Note on the $O(s, t)$ $\gamma$-matrix
Taichiro Kugo$^*$

\S 1. Euclidean Case: $SO(d = 2n)$

1.1. Clifford algebra $\mathcal{C}$

The Clifford algebra $\mathcal{C}$ is generated by $\gamma_\mu$ ($\mu = 1, \cdots, 2n$):

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}, \quad \gamma_\mu^\dagger = \gamma_\mu.$$ (1.1)

Define creation and annihilation operators of $n = d/2$ fermions:

$$a_k^\dagger = \frac{1}{2}(\gamma_{2k-1} + i\gamma_{2k}) \quad \gamma_{2k-1} = a_k^\dagger + a_k$$
$$a_k = \frac{1}{2}(\gamma_{2k-1} - i\gamma_{2k}) \quad \gamma_{2k} = (a_k^\dagger - a_k)/i$$ (1.2)

For the case of a single space of fermion, the creation and annihilation operators $a, a^\dagger$ are represented by the Pauli matrices as follows:

$$\{1\} = \{1\}, \quad a^\dagger |+\rangle = 0, \quad a^\dagger |-\rangle = |+\rangle$$
$$a^\dagger (|+\rangle, |-\rangle) = (|+\rangle, |-\rangle) \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) = (|+\rangle, |-\rangle) \frac{1}{2}(\sigma_1 + i\sigma_2)$$
$$a (|+\rangle, |-\rangle) = (|+\rangle, |-\rangle) \frac{1}{2}(\sigma_1 - i\sigma_2)$$ (1.3)

The representation space in this case is, therefore, given by:

$$\left\{ |\pm, \pm, \cdots, \pm \rangle \right\} = \left\{ |s_1, s_2, \cdots, s_n \rangle = a_1^{-1-1} a_2^{1-1} \cdots a_n^{1-1} |+, +, \cdots, + \rangle \right\}$$ (1.4)

On this basis, $\gamma$ matrices are represented as (Standard Representation)

$$\gamma_1 = \sigma_1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$$
$$\gamma_2 = \sigma_2 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$$
$$\gamma_3 = \sigma_3 \otimes \sigma_1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$$
$$\gamma_4 = \sigma_3 \otimes \sigma_2 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$$
$$\vdots$$
$$\gamma_{2n-1} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1$$
$$\gamma_{2n} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2$$ (1.5)

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Then $\Gamma_5 = \gamma_{2n+1}$ is defined by
\[
\Gamma_5 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \\
= i^{-n} \gamma_1 \gamma_2 \cdots \gamma_{2n} \equiv \gamma_{2n+1}. \tag{1-6}
\]

1.2. Charge conjugation matrix

\[
C^{-1} \gamma_{\mu} C = \eta' \gamma_{\mu}^T \quad (\eta' = \pm 1) \\
C^T = \varepsilon^C, \quad C^\dagger C = 1. \tag{1-7}
\]

In even dimension, either sign for $\eta'$ can be chosen, but it is determined in the odd dimension $d = 2n + 1$: indeed, the relation $C^{-1} \gamma_{\mu} C = \eta' \gamma_{\mu}^T$ should hold also for $\mu = 2n + 1$, so
\[
C^{-1} \gamma_{2n+1} C = (\eta')^{2n} i^{-n} \gamma_{2n+1} \gamma_{2n} \cdots \gamma_1 \gamma_2 = (\eta')^{2n} i^{-n} (\gamma_{2n} \gamma_{2n-1} \cdots \gamma_1)^T \\
= (\eta')^{2n} i^{-n} (-1)^{n(2n-1)} (\gamma_1 \gamma_2 \cdots \gamma_{2n})^T = (-1)^n \gamma_{2n}^T 
\]
so that
\[
\eta' = (-1)^n = (-1)^{\frac{d}{2}} \quad \text{in } d = 2n + 1 \text{ dimension.} \tag{1-9}
\]

Noting
\[
\sigma_1 \sigma_i \sigma_1 = + \sigma_i^T \quad \text{and} \quad \sigma_2 \sigma_i \sigma_2 = - \sigma_i^T \quad \text{for } i = 1, 2 \\
\sigma_1 \sigma_3 \sigma_1 = - \sigma_3^T \quad \text{and} \quad \sigma_2 \sigma_3 \sigma_2 = - \sigma_3^T, \tag{1-10}
\]

We see that $C$ is explicitly given by in the Standard Representation:
\[
\begin{cases}
C = \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \cdots \quad \text{for } \eta' = +1 \\
C = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \cdots \quad \text{for } \eta' = -1
\end{cases} \tag{1-11}
\]

Eqs. (1-10) and (1-6) imply that the last factor of $C$ has to be $\sigma_2$ in the case of odd dimensions, and this requires again $\eta' = (-1)^n$, i.e., Eq. (1-9). Note that this explicit $C$ in the standard repr. is clearly unitary. The transpose is found to be:
\[
\begin{cases}
C^T = \sigma_1 \otimes - \sigma_2 \otimes \sigma_1 \otimes \cdots \quad \text{for } \eta' = +1 \\
C^T = - \sigma_2 \otimes \sigma_1 \otimes - \sigma_2 \otimes \cdots \quad \text{for } \eta' = -1
\end{cases} \tag{1-12}
\]
so that
\[
C^T = C \times \begin{cases}
+ \quad \text{for } n = 1 \\
- \quad \text{for } n = 2 \\
- \quad \text{for } n = 3 \\
+ \quad \text{for } n = 4 
\end{cases} \quad \text{for } \eta' = +1 \\
C^T = C \times \begin{cases}
- \quad \text{for } n = 1 \\
- \quad \text{for } n = 2 \\
+ \quad \text{for } n = 3 \\
+ \quad \text{for } n = 4 
\end{cases} \quad \text{for } \eta' = -1
\]
\[
\text{for } \eta' = +1 \\
\text{for } \eta' = -1
\]
Thus the sign $\epsilon'$ of $C^T = \epsilon' C$ is given in dimension $d = 2n$ and $2n + 1$ by

$$
\epsilon' = \cos \frac{\pi}{2} n + \eta' \sin \frac{\pi}{2} n.
$$

(1-14)

The symmetry property of rank $r$ gamma tensor $\gamma_{\mu_1\mu_2\ldots\mu_r} C$ can be seen as follows:

$$
\gamma_{\mu_1\mu_2\ldots\mu_r} C = \gamma_{\mu_1} \gamma_{\mu_2} \ldots \gamma_{\mu_r} C = (\eta')^r C_{\gamma_{\mu_1} \gamma_{\mu_2} \ldots \gamma_{\mu_r}}^T
$$

$$
= (\eta')^r \epsilon' C^T \gamma_{\mu_1} \gamma_{\mu_2} \ldots \gamma_{\mu_r} C = (\eta')^r \epsilon' (\gamma_{\mu_1} \gamma_{\mu_2} \ldots \gamma_{\mu_r} C)^T
$$

$$
= (\eta')^r \epsilon' (\gamma_{\mu_1} \gamma_{\mu_2} \ldots \gamma_{\mu_r} C)^T
$$

(1-15)

Table I. Possible signs for $\eta'$ and $\epsilon'$: $C^{-1} \gamma_{\mu} C = \eta' \gamma_{\mu}^T$, $C^T = \epsilon' C$

<table>
<thead>
<tr>
<th>dimension (mod 8)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta'$</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\epsilon'$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

Table II. The rank $r$ of $\gamma_{\mu_1\mu_2\ldots\mu_r}$ for which $\gamma_{\mu_1\mu_2\ldots\mu_r} C$ are symmetric and anti-symmetric matrices.

<table>
<thead>
<tr>
<th>dimension $d$</th>
<th>$\eta'$</th>
<th>$\epsilon'$</th>
<th>$r$ of Symmetric $\gamma_{\mu_1\mu_2\ldots\mu_r} C$</th>
<th>$r$ of Anti-symmetric $\gamma_{\mu_1\mu_2\ldots\mu_r} C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>+</td>
<td>+</td>
<td>0,1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>1,2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>+</td>
<td>-</td>
<td>2,3</td>
<td>0,1,4</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>1,2</td>
<td>0,3,4</td>
</tr>
<tr>
<td>5</td>
<td>+</td>
<td>-</td>
<td>2</td>
<td>0,1</td>
</tr>
<tr>
<td>6</td>
<td>+</td>
<td>-</td>
<td>2,3,6</td>
<td>0,1,4,5</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>+</td>
<td>0,3,4</td>
<td>1,2,5,6</td>
</tr>
<tr>
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<td>-</td>
<td>+</td>
<td>0,3</td>
<td>1,2</td>
</tr>
<tr>
<td>8</td>
<td>+</td>
<td>+</td>
<td>0,1,4,5,8</td>
<td>2,3,6,7</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>+</td>
<td>0,3,4,7,8</td>
<td>1,2,5,6</td>
</tr>
<tr>
<td>9</td>
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<td>+</td>
<td>0,1,4</td>
<td>2,3</td>
</tr>
<tr>
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<td>+</td>
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<td>2,3,6,7,10</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>1,2,5,6,9,10</td>
<td>0,3,4,7,8</td>
</tr>
<tr>
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<td>-</td>
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<td>0,3,4</td>
</tr>
<tr>
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<td>-</td>
<td>2,3,6,7,10,11</td>
<td>0,1,4,5,8,9,12</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>1,2,5,6,9,10</td>
<td>0,3,4,7,8,11,12</td>
</tr>
</tbody>
</table>
§2. Clifford 代数の表現および\(\eta', \varepsilon'\)の一意性

任意の表現の\(\gamma\)行列を持ってきたとき、それらから fermion 演算子 \(a_k, a_k^\dagger\) を式 (1.2) のように作れば、そのすべての生成演算子で消える状態 \(+, +, \cdots, +\) を必ず作れる。これがいくつもあれば、直交化して独立にしておく。そのそれぞれの上で、式 (1.4) の部分空間を作れて、その base に関しては、元の\(\gamma\)行列は、標準表示の (1.5) で表現される。よって、既約表示では、状態 \(+, +, \cdots, +\) は一意的である。この時、base (1.4) は正規直交系であるから、あるユニタリ行列 \(U\) が存在して、

\[
\gamma_{\mu}^{\text{std}} = U^{-1} \gamma_\mu U
\]

(2.1)

と書ける。

この式より、また、\(\eta', \varepsilon'\) の一意性が言える。実際、一般の表示の\(\gamma\)行列に対する、charge conjugation matrix \(C\) と、標準表示のそれ \(C^{\text{std}}\)との関係は、それぞれの定義を比較して

\[
C = UC^{\text{std}}U^T
\]

となり、\(\eta', \varepsilon'\)は、両者が共通である事がわかる。また、\(C^{\text{std}}\) のユニタリ性から \(C\) のユニタリ性もである。

From the transformation property \(C = UC^{\text{std}}U^T\) under the change of the basis, we note that \(C\) can always be taken to be 1 whenever \(\varepsilon' = 1\), i.e., \(C^T = C\). Indeed, for such cases, \(C^{\text{std}}\) is symmetric and contains even number of \(\sigma_2\) factors. So it is real symmetric matrix and can be be diagonalized by an orthogonal matrix \(O\). But the eigenvalues are 1 and \(-1\):

\[
OC^{\text{std}}O^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = J \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} J
\]

\[
J \equiv \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = J^T.
\]

(2.3)

Therefore \((JO)C^{\text{std}}(JO)^T\) becomes a unit matrix 1.

The explicit construction of such \(\gamma\) matrix representation for which \(C\) becomes unit matrix is as follows: in 9 dimensions, \(\gamma\) matrices have to be symmetric by themselves if \(C = 1\).

\[
\begin{align*}
\gamma_1 &= \sigma_3 \otimes 1 \otimes 1 \otimes 1 \\
\gamma_2 &= \sigma_1 \otimes 1 \otimes 1 \otimes 1 \\
\gamma_3 &= \sigma_2 \otimes \sigma_3 \otimes 1 \otimes \sigma_2 \\
\gamma_4 &= \sigma_2 \otimes \sigma_1 \otimes 1 \otimes \sigma_2 \\
\gamma_5 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes 1 \\
\gamma_6 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \otimes 1 
\end{align*}
\]
\begin{align*}
\gamma_7 &= \sigma_2 \otimes 1 \otimes \sigma_2 \otimes \sigma_3 \\
\gamma_8 &= \sigma_2 \otimes 1 \otimes \sigma_2 \otimes \sigma_1 \\
\gamma_9 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\end{align*}

(2.4)

We can go down to 7 dimension, by throwing away the first two gamma matrices \( \gamma_1, \gamma_2 \) and simultaneously the first column of the tensor product. Then the resultant seven \( \gamma \) matrices become antisymmetric by losing their first \( \sigma_2 \) factors, being in accord with \( \eta' = -1 \) in seven dimension. Clearly these can be repeated for \( 8n + 1 \) dimensions by using these \( \gamma_1 - \gamma_8 \) blocks and \( \otimes^4 \sigma_2 \) and \( \otimes^4 1 \) as a building blocks of tensor product.
§3. (General) Minkowski Case: \(SO(t, s)\)

3.1. Clifford algebra \(C\)

The Clifford algebra \(C\) in this \(SO(t, s)\) case is:

\[
\gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = 2\eta_{\mu\nu}, \quad \eta_{\mu\nu} = \text{diag}(+\cdots+, -\cdots-, -\cdots+),
\]

\[
\gamma_{\mu}^\dagger = \begin{cases} 
\gamma_{\mu} & \text{for } \mu = 1, \cdots, t \\
-\gamma_{\mu} & \text{for } \mu = t + 1, \cdots, t + s
\end{cases}.
\]  \(\text{(3-1)}\)

The standard representation in this case is simply given by

\[
\gamma_{\mu} \equiv \begin{cases} 
\gamma_{\mu}^E & \text{for } \mu = 1, \cdots, t \\
i^{-1}\gamma_{\mu}^E & \text{for } \mu = t + 1, \cdots, t + s
\end{cases}.
\]  \(\text{(3-2)}\)

by putting \(i^{-1}\) to the space components from the previous Euclidean one \(\gamma_{\mu}^E\). Since the change from Euclidean to Minkowskian cases is only the multiplicatin of the non-matrix factor \(i\), the same charge conjugation matrix \(C\) as before satisfies

\[
C^{-1}\gamma_{\mu}C = \eta'\gamma_{\mu}^T \quad (\eta' = \pm 1).
\]  \(\text{(3-3)}\)

So the signs of \(\eta'\) and \(\varepsilon'\) in Table I and the symmetry properties of gamma matrices in Table II are also valid in the general Minkowskian cases.

3.2. \(B\)-Conjugation

With a matrix

\[
\Gamma_0 \equiv \gamma_1 \gamma_2 \cdots \gamma_t, \quad \Gamma_0 \Gamma_0^\dagger = 1.
\]  \(\text{(3-4)}\)

we define the Dirac conjugate field \(\bar{\psi}\) by

\[
\bar{\psi} = \psi^\dagger \Gamma_0^{-1}
\]  \(\text{(3-5)}\)

and Dirac conjugation by

\[
(\bar{\psi} \gamma_{\mu} \chi)^\dagger \equiv \bar{\chi} \gamma_{\mu} \psi \quad \Rightarrow \quad \bar{\gamma}_{\mu} = \Gamma_0 \gamma_{\mu}^T \Gamma_0
\]  \(\text{(3-6)}\)

for which we have

\[
\Gamma_0 \gamma_{\mu}^T \Gamma_0 = (-1)^{\frac{1}{2}t+1} \gamma_{\mu}.
\]  \(\text{(3-7)}\)

For the existence of Majorana(-Weyl) spinor, however, more important than the charge conjugation matrix \(C\) is the following matrix \(B\):

\[
B^{-1}\gamma_{\mu}B = \eta \gamma_{\mu}^*, \quad (\eta = \pm 1),
\]

\[
B^T = \varepsilon B, \quad B^\dagger B = 1.
\]  \(\text{(3-8)}\)
Indeed, we define the charge conjugation by

\[ \psi^c = C\psi^T (= C\Gamma_0^*\psi^*) \]  \hspace{1cm} (3.9)

and also write it into the form

\[ \psi^f = B\psi^* . \]  \hspace{1cm} (3.10)

Then, comparing the two expressions, we find the relation between \( B \) and \( C \) as

\[ B = C\Gamma_0^* . \]  \hspace{1cm} (3.11)

Indeed, then, using the unitarity of \( C \) and \( \Gamma_0 \), we have the properties Eq. (3.8) of \( B \):

\[ B^\dagger\Gamma_0^T C^\dagger C\Gamma_0^* = (\Gamma_0^\dagger\Gamma_0)^* = 1, \]
\[ B^{-1}\gamma_\mu B = \Gamma_0^T C^{-1}\gamma_\mu C\Gamma_0^* = \eta'(\Gamma_0^\dagger\Gamma_0)^* = \eta(-1)^{|t|/2}(\Gamma_0^\dagger\Gamma_0)^* = \eta(-1)^{|t|+1}\gamma_\mu^*, \]
\[ B^T = \Gamma_0^\dagger C^T = \epsilon'\Gamma_0^\dagger C = \epsilon'\gamma_1 \cdots \gamma_l C \]
\[ \begin{align*}
\eta &= \eta(-1)^{|t|+1}, \\
\epsilon &= \epsilon'(-1)^{|t|/2} \epsilon'(-1)^{|t|/2}. 
\end{align*} \hspace{1cm} (3.12)

so that we find

\[ \eta = \eta(-1)^{|t|+1} , \quad \epsilon = \epsilon'(-1)^{|t|/2} \epsilon'(-1)^{|t|/2} . \]  \hspace{1cm} (3.13)

\( \epsilon \) is a mod 4 function of \( t \). Examining all cases by using the expression Eq. (1.14), we find

\[ \epsilon = \cos \frac{\pi s - t}{2} - \eta \sin \frac{\pi s - t}{2} . \]  \hspace{1cm} (3.14)

As the previous Eq. (1.14) does, this applies only to even dimensions \( d = s + t = 2n \) for which \((s - t)/2\) is an integer. This is again a mod 4 function of \((s - t)/2\).

We thus obtain the results for the signs \( \eta \) and \( \epsilon \) as follows:

\[ \begin{align*}
\epsilon &= +1, \quad \eta = -1 : \quad s - t = 1, 2, 8, \text{mod 8} \\
\epsilon &= +1, \quad \eta = +1 : \quad s - t = 6, 7, 8, \text{mod 8} \\
\epsilon &= -1, \quad \eta = -1 : \quad s - t = 4, 5, 6, \text{mod 8} \\
\epsilon &= -1, \quad \eta = +1 : \quad s - t = 2, 3, 4, \text{mod 8} 
\end{align*} \hspace{1cm} (3.15)

This is summarized more explicitly in Table III.
Table III. Possible signs for $\eta$ and $\varepsilon$: $B^{-1}\gamma_\mu B = \eta\gamma^\mu_\mu$, $B^T = \varepsilon B$, together with the signs for $\eta'$ and $\varepsilon'$: $C^{-1}\gamma_\mu C = \eta'\gamma^\mu_\mu$, $C^T = \varepsilon'C$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\eta$</th>
<th>$\varepsilon$</th>
<th>$\eta'$</th>
<th>$\varepsilon'$</th>
<th>$\eta'$</th>
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</tr>
</tbody>
</table>

Before closing this subsection, we note that the charge conjugation $\psi^c$ can also be written in the form:

$$\bar{\psi}^c = \psi^T \Gamma^T \mathbf{I}^\dagger \cdot \Gamma^{-1} = (\eta')^t(-)^{[\frac{d}{2}]} C^{-1} \cdot \mathbf{I}_0 \cdot \eta^t \cdot \eta^-[\frac{d}{2}] \cdot \Gamma_0^{-1} \cdot \eta^t. \quad (3.16)$$

Namely,

$$\bar{\psi}^c = \tau \psi^T C^{-1}, \quad \tau \equiv (\eta')^t(-)^{[\frac{d}{2}]} = \eta^t(-)^{[\frac{d}{2}]} \eta^t. \quad (3.17)$$

We are now in a position to discuss separately when Weyl, Majorana and Majorana-Weyl spinors can exist.

3.3. Weyl spinor

When the dimension $d = s + t$ is even, we can define $\Gamma^5_5$ in this general Minkowskian case as

$$\Gamma^5_5 = \gamma^5 = \gamma^i \gamma^j \cdots \gamma^d = i^{(d/2)\gamma^i_1 \gamma^j_2 \cdots \gamma^j_d}, \quad \Gamma^2_5 = 1, \quad (\Gamma^T = \Gamma^\dagger) \quad (3.18)$$

This $\Gamma^5_5$ has the $B$ conjugation property:

$$B^{-1} \Gamma^5_5 B = (-)^{(s-t)/2} \Gamma^5_5 \quad (3.19)$$

(For odd dimension $d + 1$, $\eta$, and hence $\varepsilon$ also, is fixed: the anti-hermitian $\gamma_{d+1}$ should be $\gamma_{d+1} = i^{-1} \Gamma^5_5$, for which Eq. (3.4) holds with $\eta = -(-)^{(s-t)/2}$.)
Whenever $d$ is even, we have Weyl spinors: we can construct the chiral projection operator

$$\mathcal{P}_\pm = \frac{1}{2} (1 \pm \Gamma_5) ,$$

and $\mathcal{P}_\pm \psi \equiv \psi_\pm$ gives the Weyl spinors.

3.4. Majorana and Pseudo-Majorana

The Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

and its hermitian conjugation and multiplication of $(-\eta)B$ gives

$$\left(i\gamma^\mu \partial_\mu - (-\eta)m\right)B\psi^* = 0$$

So $B\psi^*$ satisfies the same equation as $\psi$ if $(-\eta)m = m$. If in addition $\varepsilon = +1$, we can equate $B\psi^*$ with $\psi$:

$$\text{Majorana: } B\psi^* = \psi \quad (\text{if } \varepsilon = +1)$$

This is possible only when $\varepsilon = +1$ because Eq. (3.23) implies $BB^* = 1$. If $\varepsilon = -1$ and we have two spinors $\psi_i (i = 1, 2)$, then we can impose instead an “SU(2) reality” condition

$$\text{SU(2) Majorana: } \varepsilon_{ij} B\psi^j = \psi_i \quad (\text{if } \varepsilon = -1)$$

where $\varepsilon_{ij}$ is the SU(2) invariant anti-symmetric tensor. If we have $2N$ spinors, we can instead impose USp(2N) reality condition $\Omega_{ij} B\psi^j = \psi_i$ by replacing the SU(2) metric $\varepsilon_{ij}$ by USp(2N) invariant (real anti-symmetric) metric $\Omega_{ij}$. In both cases of Majorana and USp(2N) Majorana, the condition $(-\eta)m = m$ means that if $\eta = +1$ we must have $m = 0$. So we put the term pseudo for $\eta = +1$.

Thus we can have the following four types of Majorana spinors for the combination of the signs of $\eta$ and $\varepsilon$:

- Majorana for $\varepsilon = +1$, $\eta = -1 \iff s - t = 1, 2, 8, \text{ mod } 8$
- pseudo-Majorana for $\varepsilon = +1$, $\eta = +1 \iff s - t = 6, 7, 8, \text{ mod } 8$
- USp(2N) Majorana for $\varepsilon = -1$, $\eta = -1 \iff s - t = 4, 5, 6, \text{ mod } 8$
- USp(2N) pseudo-Majorana for $\varepsilon = -1$, $\eta = +1 \iff s - t = 2, 3, 4, \text{ mod } 8$

3.5. Majorana-Weyl Spinors

The Weyl spinors $\psi_\pm \equiv \mathcal{P}_\pm \psi$ satisfying

$$\Gamma_5 \psi_\pm = \pm \psi_\pm$$

(3.26)
always exist for even dimension \( d \). But this is compatible with the [SU(2)] Majorana condition, Eq. (3·23) or Eq. (3·24), only if

\[
\sigma \equiv (-1)^{(s-t)/2} = 1 \implies s - t = 0 \mod 4 \tag{3·27}
\]

This is because we have from the B conjugation property Eq. (3·19) of \( T_5 \)

\[
B^{-1} \mathcal{P}_\pm B = \mathcal{P}_\pm^* \tag{3·28}
\]

with \( \sigma = (-1)^{(s-t)/2} \). So, applying the chiral projection \( \mathcal{P}_\pm \) to the Majorana spinor condition Eq. (3·23), for instance, we would get

\[
\psi = B\psi^* \implies \mathcal{P}_\pm \psi = \mathcal{P}_\pm B\psi^*
\]

\[
\psi_\pm = B\mathcal{P}_\pm^* \psi^* = B(\psi_{\pm\sigma})^* \tag{3·29}
\]

We, therefore, see that such a reality condition on Weyl spinors can be a closed condition only if \( \sigma = +1 \).

Noting that \( \varepsilon = -1 \) for \( s - t = 4 \mod 8 \), and \( \varepsilon = +1 \) for \( s - t = 8 \mod 8 \), we thus find that we can have

\[
\begin{align*}
(\text{pseudo-}) \text{ Majorana-Weyl} & \quad \text{for} \quad s - t = 8 \mod 8 \\
\text{USp}(2N) \ (\text{pseudo-}) \text{ Majorana-Weyl} & \quad \text{for} \quad s - t = 4 \mod 8 \tag{3·30}
\end{align*}
\]