CMB bispectrum generated from primordial magnetic fields

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❖collaborators:
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based on MS + (2010, 2011)
new
[1003.2096, 1009.3632, 1012.1079, 1101.5287, 1103.4103]

formulation

analysis

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1. Introduction of the cosmological magnetic fields
2. Formulation of CMB bispectrum
3. CMB bispectrum induced from primordial magnetic fields

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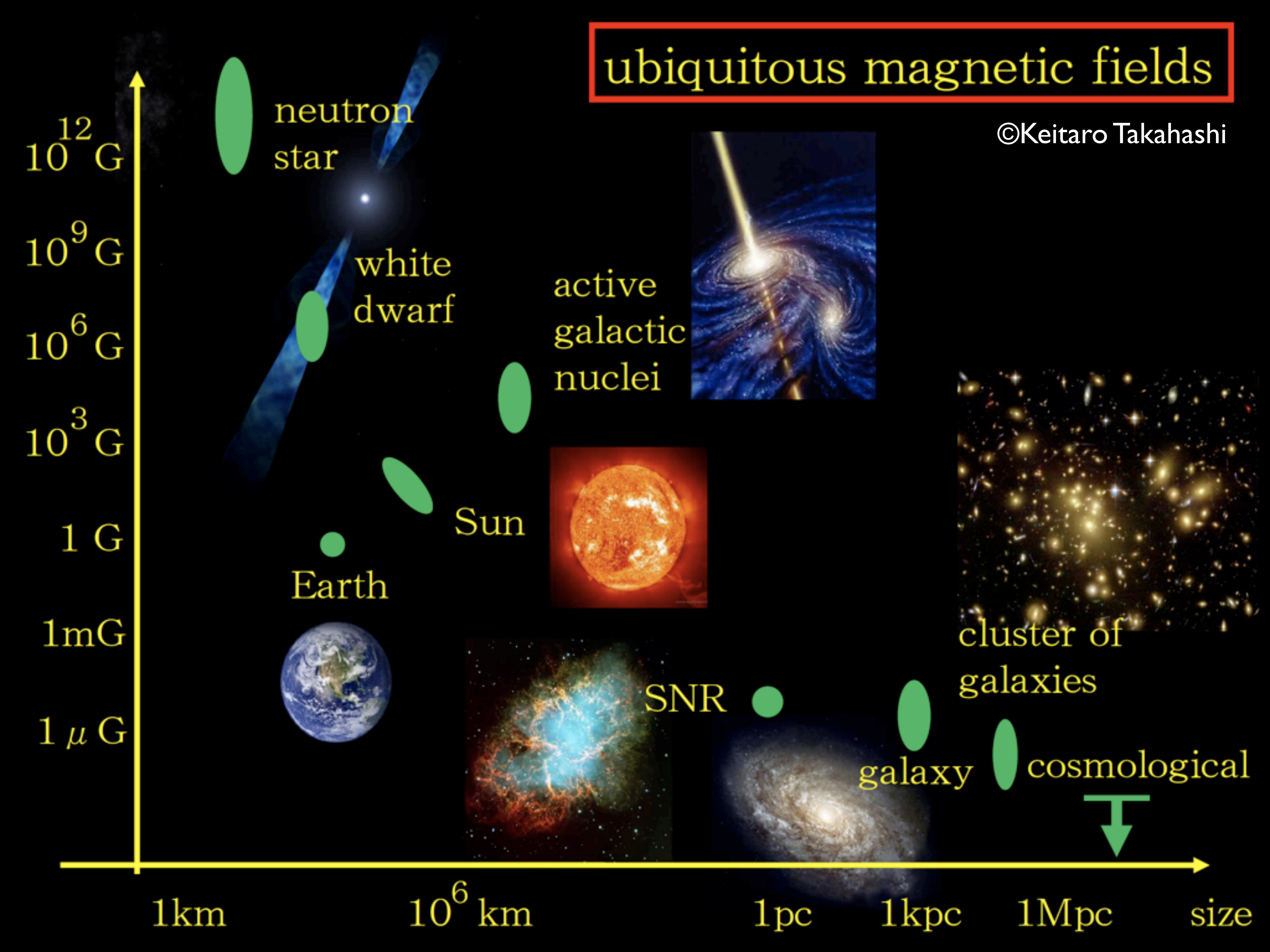
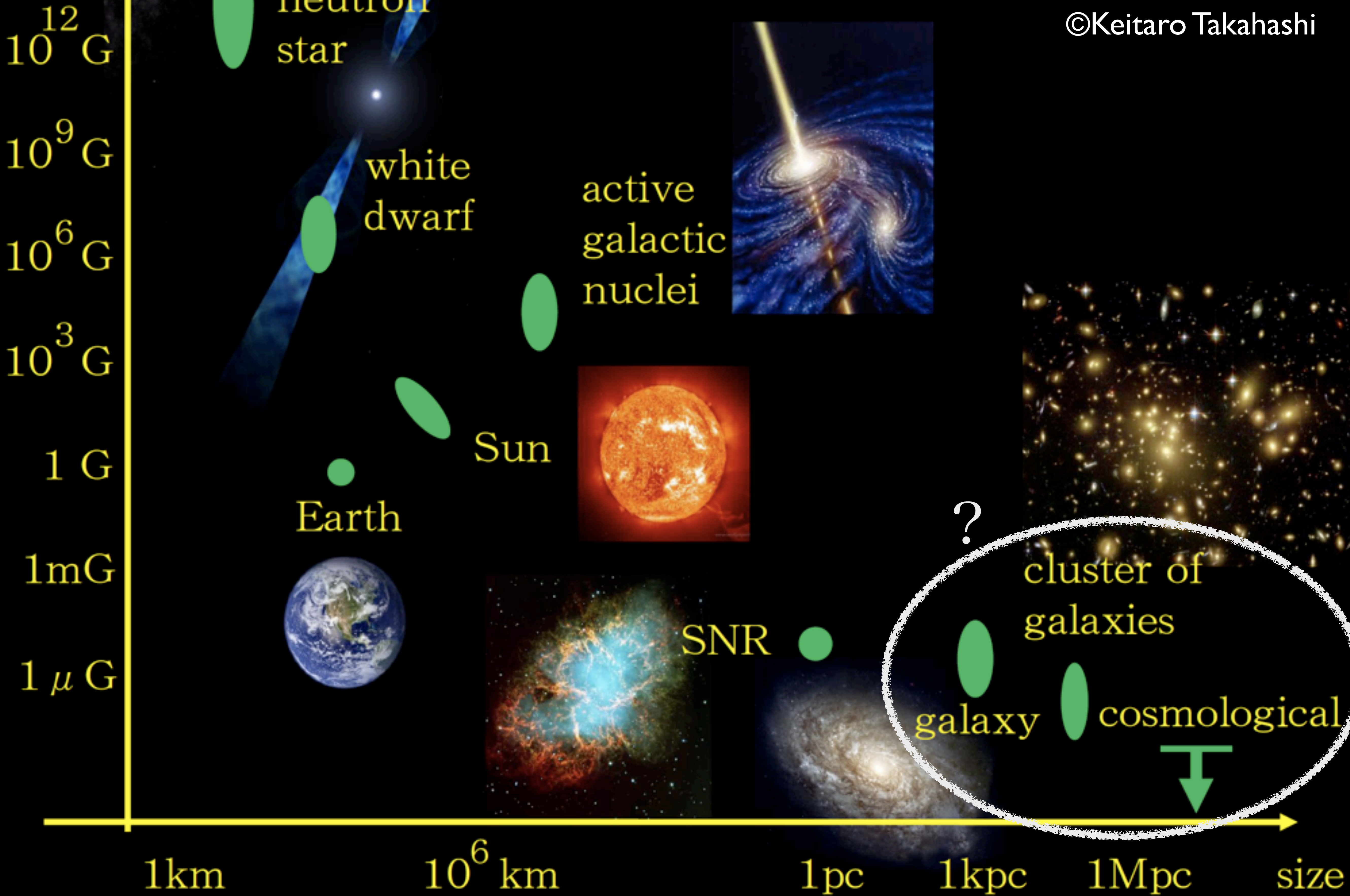
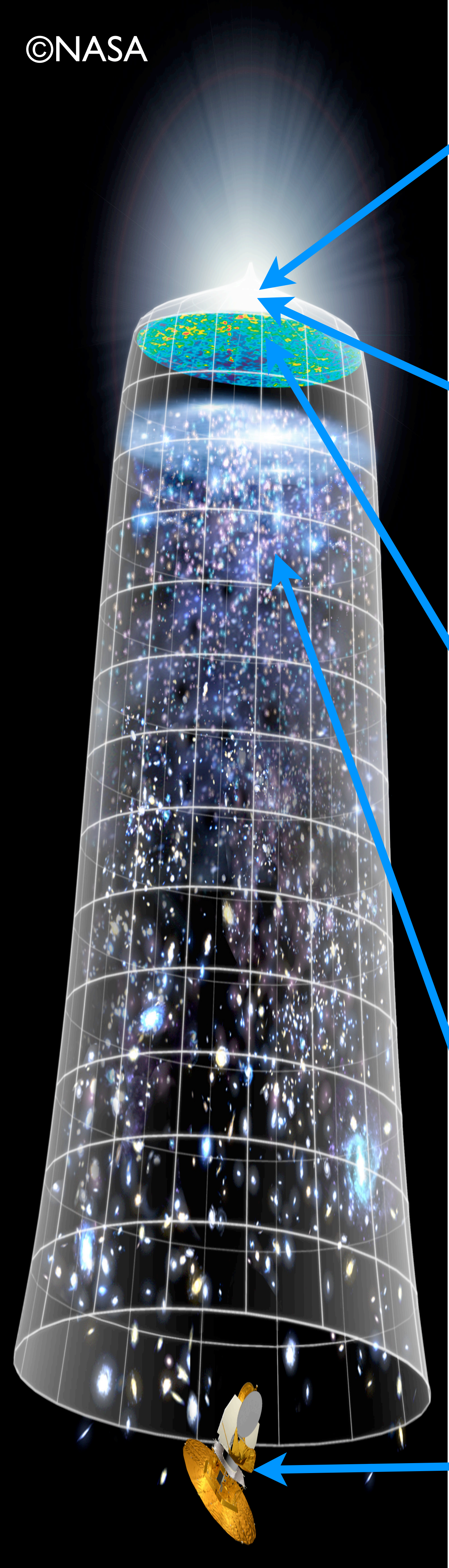

ubiquitous magnetic fields
©Keitaro Takahashi
10^{12} G
10^{9} G
10^{6} G
10^{3} G
1 G
1mG
1 μ G
neutron star
white dwarf
active galactic nuclei
Sun
Earth
SNR
cluster of galaxies
galaxy
cosmological
1km
10^{6} km
1pc
1kpc
1Mpc
size

ubiquitous magnetic fields





- inflation
- phase transition
  (t < 10^{-10}sec)

nucleosynthesis
of light elements
(t ~ 1sec)

H, He
recombination
(t ~ 0.4Myr)

beginning of galaxy
formation

present

coupling between vector field
& inflaton, electroweak or
QCD phase transition?

Martin & Yokoyama [0711.4304],
Bamba & Sasaki [0611701], ...

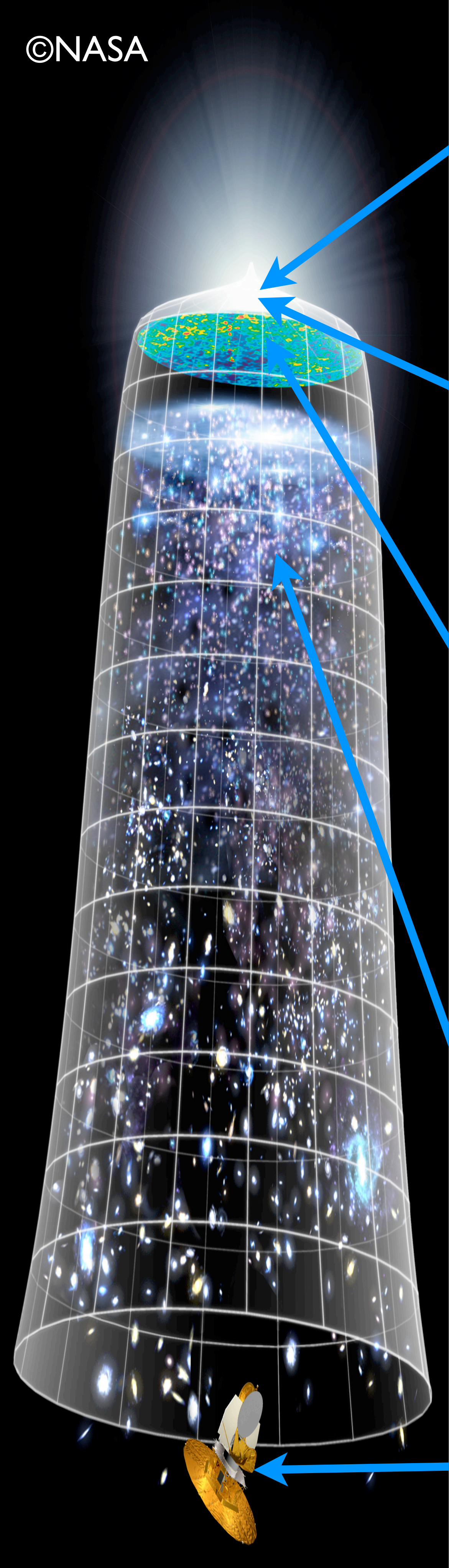
2nd order perturbation?

Ichiki + [0603631], ...

$B_{1\mathrm{Mpc}} \sim 10^{-20}\mathrm{G}$

amplified by
astrophysical processes
(dynamo mechanism?)

$B_{1\mathrm{Mpc}} \sim O(1\mu\mathrm{G})$



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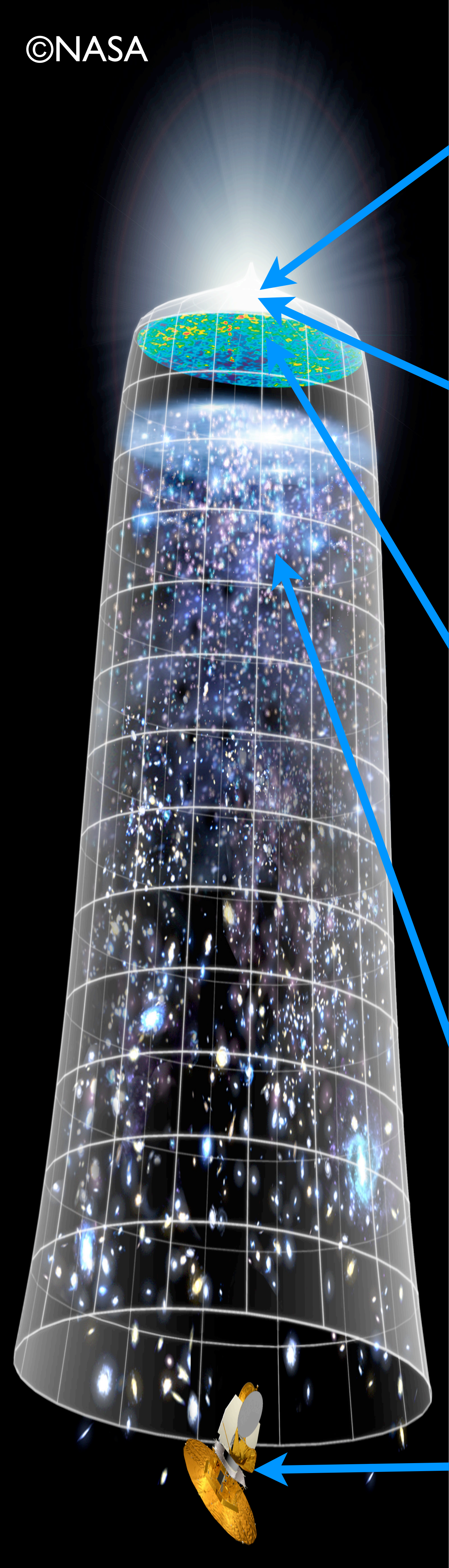

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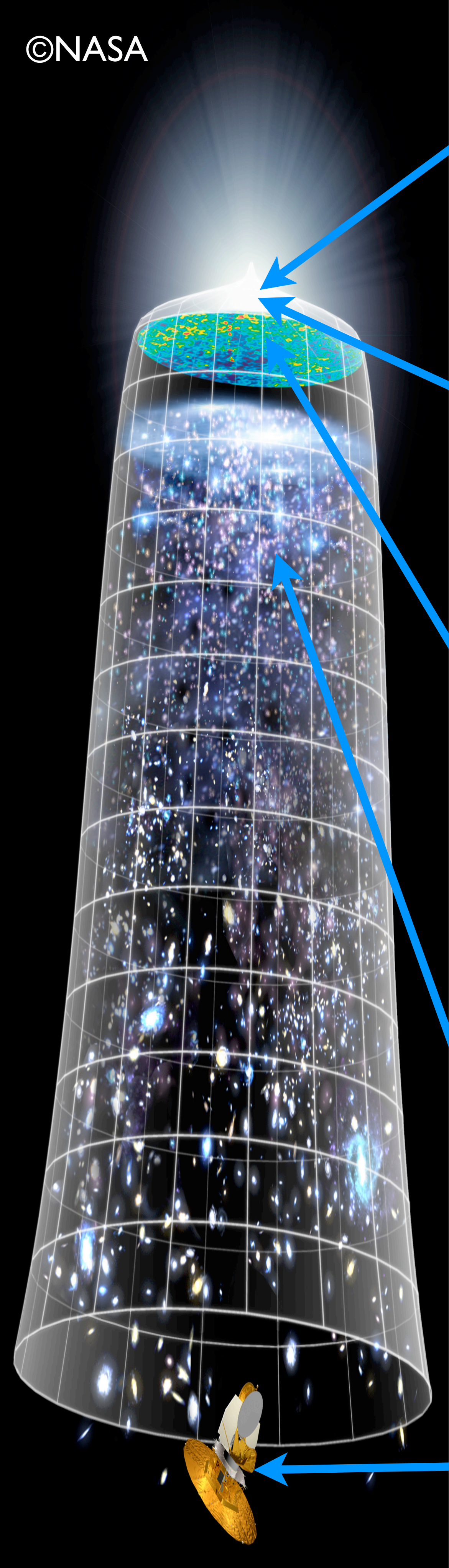

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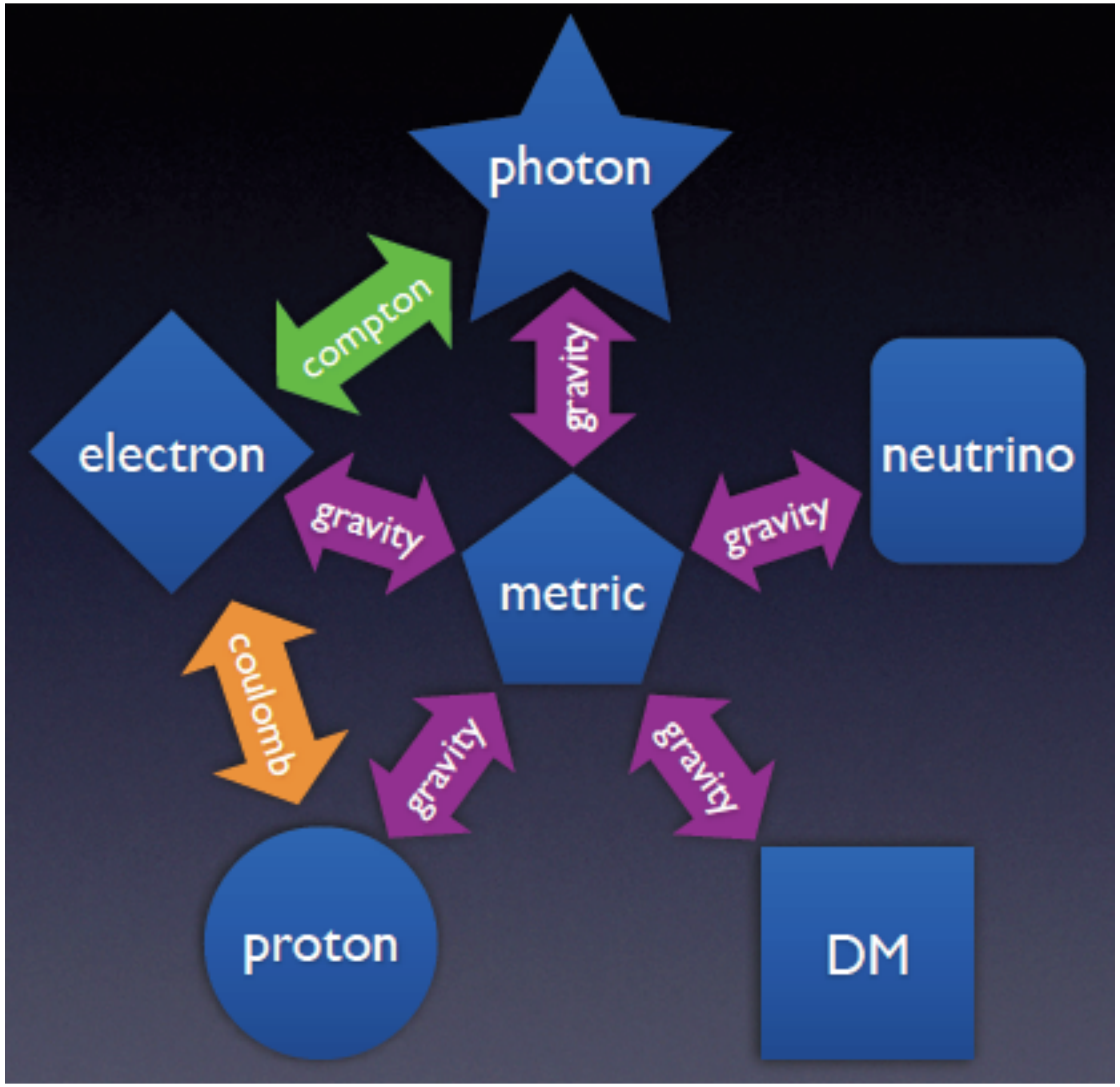
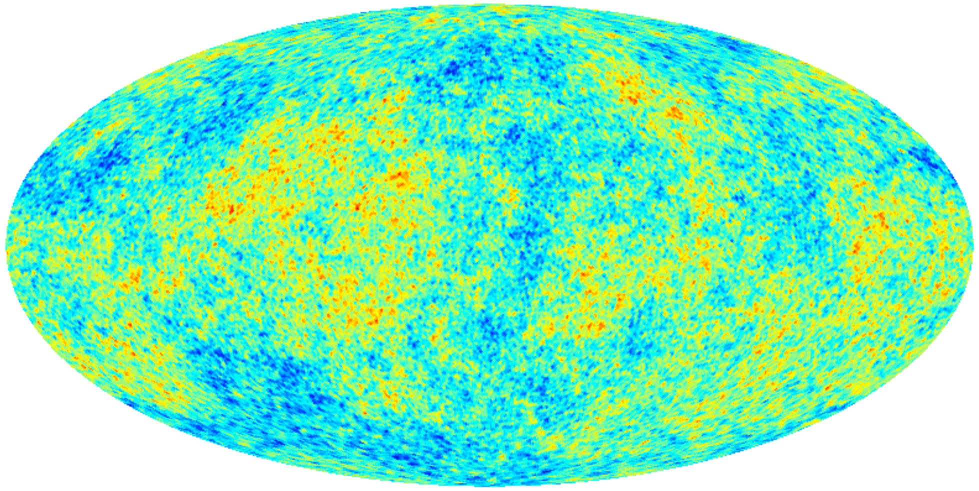
CMB fluctuation sourced from primordial magnetic fields (PMFs)



if PMF exists...



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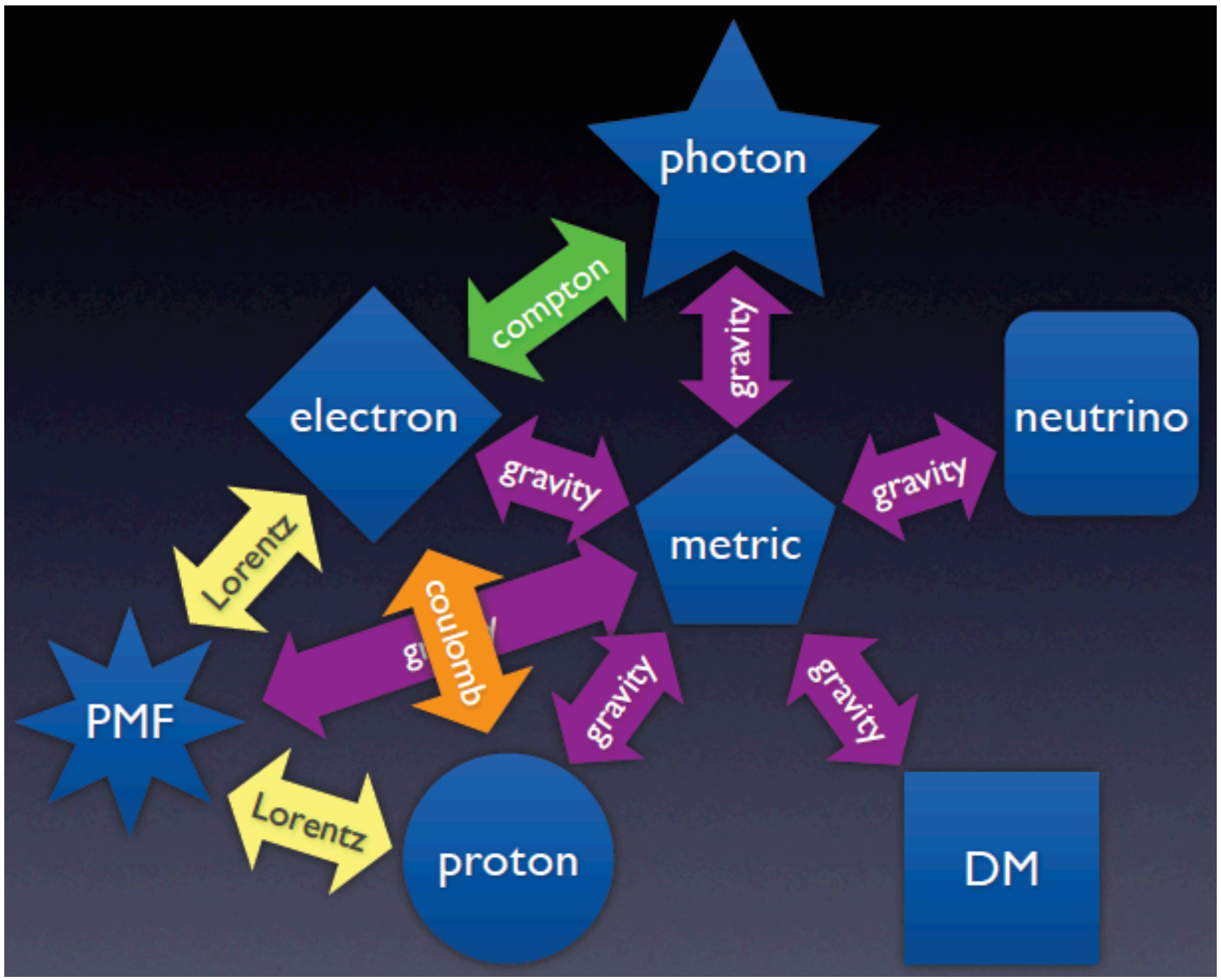

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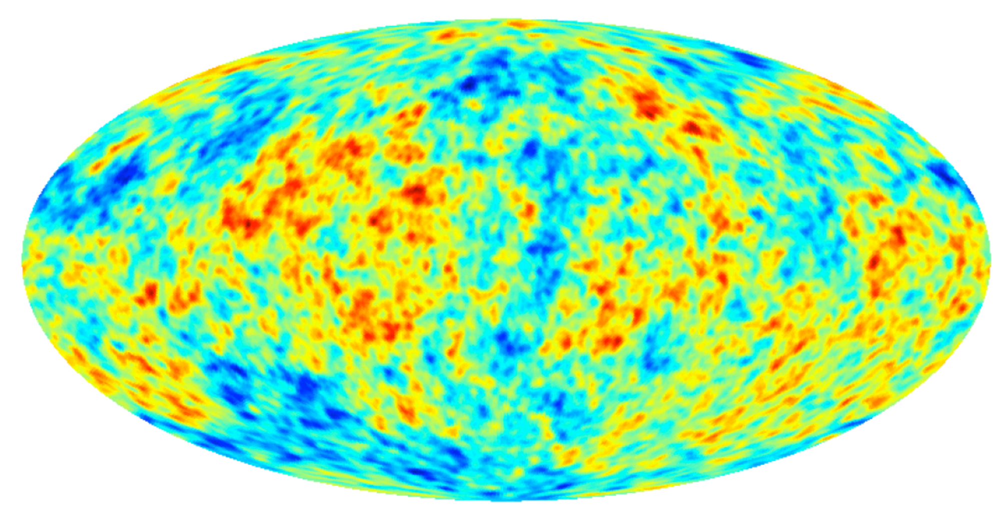

Current bounds on the primordial magnetic field

- using CMB power spectrum,
  - $B_{1Mpc} < 5.0nG$, $n_B < -0.12$ (WMAP7+ACBAR+BICEP+QUAD) : Paoletti + [1005.0148]
- using CMB + matter power spectrum
  - $B_{1Mpc} < 2.98nG$, $n_B < -0.25$ (WMAP5+ACBAR+CBI+Boomerang+2dFDR) : Yamazaki + [1001.2012]
  - $B_{1Mpc} < 6.4nG$ (CMB+BAO+HST+BBN+SN), $< 1.3nG$ (+SDSS Ly-α) : Shaw + [1006.4242]

$B_{1Mpc} < O(1)nG$, $n_B \sim -3$ (nearly scale invariant spectrum)

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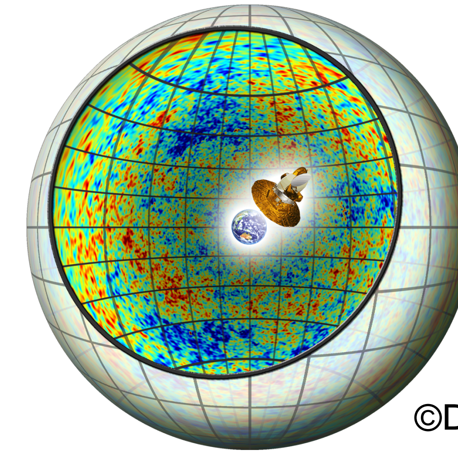
CMB anisotropy



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expand with spin-weighted spherical harmonics:

$$\Delta_I(\hat{n}) = \sum_{\ell m} a^{(Z)}_{I,\ell m} Y_{\ell m}(\hat{n})$$

$$(\Delta_Q \pm i\Delta_U)(\hat{n}) = \sum_{\ell m} a^{(Z)}_{\pm 2,\ell m}\, {}_{\pm 2}Y_{\ell m}(\hat{n})$$

$$a^{(Z)}_{E,\ell m} \equiv -\frac{a^{(Z)}_{2,\ell m} + a^{(Z)}_{-2,\ell m}}{2}$$

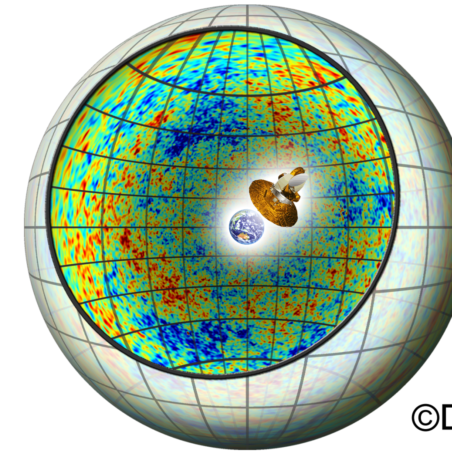
$$a^{(Z)}_{B,\ell m} \equiv -\frac{a^{(Z)}_{2,\ell m} - a^{(Z)}_{-2,\ell m}}{2}$$

- line-of-sight integration for ***k*** (|| ***z***)
- transform ***k*** (|| ***z)*** into the arbitrary direction

$$a^{(Z)}_{X,\ell m} = 4\pi(-i)^\ell \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda [\mathrm{sgn}(\lambda)]^{\lambda + x}\, {}_{-\lambda}Y^*_{\ell m}(\hat{\mathbf{k}}) \xi^{(\lambda)}(\mathbf{k}) \mathcal{T}^{(Z)}_{X,\ell}(k)$$

- Z = S (: scalar), = V (: vector), = T (: tensor)
- X = I (: intensity), = E, B (: polarization)
- x = 0 (: X = I, E), =1 (: X = B)

CMB anisotropy



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- primordial perturbation

$$\xi^{(\lambda)}(\mathbf{k}) \equiv \sum_{\ell' m'} \xi_{\ell' m'}^{(\lambda)}(k) {}_{-\lambda}Y_{\ell' m'}(\hat{k})$$

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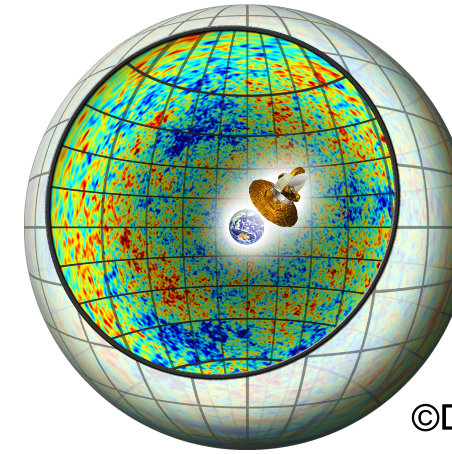
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- transform ***k*** (|| ***z)*** into the arbitrary direction

- transfer function (derived from CPT)

$$a^{(Z)}_{X,\ell m} = 4\pi(-i)^\ell \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda [\mathrm{sgn}(\lambda)]^{\lambda+x}\, {}_{-\lambda}Y^*_{\ell m}(\hat{\mathbf{k}}) \boxed{\xi^{(\lambda)}(\mathbf{k})} \boxed{\mathcal{T}^{(Z)}_{X,\ell}(k)}$$

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CMB Bispectrum

define initial bispectrum:

$$\left\langle \prod_{i=1}^{3} \xi^{(\lambda_i)}(\mathbf{k_i}) \right\rangle \equiv (2\pi)^3 F^{\lambda_1\lambda_2\lambda_3}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3})\delta\left(\sum_{i=1}^{3}\mathbf{k_i}\right)$$

define CMB angle-averaged bispectrum:

$$B^{(Z_1Z_2Z_3)}_{X_1X_2X_3,\ell_1,\ell_2,\ell_3} \equiv \sum_{m_1m_2m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$\boxed{\begin{aligned} B^{(Z_1Z_2Z_3)}_{X_1X_2X_3,\ell_1,\ell_2,\ell_3} = & \left[\prod_{n=1}^{3} 4\pi(-i)^{\ell_n}\int_0^\infty \frac{k_n^2 dk_n}{(2\pi)^3}\mathcal{T}^{(Z_n)}_{X_n,\ell_n}(k_n)\sum_{\lambda_n}[\mathrm{sgn}(\lambda_n)]^{\lambda_n+x_n}\right] \\ & \times(2\pi)^3\mathcal{F}^{\lambda_1\lambda_2\lambda_3}_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3)\ . \end{aligned}}$$

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Only when primordial perturbations deviate from Gaussian statistics, F has non zero value.

CMB bispectrum is a good tool for constraining on non Gaussianity!!

(e.g.) the primordial curvature perturbation

$$\Phi(\mathbf{x}) \equiv \Phi_{\rm L}(\mathbf{x}) + \Phi_{\rm NL}(\mathbf{x})$$
$$\Phi_{\rm NL}(\mathbf{x}) \equiv f_{\rm NL}[\Phi_{\rm L}(\mathbf{x})^2 - \langle \Phi_{\rm L}(\mathbf{x})^2 \rangle]$$

❖from theories

in all single-field inflation models

$$f_{\rm NL}^{\rm local} = \frac{5}{12}(1-n_s) \simeq 0.015$$

Creminelli & Zaldarriaga [0407059]

on the other hand, large PNG can be induced from specific models (multifield inflation, cosmic string, ...)

COMPARE

❖from observation

$$-10 < f_{\rm NL}^{\rm local} < 74 \ (95\% CL)$$

from WMAP-7yr data

Komatsu + [1001.4538]

we may obtain finer information on the early Universe!!

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Magnetic fields also induce large non-Gaussianity due to $a_{\ell m} \propto$ (Gaussian B)^2

Brown, Crittenden [0506570]

Previous bounds from CMB bispectrum

- $B_{1\mathrm{Mpc}} < 40\mathrm{nG}$ (from $f_{NL} < 100$) : Seshadri + [0902.4066]
- $B_{1\mathrm{Mpc}} < O(10)\mathrm{nG}$ (from $f_{NL} < 100$) : Caprini + [0903.1420]
- $B_{1\mathrm{Mpc}} < 2 - 4\mathrm{nG}$ (from $|f_{NL}| < 10$) : Trivedi + [1009.2724]

"They neglect the complicated calculation (the angular dependence) or don't consider the vector- and tensor-mode contribution"

We aim to find the exact formulae, curves of the scalar, vector, and tensor CMB bispectrum and constraint on PMFs

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Settings

metric (FLRW + perturbation):

$$ds^2 = a(\tau)^2[-d\tau^2 + 2h_{0b}d\tau dx^b + (\delta_{bc} + h_{bc})dx^b dx^c]$$

scalar-vector-tensor decomposition:

$$\eta(\mathbf{k}) = \eta^{(0)}(\mathbf{k}),$$
$$\omega_a(\mathbf{k}) = \omega^{(0)}(\mathbf{k})O_a^{(0)} + \sum_{\lambda=\pm1} \omega^{(\lambda)}(\mathbf{k})O_a^{(\lambda)} ,$$
$$\chi_{ab}(\mathbf{k}) = \chi^{(0)}(\mathbf{k})O_{ab}^{(0)} + \sum_{\lambda=\pm1} \chi^{(\lambda)}(\mathbf{k})O_{ab}^{(\lambda)} + \sum_{\lambda=\pm2} \chi^{(\lambda)}(\mathbf{k})O_{ab}^{(\lambda)}$$

projection operator:

$$O_a^{(0)} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv k^{-1}\nabla_a e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{k}_a e^{i\mathbf{k}\cdot\mathbf{x}} ,$$
$$O_{ab}^{(0)} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv \left(k^{-2}\nabla_a\nabla_b + \frac{\delta_{a,b}}{3}\right) e^{i\mathbf{k}\cdot\mathbf{x}} = \left(-\hat{k}_a\hat{k}_b + \frac{\delta_{a,b}}{3}\right) e^{i\mathbf{k}\cdot\mathbf{x}} ,$$
$$O_a^{(\pm1)} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv -i\epsilon_a^{(\pm1)}(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} ,$$
$$O_{ab}^{(\pm1)} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv k^{-1}\nabla_a O_b^{(\pm1)} e^{i\mathbf{k}\cdot\mathbf{x}} = \hat{k}_a\epsilon_b^{(\pm1)}(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} ,$$
$$O_{ab}^{(\pm2)} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv e_{ab}^{(\pm2)}(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} ,$$

assume rotational-invariant Gaussian PMF:

$$\langle B_a(\mathbf{k})B_b(\mathbf{p})\rangle = (2\pi)^3 \frac{P_B(k)}{2} P_{ab}(\hat{\mathbf{k}})\delta(\mathbf{k} + \mathbf{p})$$
$$P_{ab}(\hat{\mathbf{k}}) \equiv \sum_{\sigma=\pm1} \epsilon_a^{(\sigma)}\epsilon_b^{(-\sigma)} = \delta_{ab} - \hat{k}_a\hat{k}_b$$
$$P_B(k) = \frac{(2\pi)^{n_B+5}}{\Gamma(n_B/2 + 3/2)k_{1\,\mathrm{Mpc}}^3} B_{1\,\mathrm{Mpc}}^2 \left(\frac{k}{k_{1\,\mathrm{Mpc}}}\right)^{n_B}$$

scaling relation of PMF:

$$B^b(\mathbf{x}, \tau) = B^b(\mathbf{x})/a^2$$

energy momentum tensor (EMT) of PMF:

$$T_0^0 = -\rho_B = -\frac{1}{8\pi a^4} B^2(\mathbf{x}) \equiv -\rho_\gamma\Delta_B,$$
$$T_c^0 = T_0^b = 0,$$
$$T_c^b = \frac{1}{4\pi a^4}\left[\frac{B^2(\mathbf{x})}{2}\delta_c^b - B^b(\mathbf{x})B_c(\mathbf{x})\right]$$
$$\equiv \rho_\gamma(\Delta_B\delta_c^b + \Pi_{Bc}^b),$$

Vector mode of a_{lm}

Mack + [0105504], ...

If fix Gauge: $\partial_\tau \mathbf{h}^{(\mathbf{V})} = 0$ and introduce: $h_{0a}^{(V)} \equiv -A_a$

Gauge-invariant vector potential and vorticity are written:

$$\mathbf{V} \equiv \mathbf{A} - \partial_\tau \mathbf{h}^{(\mathbf{V})} = \mathbf{A}$$
$$\mathbf{\Omega} \equiv \mathbf{v} - \mathbf{A} = \mathbf{v} - \mathbf{V}$$

v: velocity perturbation

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v: velocity perturbation

Consider tight coupling limit: $v_\gamma \sim v_b \equiv v$,
namely $\Omega_\gamma \sim \Omega_b \equiv \Omega$

Einstein eq.:

$$\mathbf{V}' + 2\frac{a'}{a}\mathbf{V} = -\frac{16\pi G \rho_{\gamma,0}(\mathbf{\Pi}_\gamma^{(\mathbf{V})} + \mathbf{\Pi}_\nu^{(\mathbf{V})} + \mathbf{\Pi}_\mathbf{B}^{(\mathbf{V})})}{a^2 k}$$

Euler eq. of photon and baryon:

$$(1+R)\mathbf{\Omega}' + R\frac{a'}{a}\mathbf{\Omega} = \frac{k\rho_{\gamma,0}\mathbf{\Pi}_\mathbf{B}^{(\mathbf{V})}}{a^4(\rho_\gamma + p_\gamma)}$$

Vector mode of a_{lm}

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Lorentz force term: $L = k\Pi^{(V)}$

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Mack + [0105504], ...

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Lorentz force term: $L = k\Pi^{(V)}$

Vorticity solution:

$$\mathbf{\Omega}(\mathbf{k},\tau) \simeq \beta(k,\tau)\mathbf{\Pi}_\mathbf{B}^{(\mathbf{V})}(\mathbf{k}) ,$$

$$\beta(k,\tau) = \begin{cases} \frac{k\tau\rho_{\gamma,0}}{(1+R)(\rho_{\gamma,0}+p_{\gamma,0})} & \text{for } k < k_S \\ \frac{5\tau_c'\rho_{\gamma,0}}{k(\rho_{\gamma,0}+p_{\gamma,0})} & \text{for } k > k_S \end{cases}$$

Vector mode of a_{lm}

Mack + [0105504], ...

If fix Gauge: $\partial_\tau \mathbf{h}^{(\mathbf{V})} = 0$ and introduce: $h^{(V)}_{0a} \equiv -A_a$

Gauge-invariant vector potential and vorticity are written:

$$\mathbf{V} \equiv \mathbf{A} - \partial_\tau \mathbf{h}^{(\mathbf{V})} = \mathbf{A}$$
$$\mathbf{\Omega} \equiv \mathbf{v} - \mathbf{A} = \mathbf{v} - \mathbf{V}$$

v: velocity perturbation

Consider tight coupling limit: $v_\gamma \sim v_b \equiv v$, namely $\Omega_\gamma \sim \Omega_b \equiv \Omega$

Einstein eq.:

$$\mathbf{V}' + 2\frac{a'}{a}\mathbf{V} = -\frac{16\pi G \rho_{\gamma,0}(\mathbf{\Pi}^{(\mathbf{V})}_\gamma + \mathbf{\Pi}^{(\mathbf{V})}_\nu + \mathbf{\Pi}^{(\mathbf{V})}_\mathbf{B})}{a^2 k}$$

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intial condition

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CMB intensity fluctuation of vector mode is sourced from Ω at recombination epoch:

Doppler ISW

$$\begin{aligned} \Delta_I(\hat{\mathbf{n}}) &= -\mathbf{v}_\gamma \cdot \hat{\mathbf{n}}|_{\tau_*}^{\tau_0} + \int_{\tau_*}^{\tau_0} d\tau \mathbf{V}' \cdot \hat{\mathbf{n}} \\ &\simeq \mathbf{\Omega}_{\tau_*} \cdot \hat{\mathbf{n}} \end{aligned}$$

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$$\simeq \mathbf{\Omega}_{\tau_*} \cdot \hat{\mathbf{n}}$$

line-of-sight integration

$$a_{I,\ell m}^{(V)} \equiv \int d^2\hat{\mathbf{n}} \Delta_I(\hat{\mathbf{n}}) Y_{\ell m}^*(\hat{\mathbf{n}})$$
$$\simeq \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int d^2\hat{\mathbf{n}} [\mathbf{\Pi}_{\mathbf{B}}^{(\mathbf{V})}(\mathbf{k}) \cdot \hat{\mathbf{n}}] Y_{\ell m}^*(\hat{\mathbf{n}})$$
$$\times \beta(k, \tau_*) e^{-i\mu_{k,n} x_*}$$

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Through the coordinate transformation and hard calculation:

$$\mathbf{\Pi}_{\mathrm{B}}^{(\mathrm{V})}(\mathbf{k}) \cdot \hat{\mathbf{n}} \to -i\sqrt{\frac{1-\mu_{k,n}^2}{2}} \times \sum_{\lambda=\pm 1} \Pi_{Bv}^{(\lambda)}(\mathbf{k}) e^{i\lambda\phi_{k,n}} ,$$
$$Y_{\ell m}^*(\hat{\mathbf{n}}) \to \sum_{m'} D_{mm'}^{(\ell)} \left(S(\hat{\mathbf{k}})\right) Y_{\ell m'}^*(\Omega_{k,n})$$
$$d^2\hat{\mathbf{n}} \to d\Omega_{k,n} ,$$

$$a_{I,\ell m}^{(V)} = 4\pi(-i)^\ell \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda=\pm 1} \lambda_{-\lambda} Y_{\ell m}^*(\hat{k}) \xi^{(\lambda)}(\mathbf{k}) \mathcal{T}_{I,\ell}^{(V)}(k)$$
$$\xi^{(\pm 1)}(\mathbf{k}) \equiv \Pi_{Bv}^{(\pm 1)}(\mathbf{k})$$

$$\mathcal{T}_{I,\ell}^{(V)}(k) \simeq \left[\frac{(\ell+1)!}{(\ell-1)!}\right]^{1/2} \frac{\beta(k,\tau_*)}{\sqrt{2}} \frac{j_\ell(x_*)}{x_*}$$

Tensor mode of a_{lm}

Lewis [0406096], ...

After neutrino decoupling, Π_ν has finite value and compensates Π_B

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decompose metric into helicity state:

$$h_{ab}(\mathbf{k},\tau) = \sum_{\lambda=\pm 2} \xi^{(\lambda)}(\mathbf{k},\tau) e_{ab}^{(\lambda)}(\hat{\mathbf{k}})$$

- $R_\gamma \sim 0.6$
- τ_ν: neutrino decoupling time
- τ_B: PMF generation time

Einstein eq.: $$\xi^{(\pm2)''}(\mathbf{k},\tau) + 2\frac{a'}{a}\xi^{(\pm2)'}(\mathbf{k},\tau) + k^2\xi^{(\pm2)}(\mathbf{k},\tau) \approx \begin{cases} 16\pi G a^2 \rho_\gamma \Pi_{Bt}^{(\pm2)}(\mathbf{k}) & (\tau_B \lesssim \tau \lesssim \tau_\nu) \\ 0 & (\tau \gtrsim \tau_\nu) \end{cases}$$

Tensor mode of $a_{\ell m}$

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Solution:
$$\xi^{(\pm2)}(\mathbf{k},\tau) = \alpha(k,\tau)\xi^{(\pm2)}(\mathbf{k})$$
$$\alpha(k,\tau) \sim \begin{cases} 1 & (k\tau < 1) \\ a(k^{-1})/a(\tau) & (k\tau > 1) \end{cases}$$

$$\boxed{\xi^{(\pm2)}(\mathbf{k}) \approx \xi^{(\pm2)}(\mathbf{k},\tau_\nu) \approx 6R_\gamma \ln\left(\frac{\tau_\nu}{\tau_B}\right)\Pi_{Bt}^{(\pm2)}(\mathbf{k})}$$

initial condition

Tensor mode of $a_{\ell m}$

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initial condition

$\xi^{(\pm 2)}$ survives passively and generates CMB anisotropy through the ISW effect = “passive mode”

CMB intensity fluctuation of tensor mode is sourced from h' (ISW):

ISW

$$\Delta_I^{(T)}(\hat{\mathbf{n}}) \approx \int_{\tau_*}^{\tau_0} d\tau \frac{1}{2} h_{ab}^{(T)\prime} \hat{n}_a \hat{n}_b$$

line-of-sight integration

$$\begin{aligned}
a_{I,\ell m}^{(T)} &\equiv \int d^2\hat{\mathbf{n}} \Delta_I^{(T)}(\hat{\mathbf{n}}) Y_{\ell m}^*(\hat{\mathbf{n}}) \\
&\simeq \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int d^2\hat{\mathbf{n}} \left[\frac{1}{2} \sum_{\lambda=\pm 2} \xi^{(\lambda)}(\mathbf{k}) e_{ab}^{(\lambda)}(\hat{\mathbf{k}}) \hat{n}_a \hat{n}_b \right] Y_{\ell m}^*(\hat{\mathbf{n}}) \\
&\quad \times \int_0^{\tau_0} d\tau \alpha'(k,\tau) e^{-i\mu_{k,n} x} \\
&= 4\pi(-i)^\ell \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda=\pm 2} {}_{-\lambda}Y_{\ell m}^*(\hat{k}) \xi^{(\lambda)}(\mathbf{k}) \mathcal{T}_{I,\ell}^{(T)}(k)
\end{aligned}$$

$$\mathcal{T}_{I,\ell}^{(T)}(k) = -\left[\frac{(\ell+2)!}{(\ell-2)!}\right]^{1/2} \int_0^{\tau_0} d\tau \frac{\alpha'(k,\tau)}{2\sqrt{2}} \frac{j_\ell(x)}{x^2}$$

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Also in scalar mode, "passive mode" dominates

$$\boxed{a_{X,\ell m}^{(Z)} \propto 4\pi(-i)^\ell \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda \mathrm{sgn}(\lambda)^{\lambda+x} {}_{-\lambda}Y_{\ell m}^*(\hat{k}) \Pi_{Bz}^{(\lambda)}(\mathbf{k}) \mathcal{T}_{X,\ell}^{(Z)}(k)}$$

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ISW

$$\Delta_I^{(T)}(\hat{\mathbf{n}}) \approx \int_{\tau_*}^{\tau_0} d\tau \frac{1}{2} h_{ab}^{(T)\prime} \hat{n}_a \hat{n}_b$$

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need the statistics of initial anisotropic stress of PMF Π_{Bz}

Bispectrum of magnetic anisotropic stress

Because B is Gaussian, bispectrum of Π_B ($\propto B^6$) is finite value:

$$\langle \Pi_{Bab}(\mathbf{k_1})\Pi_{Bcd}(\mathbf{k_2})\Pi_{Bef}(\mathbf{k_3})\rangle = (-4\pi\rho_{\gamma,0})^{-3}\left[\prod_{n=1}^{3}\int_0^{k_D} k_n'^2 dk_n' P_B(k_n') \int d^2\hat{\mathbf{k}}_n'\right]$$
$$\times\delta(\mathbf{k_1}-\mathbf{k}_1'+\mathbf{k}_3')\delta(\mathbf{k_2}-\mathbf{k}_2'+\mathbf{k}_1')\delta(\mathbf{k_3}-\mathbf{k}_3'+\mathbf{k}_2')$$
$$\times\frac{1}{8}[P_{ad}(\hat{\mathbf{k}}_1')P_{be}(\hat{\mathbf{k}}_3')P_{cf}(\hat{\mathbf{k}}_2') + \{a\leftrightarrow b \text{ or } c\leftrightarrow d \text{ or } e\leftrightarrow f\}]$$

the symmetric 7 terms under the permutations of indices

scalar, vector and tensor parts of Π_B:

$$\left\langle \prod_{n=1}^{3}\Pi_{Bs}^{(0)}(\mathbf{k_n})\right\rangle = \langle \Pi_{Bab}(\mathbf{k_1})\Pi_{Bcd}(\mathbf{k_2})\Pi_{Bef}(\mathbf{k_3})\rangle$$
$$\times\frac{27}{8}(-\hat{k}_{1a}\hat{k}_{1b}+\frac{1}{3}\delta_{a,b})(-\hat{k}_{2c}\hat{k}_{2d}+\frac{1}{3}\delta_{c,d})(-\hat{k}_{3e}\hat{k}_{3f}+\frac{1}{3}\delta_{e,f})$$

$$\left\langle \prod_{n=1}^{3}\Pi_{Bv}^{(\lambda_n)}(\mathbf{k_n})\right\rangle = \langle \Pi_{Bab}(\mathbf{k_1})\Pi_{Bcd}(\mathbf{k_2})\Pi_{Bef}(\mathbf{k_3})\rangle$$
$$\times\hat{k}_{1a}\epsilon_b^{(-\lambda_1)}(\hat{\mathbf{k}}_1)\hat{k}_{2c}\epsilon_d^{(-\lambda_2)}(\hat{\mathbf{k}}_2)\hat{k}_{3e}\epsilon_f^{(-\lambda_3)}(\hat{\mathbf{k}}_3) \quad (\text{for } \lambda_n=\pm 1)$$

$$\left\langle \prod_{n=1}^{3}\Pi_{Bt}^{(\lambda_n)}(\mathbf{k_n})\right\rangle = \langle \Pi_{Bab}(\mathbf{k_1})\Pi_{Bcd}(\mathbf{k_2})\Pi_{Bef}(\mathbf{k_3})\rangle$$
$$\times\frac{1}{8}e_{ab}^{(-\lambda_1)}(\hat{\mathbf{k}}_1)e_{cd}^{(-\lambda_2)}(\hat{\mathbf{k}}_2)e_{ef}^{(-\lambda_3)}(\hat{\mathbf{k}}_3) \quad (\text{for } \lambda_n=\pm 2)$$

CMB Bispectrum from PMF

$$B^{(Z_1Z_2Z_3)}_{X_1X_2X_3,\ell_1,\ell_2,\ell_3} = \left[\prod_{n=1}^{3} 4\pi(-i)^{\ell_n}\int_0^\infty \frac{k_n^2 dk_n}{(2\pi)^3}\mathcal{T}^{(Z_n)}_{X_n,\ell_n}(k_n)\sum_{\lambda_n}[\mathrm{sgn}(\lambda_n)]^{\lambda_n+x_n}\right] \times (2\pi)^3\mathcal{F}^{\lambda_1\lambda_2\lambda_3}_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3) \ .$$

$$\mathcal{F}^{\lambda_1\lambda_2\lambda_3}_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3) = \sum_{m_1m_2m_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3\end{pmatrix}\left[\prod_{n=1}^{3}\int d^2\hat{\mathbf{k}}_{\mathbf{n}\,-\lambda_n}Y^*_{\ell_n m_n}(\hat{\mathbf{k}}_{\mathbf{n}})\right]\left\langle\prod_{n=1}^{3}\xi^{(\lambda_n)}(\mathbf{k_n})\right\rangle/(2\pi)^3$$

~~~~~~~~~~ *calculation procedure* ~~~~~~~~~~

1. expand all angular dependencies with ${}_sY_{lm}$

$$\Pi^{(\lambda)}_{Bz}(\mathbf{k}) = \sum_{\ell'm'}\Pi^{(\lambda)}_{Bz,\ell'm'}(k)_{-\lambda}Y_{\ell'm'}(\hat{\mathbf{k}})$$

$$\epsilon^{(\pm1)}_a(\hat{\mathbf{k}}) = \mp\sum_m \alpha_a^m {}_{\pm1}Y_{1m}(\hat{\mathbf{k}})$$

$$e^{(\pm2)}_{ab}(\hat{\mathbf{k}}) = \frac{3}{\sqrt{2\pi}}\sum_{Mm_am_b} {}_{\mp2}Y^*_{2M}(\hat{\mathbf{k}})\alpha_a^{m_a}\alpha_b^{m_b}\begin{pmatrix}2 & 1 & 1 \\ M & m_a & m_b\end{pmatrix}$$

$$\alpha_a^m\alpha_a^{m'} = \frac{4\pi}{3}(-1)^m\delta_{m,-m'}$$

$$\delta\left(\sum_{i=1}^{3}\mathbf{k_i}\right) = 8\int_0^\infty y^2dy\left[\prod_{i=1}^{3}\sum_{L_iM_i}(-1)^{L_i/2}j_{L_i}(k_iy)Y^*_{L_iM_i}(\hat{\mathbf{k_i}})\right]I^{0\,0\,0}_{L_1L_2L_3}\begin{pmatrix}L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3\end{pmatrix}$$
~~~~~~~~~~

2. express their integrals with the Wigner symbols

$$\int d^2\hat{\mathbf{k}} \prod_{i=1}^{4} {}_{s_i}Y_{l_i m_i}(\hat{\mathbf{k}}) = \sum_{l_5 m_5 s_5} I_{l_1\ l_2\ l_5}^{-s_1 -s_2\ s_5} I_{l_3\ l_4\ l_5}^{-s_3 -s_4 -s_5} \begin{pmatrix} l_1 & l_2 & l_5 \\ m_1 & m_2 & m_5 \end{pmatrix} \begin{pmatrix} l_3 & l_4 & l_5 \\ m_3 & m_4 & m_5 \end{pmatrix}$$

3. sum up them over multipoles

$$\sum_{\substack{m_4 m_5 m_6 \\ m_7 m_8 m_9}} \begin{pmatrix} l_4 & l_5 & l_6 \\ m_4 & m_5 & m_6 \end{pmatrix} \begin{pmatrix} l_7 & l_8 & l_9 \\ m_7 & m_8 & m_9 \end{pmatrix} \begin{pmatrix} l_4 & l_7 & l_1 \\ m_4 & m_7 & m_1 \end{pmatrix} \begin{pmatrix} l_5 & l_8 & l_2 \\ m_5 & m_8 & m_2 \end{pmatrix} \begin{pmatrix} l_6 & l_9 & l_3 \\ m_6 & m_9 & m_3 \end{pmatrix}$$
$$= \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{Bmatrix} ,$$

$$\sum_{m_4 m_5 m_6} (-1)^{\sum_{i=4}^{6} l_i - m_i} \begin{pmatrix} l_5 & l_1 & l_6 \\ m_5 & -m_1 & -m_6 \end{pmatrix} \begin{pmatrix} l_6 & l_2 & l_4 \\ m_6 & -m_2 & -m_4 \end{pmatrix} \begin{pmatrix} l_4 & l_3 & l_5 \\ m_4 & -m_3 & -m_5 \end{pmatrix}$$
$$= \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{Bmatrix}$$

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$$= (-8\pi^2 \rho_{\gamma,0})^{-3} \left[\prod_{n=1}^{3} \int_0^{k_D} k_n'^2 dk_n' P_B(k_n') \right]$$

$$\times \sum_{LL'L''} \sum_{S,S',S''=\pm 1} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ L' & L'' & L \end{Bmatrix} f^{S''S\lambda_1}_{L''L\ell_1}(k_3', k_1', k_1) f^{SS'\lambda_2}_{LL'\ell_2}(k_1', k_2', k_2) f^{S'S''\lambda_3}_{L'L''\ell_3}(k_2', k_3', k_3)$$

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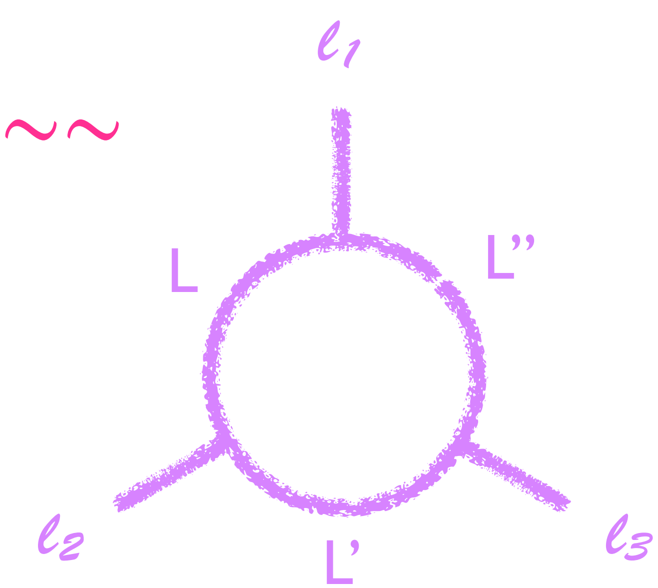
$$\sum_{\substack{m_4 m_5 m_6 \\ m_7 m_8 m_9}} \begin{pmatrix} l_4 & l_5 & l_6 \\ m_4 & m_5 & m_6 \end{pmatrix} \begin{pmatrix} l_7 & l_8 & l_9 \\ m_7 & m_8 & m_9 \end{pmatrix} \begin{pmatrix} l_4 & l_7 & l_1 \\ m_4 & m_7 & m_1 \end{pmatrix} \begin{pmatrix} l_5 & l_8 & l_2 \\ m_5 & m_8 & m_2 \end{pmatrix} \begin{pmatrix} l_6 & l_9 & l_3 \\ m_6 & m_9 & m_3 \end{pmatrix}$$
$$= \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{Bmatrix} ,$$

$$\sum_{m_4 m_5 m_6} (-1)^{\sum_{i=4}^{6} l_i - m_i} \begin{pmatrix} l_5 & l_1 & l_6 \\ m_5 & -m_1 & -m_6 \end{pmatrix} \begin{pmatrix} l_6 & l_2 & l_4 \\ m_6 & -m_2 & -m_4 \end{pmatrix} \begin{pmatrix} l_4 & l_3 & l_5 \\ m_4 & -m_3 & -m_5 \end{pmatrix}$$
$$= \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{Bmatrix}$$



$$= (-8\pi^2 \rho_{\gamma,0})^{-3} \left[\prod_{n=1}^{3} \int_0^{k_D} k_n'^2 dk_n' P_B(k_n') \right]$$
$$\times \boxed{\sum_{L L' L''} \sum_{S,S',S''=\pm 1} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ L' & L'' & L \end{Bmatrix} f^{S'' S \lambda_1}_{L'' L \ell_1}(k_3', k_1', k_1) f^{S S' \lambda_2}_{L L' \ell_2}(k_1', k_2', k_2) f^{S' S'' \lambda_3}_{L' L'' \ell_3}(k_2', k_3', k_3)}$$

scalar mode: $$f^{S''S0}_{L''L\ell}(r_3,r_2,r_1) \equiv -16\pi R_\gamma \ln(\tau_\nu/\tau_B) \sum_{L_1L_2L_3} \int_0^\infty y^2 dy j_{L_3}(r_3y) j_{L_2}(r_2y) j_{L_1}(r_1y)$$
$$\times (-1)^{L_2} (-1)^{\frac{L_1+L_2+L_3}{2}} I^{0\,0\,0}_{L_1L_2L_3} I^{0S''-S''}_{L_31L''} I^{0S-S}_{L_21L}$$
$$\times \left[\sum_{L_k=0,2} \frac{4\pi}{3} (-1)^{L_3} I^{0\,0\,0}_{L_1\ell L_k} I^{000}_{11L_k} \left\{ \begin{matrix} L'' & L & \ell \\ L_3 & L_2 & L_1 \\ 1 & 1 & L_k \end{matrix} \right\} - \frac{1}{3}(-1)^L \left\{ \begin{matrix} L & L'' & \ell \\ L_3 & L_2 & 1 \end{matrix} \right\} \delta_{\ell,L_1} \right]$$

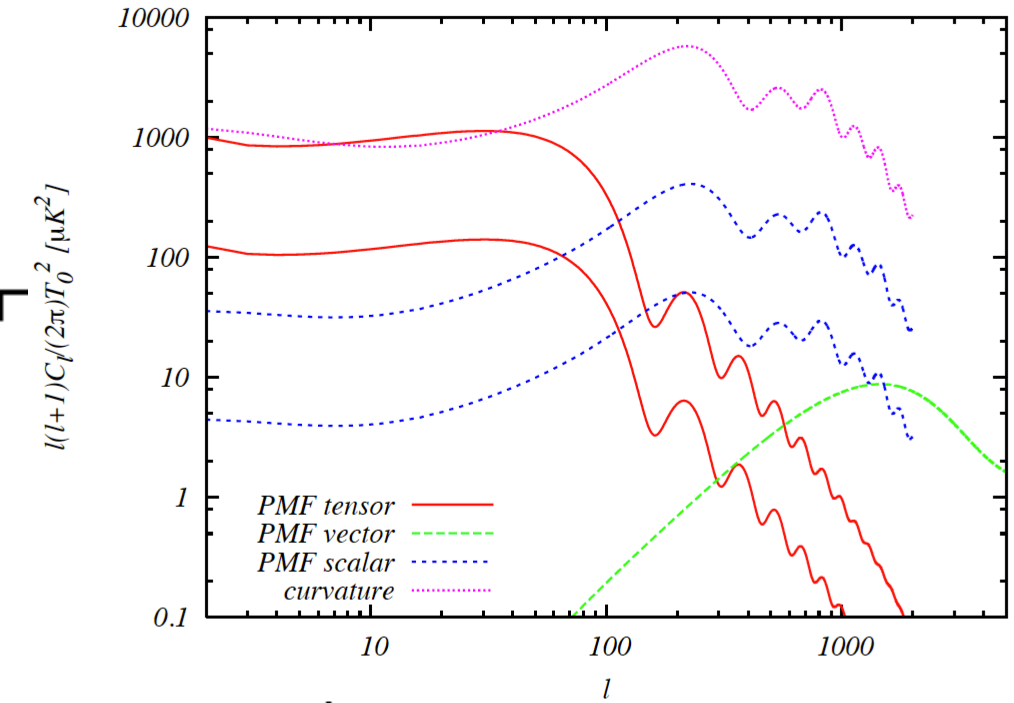
vector mode (λ = ±1): $$f^{S''S\lambda}_{L''L\ell}(r_3,r_2,r_1) \equiv \frac{2(8\pi)^{3/2}}{3} \sum_{L_1L_2L_3} \int_0^\infty y^2 dy j_{L_3}(r_3y) j_{L_2}(r_2y) j_{L_1}(r_1y)$$
$$\times \lambda (-1)^{\ell+L_2+L_3} (-1)^{\frac{(L_1+L_2+L_3)}{2}} I^{0\,0\,0}_{L_1L_2L_3} I^{0S''-S''}_{L_31L''} I^{0S-S}_{L_21L} I^{0\lambda-\lambda}_{L_1\ell 2} \left\{ \begin{matrix} L'' & L & \ell \\ L_3 & L_2 & L_1 \\ 1 & 1 & 2 \end{matrix} \right\}$$

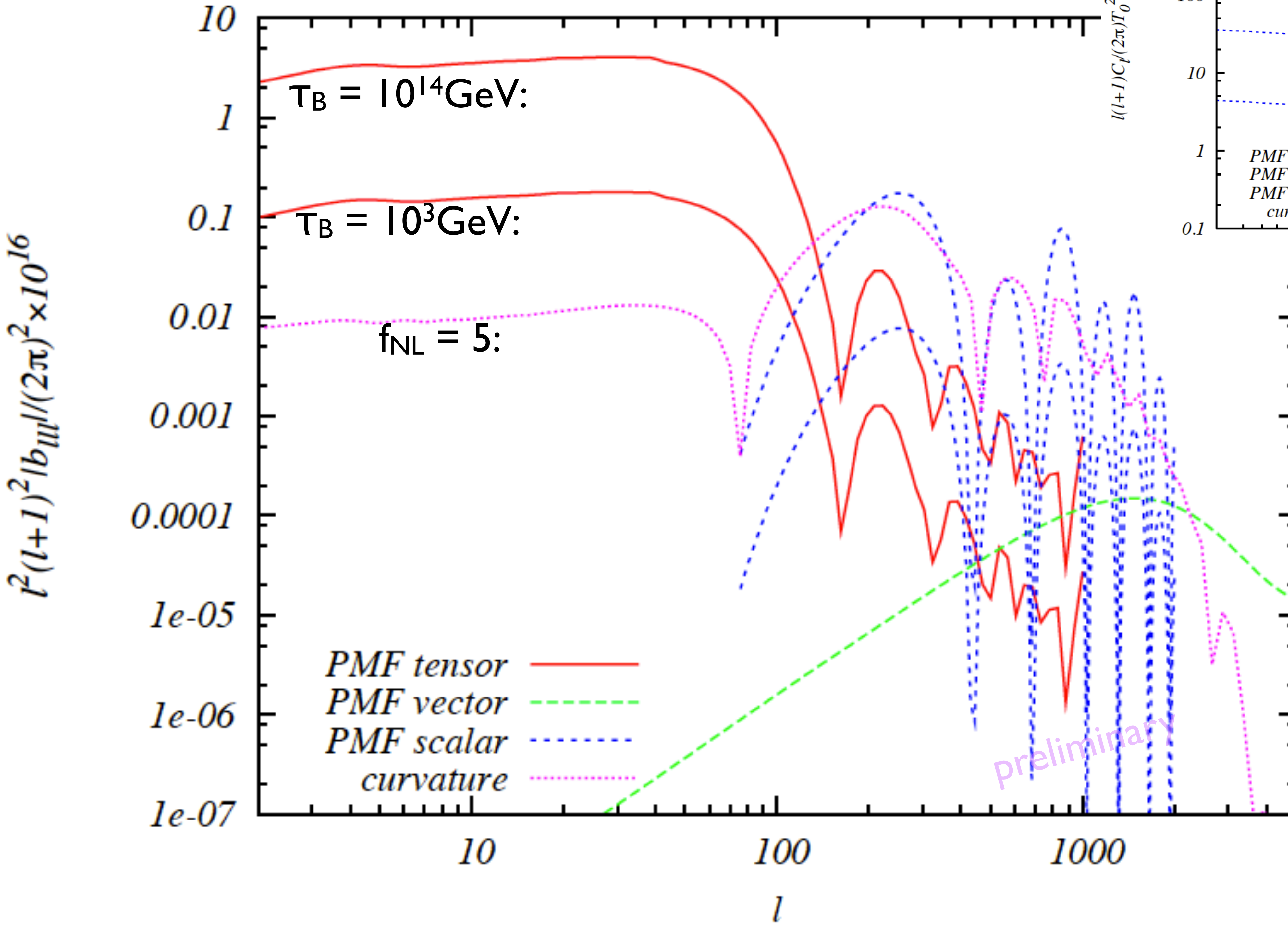
tensor mode (λ = ±2): $$f^{S''S\lambda}_{L''L\ell}(r_3,r_2,r_1) \equiv -4(8\pi)^{3/2} R_\gamma \ln(\tau_\nu/\tau_B) \sum_{L_1L_2L_3} \int_0^\infty y^2 dy j_{L_3}(r_3y) j_{L_2}(r_2y) j_{L_1}(r_1y)$$
$$\times (-1)^{\ell+L_2+L_3} (-1)^{\frac{L_1+L_2+L_3}{2}} I^{0\,0\,0}_{L_1L_2L_3} I^{0S''-S''}_{L_31L''} I^{0S-S}_{L_21L} I^{0\lambda-\lambda}_{L_1\ell 2} \left\{ \begin{matrix} L'' & L & \ell \\ L_3 & L_2 & L_1 \\ 1 & 1 & 2 \end{matrix} \right\}$$

Summation ranges are restricted by selection rules

$$I^{s_1s_2s_3}_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

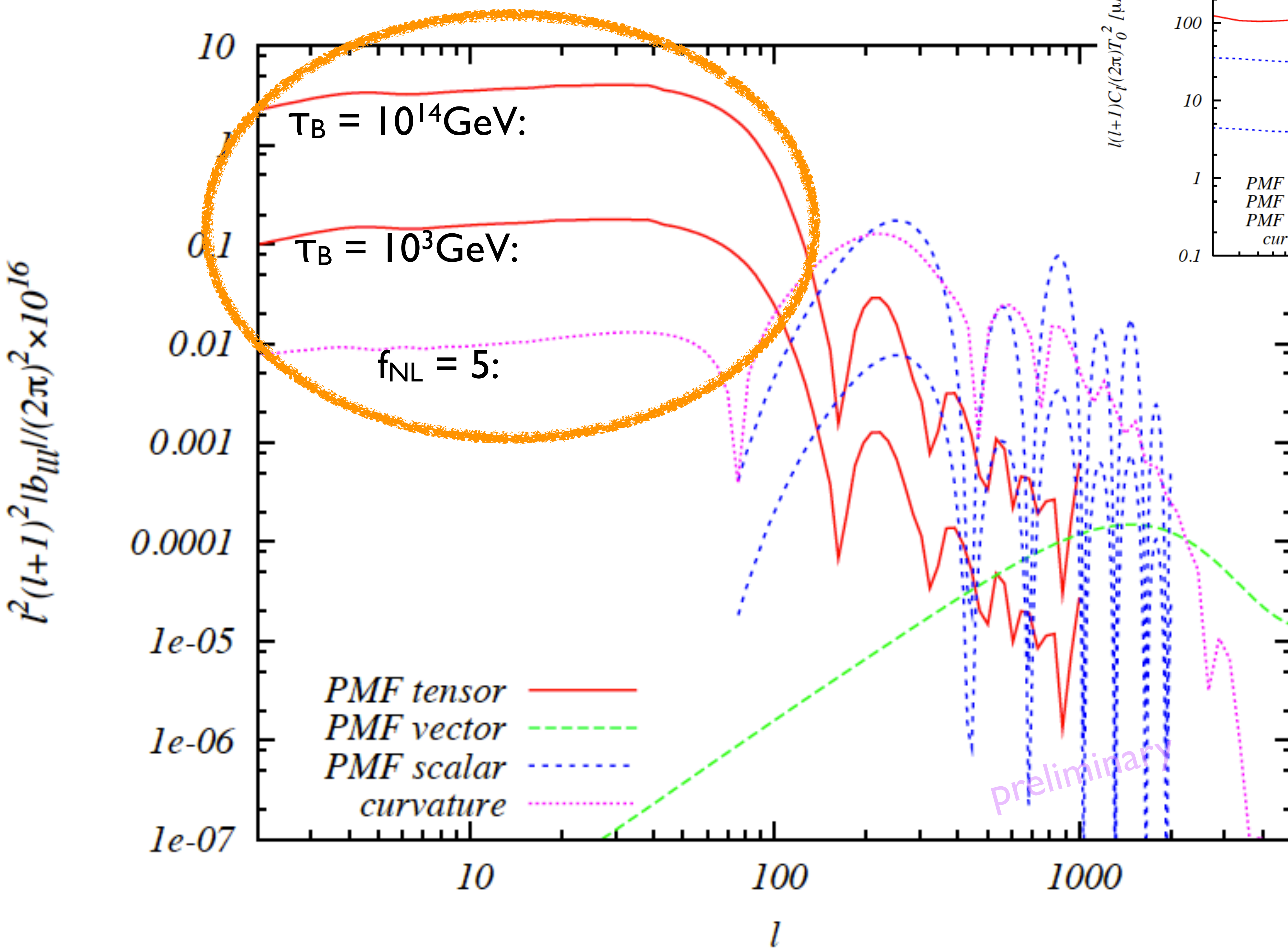
CMB reduced bispectra $b_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(4\pi)}{(2\ell_1+1)(2\ell_1+2)(2\ell_1+3)}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} B_{\ell_1\ell_2\ell_3}$ of intensity mode for $\ell_1 = \ell_2 = \ell_3$

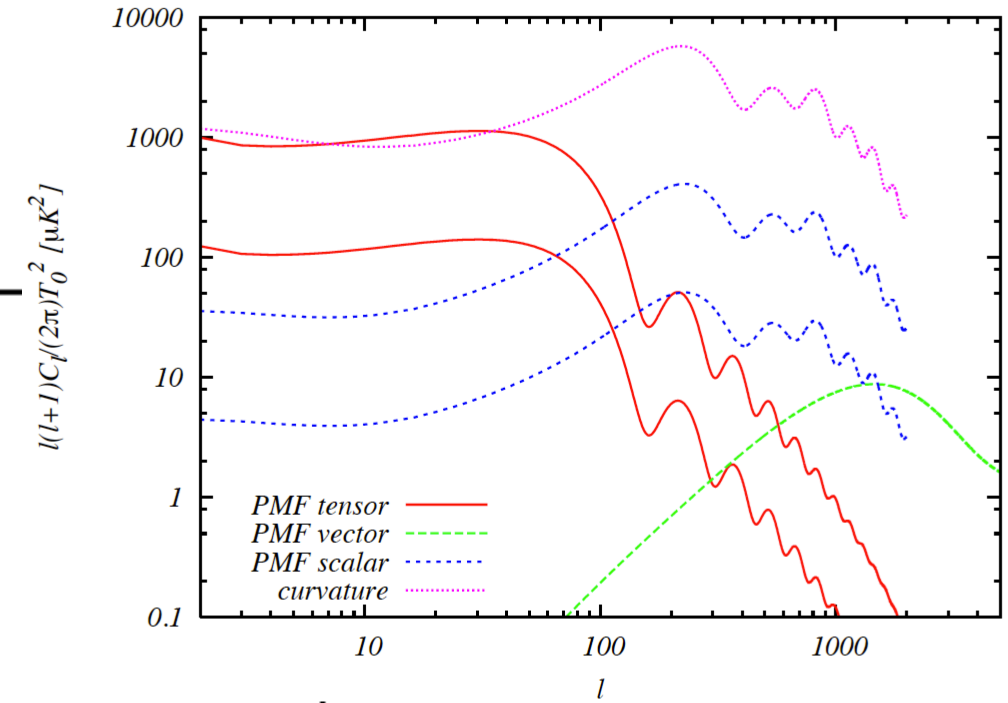




n_B = -2.9, B_{1Mpc} = 4.7nG:

CMB reduced bispectra $b_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(4\pi)}{(2\ell_1+1)(2\ell_1+2)(2\ell_1+3)}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} B_{\ell_1\ell_2\ell_3}$ of intensity mode for $\ell_1 = \ell_2 = \ell_3$





$n_B = -2.9$, $B_{1Mpc} = 4.7$nG:

CMB reduced bispectra $b_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(4\pi)}{(2\ell_1+1)(2\ell_1+2)(2\ell_1+3)}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} B_{\ell_1\ell_2\ell_3}$ of tensor intensity mode for $\ell_1 = \ell_2 \neq \ell_3$

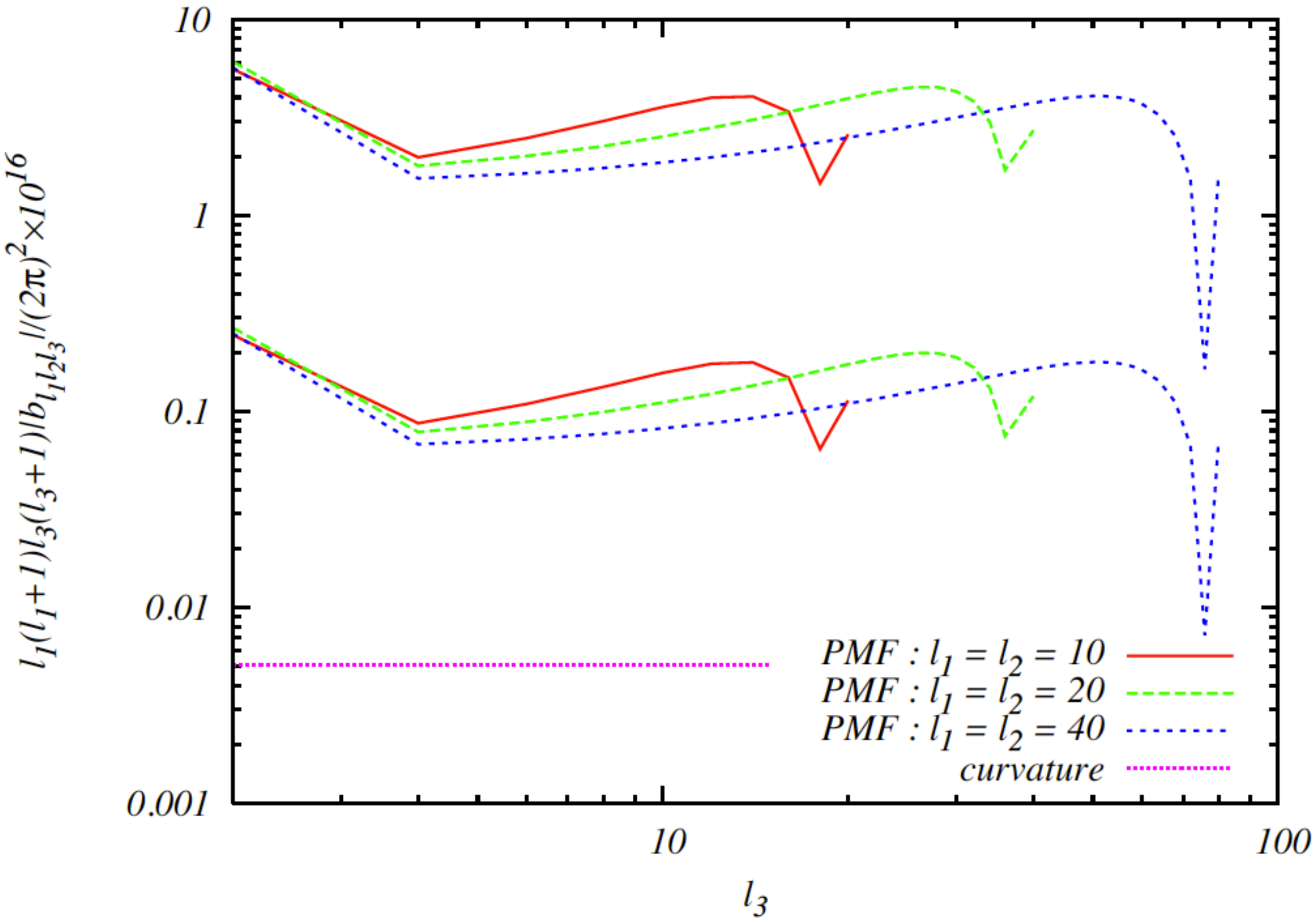
$$\ell_1(\ell_1+1)\ell_3(\ell_3+1)|b_{\ell_1\ell_2\ell_3}|$$

$$\sim (130-6)\times 10^{-16}\left(\frac{B_{1\mathrm{Mpc}}}{4.7\mathrm{nG}}\right)^6$$

COMPARE

$$\ell_1(\ell_1+1)\ell_3(\ell_3+1)b_{\ell_1\ell_2\ell_3} \sim 4\times 10^{-18} f_{\mathrm{NL}}^{\mathrm{local}}$$

$$\left(\frac{B_{1\mathrm{Mpc}}}{1\mathrm{nG}}\right) \sim (1.22-2.04)|f_{\mathrm{NL}}^{\mathrm{local}}|^{1/6}$$



obs result: $|f_{NL}| < 100$ ☞ $B_{1Mpc} < 2.6$ - 4.4nG

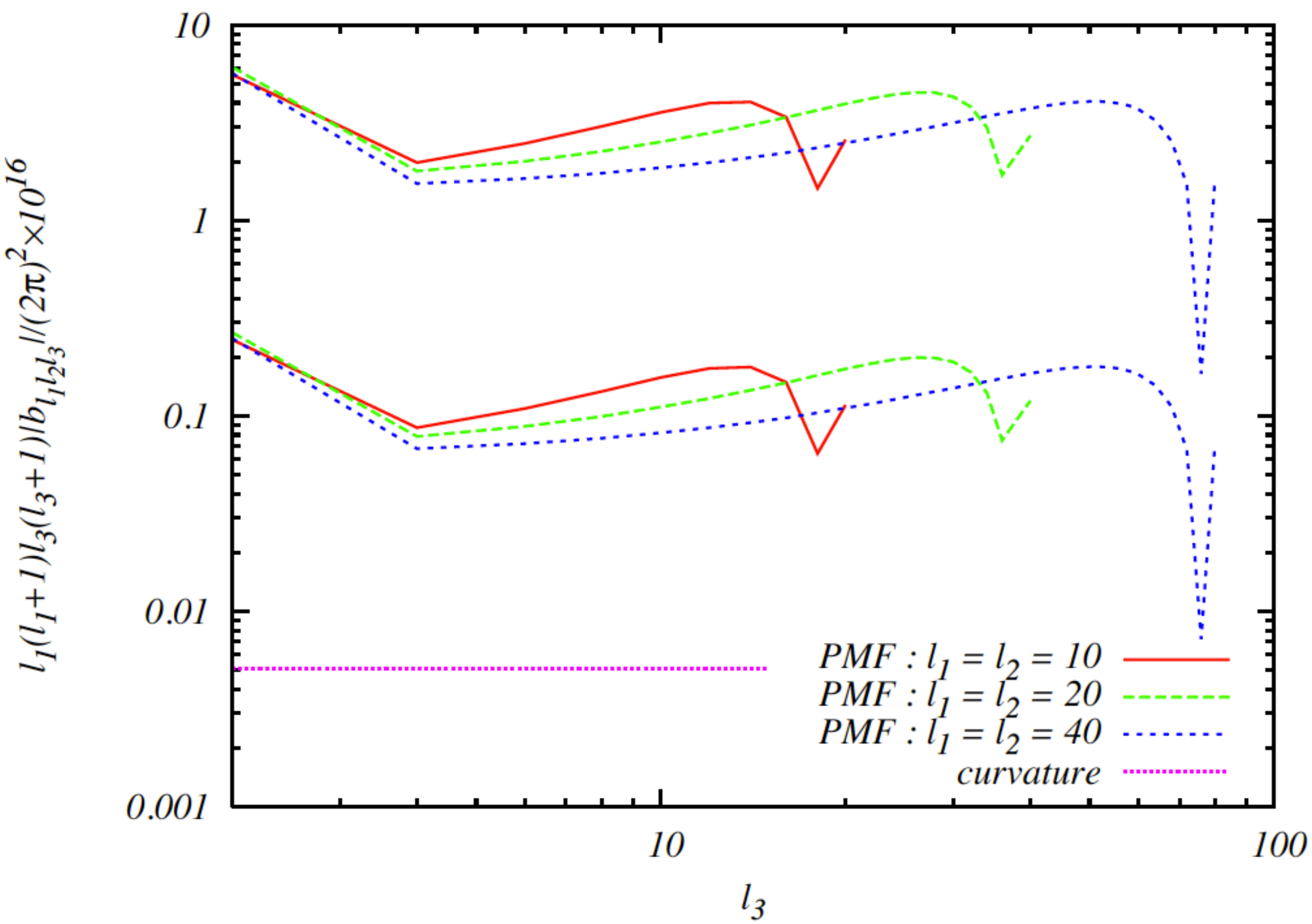
CMB reduced bispectra $b_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(4\pi)}{(2\ell_1+1)(2\ell_1+2)(2\ell_1+3)}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} B_{\ell_1\ell_2\ell_3}$ of tensor intensity mode for $\ell_1 = \ell_2 \neq \ell_3$

$$\ell_1(\ell_1+1)\ell_3(\ell_3+1)|b_{\ell_1\ell_2\ell_3}|$$
$$\sim (130-6)\times 10^{-16}\left(\frac{B_{1\mathrm{Mpc}}}{4.7\mathrm{nG}}\right)^6$$

COMPARE

$$\ell_1(\ell_1+1)\ell_3(\ell_3+1)b_{\ell_1\ell_2\ell_3} \sim 4\times 10^{-18} f_{\mathrm{NL}}^{\mathrm{local}}$$

$$\left(\frac{B_{1\mathrm{Mpc}}}{1\mathrm{nG}}\right) \sim (1.22-2.04)|f_{\mathrm{NL}}^{\mathrm{local}}|^{1/6}$$



obs result: $|f_{NL}| < 100$ ☞ $B_{1Mpc} < 2.6 - 4.4$nG

✳tighter than power spectrum constraints

✳tighter by factor of 4 - 2 than Seshadri + [0902.4066]

Summary

- present the CMB bispectrum induced from the scalar, vector and tensor modes of PMFs by taking into account the full angular dependence
- find the roughly constraint : $B_{1\mathrm{Mpc}} < 2.6 - 4.4\mathrm{nG}$ for $n_B \sim -3$ from the current observational data
- future works
  - if $n_B \neq -3$...
  - consider mode-coupling terms, polarizations