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The rigid body dynamics of unidirectional spin

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A toy consists of a boat-shaped body showing great preference for spin in one direction only. Its sophisticated rigid body dynamics is examined in some detail, and fully accounts for this curious behaviour.

1. INTRODUCTION

A toy available commercially in the U.S.A. or alternatively manufactured by hand consists of a solid boat-shaped body (of plastic or wood)*. When placed on a reasonably smooth horizontal table, it rests comfortably on its curved underneath, with the long direction of the body approximately horizontal. If the body is spun about its vertical axis of rest in one direction (say counterclockwise), it spins smoothly, whereas if spun in the opposite (clockwise) sense, the spin soon diminishes to zero while a strong, roughly longitudinal, oscillation takes over, and then the body begins to spin in its preferred (counterclockwise) sense, with the oscillation rapidly dying away.

This remarkable behaviour is not often discussed and receives only passing mention in even as comprehensive a text as Routh's *Advanced rigid dynamics*. In its later editions (e.g. Cambridge University Press, 1905) reference is given to a paper by G. T. Walker (1896), which is the only analysis that I have found. Though Walker's paper describes several of the principal features of the problem, it is flawed by a number of omissions and by some unnecessary restrictions. In the present paper I attempt to give a more complete analysis of this problem, a problem that displays the great sophistication of rigid body dynamics even in as simple a case as a body rolling on a horizontal surface.

2. PRELIMINARIES

Many people, even trained scientists, find it hard to understand that the behaviour of the toy does not violate the principle of conservation of angular momentum. The situation becomes clear as soon as one sets up the equations of this *rolling* motion of the toy (i.e. the point of the body in contact with the table has zero velocity), and remembers that the angular momentum equation applies solely about the centre of mass.

The following notation will be used:

$\boldsymbol{s}$, position vector of centre of mass, $\mathrm{d}\boldsymbol{s}/\mathrm{d}t = \boldsymbol{v}$;
$\boldsymbol{r}$, vector from centre of mass to point of contact;
$\boldsymbol{F}$, force exerted by the table on the body at the point of contact;
$\boldsymbol{\omega}$, angular velocity of the body;

* I owe mine to the courtesy of Mr Chris Elliott, County Science Advisor, Hertfordshire.

$\boldsymbol{h}$, angular momentum of the body;
M, its mass;
$\boldsymbol{u}$, upward vertical unit vector.

Then the relevant equations are evidently

$$M\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t = \boldsymbol{F} - Mg\boldsymbol{u}, \tag{1}$$

$$\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r} = 0, \tag{2}$$

$$\mathrm{d}\boldsymbol{h}/\mathrm{d}t = \boldsymbol{r} \times \boldsymbol{F}. \tag{3}$$

Therefore

$$\mathrm{d}\boldsymbol{h}/\mathrm{d}t = M\boldsymbol{r} \times [\mathrm{d}\boldsymbol{v}/\mathrm{d}t + g\boldsymbol{u}], \tag{4}$$

and thus for the vertical component of the angular momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}\cdot\boldsymbol{h}) = M(\boldsymbol{u}\times\boldsymbol{r})\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}. \tag{5}$$

With a rolling sphere, or with a general body spinning with the vector $\boldsymbol{r}$ of contact strictly vertical, $\boldsymbol{u}\times\boldsymbol{r}$ vanishes, and thus the vertical component of angular momentum is unchanging. Even if $\boldsymbol{u}\times\boldsymbol{r}$ does not vanish, but does not have a *systematic* relation to $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$, the vertical angular momentum only fluctuates imperceptibly, and these are the cases we are familiar with. But if the small vector $\boldsymbol{u}\times\boldsymbol{r}$ is systematically related to $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$, large changes of the vertical angular momentum can readily occur.

But how can the body show a chirality, a preference for one direction of spin over the other? The body has principal axes of inertia about its centre, and a surface of contact, which in the second approximation must be a quadric, with its own principal axes. If these two sets of axes are at an angle to each other, a sense of rotation is readily defined.

The next step must be to describe the surface of contact. In general there is no need for the vector $\boldsymbol{r}$ in the rest position to have any particular relation to the principal axes of inertia. However, it is difficult to conceive of a smooth spin unless there is in fact coincidence, and I shall make this assumption, just as G. T. Walker did.

Choosing now coordinates fixed in the body, centred at the centre of mass, along the principal axes of inertia, with moments of inertia A, B, C (the choice is made so that $A > B$), the C axis (z) being vertically downwards, I can describe the surface of the body in second approximation by

$$z = a\left[1 - \tfrac{1}{2}p\left(\frac{x}{a}\right)^2 - q\frac{xy}{a^2} - \tfrac{1}{2}s\left(\frac{y}{a}\right)^2\right], \tag{6}$$

where a is the rest distance from the centre of mass to the point of contact, and the first-order terms vanish because $x = y = 0$ is the rest position. The surface will be convex if

$$p > 0, \quad s > 0, \quad ps > q^2, \tag{7}$$

and the rest position will be stable if no radius of curvature is less than a, so that we must have

$$1 > p, \quad 1 > s, \quad (1-p)(1-s) > q^2. \tag{8}$$

Clearly there is chirality unless $q = 0$.

3. The equations of motion

With three unequal moments of inertia, it appears best to use the axes fixed in the body (x, y, z above) and to denote time differentiation with respect to these moving axes by a dot.

If the point of contact has coordinates x, y (z being given by (6)) so that $\boldsymbol{r} = (x, y, z)$, then since the normal to the surface at the point of contact must be parallel to $\boldsymbol{u}$,

$$\boldsymbol{u} = -w\left(\frac{px+qy}{a}, \frac{qx+sy}{a}, 1\right), \tag{9}$$

where

$$w^{-2} = \left(\frac{px+qy}{a}\right)^2 + \left(\frac{qx+sy}{a}\right)^2 + 1, \quad w > 0, \tag{10}$$

to ensure the unit character of $\boldsymbol{u}$.

Since $\boldsymbol{u}$ is fixed in space,

$$0 = \mathrm{d}\boldsymbol{u}/\mathrm{d}t = \dot{\boldsymbol{u}} + \boldsymbol{\omega}\times\boldsymbol{u} \tag{11}$$

and thus

$$\boldsymbol{\omega} = \dot{\boldsymbol{u}}\times\boldsymbol{u} + n\boldsymbol{u}, \tag{12}$$

where n is the spin.

From this point the manipulation becomes too heavy if all terms are kept in. Supposing x, y to be reasonably small, terms beyond the second order will be neglected. The spin n will not be assumed to be small.

Then

$$\left.\begin{aligned} \omega_1 &= \frac{1}{a}[q\dot{x} + s\dot{y} - n(px+qy)],\\ \omega_2 &= \frac{1}{a}[-p\dot{x} - q\dot{y} - n(qx+sy)],\\ \omega_3 &= \frac{1}{a^2}(ps-q^2)(\dot{x}y - x\dot{y}) - n\left[1 - \frac{1}{2}\left(\frac{px+qy}{a}\right)^2 - \frac{1}{2}\left(\frac{qx+sy}{a}\right)^2\right]. \end{aligned}\right\} \tag{13}$$

Evidently

$$h_1 = A\omega_1, \quad h_2 = B\omega_2, \quad h_3 = C\omega_3. \tag{14}$$

From (2),

$$\left.\begin{aligned} v_1 &= p\dot{x} + q\dot{y} + n[qx - (1-s)y],\\ v_2 &= q\dot{x} + s\dot{y} + n[(1-p)x - qy],\\ v_3 &= \text{second order (and, it turns out, contributing only}\\ &\quad\ \text{to third order in the later equations).} \end{aligned}\right\} \tag{15}$$

Substituting in (4), and remembering that $\mathrm{d}\boldsymbol{P}/\mathrm{d}t = \dot{\boldsymbol{P}} + \boldsymbol{\omega}\times\boldsymbol{P}$ (where $\boldsymbol{P}$ is any vector) and that third and higher orders are neglected,

$$\begin{aligned} A\dot{\omega}_1 - (B-C)\,\omega_2\omega_3 &= A[q\ddot{x} + s\ddot{y} - n(p\dot{x} + q\dot{y}) - \dot{n}(px+qy)]/a\\ &\quad - n(B-C)\,[p\dot{x} + q\dot{y} + n(qx+sy)]/a\\ &= -Ma\{q\ddot{x} + s\ddot{y} + n(1-2p)\,\dot{x} - 2nq\dot{y} + \dot{n}[(1-p)\,x - qy]\\ &\quad - n^2[qx - (1-s)\,y]\} + Mga[qx - (1-s)\,y], \end{aligned} \tag{16}$$

$$\begin{aligned}B\dot{\omega}_2-(C-A)\,\omega_3\,\omega_1 &= -B[p\ddot{x}+q\ddot{y}+n(q\dot{x}+s\dot{y})+\dot{n}(qx+sy)]/a\\ &\quad +(C-A)\,n[q\dot{x}+s\dot{y}-n(px+qy)]/a\\ &= Ma\{p\ddot{x}+q\ddot{y}+n[2q\dot{x}-(1-s)\,\dot{y}]+\dot{n}[qx-(1-s)\,y]\\ &\quad +n^2[(1-p)\,x-qy]\}+Mga[x(1-p)-qy],\end{aligned} \tag{17}$$

$$\begin{aligned}C\dot{\omega}_3-(A-B)\,\omega_1\,\omega_2 &= -C\dot{n}\left[1-\frac{1}{2}\left(\frac{px+qy}{a}\right)^2-\frac{1}{2}\left(\frac{qx+sy}{a}\right)^2\right]\\ &\quad +\frac{1}{a^2}Cn[(px+qy)\,(p\dot{x}+q\dot{y})+(qx+sy)\,(q\dot{x}+s\dot{y})]\\ &\quad +\frac{1}{a^2}C(ps-q^2)\,(\ddot{x}y-x\ddot{y})\\ &= Mx\{q\ddot{x}+s\ddot{y}+\dot{n}[(1-p)\,x-qy]\}-My\{p\ddot{x}+q\ddot{y}+n(q\dot{x}+s\dot{y})\\ &\quad +\dot{n}[qx-(1-s)\,y]\}+Mn\{x[(1-2p)\,\dot{x}-2q\dot{y})]\\ &\quad -y[2q\dot{x}-(1-2s)\,\dot{y}]-xn[qx-(1-s)\,y]\\ &\quad -yn[(1-p)\,x-qy]\}-M\frac{g}{a}[q(x^2+y^2)+(p-s)\,xy].\end{aligned} \tag{18}$$

Equation (18) shows that $\dot{n}$ is of second order, there being no first-order terms in this equation. Equations (16) and (17) are thus of first order and can be further simplified, with the $\dot{n}$ term omitted, using the notation

$$A+Ma^2 = \alpha Ma^2, \quad B+Ma^2 = \beta Ma^2, \quad C = \gamma Ma^2 \tag{19}$$

(so that $\alpha > 1$, $\beta > 1$, $\gamma > 0$, $\alpha+\beta-\gamma > 2$, $\beta+\gamma-\alpha > 0$, $\gamma+\alpha-\beta > 0$),

to become

$$\begin{aligned}\alpha(q\ddot{x}+s\ddot{y})-(\alpha+\beta-\gamma)\,n(p\dot{x}+q\dot{y})-(\beta-\gamma)\,n^2(qx+sy)+n\dot{x}+n^2y&\\ = [qx-(1-s)\,y]\,g/a,&\end{aligned} \tag{16'}$$

$$\begin{aligned}\beta(p\ddot{x}+q\ddot{y})+(\alpha+\beta-\gamma)\,n(q\dot{x}+s\dot{y})-(\alpha-\gamma)\,n^2(px+qy)-n\dot{y}+n^2x&\\ = [qy-(1-p)\,x]\,g/a.&\end{aligned} \tag{17'}$$

The complicated equation (18) will only be required for small n, when it becomes

$$a^2\gamma\dot{n} = \gamma(ps-q^2)\,(\ddot{x}y-\ddot{y}x)+[q(y\ddot{y}-x\ddot{x})+py\ddot{x}-s\ddot{y}x]+[(p-s)\,xy+q(x^2+y^2)]\,g/a. \tag{18'}$$

4. The spinning motion

The next step is to consider the stability of the spinning motion. Suppose that in (16′) and (17′) both x and y vary as $\exp(\sigma t)$. Then after some manipulation one arrives at the *characteristic equation*

$$\begin{aligned}(\sigma^2+n^2)\,\{&\alpha\beta(ps-q^2)\,\sigma^2+(\alpha-\beta)\,qn\,\sigma+n^2\,[1-p(\alpha-\gamma)-s(\beta-\gamma)+(ps-q^2)\\ &\times(\alpha-\gamma)\,(\beta-\gamma)]+(g/a)\,[\alpha s+\beta p-(ps-q^2)\,(\alpha+\beta)]\}\\ &+n^2(g/a)\,[2-(1+\alpha+\beta-\gamma)\,(p+s)+2(\alpha+\beta-\gamma)\,(ps-q^2)]\\ &+(g/a)^2\,[(1-p)\,(1-s)-q^2] = 0.\end{aligned} \tag{20}$$

Several points emerge immediately from this complicated equation, assuming (to avoid trivialities) that $q \neq 0$, $n \neq 0$ (and of course, $\alpha \neq \beta$).

(i) If certain values of σ are the roots of the quartic for some value of n, then minus these values of σ will be the roots for $-n$.

(ii) There can be no non-zero purely imaginary roots of (20) unless $\sigma = \pm in$, in which case n is determined by the vanishing of the sum of the last two terms of (20).

(iii) The real parts of σ do not arise from any dissipation (none has been put into this analysis) but correspond to a flow of energy from or into the spin into or from the x, y oscillations determined by (20). After all, with small x, y, but finite n, only a small change in the spin and its energy is required to compensate for a large fractional change in the small x, y motion.

Extrapolating somewhat to appreciable x, y motions the following possibilities arise:

(*a*) All the roots of (20) have real parts of the same sign ('concordance'). Accordingly, one sign of n will give a stable spin. Any oscillations initially present will die away, their energy enhancing the spin. The opposite spin will be unstable, and its kinetic energy will flow largely into the oscillation with the σ having the largest real part. The speed of spin will diminish.

(*b*) Some of the roots of (20) have real parts of one sign, the other(s) of the opposite sign ('discordance'). There can be no truly stable spin in either direction.

(*b*′) Yet suppose that with a particular direction of spin, one conjugate complex pair of roots has a certain negative real part, while the other root(s) have very much smaller positive real parts. Then if the initial conditions involve a sizeable oscillation corresponding to the roots with the large negative real part, this oscillation will die down, its energy flowing much more rapidly into the spin motion than it can flow out from it into the other mode(s). Thus the body will be spun up.

In fact the spin enters (20) importantly. Accordingly, as the spin changes, the roots may switch between régimes (*a*) (*b*) and (*b*′).

Note that (20) involves n, g/a, and six fixed non-dimensional coefficients, α, β, γ describing the inertial properties of the body, and p, q, s describing the shape of the rolling surface. Nonetheless several clear features will emerge in the study of the roots.

Since the roots depend continuously on the values of the terms in (20), the real parts of roots cannot change sign without being either purely imaginary ($\sigma = \pm in$ being the only possibility) or vanishing. Thus particular attention will be paid to these switch points.

It is convenient to replace σ by the non-dimensional $\rho = \sigma/n$, and to characterize the spin by the non-dimensional parameter $\Omega = g/an^2$. There is also some advantage in replacing p, q, s by θ, φ, ψ, where

$$\left.\begin{aligned}\theta+\varphi &= p+s,\\ \theta\varphi &= ps-q^2,\\ \psi(\theta-\varphi) &= p-s\end{aligned}\right\} \tag{21}$$

so that

$$\left.\begin{aligned} p &= \tfrac{1}{2}(\theta+\varphi)+\tfrac{1}{2}\psi(\theta-\varphi),\\ q &= \tfrac{1}{2}(\theta-\varphi)(1-\psi^2)^{\frac{1}{2}},\\ s &= \tfrac{1}{2}(\theta+\varphi)-\tfrac{1}{2}\psi(\theta-\varphi).\end{aligned}\right\} \tag{21'}$$

The shape inequalities (6) and (7) can be satisfied without loss of generality by

$$0<\varphi\leqslant\theta<1,\quad -1\leqslant\psi\leqslant 1. \tag{22}$$

Then (20) becomes

$$(\rho^2+1)(\rho^2+\chi\rho+\kappa+\Omega\lambda)+\Omega\mu+\Omega^2\nu=0, \tag{23}$$

with

$$\alpha\beta\theta\varphi\chi=\tfrac{1}{2}(\alpha-\beta)(\theta-\varphi)(1-\psi^2)^{\frac{1}{2}}\ \text{(assumed to be positive)}, \tag{24}$$

$$\alpha\beta\theta\varphi\kappa=1-\tfrac{1}{2}(\alpha+\beta-2\gamma)(\theta+\varphi)+(\alpha-\gamma)(\beta-\gamma)\theta\varphi-\tfrac{1}{2}(\alpha-\beta)(\theta-\varphi)\psi, \tag{25}$$

$$\alpha\beta\theta\varphi\lambda=\tfrac{1}{2}(\alpha+\beta)(\theta+\varphi-2\theta\varphi)-\tfrac{1}{2}(\alpha-\beta)(\theta-\varphi)\psi, \tag{26}$$

$$\alpha\beta\theta\varphi\mu=2-(\theta+\varphi)-(\alpha+\beta-\gamma)(\theta+\varphi-2\theta\varphi), \tag{27}$$

and

$$\alpha\beta\theta\varphi\nu=(1-\theta)(1-\varphi). \tag{28}$$

It can readily be shown that by virtue of (22) both λ and ν are necessarily positive. For small values of θ and φ both κ and μ are positive, but for larger θ, φ invariably μ becomes negative and, provided that $\alpha-\gamma>1$, κ may also become so.

5. The roots of the characteristic equation

These need to be studied in detail to see under what conditions cases 4a, b and b' may arise.

For large Ω (small spin),

$$\rho=\mathrm{i}l\Omega^{\frac{1}{2}}+m+O(\Omega^{-\frac{1}{2}}), \tag{29}$$

where

$$2l^2=\lambda\pm(\lambda^2-4\nu)^{\frac{1}{2}} \tag{30}$$

and

$$m=\tfrac{1}{4}\chi[-1\mp\lambda(\lambda^2-4\nu)^{-\frac{1}{2}}].$$

Since† $\lambda^2>4\nu>0$, both values of l are real, as are both values of m. However, one m is positive, the other, larger, m is negative and thus the area of large Ω never has concordance. In relation to 4b', note that the ratio of the absolute values of m depends only on the ratio $\lambda(4\nu)^{-\frac{1}{2}}$ (as shown in the footnote). Evidently this ratio can easily be made very large, essentially by having θ much larger than φ, with ψ preferably negative.

† From (26) and (28),

$$\lambda(4\nu)^{-\frac{1}{2}}=\tfrac{1}{4}[\alpha\beta\theta\varphi(1-\theta)(1-\varphi)]^{-\frac{1}{2}}\{(\alpha+\beta)[\theta(1-\varphi)+\varphi(1-\theta)]-(\alpha-\beta)\psi[\theta(1-\varphi)-\varphi(1-\theta)]\}.$$

Putting $\theta(1-\varphi)/\varphi(1-\theta)=Q^2>1$, $\alpha/\beta=S^2>1$, this simplifies to

$$\lambda(4\nu)^{-\frac{1}{2}}=\tfrac{1}{4}[(S+S^{-1})(Q+Q^{-1})-\psi(S-S^{-1})(Q-Q^{-1})].$$

The smallest value of this will arise for $\psi=1$ when

$$\lambda(4\nu)^{-\frac{1}{2}}=\tfrac{1}{2}[SQ^{-1}+QS^{-1}]>1.$$

The ratio of the two values of m is important, and is determined by $\lambda(4\nu)^{-\frac{1}{2}}$. To give some numerical examples, here is a short table:

α	β	θ	φ	ψ	$\lambda(4\nu)^{-\frac{1}{2}}$	$-m_-/m_+$
5	2	0.3	0.2	0	1.1472	2.9220
5	2	0.6	0.2	−0.8	7.8747	246
5	2	0.6	0.05	−0.8	16.1411	1042

As Ω diminishes, concordance can only arise if ρ passes through i or through zero. If $\rho=\mathrm{i}$, then by (23) we must have

$$\mu\Omega+\nu\Omega^2=0, \tag{31}$$

an equation that other than at $\Omega=0$ can only be satisfied if μ is negative since Ω and ν are necessarily positive. Note that if Ω takes such a value $\Omega_1=-\mu/\nu$, two conjugate complex roots change the signs of their real parts there. Next, $\rho=0$ implies

$$\kappa+(\lambda+\mu)\Omega+\nu\Omega^2=0, \tag{32}$$

an equation that may have two, one or no roots in the relevant region (Ω real and positive). In fact there can be no such root if μ is positive, for then by implication (see below) κ is also positive. Thus concordance can only occur for negative μ, in which case there will be a value $\Omega_1=-\mu/\nu$ which, it can be shown, is necessarily larger than any root of (32).

It may now be helpful to examine the signs of μ and κ in the θ, φ triangle (note constraints (22)) (figure 1). In the part near the origin both are necessarily positive. The hyperbola $\mu=0$ intersects $\theta=\varphi$ at $\theta=(\alpha+\beta-\gamma)^{-1}$, which is less than $\frac{1}{2}$, and $\varphi=0$ at $\theta=2/(\alpha+\beta-\gamma+1)$, which is further to the right but less than 2/3.

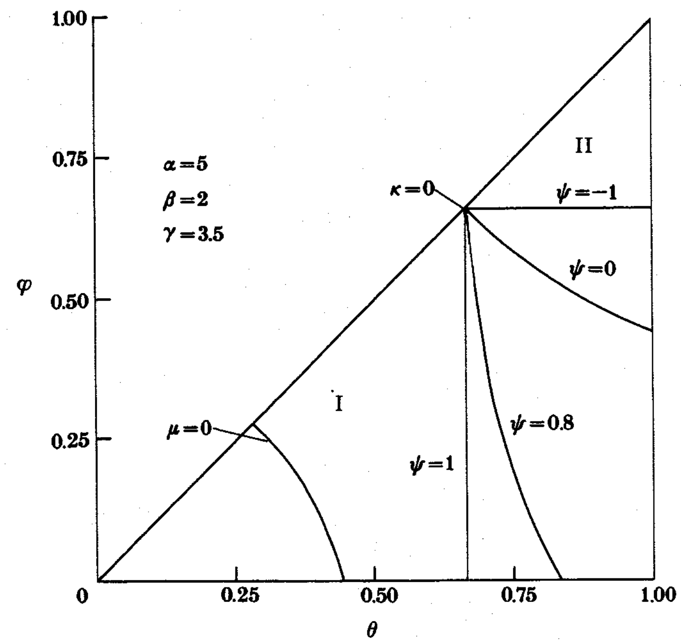

FIGURE 1. The θ–ϕ triangle.

For many shapes of the body $\alpha > \gamma+1$, in which case the curve $\kappa = 0$ exists in the triangle. It starts at $\theta = \varphi = (\alpha-\gamma)^{-1}$ and is straight and vertical for $\psi = 1$, straight and horizontal for $\psi = -1$, and runs as a hyperbola between these two extreme shapes otherwise (figure 1). Note that this line is invariably to the right of $\mu = 0$, so that, proceeding right from the origin, the zone in which κ and μ are positive is followed by a zone (zone I) of negative μ and positive κ, which in turn, for $\alpha > \gamma+1$, is followed by a final zone II in which both are negative.

For completeness it should be said that if also $\beta > \gamma+1$ (this is true only for a body in the shape of a vertical spindle) another branch of $\kappa = 0$ will intervene towards the top right hand corner of the triangle, beyond which κ is again positive.†

Note that in general throughout zone II $\lambda > \nu$ and $-\mu > -\kappa$.

If we then follow through the roots of the characteristic equation in the upper half of the complex plane, starting with large Ω and working downward, then initially we descend at uneven speed along curves asymptotically parallel to the imaginary axis, one on each side, with distances given by (30).

For positive μ (and therefore κ) each curve stays on its side of the imaginary axis to their terminations at $\Omega = 0$, when the one on the right hand side ends at $\rho = \mathrm{i}$. Thus in such a case spin in either direction is unstable except in the limiting case of infinite spin.

If, however, the shape is represented in zone I or in zone II, the curve from the right hand side crosses the imaginary axis at $\rho = \mathrm{i}$ for $\Omega_{\mathrm{i}} = -\mu/\nu$ and then the two are both on the left hand side, giving concordance. There is, however, a major difference between the zone I and the zone II cases. In the first there is no further crossing of the imaginary axis. For all Ω less than Ω_{i} (and therefore spins above a certain critical spin) there is concordance, and hence spin in one direction is stable, and in the other unstable.

If the shape is represented in zone II, however, there will be at least one, or possibly two, crossings of the imaginary axis at $\rho = 0$. This will indeed occur through one of the roots, with diminishing Ω, descending to the real axis and splitting into two real roots, one (or both) of which cross over to the positive side (figure 2). Thus, for very high spin, concordance ceases, and both spin directions are unstable. It is perhaps unusual that for ordinary (as opposed to secular) stability there is an upper limit of spin for stability, and this feature might merit further investigation.

6. Conclusions

It thus emerges that the observed phenomenon can be well described within the terms of this paper as the rolling motion (without rolling friction) of a rigid body. It is patently necessary that the body should not have inertial symmetry about the vertical (axis 3), i.e. $A > B$, and that the quadric of contact should not be aligned with the inertia quadric ($q \neq 0$). But the actual value of this asymmetry is of no great interest. As long as $\chi \neq 0$, it will have the effects needed in the

† In this 'upper zone I' in fact $\lambda+\mu$ is negative, so that ρ vanishes for two positive value of Ω, leading to the track of the roots of (23) shown in figure 2 (vi), with two separate zones of concordance.

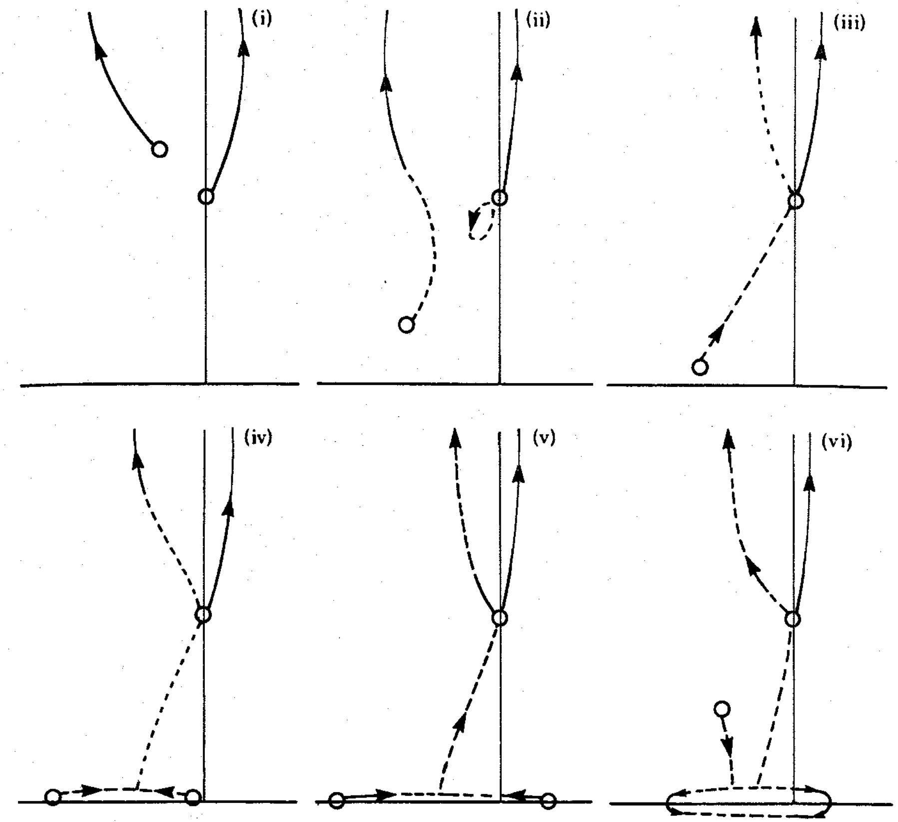

Figure 2. Various cases of the movements of the roots of the characteristic equation (23) with Ω increasing in the arrowed directions from 0 (marked ○). Schematic representation of upper half plane only. In zones of concordance the lines are broken. (i) $\kappa > 0$, $\mu > 0$; (ii) $\kappa > \frac{1}{4}\chi^2$, $\mu > 0$ (zone I); (iii) $\kappa > \frac{1}{4}\chi^2$, $\mu > 0$ (zone I); (iv) $\frac{1}{4}\chi^2 > \kappa > 0 > \mu$ (zone I); (v) $0 > \kappa$, $0 > \mu$ (zone II); (vi) $\kappa > 0 > \mu$ (upper zone I).

N.B. For cases (iii) to (vi), as opposed to the loop of (ii), the following conditions have to apply:

$$\chi < 4, \quad \kappa < 1+\tfrac{1}{4}\chi^2, \quad \tfrac{1}{4}\chi^2+\mu\Omega_3+\nu\Omega_3^2 < 0,$$

where

$$\lambda\Omega_3 = 1+\tfrac{1}{4}\chi^2-\kappa > 0.$$

non-dissipative case alone examined. In practice, with some dissipation present, too small a value of χ may mean that the instabilities in question do not show up before the spin has decayed.

Next, it is remarkable that the shape of the surface of contact is far more significant than the moments of inertia, though these do affect the boundaries of zones I and II in the θ–φ diagram. But with too small a curvature of the surface, there will always be instability in both directions of spin.

It is also clear that with the spin not too small (and, as regards zone II, not too high), spin in one direction will be stable, in the other unstable. The evolution of the unstable motion, however, needs fuller consideration. Since in this case of concordance, there are two complex roots with significant positive real parts, energy is evidently fed from the spin into these motions, which are oscillations, the one with larger real part gaining the bulk of it. (Although for small χ one of the oscillations could be called transverse and the other longitudinal, for large χ there is no such easy classification.) Indeed, so much spin energy will go into this oscillation that the spin will diminish to the value (corresponding to Ω_1) where the root corresponding to the minor oscillation crosses over to the other side of the imaginary axis, meaning that it too will diminish, and all the energy now flows into the major one. Extrapolating towards $\Omega = \infty$, the spin will be reduced to zero. Is this asymptotic and does the spin, once it reaches zero, remain so? Equation (18′) certainly does not suggest this. Putting any oscillation into x and y will yield a non-oscillatory part of $\dot{n}$, though without detailed calculation this does not indicate whether the spin tends to the previous (unstable) direction, or to the opposite one. But to go back to the spin that has just diminished to zero is most implausible. Therefore there will be at least an initial growth of the spin opposite to the one before. This reverses the significance of the roots of the characteristic equation, which for such a small spin are on opposite sides. The previously violently growing major oscillation, in which all the energy of the motion resides, will now diminish rapidly, feeding its energy into the new spin, while the previously diminishing minor oscillation will now have a tendency to grow. If, however, as mentioned in case $4b'$ above, the decay rate of the major oscillation far exceeds the growth rate of the minor one, the spin will increase sufficiently for Ω to pass Ω_1, and thus a stable domain of concordance will be entered, with the spin opposite to that at the start.

The designer of such a toy, in addition to ensuring that the shape is firmly in zones I or II, must also see to it that the ratio of the growth rates is high, a point discussed in the footnote on page 270.

It is interesting to see how complex a subject the rolling motion of a rigid body is. Yet more general cases, where the rest position does not have a principal axis of inertia vertical, await examination.

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Formation energies of small defects in non-stoichiometric rutile

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The enthalpies of formation of traditional and reconstructed small defects, having both oxygen vacancy and cation interstitial natures, have been calculated for non-stoichiometric rutile. Reconstructed interstitial defects, consisting of two pairs of face-shared octahedra linked by a string of cation Frenkel defects, prove to have significantly lower formation enthalpies (per eliminated oxygen), confirming recent interpretations of electrical conductivity and high-resolution electron microscopic results.

1. Introduction

Analysis of electrical conductivity measurements, over a wide range of temperatures (T) and partial pressures of oxygen (p_{O_2}) led to the identification of three régimes within the non-stoichiometric phase TiO_{2-x}, where the predominant non-stoichiometric defects are assumed to be: I, interstitial Ti^{4+}, prevailing for $T > 900$ °C and for relatively low partial pressures of oxygen; II, doubly ionized oxygen vacancies V_0 prevailing for relatively lower temperatures and for increasing p_{O_2} and III, a second oxygen vacancy type, associated with charge compensation of aliovalent Me^{3+} impurities, prevailing for $x \leqslant 0.0016$, independently of p_{O_2} or T (Marucco *et al.* 1981).

The observation of precipitation of pairs of crystallographic shear planes (CSP) in TiO_{2-x} ($0 \leqslant x \leqslant 0.01$) and $(Ti, Cr)O_{2-x}$ ($0 \leqslant x \leqslant 0.05$) specimens, after they had been cooled slowly across the phase boundary, strongly suggested new models for small linear defects in régime I (Blanchin *et al.* 1984; Bursill *et al.* 1984). These linear defects are in fact straightforward reconstructions of the traditional cation interstitial model, but which should allow for much lower formation energies, because they are electrostatically neutral and satisfy the well known bonding characteristics for incorporation of trivalent ions in face-shared pairs of octahedral sites (Bursill & Blanchin 1983). Later, reconstructed oxygen vacancy models were derived, in order to explain electron spin resonance (ESR) and other physical property measurements relevant to régimes II and III (Bursill & Blanchin 1984).

Catlow & James (1982) presented a theoretical survey of defect energies in TiO_{2-x}, predicting that oxygen vacancies should be the most stable non-stoichiometric defect. They constructed a realistic interatomic potential and included relaxation of ionic positions surrounding the vacancy or interstitial defects, as well as including short-range and ionic polarization energies. It was concordance, there are two complex roots with significant positive real parts, energy is evidently fed from the spin into these motions, which are oscillations, the one with larger real part gaining the bulk of it. (Although for small χ one of the oscillations could be called transverse and the other longitudinal, for large χ there is no such easy classification.) Indeed, so much spin energy will go into this oscillation that the spin will diminish to the value (corresponding to Ω_1) where the root corresponding to the minor oscillation crosses over to the other side of the imaginary axis, meaning that it too will diminish, and all the energy now flows into the major one. Extrapolating towards $\Omega = \infty$, the spin will be reduced to zero. Is this asymptotic and does the spin, once it reaches zero, remain so? Equation (18′) certainly does not suggest this. Putting any oscillation into x and y will yield a non-oscillatory part of $\dot{n}$, though without detailed calculation this does not indicate whether the spin tends to the previous (unstable) direction, or to the opposite one. But to go back to the spin that has just diminished to zero is most implausible. Therefore there will be at least an initial growth of the spin opposite to the one before. This reverses the significance of the roots of the characteristic equation, which for such a small spin are on opposite sides. The previously violently growing major oscillation, in which all the energy of the motion resides, will now diminish rapidly, feeding its energy into the new spin, while the previously diminishing minor oscillation will now have a tendency to grow. If, however, as mentioned in case 4b' above, the decay rate of the major oscillation far exceeds the growth rate of the minor one, the spin will increase sufficiently for Ω to pass Ω_1, and thus a stable domain of concordance will be entered, with the spin opposite to that at the start.

The designer of such a toy, in addition to ensuring that the shape is firmly in zones I or II, must also see to it that the ratio of the growth rates is high, a point discussed in the footnote on page 270.

It is interesting to see how complex a subject the rolling motion of a rigid body is. Yet more general cases, where the rest position does not have a principal axis of inertia vertical, await examination.

Reference

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Formation energies of small defects in non-stoichiometric rutile

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