcal{A},\, \psi_{\mathrm{R}},\, \mathcal{F}]$, the extraneous D, A, S, K_m gauges are fixed by the following conditions:

$$\begin{aligned} D:\quad \mathrm{Re}\mathcal{A} = \sqrt{3}, \qquad A:\quad \mathrm{Im}\mathcal{A} = 0,\\ S:\quad \psi_{\mathrm{R}} = 0, \qquad K_m:\quad b_\mu = 0, \end{aligned} \tag{173}$$

where the last b_μ is the Weyl (D) gauge field. Then, writing $\mathcal{F} = \frac{1}{\sqrt{3}}(S - iP)$ and $A_\mu = -\frac{2}{3}A_\mu^{\text{aux}}$, $\Sigma\bar{\Sigma}$ takes the form

$$\Sigma\bar{\Sigma} = [\,3,\ 0,\ -2S,\ 2P,\ -2A_m^{\text{aux}},\ 0,\ -\tfrac{1}{3}(S^2 + P^2 - A_m^{\text{aux}\,2})\,] \tag{174}$$

Substituting this components expression into Eq. (172) and applying the D-term formula, we actually obtain the following action of pure supergravity:

$$\mathcal{L}_{\text{pure SUGRA}} = e\big[R + e^{-1}\bar{\psi}_\mu R^\mu - \tfrac{1}{3}(S^2 + P^2 - A_m^{\text{aux}\,2})\big]. \tag{175}$$

S, P and A_μ^{aux} constitute the well-known minimal set of auxiliarly fields, hence the name of minimal Poincaré supergravity.

If one considers more general matter coupled system, the Lagrangian would take the form

$$\mathcal{L} = \big[\Sigma\bar{\Sigma}\,\tilde{\Phi}(\phi, \bar{\phi})\big]_D + \big[\Sigma^3 W(\phi)\big]_F\ , \tag{176}$$

omitting the possible gauge fields. Here ϕ denotes a set of matter multiplets $\{\,\phi_i\,\}$. Now we can explain another virtue of our superconformal tensor calculus, as promised above.

First, we note that we can eliminate the superpotential term by redefining the compensator as $W^{1/3}(\phi)\Sigma \to \Sigma$, and rewrite the Lagrangian into the following form using $\Phi \equiv \tilde{\Phi}/\left|W\right|^{2/3}$:

$$\mathcal{L} = \left[\Sigma\bar{\Sigma}\,\Phi(\phi,\bar{\phi})\right]_D + \left[\Sigma^3\right]_F \ , \tag{177}$$

In this matter coupled system, the multiplet $\Sigma\bar{\Sigma}\,\Phi(\phi,\bar{\phi}) \equiv V$ in the D-term has the following first two components:

$$\begin{aligned} C(V) &= \left|\mathcal{A}\right|^2 \Phi(z,z^*) \\ \tfrac{1}{2}Z(V) &= i\left|\mathcal{A}\right|^2 \left(\Phi_i\chi^i_{\mathrm{L}} - \Phi^i\chi_{\mathrm{R}i}\right) + i\Phi\left(\mathcal{A}\psi_{\mathrm{L}} - \mathcal{A}^*\psi_{\mathrm{R}}\right) \ , \end{aligned} \tag{178}$$

with notation $\Phi^i \equiv \partial\Phi(z,z^*)/\partial z_i$, $\Phi_i \equiv \partial\Phi(z,z^*)/\partial z^{*i}$. Therefore, to obtain the canonical form of Einstein-Hilbert as well as Rarita-Schwinger action $R + e^{-1}\bar{\psi}_\mu R^\mu$, it would be best to take the gauge conditions for the extraneous gauges D, A, S, K_m as[?]

$$\begin{aligned} D:\quad \mathrm{Re}\mathcal{A} &= \sqrt{3}\Phi^{-1/2}, \qquad A:\quad \mathrm{Im}\mathcal{A} = 0, \\ S:\quad \psi_{\mathrm{R}} &= -\mathcal{A}\Phi^{-1}\Phi^i\chi_{\mathrm{R}i}, \qquad K_m:\quad b_\mu = 0. \end{aligned} \tag{179}$$

Indeed, in this superconformal gauge, we have $C(V) = 3$ and $Z(V) = 0$, yielding the desired canonical Einstein-Hilbert and Rarita-Schwinger action $R + e^{-1}\bar{\psi}_\mu R^\mu$ from the beginning, as is seen from the D-term action formula. Note that this is really the power of superconformal tensor calculus. In the Poincaré tensor calculus, there is no freedom of choosing those gauges! From the superconformal viewpoint, the Poincaré tensor calculus is just the tensor calculus obtained from the superconformal one by choosing the Poincaré gauge fixing conditions Eq. (173). It is a good gauge conditions for pure supergravity system, but is ridiculous one for the matter coupled system. There is, however, no other way in the Poincaré tensor calculus, since there are no extraneous gauge freedom. Compare this simplification with the big calculation performed by Cremmer, Ferrara, Girardello and Van Proeyen[?] using the Poincaré tensor calculus. The first thing the latter authors had to do was 1) Weyl rescaling of the vierbein and other fields, 2) chiral rotations of the fermion fields, and 3) recombination of $\tilde{\Phi}$ and the superpotential W into the Kähler potential $\frac{1}{3}K = \ln(\tilde{\Phi}/\left|W\right|^{2/3})$. The first and second tasks are simply bypassed here by the above D and A gauge conditions and the third was the task performed in one line already in Eq. (177).