Numerical studies on cosmological perturbations in braneworld

Takashi Hiramatsu

submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Science

Department of Physics, The University of Tokyo

December 20, 2006
Abstract

We study the evolution of scalar (inflaton) and tensor (inflationary gravitational wave background; IGWB) perturbations in a cosmological setup based on the Randall–Sundrum single brane model in which we live on a three-dimensional hypersurface, brane, embedded in a five-dimensional anti-de Sitter spacetime. In this setup, two high-energy effects play an important role in the evolutions of the perturbations in the high-energy regime of the universe. One of the effects comes from the high-energy correction to the Friedmann equation, leading to a peculiar cosmic expansion at the early stage of the universe; the other is due to the existence of the extra-dimensional (bulk) metric perturbations, which can be interpreted as the energy loss of the perturbations on the brane.

As for the IGWB, we focus on its evolution during the high-energy radiation-dominated epoch, in which the high-energy correction term of the Friedmann equation makes a dominant contribution. We set initially wave-length of the IGWB to be sufficiently larger than the Hubble horizon, then solve numerically the five-dimensional version of the evolution equation by means of the spectral collocation method. As a result, we find that the two high-energy effects cancel each other, resulting in the same observed spectrum as the standard four-dimensional prediction. Further simulations with different cosmological setup reveal that the cancellation occurs only in the radiation-dominated epoch. Moreover, we find a universal relation between the amount of the KK-mode excitations and a Lorentz factor of the brane velocity in the bulk. The effect of the brane velocity using a simple toy model is discussed.

As for the inflaton perturbation, we investigate its evolution during the inflation with a specific brane inflation model proposed by R. M. Hawkins and J. E. Lidsey. The inflaton perturbations on the brane couples with the bulk metric perturbations. In order to quantify the effect of the bulk metric perturbations, we adopt a simplified initial condition, in which no bulk metric perturbation is present at the initial time. By means of the spectral collocation method, we solve numerically the coupled equations of the metric perturbations and the inflaton perturbations. As a result, we find extra suppressions of the amplitudes of the comoving curvature perturbations on small scales associated with the bulk metric perturbations, while the amplitudes on super-horizon scales become constant as in the four-dimensional cases. In addition, we discuss the observational implications of this result.
Contents

1 Introduction 1

2 Cosmological Perturbation in Four-dimensional Theory 5
  2.1 Some puzzles of Big Bang theory ................................. 5
  2.2 Inflation ......................................................... 8
  2.3 Perturbation theory ............................................... 11
    2.3.1 Cosmological perturbation and gauge freedom ............ 11
    2.3.2 Tensor decomposition ....................................... 12
    2.3.3 Metric and matter perturbations .......................... 13
    2.3.4 Gauge freedom .............................................. 14
    2.3.5 Gauge invariant variables .................................. 16
    2.3.6 Evolution equation of matter perturbations ............... 17
    2.3.7 Evolution equation for inflaton perturbation .............. 17
    2.3.8 Quantum-mechanical treatment of inflaton perturbations 19
    2.3.9 Quantum generation of gravitational waves ................. 22
  2.4 Observables ..................................................... 23
    2.4.1 Consistency relation ....................................... 23
    2.4.2 Inflationary Gravitational Wave Background ............... 25

3 Braneworld 31
  3.1 Motivations and history ........................................ 31
  3.2 Randall–Sundrum single brane model .......................... 34
    3.2.1 Schematic description ..................................... 34
    3.2.2 Coordinate systems ....................................... 35
    3.2.3 Effective Einstein equations ............................... 37
  3.3 Cosmological setup based on RSII model ........................ 40
    3.3.1 Friedmann equation ........................................ 40
    3.3.2 Bulk solutions .......................................... 42
  3.4 Cosmological solutions ......................................... 44
    3.4.1 Perfect fluid on the brane ................................ 44
    3.4.2 Slow-roll inflation on the brane ......................... 45
    3.4.3 Hawkins–Lidsey model .................................... 47
4 Cosmological Perturbations in Braneworld

4.1 Metric perturbations and their gauge transformation
- 4.1.1 scalar perturbations
- 4.1.2 vector perturbations
- 4.1.3 tensor perturbations

4.2 Evolution equations of metric perturbations
- 4.2.1 Scalar part
- 4.2.2 Vector part
- 4.2.3 Tensor part

4.3 Brane bending

4.4 Junction conditions
- 4.4.1 Scalar part
- 4.4.2 Vector part
- 4.4.3 Tensor part

4.5 Evolution equations of matter perturbations

4.6 Quantisation of perturbations
- 4.6.1 Scalar perturbations
- 4.6.2 Tensor perturbations

5 Numerical Studies on Tensor Perturbations – Gravitational Wave Background

5.1 Introduction

5.2 High-energy effects in the braneworld cosmology

5.3 Evolution equation and initial conditions
- 5.3.1 Evolution equation of GWs
- 5.3.2 Initial conditions

5.4 Setup and parameters for numerical simulations

5.5 Code check and qualitative behaviour of GWs
- 5.5.1 Code check
- 5.5.2 Behaviour of GWs in the bulk and the validity check of the initial condition

5.6 IGWB spectra
- 5.6.1 Comparison with reference waves
- 5.6.2 IGWB spectrum in the five-dimensional cosmology
- 5.6.3 Initial time dependence
- 5.6.4 Dependence on equation of state

5.7 Conclusion

6 Numerical Studies on Scalar Perturbations – Inflaton Perturbations

6.1 Introduction

6.2 Summary of basic equations
6.2.1 Background spacetime .............................................. 90
6.2.2 Bulk metric perturbations ........................................... 91
6.2.3 Inflaton perturbations ............................................... 91
6.2.4 Junction conditions .................................................. 92
6.3 Evolution of curvature perturbations ................................. 92
   6.3.1 Initial conditions and boundary conditions .................... 92
   6.3.2 Evolution of curvature perturbations ............................ 94
   6.3.3 Check of numerical calculations ................................. 95
6.4 Conclusion and discussion ............................................ 96

7 Conclusion ........................................................................ 101
   7.1 Summary ....................................................................... 101
   7.2 Discussion and future prospects .................................... 102

A Notations ......................................................................... 107

B Calculations of Some Tensor Quantities ............................... 115
   B.1 Extrinsic curvature ...................................................... 115
   B.2 4D Christoffel symbol ................................................ 116
   B.3 5D Perturbed Einstein tensor ....................................... 116
   B.4 4D perturbed Einstein tensor ....................................... 118
   B.5 Quadratic energy–momentum tensor ......................... 118
   B.6 Projected Weyl tensor ................................................ 119

C The Spectral Collocation Method ....................................... 121
   C.1 Reduction of evolution equations ................................. 121
   C.2 Temporal methods ..................................................... 124
      C.2.1 The predictor–corrector method with Adams–Bashforth and Adams–Moulton formulae ...................................... 124
      C.2.2 Runge–Kutta method ............................................. 127

D Moving Brane in Minkowski Bulk ..................................... 129
Chapter 1

Introduction

Marvellous images of the universe’s childhood projected on the oldest screen has made a great impression on cosmologists. Incredibly precise observations based on the Cosmic Microwave Background (CMB) experiments show important relic images taken in the very early universe. The CMB observations gave us an enormous amount of information; for instance, its age, its components, and the origin of the seeds of the rich cosmic structures we see today. Moreover, in near future, we may get closer and closer to the newborn universe through the Inflationary Gravitational Wave Background (IGWB), and may have the answer of the question how the seeds were created during the early stage of the universe.

A theoretical framework to give a simple mechanism of the generation of cosmic structures is known as the inflationary theory. In this framework, the accelerated phase of the cosmic expansion took place and the spacetime inhomogeneity was generated quantum-mechanically in the course of the accelerated expansion, called the inflation. The quantum-generated fluctuations may transit to classical fluctuations, namely, the fluctuations of the spacetime curvature and the IGWB. The curvature fluctuations evolves and transferred to the density perturbations of baryons and photons filling the early universe. When the size of the universe becomes large enough to decouple the interactions (called decoupling), the photons started to travel freely through the universe, whose surface viewed from us is called Last Scattering Surface. We observe the photons as the CMB. The CMB was discovered by A. A. Penzias and R. W. Wilson in 1965 as an isotropic radiation from any directions [1]. This detection became a strong evidence of the Big Bang theory [2]. After that, a lot of efforts has been made to improve the sensitivities of observatories to the CMB. Then, in 1992, COsmic Background Explorer (COBE) launched in 1989 has firstly detected the anisotropies of the CMB with its angular resolution of 7 degrees [3,4] suggesting on evidences for inflation [5,6]. The anisotropies results from the curvature inhomogeneity before the decoupling, giving us many kinds of information about the inflation. After some experiments\footnote{For example, MAXIMA (Millimeter-wave Anisotropy Experiment Imaging Array) [7], DASI (Degree Angular Scale Interferometer) [8], BOOMERanG (Balloon Observations Of Millimeter Extragalactic Radiation and Geophysics) [9,10].}, tremendously precise measurement of the CMB has been done by the Wilkinson Microwave
Anisotropy Probe (WMAP) satellite launched at 2001 [11, 12]. Together with theoretical results from the inflationary theories, we obtained a large number of detailed information about the universe. Recent attention has begun to be focused on the polarisations of the CMB with future missions, such as PLANCK [13] and CMBPOL [14].

Meanwhile the interaction of the IGWB with the matter and photons may be quite week. This means that we may be able to directly observe the images of the universe beyond the last scattering surface. Catching the IGWB enables us to observe the very early universe at $10^{-38}$ sec after its born in principle [15]. The direct observation of the IGWB is one of important observational goals in future space-based missions, such as Laser Interferometer Space Antenna (LISA) [16], DECI-hertz Gravitational-wave Observatory (DECIGO) [17] and Big-Bang Observer (BBO) [18,19], while it is still a challenging problem for the current laser interferometers, such as Laser Interferometer Gravitational-wave Observatory (LIGO) [20] and TAMA [21].

Evolution of such tiny fluctuations is treated theoretically by cosmological perturbations, in which we consider the spacetime fluctuation coupled with matter in the homogeneous and isotropic cosmology [22–25]. The cosmological perturbation theory has a very long history. There are many kinds of ways to deal with the perturbations and many kinds of notations. In this thesis, we basically follow Refs. [24,25].

Observing primordial fluctuations generated during inflationary epoch may allow us to ask a fundamental question: “Why is our universe described as a four-dimensional spacetime ?” or “Is there a possibility that we live in a higher-dimensional spacetime ?”. This kind of questions emerges from the unified theory such as string theories. In string theory, our universe is described by a 10 dimensional spacetime. There is so far no experimental and observational evidence that our universe is such a higher-dimensional spacetime. With growing observational technologies, the day may be not far off that future observations stand our vision of the universe on its head. Hence it is very important and urgent to study the cosmological perturbations in the framework of higher-dimensional cosmologies.

What makes the theoretical investigation possible is the appearance of the Randall–Sundrum (RS) braneworld model [26,27]. A concept of braneworld itself has been originally investigated in the particle physics since early 1980’s [28, 29], in which we live in a spatial hypersurface embedded in a higher-dimensional spacetime. Recently, especially since the RS model was proposed, observational predictions in braneworld scenarios have been discussed by cosmologists. In this model, we live on a brane, which is a three-dimensional hypersurface embedded in a five-dimensional anti-de Sitter bulk spacetime (AdS$_5$). There are originally two types of the model, but this thesis mainly deals with the single brane model called Randall–Sundrum type II model [27]. In the RS single brane model, two high-energy effects play an important role in the evolutions of the primordial perturbations. One of them comes from the high-energy correction to the Friedmann equation, leading to a peculiar expansion of the universe; the other is the existence of the extra-dimensional (bulk) metric perturbations.

The behaviour of the IGWB with those high-energy effects in the braneworld has been
well studied in an analytic way [30–34] [35,36] and by means of numerical simulations [33,37–41]. Contrary, as for the curvature perturbations, we could not well treat the bulk metric perturbations because of technical difficulties. Refs. [42–48] investigated the correction to the standard inflation results only taking into account the peculiar cosmic expansion. Although some others have recently studied them effectively taking into account the bulk metric perturbations [49–51], the predictability of these analyses is limited because of the approximations used in them.

In this thesis, we focus on the evolution of curvature perturbations during the inflation and the evolution of IGWB after the inflation in a cosmological setup based on the RS single brane model. The final goal is to predict the spectrum of the IGWB as a direct observable of the gravitational wave experiments, and to clarify the effects of the bulk metric perturbations on the inflaton perturbations during the inflation. For these purposes, we directly solve the evolution equations containing the bulk metric perturbations, especially implementing a numerical method referred to the *spectral collocation method* [52,53].

So far, the spectrum of the IGWB has been thought to be enhanced by the high-energy correction to the Friedmann equation, in contrast to the standard results predicted in the four-dimensional theory. However, the full treatment of the bulk metric perturbations reveals that the two high-energy effects cancel each other, resulting in the almost same spectrum as one predicted in the standard four-dimensional theory. Moreover, we found that there is a universal relation for the amount of the KK-mode excitations leading to the extra suppression of the IGWB amplitude on the brane.

As for the inflaton perturbations, many observational predictions in the braneworld scenario have been made neglecting the bulk metric perturbations. In the four-dimensional cases, we can safely neglect a coupling of the inflaton perturbations to the gravity on the small scales. As a consequence, the Bunch–Davis vacuum as an initial state for the perturbations can be validated as a natural choice of the vacuum state. However, this treatment cannot be applied to the five-dimensional cases, since the behaviour of the gravity becomes essentially five-dimensional on the small scales. Our work is the first numerical result for the evolution of (classical) scalar curvature perturbations to take into account the backreaction of the metric perturbations correctly. Using the brane inflation model proposed by R. M. Hawkins and J. E. Lidsey, we found the damping feature of the curvature perturbations on the sub-horizon scales during the inflation, due to the five-dimensional behaviour of the gravity. This result leads to a small deviations of the curvature perturbations from the standard four-dimensional predictions. In other words, we found that we cannot simply assume the Bunch–Davies vacuum in a braneworld scenario.

This thesis is organised as follows:

**Chapter 2**

We start with reviewing the inflation scenarios in the four-dimensional universe. After that, we survey the cosmological perturbation theory and discuss its connection to observations.
Chapter 3

We first introduce braneworld scenarios with their motivations and histories. Then, focusing on the Randall–Sundrum single brane model, we show its background solutions by solving the five-dimensional Einstein equation.

Chapter 4

This chapter is devoted to prepare the formalisms used in the following chapters. Introducing the five-dimensional metric perturbations and matter perturbations confined to the brane, we derive their evolution equations and the junction conditions describing the coupling of each perturbation.

Chapter 5

We focus on the evolution of tensor perturbations during the high-energy radiation dominated universe. The original contribution is based on


Chapter 6

We turn to focus on the evolution of inflaton perturbations during the inflation with the Hawkins–Lidsey brane inflation model. The original contribution is based on


Chapter 7

This chapter is devoted to summarise this thesis and discuss the physical implications drawn from our works.

In appendix A, we summarise notations and some useful quantities used in this thesis. In appendix B, we list components of tensorial quantities such as curvature tensors. In appendix C, we give details of our numerical methods applied to perform simulations in Ch. 5 and Ch. 6. Finally, in appendix D, we explain an extra simulation performed in discussions in Ch. 7.
Chapter 2
Cosmological Perturbation in Four-dimensional Theory

The cosmological perturbation theory plays an important role in the recent ‘precision cosmology’ based on many kinds of observations with significant accuracies. This chapter consists of four sections: In Sec. 2.1, we outline some puzzles inherent in the Big Bang theory. In Sec. 2.2, we introduce a mechanism of inflation for the sake of resolving the problems. In Sec. 2.3, we will survey the perturbation theory in the Friedmann–Lemaître–Robertson–Walker universe. In Sec. 2.4, we focus on the connection of the perturbation theory to some observations.

2.1 Some puzzles of Big Bang theory

In this section, we review the inflationary scenario. To begin with, let us get an overview of the standard model studied by A. A. Friedmann [56], G.-H. Lemaître [57], H. P. Robertson [58] and A. G. Walker [59], so-called Friedmann–Lemaître–Robertson–Walker (FLRW) model. FLRW model is an exact solution of the Einstein equation, which has been used as a mathematical expression of the standard expanding universe model.

The metric tensor for an isotropic and homogeneous universe is given by

\[ g_{\mu\nu} = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j, \tag{2.1} \]

where \( \gamma_{ij} \) represents a maximally symmetric three-dimensional metric on a spatial hypersurface whose spatial curvature is given by

\[ (3)R_{ijkl} = \frac{K}{a^2}(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{2.2} \]

When \( K < 0, = 0, > 0 \), the spatial hypersurface becomes a hyperbolic, a flat, a spherical manifold, respectively.

We assume that the universe is filled with a perfect fluid whose energy–momentum tensor is given by

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{2.3} \]
where \( \rho \) and \( p \) denote the energy density and the pressure of the universe. In the isotropic and homogeneous universe, the 4-velocity of the fluid is given by

\[
u^\mu = (1, 0, 0, 0),
\]

(2.4)

So the non-zero components of the energy–momentum tensor are

\[
T^{0}_0 = -\rho, \\
T^{i}_j = p\delta^{i}_j.
\]

(2.5)

(2.6)

Substituting these equations into the Einstein equation, \( G_{\mu\nu} = 8\pi G T_{\mu\nu} \), we then obtain two equations with respect to the scale factor \( a(t) \) :

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3M_{\text{Pl}}^2} \rho - \frac{K}{a^2}, \\
\frac{\ddot{a}}{a} = -\frac{4\pi}{3M_{\text{Pl}}^2}(\rho + 3p),
\]

(2.7)

(2.8)

where \( M_{\text{Pl}} = G^{-1/2} \approx 1.2 \times 10^{19} \text{ GeV} \) is the Planck mass. The first equation is called the *Friedmann equation* which describes the rate of cosmic expansion. The latter one called the *acceleration equation* which describes the acceleration of the expansion. In addition, The energy conservation law \( \nabla_{\mu} T^{\mu}_\nu = 0 \), derived from the Bianchi identity and the Einstein equation, yields

\[
\dot{\rho} = -3(\rho + p) \frac{\dot{a}}{a},
\]

(2.9)

which describes the time-evolution of the matter in the universe.

The Big Bang theory can explain that huge numbers of Hydrogen, Helium and Lithium nuclei formed at the early stage of the universe (Big Bang Nucleosynthesis, BBN) [15], and the fact that observed light spectra of very far galaxies’ are shifted towards red, or *redshifted*. Nevertheless, some observational problems are inherent in the theory. Let us mention three representative ones.

– Horizon problem

Recently, COsmic Background Explorer (COBE) launched in 1989 observed the temperature as \( 2.73 \pm 0.06 \text{ K} \) in any direction [5] \footnote{According to Ref. [60], the accuracy has been improved to \( 2.725 \pm 0.002 \text{ K} \).}. The isotropy of CMB temperature conflicts with the fact that the region where the causality works at the epoch of decoupling were sufficiently smaller than the Hubble radius at the present time.

This can be shown by following calculation. The radius of particle horizon where the light travels from the beginning of the universe \( t = t_a \) is defined by

\[
D_H(t) = a(t)r,
\]

(2.10)
2.1 Some puzzles of Big Bang theory

where $r$ is comoving length given by

$$ds^2 = -dt^2 + a(t)^2 dr^2 = 0 \iff r = \int_{t_0}^{t} \frac{dt}{a}. \quad (2.11)$$

The particle horizon at the decoupling phase $D_H(t_{\text{dec}})$ corresponds to the region where photons could have contacted causally at that time. If we set $t_0 = 0$, we obtain $r = 3t^{1/3}$, leading to $D_H(t) = 3t$ in the matter-dominated epoch. Therefore, the ratio of the particle horizon size at the decoupling phase to the one at present time becomes

$$\frac{D_H(t_{\text{dec}})}{D_H(t_0)} \approx \left(\frac{t_0}{t_{\text{dec}}}\right)^{1/3} \approx 10^{-2}. \quad (2.12)$$

This scale is almost the same size as the apparent diameter of the moon observed from the Earth. Nevertheless, we observed thermalized photons in all sky direction. This puzzle is called \textit{Horizon problem}.

- \textbf{Flatness problem}

  From recent observations of CMB, we know that our universe is almost flat. Defining new variables,

  $$\Omega = \frac{\rho}{\rho_c}, \quad \rho_c = \frac{3M_{\text{Pl}}^2 H^2}{8\pi}, \quad (2.13)$$

  the Friedmann equation (2.7) can be rewritten as

  $$\Omega - 1 = \frac{K}{a^2 H^2}. \quad (2.15)$$

  Note that the quantity $\Omega$ represents the flatness of the universe. The quantity $a^2 H^2$ is a decreasing function of time if the component of our universe is dominated by ordinary matter like radiation or dust. Hence, if we set $\Omega \approx 1$ at the present time, the value of $\Omega$ is tremendously close to 1 at the early time. For example, we require $|\Omega - 1| < \mathcal{O}(10^{-16})$ at the epoch of nucleosynthesis [61], and $|\Omega - 1| < \mathcal{O}(10^{-64})$ at the Planck epoch [62]. This extremely fine-tuning problem is called \textit{flatness problem}.

- \textbf{Unwanted relics}

  From the modern viewpoint of particle physics, a lot of unobserved particles, such as gravitino, Kaluza–Klein particles, moduli particles, etc., are predicted in superstring and supergravity theories. Moreover, depending on the theory, the spontaneous symmetry breaking at the energy scale of the Grand Unified Theory may produce many kinds of topological defects, e.g., magnetic monopoles, cosmic strings, domain walls, etc. These unwanted relics may conflict with the observational results of nucleosynthesis and experiments with particle accelerators.
2.2 Inflation

In order to resolve these problems, K. Sato and A. Guth proposed the basic ideas of inflation, an accelerating expansion phase at the very early universe [63,64], which is now termed as old inflation. The revised version of inflation was proposed by A. Linde [65], and A. Albrecht and J. Steinhardt [66] in 1982 referred as new inflation. However, these scenarios suffer from theoretical problems about the duration of inflation and initial conditions. In 1983, A. Linde considered the case that the initial conditions for scalar field driving inflation may be chaotic, which is called chaotic inflation [67]. This inflation model can resolve the remaining problems. After that, many kinds of inflation models have been proposed (for review, see [61,68,69], and also see [70,71]).

The mechanism of inflation can be easily understood from the acceleration equation (2.8), which indicates an accelerating expansion if

\[ \rho + 3p < 0. \tag{2.16} \]

The energy density of the universe \( \rho \) is always positive. Hence, if the pressure is negative, the accelerating expansion is realised. While the pressure of ordinary matters does not become negative, a scalar field can realise such peculiar situation.

Let us consider a homogeneous and isotropic scalar field, \( \phi(t) \), filled in our universe, whose Lagrangian density is given as [61]

\[ \mathcal{L} = -\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi). \tag{2.17} \]

The energy–momentum tensor is therefore [25,61]

\[ T^\mu_{\nu} = g^{\mu\lambda} \phi_{,\lambda} \phi_{,\nu} - \delta^\mu_{\nu} \left( \frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + V(\phi) \right). \tag{2.18} \]

From this, we can identify the energy density and pressure of the scalar field as

\[ \rho = T^0_0 = \frac{1}{2} \dot{\phi}^2 + V(\phi), \tag{2.19} \]
\[ p = \frac{1}{3} T^k_k = \frac{1}{2} \dot{\phi}^2 - V(\phi). \tag{2.20} \]

Hence the Friedmann equation (2.7) with \( K = 0 \) becomes

\[ H^2 = \frac{8\pi}{3M_{Pl}^2} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right]. \tag{2.21} \]

The equation of motion for the scalar field can be derived from the energy conservation law which comes from the equation \( \nabla_\mu T^\mu_{\nu} = 0 \):

\[ \dddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0. \tag{2.22} \]
In the case that the potential energy \( V(\phi) \) dominates the kinetic energy,

\[
|V(\phi)| \gg |\dot{\phi}|^2,
\]

(2.23)

the pressure of our universe becomes negative. Defining the slow-roll parameters \([61]\),

\[
\epsilon_s = \frac{M_{Pl}^2}{16\pi} \left( \frac{V'}{V} \right)^2,
\]

(2.24) \\
\[
\eta_s = \frac{M_{Pl}^2}{8\pi} \frac{V''}{V},
\]

(2.25)

this condition can be recasted as

\[
\epsilon_s < 1, \quad |\eta_s| < 1.
\]

(2.26)

Under these conditions, the Friedmann equation (2.21) becomes

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3M_{Pl}^2} V(\phi),
\]

(2.27)

and the equation of motion yields

\[3H\dot{\phi} \approx -V'(\phi).\]

(2.28)

Rewriting the temporal derivative in the left-hand side to a derivative with respect to the scalar field \( d/d\phi \), and integrating by \( \phi \), the scale factor becomes

\[a(\phi) = \exp \left( -\frac{8\pi}{M_{Pl}^2} \int \frac{V}{V'} \, d\phi \right).\]

(2.29)

An exponential expansion caused by the slowly rolling scalar field (inflaton) satisfying the conditions (2.23) is called slow-roll inflation. The logarithm of the ratio of scale factor at the end of inflation to its value at some initial time gives a duration of inflation:

\[N(t) = \log \frac{a(t_{end})}{a(t)} \approx \frac{8\pi}{M_{Pl}^2} \int_{\phi(t_{end})}^{\phi(t)} \frac{V}{V'} \, d\phi,
\]

(2.30)

which is referred as e-folding number. In order to resolve the flatness problem and the horizon problem, we need \( N \sim 70 \) \([61]\].

While the slow-roll parameters (2.24) and (2.25) are useful for us to discuss qualitatively the observational results of the inflation, there is another definition of them, which enables us to clearly relate the inflaton potential with the observables. The new slow-roll parameters are defined in the context of Hamilton–Jacobi formalism \([61,72]\), in which we treat the scalar field as the time variable. From Eqs.(2.21) and (2.22), we obtain

\[
\dot{H} = -\frac{4\pi}{M_{Pl}^2} \dot{\phi}^2,
\]

(2.31)
yielding

\[ \frac{dH}{d\phi} = \frac{\dot{H}}{\phi} = -\frac{4\pi}{M_{Pl}^2} \dot{\phi}. \] (2.32)

Using this, the Friedmann equation (2.21) and the energy conservation law (2.9) in a flat universe can be rewritten as

\[ \left(\frac{dH}{d\phi}\right)^2 - \frac{12\pi}{M_{Pl}^2} H^2 = -\frac{32\pi^2}{M_{Pl}^4} V(\phi), \] (2.33)

\[ \frac{d\rho}{d\phi} = -3H\dot{\phi}. \] (2.34)

The former equation is called Hamilton–Jacobi equation [61]. A relation \( da/d\phi = aH/\dot{\phi} \) with Eqs. (2.21) (2.34) yields a differential equation for \( a(\phi) \), whose integration becomes

\[ a(\phi) = \exp\left(-\frac{4\pi}{M_{Pl}^2} \int_{\phi_0}^\phi \frac{H}{dH/d\phi} d\phi\right). \] (2.35)

This equation implies that, once the functional form of a geometrical quantity \( H(\phi) \) or the potential \( V(\phi) \) has been specified, the cosmological dynamics is determined. For example, \( H \sim \exp(\phi) \) gives the power-law inflation [61].

Following the Hamilton–Jacobi formalism, we can introduce a different set of slow-roll parameters:

\[ \epsilon_H(\phi) = \frac{M_{Pl}^2}{4\pi} \left(\frac{H'(\phi)}{H(\phi)}\right)^2 = 3\frac{\dot{\phi}^2/2}{V + \phi^2/2}, \] (2.36)

\[ \eta_H(\phi) = \frac{M_{Pl}^2}{4\pi} \frac{H''(\phi)}{H(\phi)} = -\frac{\ddot{\phi}}{H\dot{\phi}}. \] (2.37)

In the slow-roll limit, these parameters coincide with \( \epsilon_s \) and \( \eta_s - \epsilon_s \) shown in Eqs. (2.24) and (2.25), respectively. Sometimes one additionally introduce the third parameter [73]:

\[ \xi_H(\phi) = \frac{M_{Pl}^2}{4\pi} \left(\frac{H'(\phi)H''(\phi)}{H(\phi)^2}\right)^{1/2} = \left[ \epsilon_H\eta_H - \left(\frac{M_{Pl}^2}{4\pi} \right)^{1/2} \eta_H(\phi) \right]^{1/2}. \] (2.38)

The most important point of these parameters is that the definition of inflation is given precisely by

\[ \ddot{a} > 0 \iff \epsilon_H < 1. \] (2.39)

The Hamilton–Jacobi formalism plays an important role to make inflation models also in braneworld scenarios discussed in Ch. 3.
2.3 Perturbation theory

2.3.1 Cosmological perturbation and gauge freedom

The inflation mechanism not only resolves many problems inherent in the Big-Bang theory, but also provides a seed of large-scale structure which we observed today (e.g., galaxies and their (super-) clusters). During the inflation, the inflaton field and the spacetime may be quantum-mechanically fluctuated. These fluctuations were expanded to the super-horizon scales and might have become classical fluctuations, namely, fluctuations of density, photon, gravitational potential and, in some cases, gravitational waves. In this section, we review the cosmological perturbation theory in the four-dimensional spacetime [23–25].

To begin with, we mention a formal definition of cosmological perturbations and gauge freedom inherent in it. Let us consider a 4D background spacetime manifold $\mathcal{M}$ which represents the isotropic and homogeneous universe described by the Friedmann equation (2.7). The physical spacetime manifold $\mathcal{M}$ corresponds the perturbation of the reference manifold $\mathcal{M}$. In these manifolds, there are appropriate coordinate system which is chosen according to the symmetry of $\mathcal{M}$. To identify between two manifolds, a one-to-one mapping $\phi : \mathcal{M} \to \mathcal{M}$ is introduced. Then the perturbation of a tensor $T(x)$ is defined by

$$\delta T(x) = T(x) - \overline{T}(x), \quad (2.40)$$

where $T$ and $\overline{T}$ are defined in $\mathcal{M}$ and $\mathcal{M}$, respectively, and $x$ represents a given set of coordinates whose values are identified via the mapping $\phi$. The perturbation is defined at the same coordinate point in this manner.

This definition has a degree of freedom for choosing a mapping function $\phi$, i.e., the coordinate system defined on $\mathcal{M}$ can be slightly changed to other one. This change of coordinate is described as

$$x^\mu \to \bar{x}^\mu = x^\mu + \delta x^\mu, \quad (2.41)$$

which is called gauge transformation. Due to this coordinate transformation, the tensor $\bar{T}$ evaluated at the new coordinate $\bar{x}^\mu$ has a slightly different value from $T$ evaluated at the old coordinate $x^\mu$. This difference is described by the Lie derivative along the vector field $\delta x^\mu$ [74] :

$$\tilde{T}(x) - T(x) = -\mathcal{L}_{\delta x}T. \quad (2.42)$$

For instance, rewriting the Lie derivative to partial derivatives, the changes of vectors $A^\mu, A_\mu$
and tensors $B^\mu\nu, B^\mu\nu, B_{\mu\nu}$ are can be described as

\begin{align*}
\bar{A}^\mu &= A^\mu - \delta x^\lambda \partial_\lambda A^\mu + (\partial_\lambda \delta x^\mu) A^\lambda, \\
\bar{A}_\nu &= A_\mu - \delta x^\lambda \partial_\lambda A_\mu - (\partial_\mu \delta x^\lambda) A_\lambda,
\end{align*}

(2.43)

\begin{align*}
\bar{B}^{\mu\nu} &= B^{\mu\nu} - \delta x^\lambda \partial_\lambda B^{\mu\nu} + (\partial_\lambda \delta x^\mu) B^{\lambda\nu} + (\partial_\lambda \delta x^\nu) B^{\mu\lambda}, \\
\bar{B}^{\mu\nu} &= B^{\mu\nu} - \delta x^\lambda \partial_\lambda B^{\mu\nu} - (\partial_\nu \delta x^\lambda) B^{\mu\lambda}, \\
\bar{B}_{\mu\nu} &= B_{\mu\nu} - \delta x^\lambda \partial_\lambda B_{\mu\nu} - (\partial_\mu \delta x^\lambda) B_{\nu\lambda} - (\partial_\nu \delta x^\lambda) B_{\mu\lambda}.
\end{align*}

(2.44) to (2.47)

Note that these expressions are equivalent to those whose right-hand sides are described in terms of covariant derivatives instead of partial derivatives, since all Christoffel symbols cancel each other.

### 2.3.2 Tensor decomposition

In this subsection, we discuss the decomposition of any vectors and 2-tensors into components according to their transformations under three-dimensional spatial rotations [22, 75]. This treatment enables us to easily describe the evolutions of perturbations.

Firstly, a 3-vector which has only spatial components can be decomposed by a scalar and a pure vector components:

\begin{equation}
\bar{X}^i = X^S_i + X^V_i, \tag{2.49}
\end{equation}

where the subscript $|i$ denotes a covariant derivative with respect to the spatial metric $\gamma_{ij}$, $X^S$ is a scalar quantity, and $X^V$ is a vector quantity which satisfies the divergence-free condition,

\begin{equation}
X^V_i |_i = 0. \tag{2.50}
\end{equation}

For later convenience, we define a vector harmonic function $\hat{e}^i$ which satisfies the divergence-free condition,

\begin{equation}
\hat{e}^i |_i = 0. \tag{2.51}
\end{equation}

Then we can write $X^V_i = X^V \hat{e}^i$. The decomposition given in Eq. (2.49) is uniquely determined as follows. Given a vector $\bar{X}^i$, the scalar $X$ can be determined by $X^S_i |_i = \bar{X}^i |_i$, and the vector $X^V_i$ is a difference between these two quantities: $X^V_i = \bar{X}^i - X^S_i$.

In the analogous way, a symmetric 3-tensor can be decomposed by a scalar, a vector, and a pure tensor components:

\begin{equation}
\bar{X}_{ij} = X^S_{|ij} + Y_{\gamma_{ij}} + \frac{1}{2}(X^V_{|ij} + X^V_{|ji}) + X^T_{ij}, \tag{2.52}
\end{equation}

where $X^T_{ij}$ is satisfying the transverse-traceless condition,

\begin{equation}
X^T_{ij} |^j = X^T_{i|} = 0. \tag{2.53}
\end{equation}
Here we can define a tensor harmonic function $\hat{e}_{ij}$ which satisfies the transverse-traceless condition,

$$\hat{e}_{ij} V^j = \hat{e}^i = 0.$$  \hfill (2.54) 

Then we can write $X^T_{ij} = X^T \hat{e}_{ij}$.

The total degree of freedom in the right-hand side of Eq. (2.52) is 11. Using the 5 conditions coming from the divergence-free condition (2.50) and the transverse-traceless condition (2.53), the following 5 equations and 1 equation (2.52) can completely and uniquely fix 6 degrees of freedom inherent in $Y, X^S, X^V_i, X^T_{ij}$:

$$\tilde{X}^i = 3Y + X^S l_i,$$

$$\tilde{X}^i_{\mid j} = Y^k + (X^S)_{\mid k} + 2KX^S l_k + \frac{1}{2} X^V l_{(j} |_{\mid i)} + KX^V_i,$$

$$\tilde{X}^i_{\mid ij} = (Y + X^S)_{\mid ij} + KX^S_{\mid ij}.$$  \hfill (2.55) 

If we consider the perturbations on a maximally symmetric spacetime, such as de Sitter, Minkowski and FLRW spacetime, we can independently treat the dynamics of the scalar, vector and tensor parts.

### 2.3.3 Metric and matter perturbations

In order to see the demonstration of the decomposition, let us consider the perturbations of the metric (2.1) and the energy–momentum tensor for a general fluid (2.3). Following the decomposition method, a general form of the perturbed metric tensor in the flat universe $(K = 0)$ becomes

$$g_{\mu\nu} = \begin{pmatrix} -(1 + 2A) & a^2 (B_i - S_i) \\ a^2 (B_j - S_j) & a^2 \left[ (1 + 2R) \delta_{ij} + 2E_{ij} + F_{ij} + F_{ji} + h_{ij} \right] \end{pmatrix},$$  \hfill (2.58) 

where 3-vectors $S_i$ and $F_i$ satisfy the divergence-free condition (2.50), a tensor $h_{ij}$ satisfies the transverse-traceless condition (2.53). The absolute values of the perturbative quantities, $A, B, E, R, S_i, F_i$ and $h_{ij}$, are assumed to be less than 1.

Assuming that the universe is described by the FLRW model, the background velocity of fluid is given by Eq. (2.4) and the energy–momentum tensor becomes Eqs. (2.5) and (2.6). Introducing its spatial perturbation $v^i \equiv u^i / u^0, |v^i| \ll 1$ for a given 4-velocity (2.4), the normalisation equation $u^\mu u_\mu = -1$ determines $u^0$ component :

$$u^\mu = (u^0, v^i), \quad g^{\mu\nu} u_\mu u_\nu = -1 \implies u^\mu = (1 - A, v^i).$$  \hfill (2.59) 

The velocity perturbation is decomposed to

$$v^i = v^S i + v^V \hat{e}^i.$$  \hfill (2.60) 

The lower index of $u^\mu$ is

$$u_0 = g_{0\nu} u^\nu = -1 - A,$$

$$u_i = g_{i\nu} u^\nu = a^2 \left\{ (B + v^S)_{,i} - (S_i - v^V_i) \right\}.$$  \hfill (2.61)
The energy–momentum tensor (2.3) can be decomposed into two parts:

\[ T_{\mu \nu} = (\rho + \delta \rho) u^\mu u_\nu + P_{\mu \alpha} P_{\nu \beta} T^{\alpha \beta}, \tag{2.62} \]

where we introduced the energy perturbation \( \delta \rho \), and the projection tensor \( P_{\mu \nu} \) is given by

\[ P_{\mu \nu} = \delta_{\mu \nu} + u^\mu u_\nu. \tag{2.63} \]

The purely spatial part of \( T_{\mu \nu} \) with its perturbations can be decomposed as

\[ T^i_j = (p + \delta p) \delta^i_j + \delta \pi^i_j, \tag{2.64} \]

where \( \delta p \) is the pressure perturbation and \( \delta \pi \) is an anisotropic stress (inhomogeneous pressure) which is same order of the perturbations.

Following Eqs. (2.62) (2.63) and (2.6), we obtain the perturbations of the energy–momentum tensor:

\[ \delta T^0_0 = -\delta \rho, \tag{2.65} \]
\[ \delta T^i_0 = \delta q_i = a^2 (\rho + p) \left\{ (B + v^S)_i - (S_i - v^V_i) \right\}, \tag{2.66} \]
\[ \delta T^i_0 = -(\rho + p) v^i, \tag{2.67} \]
\[ \delta T^i_j = \delta p \delta^i_j + \delta \pi^i_j, \tag{2.68} \]
\[ \delta T^{i i} = \delta T^{i i} = \delta p \delta^i_j + \delta \pi^i_j, \tag{2.69} \]

The perturbed momentum \( \delta q_i \) defined here can be decomposed as

\[ \delta q_i = \delta q^S_i + \delta q^V \hat{e}_i, \tag{2.70} \]

and the anisotropic stress \( \delta \pi \) is decomposed as

\[ \delta \pi^i_j = \left( \nabla^i \nabla^j - \frac{1}{3} \delta^i_j \nabla^2 \right) \delta \pi^S + \delta \pi^V (\hat{e}^i_j + \hat{e}^j_i) + \delta \pi^T \hat{e}^i_j, \tag{2.71} \]

where \( \nabla_i \) denotes a covariant derivative operator for the spatial metric \( \gamma_{ij} \) which is identical to \( \partial_i \) in the present case \((K = 0)\).

### 2.3.4 Gauge freedom

The metric tensor with the scalar perturbations in the flat universe \((K = 0)\),

\[ g_{\mu \nu} = \begin{pmatrix} -(1 + 2A) & a^2 B_j^i \\ a^2 B_j^i & a^2 [(1 + 2R) \delta_{ij} + 2E_{ij}] \end{pmatrix}, \tag{2.72} \]

has a gauge freedom associated with Eq. (2.41). The gauge transformation Eq. (2.41) can be decomposed into temporal and spatial parts:

\[ t \rightarrow \tilde{t} = t + \delta t, \tag{2.73} \]
\[ x^i \rightarrow \tilde{x}^i = x^i + \delta x^S i + \delta x^V i, \tag{2.74} \]
According to the gauge transformation, the perturbative quantities $A$, $\mathcal{R}$, $B$ and $E$ are transformed as Eq. (2.47). The explicit forms are

$$A \rightarrow \tilde{A} = A - \delta t,$$  
$$\mathcal{R} \rightarrow \tilde{\mathcal{R}} = \mathcal{R} - \frac{\dot{a}}{a} \delta t,$$  
$$B \rightarrow \tilde{B} = B + \frac{1}{a^2} \delta t - \dot{x}^S,$$  
$$E \rightarrow \tilde{E} = E - \delta x^S. \quad (2.78)$$

We can define a quantity called shear from a combination of $B$ and $E$,

$$\sigma = -B + \dot{E}, \quad (2.79)$$

which is transformed by Eq. (2.47) as

$$\sigma \rightarrow \tilde{\sigma} = \sigma - \frac{1}{a^2} \delta t. \quad (2.80)$$

In a similar way using Eq. (2.46), the perturbations of the energy–momentum tensor (2.65)–(2.69) changes their values as

$$\delta \rho \rightarrow \tilde{\delta} \rho = \delta \rho - \dot{\rho} \delta t,$$  
$$\delta v^S \rightarrow \tilde{\delta} v^S = \delta v^S + \dot{x}^S,$$  
$$\delta v^V \rightarrow \tilde{\delta} v^V = \delta v^V + \dot{x}^V,$$  
$$\delta p \rightarrow \tilde{\delta} p = \delta p - \dot{\rho} \delta t,$$  
$$\delta \pi^i_j \rightarrow \tilde{\delta} \pi^i_j = \delta \pi^i_j. \quad (2.85)$$

There are several ways to fix the gauge freedom, namely, to fix the functional forms of $\delta t$ and $\delta x^i$. Here we present three of them.

– longitudinal gauge (Newtonian gauge)

In the longitudinal gauge, $\overline{B} = \overline{E} = 0$. the gauge transformation of the perturbative quantities from an arbitrary gauge is completely determined by setting Eqs. (2.77) and (2.78) to be zero :

$$\delta t = a^2 (\dot{E} - B),$$  
$$\delta x^S = E. \quad (2.87)$$

In this gauge, the line element becomes a simple form as

$$ds^2 = -(1 + 2\overline{A}) dt^2 + a^2 (1 + 2\overline{\mathcal{R}}) \delta_{ij} dx^i dx^j. \quad (2.88)$$

– comoving gauge
In the comoving gauge, $\dot{B} + v^S = 0$, which yields $\delta q^S = 0$ from Eq. (2.66). Additionally setting $\dot{E} = 0$, the gauge transformation from an arbitrary gauge becomes

$$\delta t = -a^2(B + v^S),$$

$$\delta x^S = E.$$  \hfill (2.89)

In this gauge, the line element is expressed by

$$ds^2 = -(1 + 2\dot{A})dt^2 - 2a^2v^S_i + a^2(1 + 2\dot{R})\delta_{ij}dx^i dx^j. \hfill (2.91)$$

– synchronous gauge

In this gauge, the line element is expressed by

$$ds^2 = -(1 + 2\dot{A})dt^2 + a^2(\delta_{ij} + h_{ij})dx^i dx^j.$$  \hfill (2.92)

This choice corresponds to setting $A = B = 0$. However this cannot completely fix the gauge freedom. While the choice fixes the functional forms of $\dot{t}$ and $\dot{x}^S$ from Eqs. (2.75) and (2.77), the values of $\delta t$ and $\delta x^S$ at a given time are not determined yet.

Throughout this thesis, we use the longitudinal gauge. Note that, while there is the 5D-longitudinal gauge in the context of the braneworld scenario, the perturbative quantities evaluated at the 5D-longitudinal gauge is generally different from ones evaluated at the 4D-longitudinal gauge due to the existence of the brane bending [see Eqs. (4.44) (4.56) and (4.57)].

**2.3.5 Gauge invariant variables**

As another approach to describe the perturbation theory, J. M. Bardeen introduced gauge-invariant variables [23]. Here we present some of these quantities, which have frequently used to treat the evolution of perturbations on super-horizon scales. From the explicit forms of the gauge transformation (2.75)–(2.78) and (2.80), if we choose combinations,

$$\Phi \equiv A - (a^2\sigma)',$$

$$\Psi \equiv \mathcal{R} - \frac{\dot{a}}{a}(a^2\sigma)',$$

which do not depend on gauge choices. These quantities are called *Bardeen potentials*. The word 'potential' comes from the fact that $\Phi$ can be identified to the Newton potential.

Taking a gauge-invariant combination of metric perturbations (2.76), (2.77) and matter perturbations (2.82), we can also define the *comoving curvature perturbation*,

$$\mathcal{R}_c \equiv \mathcal{R} + a^2H(B + v^S) = \mathcal{R} + \frac{H}{\rho + p}\delta q^S,$$  \hfill (2.95)
which coincides with the curvature perturbation in the comoving gauge \((\delta q^S = 0)\). Alternatively, we can define another gauge-invariant combination of Eqs. (2.76) and (2.81),

\[
\zeta = \mathcal{R} - \frac{H}{\rho} \delta \rho = \mathcal{R} + \frac{\delta \rho}{3(\rho + P)},
\]

(2.96)

where we used the energy conservation law (2.9) in the second equality. This is called the \textit{Bardeen parameter}, representing the curvature perturbation on the uniform density surface \(\delta \rho = 0\). According to Ref. [76], the two curvature perturbations are identical to each other in the case of the perturbations of a single scalar field which will be discussed in next section. We will use the comoving curvature perturbation, \(\mathcal{R}_c\), in our numerical analysis of the evolution of inflaton perturbations in a braneworld scenario in Ch. 6.

### 2.3.6 Evolution equation of matter perturbations

Evolution equations of matter perturbations, \(\delta \rho\) and \(\delta q_i\), are obtained from the conservation law of matter,

\[
\delta (\nabla_\mu T^{\mu \nu}) = 0.
\]

(2.97)

For simplicity, we here focus on the scalar perturbations. In the 4D-longitudinal gauge, the Christoffel symbols are calculated as Eqs. (B.12)–(B.16). With them, the 0th-order energy–momentum tensor (2.5) and (2.6) with its scalar perturbations (2.65)–(2.69) enable us to calculate \(\nu = 0\) component of Eq. (2.97) as [77]

\[
\delta \rho - \frac{k^2}{a^2} \delta q^S + 3H(\delta \rho + \delta p) + 3(\rho + p) \dot{\mathcal{R}} = 0,
\]

(2.98)

and \(\nu = i\) components become

\[
\left\{\delta q^S + 3H \delta q^S + \delta p - \frac{2}{3} k^2 \delta \pi^S + (\rho + p) A\right\}_{,i} = 0.
\]

(2.99)

Note that one has to specify the equation-of-state to observe the evolution of \(\delta p\). Furthermore, to complete the formalism, we must impose an additional condition on the anisotropic stress \(\delta \pi\). In later analyses, we will focus on the tensor perturbations in the radiation-dominated universe in a braneworld scenario (Ch. 5), and the scalar perturbations during the brane inflation (Chap. 6). In the former case, it is not necessary to consider the relation between \(\delta p\) and \(\delta \rho\) since the tensor perturbations are decoupled to them as we will see in Eq. (4.72). Moreover, we will assume that there is not an anisotropic stress on the brane for simplicity. In the latter case, namely, inflaton perturbations, we will see that the inflaton perturbations do not contribute to the anisotropic stress [Eq. (2.108)].

### 2.3.7 Evolution equation for inflaton perturbation

Following the above mathematical preparation, we shall give a concrete example of the way Eqs. (2.65)–(2.69) and their evolution equation (2.98) described by a gauge-invariant
variable are used. Let us consider a case that our universe is expanded by a homogeneous scalar field $\phi_0(t)$, inflaton. Its perturbation is defined by

$$\delta \phi(t, x) = \phi(t, x) - \phi_0(t).$$ \hspace{1cm} (2.100)

The background energy density and pressure on the brane is given by Eqs. (2.19) and (2.20), and the energy–momentum tensor of scalar field $\delta \phi(t, x)$ is given as Eq. (2.18). To first order of the perturbation, we obtain the perturbed energy–momentum tensor:

$$\delta T^0_0 = -(-\dot{\phi}^2 A + \dot{\phi} \delta \phi + V'(\phi) \delta \phi),$$ \hspace{1cm} (2.101)

$$\delta T^i_0 = -\dot{\phi} \delta \phi_i,$$ \hspace{1cm} (2.102)

$$\delta T^i_j = (-\dot{\phi}^2 A + \dot{\phi} \delta \phi - V'(\phi) \delta \phi) \delta_{ij}.$$ \hspace{1cm} (2.103)

Following the definitions of $\delta \rho$, $\delta p$, $\delta q$ and $\delta \pi$ in Eqs. (2.65)–(2.69), we can read

$$\delta \rho = -\dot{\phi}^2 A + \dot{\phi} \delta \phi + V'(\phi) \delta \phi,$$ \hspace{1cm} (2.105)

$$\delta p = -\dot{\phi}^2 A + \dot{\phi} \delta \phi - V'(\phi) \delta \phi,$$ \hspace{1cm} (2.106)

$$\delta q = -\delta \phi,$$ \hspace{1cm} (2.107)

$$\delta \pi = 0.$$ \hspace{1cm} (2.108)

Therefore the behaviour of $\delta p$ and $\delta \rho$ is completely specified by the inflaton perturbations without imposing an equation-of-state. Furthermore, the inflaton perturbations do not produce an anisotropic stress as we mentioned in the last paragraph of Sec. 2.3.6.

The perturbed energy-conservation-law (2.98) yields the evolution equation of the inflaton perturbation,

$$\ddot{\phi} + 3H \dot{\phi} + \frac{k^2}{a_0^2} \delta \phi = -V''(\phi) \delta \phi - 3\dot{\phi} \ddot{\phi} + \dot{\phi} \dot{\phi} - 2V'(\phi) A.$$ \hspace{1cm} (2.109)

Note that Eq. (2.99) is automatically satisfied in the present case. To eliminate $A$, we use the perturbed Einstein equation, $\delta G_{\mu\nu} = 8\pi G \delta T^\mu_{\nu}$, where $\delta G_{\mu\nu}$ is given in Eqs. (B.30)–(B.34) replacing $\Delta$ by $A$ and $\bar{R}$ by $\delta \bar{R}$ and $\bar{R}$, respectively. The off-diagonal part of the equation yields $A + R = 0$ in the present case. Then we introduce a gauge-invariant variable, the Mukhanov–Sasaki variable, defined as [25,78,79]

$$Q = -\frac{\dot{\phi}}{H} R_c = \frac{\delta \phi}{H} \bar{R}.$$ \hspace{1cm} (2.110)

Rewriting Eq. (2.109) in terms of the gauge-invariant variable and eliminating $\ddot{\bar{R}}$ and $\dot{\bar{R}}$ by the perturbed Einstein equation, we obtain

$$\ddot{Q} + 3H \dot{Q} + \frac{k^2}{a_0^2} Q + \left\{ \frac{H}{2} \frac{H V' \phi}{\phi} - 2 \left( \frac{H}{H} \right)^2 + V''(\phi) \right\} Q = 0,$$ \hspace{1cm} (2.111)
which is called Mukhanov–Sasaki equation [25]. For the inflaton perturbations in a braneworld scenario which will be discussed in Ch. 6, there is a non-zero term in the right-hand side of the Mukhanov–Sasaki equation due to the existence of the extra-dimensional metric perturbations [see Eq. (4.73)].

2.3.8 Quantum-mechanical treatment of inflaton perturbations

Within the classical theory, we cannot determine the initial conditions of the perturbations. To calculate the primordial spectrum of the perturbations, we have to rely on the quantum theory.

The inflaton perturbation, whose energy density and pressure are given by Eqs. (2.19) and (2.20), is coupled to the scalar part of the metric perturbations. The perturbed metric in the 4D-longitudinal gauge is given by Eq. (2.88), which is rewritten in terms of the conformal time $\eta$:

$$ds^2 = a^2(\eta) \left\{ -\left(1 + 2\overline{A}\right)d\eta^2 + \left(1 + 2\overline{\mathcal{R}}\right)\delta_{ij}dx^i dx^j \right\}. \quad (2.112)$$

From the definitions of the comoving curvature perturbation (2.95) and the momentum perturbation (2.66) with its value induced by the inflaton perturbation (2.107), the comoving curvature perturbation becomes

$$\mathcal{R}_c = \overline{\mathcal{R}} - \frac{H}{\dot{\phi}} \delta\phi. \quad (2.113)$$

The spectrum of the comoving curvature perturbation $\mathcal{P}_\mathcal{R}$ is defined by

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_\mathcal{R} \delta^3(k-k'), \quad (2.114)$$

where $\mathcal{R}_k$ is a Fourier component of the comoving curvature perturbation:

$$\mathcal{R}_c = \int \frac{d^3k}{(2\pi)^{3/2}} \mathcal{R}_k(\eta)e^{ik \cdot x}. \quad (2.115)$$

Next, we consider the quantisation of the inflaton perturbations. The action during the inflation is given by

$$S = \int \left[ \frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu}\phi,_{\mu}\phi,_{\nu} - V(\phi) \right] \sqrt{-g} d^4 x. \quad (2.116)$$

The second-order perturbation of the action omitting the surface terms is calculated as [25]

$$\delta S = \frac{1}{2} \int \left( (u')^2 - u^4u,_{i} + \frac{z''}{z} u^2 \right) d\eta d^3 x, \quad (2.117)$$

where $z = a\dot{\phi}/H$, a prime denotes the derivative with respect to $\eta$, and $u$ is defined by

$$u = z\mathcal{R}_c, \quad (2.118)$$
which coincides with $aQ$. Here we consider the quantisation of the scalar field $\hat{u}(\eta, \mathbf{x})$. The field operator is decomposed into the positive and negative frequency modes:

$$\hat{u}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3/2} \left( u_k(\eta)\hat{a}_k e^{i\mathbf{k}\cdot\mathbf{x}} + u^*_k(\eta)\hat{a}^+_k e^{-i\mathbf{k}\cdot\mathbf{x}} \right),$$

(2.119)

where $\hat{a}_k$ and $\hat{a}^+_k$ are an annihilation and a creation operator satisfying

$$[\hat{a}_k, \hat{a}^+_k] = \delta^3(\mathbf{k} - \mathbf{k}'), \quad [\hat{a}_k, \hat{a}_k] = [\hat{a}^+_k, \hat{a}^+_k] = 0. \quad (2.120)$$

The vacuum state of the scalar field is defined by $\hat{a}_k |0\rangle = 0$. The perturbed Einstein equation yields the equation of motion for the mode functions $u_k$:

$$u''_k + \left( k^2 - \frac{z''}{z} \right) u_k = 0. \quad (2.121)$$

The effective mass term can be described in terms of slow-roll parameters (2.36)–(2.38) [73],

$$\frac{z''}{z} = 2a^2H^2 \left[ 1 + \epsilon_H - \frac{3}{2} \eta_H + \epsilon_H^2 - 2\epsilon_H\eta_H + \frac{1}{2} \eta_H^2 + \frac{1}{2} \xi_H^2 \right]. \quad (2.122)$$

In order to relate $a^2H^2$ to the conformal time and the slow-roll parameters, we perform a partial integration of the definition of the conformal time:

$$\eta = \frac{da}{a^2H} = -\frac{1}{aH} + \int \epsilon_H \frac{da}{a^2H},$$

(2.123)

where we used Eq. (2.36). To proceed the partial integration, we obtain

$$\eta = -\frac{1}{aH} \frac{1}{1 - \epsilon_H} + \mathcal{O}(\epsilon_H \zeta, \zeta^2), \quad (2.124)$$

where $\zeta = \epsilon_H - \eta_H$ satisfying $\epsilon_H/\eta_H = 2\epsilon_H \zeta$. In the case of power-law inflation, which is realised by setting the Hubble parameter to a exponential form, $H(\phi) = \exp(\sqrt{4\pi/p\phi}/M_{Pl})$ [80], $\epsilon_H$ becomes constant, leading to $\zeta = 0$. In this case, the first term in Eq. (2.124) becomes an exact solution. To first order of $\epsilon_H$, we obtain

$$\eta = -\frac{1}{aH} (1 + \epsilon_H) + \mathcal{O}(\epsilon_H^2, \eta_H^2). \quad (2.125)$$

Then the effective mass (2.122) becomes

$$\frac{z''}{z} = \frac{2}{\eta_H^2} \left[ 1 + 3\epsilon_H - \frac{3}{2} \eta_H + \mathcal{O}(\epsilon_H^2, \eta_H^2) \right]$$

$$= \frac{1}{\eta_H^2} \left( \nu^2 - \frac{1}{4} \right), \quad (2.126)$$

where

$$\nu = \frac{3}{2} + 2\epsilon_H - \eta_H + \mathcal{O}(\epsilon_H^2, \eta_H^2). \quad (2.127)$$
2.3 Perturbation theory

With these expressions, the equation of motion (2.121) can be solved analytically,

\[ u_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\pi/2}(-\eta)^{1/2} H^{(1)}_{\nu}(-k\eta), \]  

(2.28)

where \( H^{(1)}_{\nu}(-k\eta) \) is the Hankel function of the first kind. Here we assumed the Bunch–Davies vacuum where perturbations stay in the Minkowski vacuum state at small scales [81], namely,

\[ u_k(\eta) \rightarrow \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad \text{for} \quad |k\eta| \rightarrow \infty, \]  

(2.29)

which determines the normalisation and functional form of the mode function (2.28).

From Eqs. (2.114), (2.118) and (2.119), we obtain the vacuum expectation value of the curvature perturbation and its spectrum evaluated at the horizon crossing, \( k = aH \) :

\[ \langle R_k R^*_{k'} \rangle = \left| \frac{u_k}{z} \right|^2 \delta^3(k - k'), \]  

(2.130)

\[ \Rightarrow \mathcal{P}_R = \frac{k^3}{2\pi^2} \left| \frac{u_k}{z} \right|_{k = aH}. \]

To evaluate the spectrum on the super-horizon scales, we consider the asymptotic behaviour of the solution (2.28). In the long wave-length limit, \( k\eta \to 0 \),

\[ u_k \to e^{i(\nu-1/2)\pi/2}e^{i\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}}(-k\eta)^{-\nu+1/2}. \]  

(2.131)

Hence the primordial power spectrum of the comoving curvature perturbation on super-horizon scales becomes [73,82]

\[ \mathcal{P}_R^{1/2} = 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (1 - \epsilon_H)^{\nu-1/2} \frac{H^2}{2\pi|\dot{\phi}|_{k = aH}}. \]  

(2.132)

Using Eq. (2.32) and evaluating the spectrum to the first order of the slow-roll parameters, we obtain [82]

\[ \mathcal{P}_R^{1/2} = \left[ 1 - (2C + 1)\epsilon_H + C\eta_H \right] \frac{2}{M_{Pl}^2} \left\{ \frac{H^2}{|dH/d\phi|} \right\}_{k = aH}, \]  

(2.133)

where \( C = -2 + \log 2 + \gamma \simeq -0.73 \).

The quantisation of the metric perturbations and inflaton perturbations enables us to determine the amplitude of the perturbations from the dynamics of inflation without any ambiguities. However, to evaluate the present spectrum, \( P(k) \), it is necessary to take into account the evolution of the perturbations after re-entry of the horizon, which is described by the transfer function, \( T_R(k) \) [61] :

\[ \frac{k^3}{2\pi^2} P(k) = \left( \frac{k}{aH} \right)^4 T_R^2(k) \mathcal{P}_R, \]  

(2.134)

where \( T_R(k) \to 1 \) for \( k \to 0 \).
2.3.9 Quantum generation of gravitational waves

We can derive also the primordial spectrum of the gravitational waves generated during the inflation in the same way. The tensor perturbations can be expressed as

$$ds^2 = a^2(\eta) \left\{ -d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j \right\}.$$  \hspace{1cm} (2.135)

The perturbed action becomes

$$\delta S = \frac{M_{Pl}^2}{64\pi} \int d\eta d^3x a^2 \partial_\mu h_{ij} \partial^\mu h^{ij}.$$  \hspace{1cm} (2.136)

To treat the tensor perturbation as a canonically normalised scalar field, we introduce a new quantity $v_\lambda^k$,

$$h_{ij} = \frac{1}{a} \frac{1}{\sqrt{\frac{32\pi}{M_{Pl}^2}}} \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{\lambda=+,\times} v_\lambda^k(\eta) \varepsilon_{ij}(k, \lambda) e^{ikx},$$  \hspace{1cm} (2.137)

where we defined a polarisation tensor satisfying

$$\varepsilon_{ij} = \varepsilon_{ji}, \quad \varepsilon_{ii} = 0, \quad k_i \varepsilon_{ij} = 0, \quad \varepsilon_{ij}(k, \lambda) \varepsilon_{ij}^*(k, \mu) = \delta_{\mu\lambda}.$$  \hspace{1cm} (2.138)

Then the action (2.136) becomes

$$\delta S = \frac{1}{2} \int \sum_{\lambda=+,\times} \left[ |v_\lambda^k|^2 - \left( k^2 - \frac{a''}{a} \right) |v_\lambda^k|^2 \right] d\eta d^3k.$$  \hspace{1cm} (2.139)

The two scalar fields $v_\lambda^k (\lambda = +, \times)$ are quantised in the same way shown in the previous subsection. The field operator $\hat{v}_\lambda^k$ is decomposed as

$$\hat{v}_\lambda^k = v_k(\eta) \hat{b}_k^\lambda + v_k^*(\eta) \hat{b}_k^\dagger \lambda,$$  \hspace{1cm} (2.140)

where $\hat{b}_k^\lambda$ and $\hat{b}_k^{\dagger \lambda}$ satisfy

$$[\hat{b}_k^\lambda, \hat{b}_k^{\dagger \lambda'}] = \delta^3(\mathbf{k} - \mathbf{k}') \delta_{\lambda\lambda'}, \quad [\hat{b}_k^\lambda, \hat{b}_k^{\dagger \lambda'}] = [\hat{b}_k^{\dagger \lambda}, \hat{b}_k^{\dagger \lambda'}] = 0.$$  \hspace{1cm} (2.141)

The vacuum state of the scalar field is defined by $|\hat{b}_k^\lambda \rangle = 0$. In analogous to Eq. (2.130), the power spectrum of the tensor perturbation is defined by

$$\langle \hat{v}_k^\lambda \hat{v}_k^{\dagger \lambda} \rangle = \frac{M_{Pl}^2 a^2 2\pi^2}{32\pi} P_\lambda(\mathbf{k} - \mathbf{k}).$$  \hspace{1cm} (2.142)

The equation of motion for the mode functions $v_k$ is

$$v_k'' + \left( k^2 - \frac{a''}{a} \right) v_k = 0.$$  \hspace{1cm} (2.143)

Combining Eqs. (2.31) with (2.32) and using a slow-roll parameter $\epsilon_H$, the effective mass term becomes

$$\frac{a''}{a} = 2a^2 H^2 \left( 1 - \frac{\epsilon_H}{2} \right).$$  \hspace{1cm} (2.144)
To rewrite $a^2H^2$, we use again Eq. (2.125), which follows
\[
\frac{a''}{a} = \frac{2}{\eta^2} \left( 1 + \frac{3}{2} \epsilon_H + \mathcal{O}(\epsilon_H^2) \right).
\]
where
\[
\mu = \frac{3}{2} + \epsilon_H + \mathcal{O}(\epsilon_H^2).
\]
The solution of Eq. (2.143) is obtained by replacing $\nu$ by $\mu$ in Eq. (2.128). Its asymptotic form is given by Eq. (2.131). Therefore we obtain the power spectrum of the gravitational waves on super-horizon scales [73,82,83]:
\[
P_{h}^{1/2} = \frac{2}{\sqrt{\pi}} 2^{\mu-1/2} \frac{\Gamma(\mu)}{\Gamma(3/2)} (1 - \epsilon_H)^{\mu-1/2} \frac{H}{M_{Pl}} |_{aH-k},
\]
where the factor '2' was multiplied due to the existence of the two polarisations. Taking the first order of the slow-roll parameter, we obtain
\[
P_{h}^{1/2} = \left[ 1 - (C + 1)\epsilon_H \right] \frac{4}{\sqrt{\pi}} \frac{H}{M_{Pl}} |_{k=aH}.
\]
The amplitude of the spectrum is suppressed by the Planck mass, which indicates that the observation of the relic gravitational waves is quite difficult.

The present amount of the relic gravitational waves is conventionally represented by means of the energy spectrum, $\Omega_{GW}$, which is defined as [84]
\[
\Omega_{GW}(f) = \frac{1}{\rho_c} \frac{d\rho_{GW}}{d\log f},
\]
where $\rho_c = 3H_0^2/8\pi G = 9.8 \times 10^{-30}$ g/cm$^3$ means the critical density of the universe and $\rho_{GW}$ the energy density of GWB. We usually multiply this by $h_0^2$ where $H_0 = 100h_0$ km/s/Mpc, yielding $h_0^2\Omega_{GW}$. This energy spectrum is related to the primordial spectrum (2.147) via the transfer function for the tensor perturbations [73]:
\[
\Omega_{GW}(f) = \frac{1}{24} T_{h}^2(k) P_{h}(f),
\]
where the transfer function describes the evolution of GWs after inflation. This quantity will be discussed in detail in next section.

2.4 Observables

2.4.1 Consistency relation

Using the slow-roll approximation, we can derive an important relation between the primordial power spectra for the curvature perturbations (2.132) and for the gravitational waves
According to Ref. [73], we define the amplitudes of these spectra [85]:

\[
A_S(k) \equiv \frac{2}{5} \mathcal{P}^{1/2}_R = \left. \frac{4}{5} \frac{H^2}{M_{Pl}^2} |H'(\phi)| \right|_{k=aH} = \sqrt{\frac{512\pi}{75M_{Pl}^2} V^{3/2}} \left|_{k=aH} \right.,
\]

\[
A_T(k) \equiv \frac{1}{10} \mathcal{P}^{1/2}_h = \left. \frac{2}{5\sqrt{\pi}} \frac{H}{M_{Pl}} \right|_{k=aH} = \sqrt{\frac{32}{75}} \mathcal{P}^{1/2}_h \left|_{k=aH} \right..
\]

Using these notations and Eq. (2.36), we can immediately show that the scalar–tensor ratio is proportional to a slow-roll parameter:

\[
\frac{A_T^2}{A_S^2} = \epsilon_H.
\]

This relation implies that the amount of the primordial gravitational waves is highly suppressed by the slow-roll parameter ($\epsilon_H \ll 1$).

Here we give a specific example using a inflaton potential, $V(\phi) = m^2 \phi^2 / 2$ [67]. The inflation ends when $\epsilon_s \sim 1$ from Eq. (2.24), namely,

\[
\phi_{\text{end}} \approx \frac{M_{Pl}}{4\pi}.
\]

The e-folding number $N$ (2.30) becomes [85]

\[
N = 2\pi \frac{\phi_{\text{init}}^2}{M_{Pl}^2} - \frac{1}{2},
\]

which gives the initial value of the inflaton field

\[
\phi_{\text{init}} \approx 3M_{Pl},
\]

when 60 e-foldings are required. Substituting the values of $\phi_{\text{init}}$ and $\phi_{\text{end}}$ into Eqs. (2.151) and (2.152), we obtain

\[
A_S \approx 12 \frac{m}{M_{Pl}},
\]

\[
A_T \approx 1.4 \frac{m}{M_{Pl}}.
\]

In order to reproduce the COBE result $A_S \approx 2 \times 10^{-5}$ neglecting contributions from the gravitational waves [6], the inflaton mass $m \approx 10^{-6} M_{Pl}$ is required. The resultant scalar–tensor ratio becomes

\[
r = \frac{A_T}{A_S} \approx 0.12.
\]

The three-year results of Wilkinson Microwave Anisotropy Probe (WMAP), launched in 2001, showed $r \lesssim 0.55$ (95%CL) assuming no running of spectral index [12].
During the slow-roll inflation, the Hubble radius is slightly varying as time goes, indicating that the amplitudes of primordial spectra (2.151) and (2.152) may depend on scales. The spectral indices of the spectra are defined as

\[ A_S^2(k) \propto k^{n_S - 1} \quad \Rightarrow \quad n_S - 1 = \frac{d \log A_S^2}{d \log k}, \quad (2.160) \]

\[ A_T^2(k) \propto k^{n_T} \quad \Rightarrow \quad n_T = \frac{d \log A_T^2}{d \log k}, \quad (2.161) \]

The case for \( n_S = 1 \) is especially called Harrison-Zel’dovich spectrum [86, 87]. Rewriting \( A_S \) and \( A_T \) in terms of the slow-roll parameters (2.36) and (2.37), the straightforward calculation shows

\[ n_S = 1 + 2\eta_H - 4\epsilon_H, \quad (2.162) \]

\[ n_T = -2\epsilon_H. \quad (2.163) \]

Thus, as long as the slow-roll inflation is concerned, inflation produces nearly scale-invariant spectra of the perturbations. Hence we obtain the lowest-order consistency relation from Eq. (2.153) :

\[ n_T = -2\frac{A_T^2}{A_S^2}, \quad (2.164) \]

which must be satisfied in the single inflaton models. If the relation is broken from observational data, such a model should be ruled out.

As for the amplitude on the curvature perturbations (2.151), the easiest way is to see the curvature perturbation on the largest scale at the last scattering (\( \eta = \eta_{LS} \)), which is related to the temperature anisotropy of CMB :

\[ \frac{\Delta T}{T} = -\frac{1}{5} R_c(\eta_{LS}), \quad (2.165) \]

which is called Sacks-Wolfe effect [61, 88]. Identifying \( R_c(\eta_{LS}) \) with \( \sqrt{P(k)} \) in Eq. (2.134), we can estimate the amplitude of primordial curvature perturbations from the temperature contrast.

### 2.4.2 Inflationary Gravitational Wave Background

Standard inflation model predicts that gravitational waves (GWs) are generated by the quantum fluctuations of spacetime. In this section, we briefly review how to evaluate the power spectrum of IGWB (see also Ref. [84]).

Gravitational waves (GWs) are ultimate probes of the untouched region of the universe. Currently, large scale ground-based interferometers (TAMA300 [21], LIGO [20], VIRGO [89], GEO600 [90]) are enthusiastically trying to detect the signals emitted from stellar objects with relativistic motion – supernovae explosions, coalescence of neutron star binaries and so on. Among numerous types of GWs, the gravitational wave background (GWB) may
possess much interesting information on the cosmology, though its detection is expected to be so challenging [91]. Especially, the inflationary GWB (IGWB), generated during the inflationary epoch by the quantum fluctuations of the spacetime [83, 84, 92–98], is thought to be one of the most fundamental predictions of the inflationary scenario [63–66]. Since the history of the cosmological expansion is imprinted in the power spectrum of the IGWB, it helps us to understand the extremely early universe if we can detect the signals by the future space-based experiments, such as DECIGO [17] and BBO [18, 19].

In Fig. 2.1, a sketch of the evolution histories of GWs with various wavelengths is shown. The vertical and the horizontal axes represent the wavelength of GWs and the cosmic time, respectively. In this figure, the solid line represents the Hubble horizon scale $H^{-1}$, and we have labelled three regimes as “Inflation”, “RD” and “MD” for the inflationary epoch, the radiation-dominated epoch and the matter-dominated epoch, respectively. During the inflation, the universe experiences accelerated expansion and the wavelength of GWs eventually exceeds the Hubble horizon scale. Then the oscillatory behaviour ceases to exist and the amplitudes of GWs become frozen. After inflation, these GWs re-enter the horizon in the decelerated expansion phase (regions “RD” and “MD”). Inside the horizon, the wavelengths are redshifted and the amplitudes are reduced by the cosmological expansion. Since the horizon re-entry time depends on the comoving wave number for each GW mode, the resultant energy spectrum of the IGWB observed at present simply reflects the expansion rate at the horizon re-entry time.

To evaluate the spectrum of IGWB, we first consider the characteristic frequencies of GWB associated with the cosmic history. We define three characteristic frequencies according to the standard four-dimensional cosmology: i) the lowest frequency $f_h$; ii) the frequency of GWs re-entering the horizon just at the matter-radiation equality time, $f_{eq}$; and iii) the cut-off frequency by the inflation, $f_{\text{inf}}$. First, the largest wavelength of IGWB observed today is definitely the horizon length which corresponds to the frequency

$$f_h \approx 2.3 \times 10^{-18} \text{ Hz} \left( \frac{H_0}{72 \text{ km/s} \cdot \text{Mpc}} \right),$$

(2.166)

where $H_0$ denotes the present value of the Hubble parameter. Second, the frequency of large-scale GWs which came into the Hubble horizon at the matter-radiation equality time $t_{eq}$ can be calculated as

$$f_{eq} = \frac{1}{2\pi} \frac{a_{eq}}{a_0} H_{eq} \approx 2.1 \times 10^{-17} \text{ Hz} \left( \frac{H_0}{72 \text{ km/s} \cdot \text{Mpc}} \right) \left( \frac{1 + z_{eq}}{3200} \right)^{1/2},$$

(2.167)

where $a$ denotes the scale factor and $z$ the redshift. The subscripts ‘0’ and ‘eq’ represent the quantities evaluated at the present time $t_0$ and at the matter-radiation equality time $t_{eq}$, respectively. Finally, the highest frequency observed today is determined from the Hubble horizon at the end of the inflation, which can be calculated in the same way as (2.167):

$$f_{\text{inf}} \approx \frac{1}{2\pi} \frac{a_{\text{inf}}}{a_{eq}} \frac{1}{a_0} H_{\text{inf}} = 1.1 \text{ GHz} \left( \frac{H_{\text{inf}}}{6 \times 10^{-5} \text{M}_\text{pl}} \right)^{1/2} \left( \frac{H_0}{72 \text{ km/s} \cdot \text{Mpc}} \right)^{1/2} \left( \frac{1 + z_{eq}}{3200} \right)^{-1/4},$$

(2.168)
where $H_{\text{inf}}$ means the energy scale of the inflation, which is constrained by the COBE observation as $H_{\text{inf}} < 6 \times 10^{-5} M_{\text{pl}}$ [84]. As a consequence, the GWs with $f_h < f < f_{\text{eq}}$ re-enter the Hubble horizon during the MD phase, while for $f_{\text{eq}} < f < f_{\text{inf}}$, GWs re-enter during the RD phase. These characteristic frequencies are shown in Fig. 2.1.

Let us focus on the shape of the IGWB spectrum. As we introduced in Eq. (2.149), the power spectrum of GWB is conventionally characterised by the energy density, $\Omega_{\text{GW}}$, instead of its amplitude. Denoting the characteristic amplitude by $h$, the above quantity is related to $h^2 \Omega_{\text{GW}} = \left( \frac{h}{1.263 \times 10^{-18}} \right)^2 \left( \frac{f}{1 \text{Hz}} \right)^2$. (2.169)

The GWs on sub-horizon scales evolve as $h \propto 1/a$, which comes from the evolution equation (2.143) in the small scale limit, $k \eta \gg 1$. Then, assuming the scale factor evolves as $a \propto t^n$ near the horizon re-entry time $t_*$, the GW amplitude observed today is related to $t_*$ as

$$h_0 = \frac{a_s}{a_0} h_* \propto t_*^{-n/2}. \quad (2.170)$$

Here $h_*$ denotes the amplitude of GWs evaluated at the time $t_*$. The amplitude $h_*$ is primarily determined by the quantum fluctuations generated during the inflation, whose spectral dependence is given by $h_*^2 \propto f^{n_T}$ (e.g., Sec. 6.5 of Ref. [61]). In a single-field model of slow-roll inflation, the spectral index $n_T$ can be expressed by the slow-roll parameter $\epsilon$ as $n_T \approx -2\epsilon$ [61, 99]. In the pure de Sitter expansion, $n_T = 0$.

In the power-law expansion, the Hubble parameter at $t = t_*$ scales as

$$H_* \propto t_*^{-1}. \quad (2.171)$$

Hence the observed frequency $f$ is related to $t_*$ as

$$f = \frac{k}{2\pi a_0} = \frac{a_s H_*}{2\pi a_0} \propto t_*^{-n-1}, \quad (2.172)$$

where $k = a_s H_*$ denotes the comoving wave number of the GW concerned. From (2.170) and (2.172), the power spectrum of the IGWB becomes

$$\Omega_{\text{GW}} \propto h_0^2 f^2 \propto f^{\frac{2n}{3(1+w)} + n_T}. \quad (2.173)$$

Particularly in cases with the matter content in the universe satisfying the equation of state (EOS) of a perfect fluid

$$p = w\rho, \quad (2.174)$$

the power-law index of the scale factor, $n$, is rewritten with

$$n = \frac{2}{3(1+w)}, \quad (2.175)$$

from Eqs. (3.79) and (3.80). Then the energy spectrum (2.173) becomes

$$\Omega_{\text{GW}} \propto f^{\frac{6w-2}{3w+1} + n_T}. \quad (2.176)$$
Now, simply assuming $n_T = 0$ and applying this formula to the standard history of the universe [RD ($w = 1/3$) phase and MD ($w = 0$) phase], the energy spectrum of the IGWB observed at present becomes [98]

$$\Omega_{GW} \propto \begin{cases} f^0 & (f_{eq} < f < f_{int}), \\ f^{-2} & (f_h < f < f_{eq}). \end{cases}$$

The resultant spectrum in the four-dimensional cosmology is shown in the solid line in Fig. 2.2. The normalisation of the spectrum is weakly constrained from the CMB observation by COBE, which leads to $h_0^2\Omega_{GW} < 6.7 \times 10^{-15}$ for the plateau region in the figure [84, 99]. This figure shows that the IGWB widely exists over the frequencies which spans approximately 30 orders of magnitude. Furthermore most of the frequency region comes from the GWs which re-enters the horizon during the RD phase.

The IGWB is one of most important observational targets, which is a challenging mission for future space-based interferometers. The design sensitivity of LISA [100] currently places $h_0^2\Omega_{GW} \sim 10^{-11}$. As a 'beyond Einstein' program proposed by NASA, Big-Bang Observer (BBO) is planning to have a sensitivity with $h_0^2\Omega_{GW} \sim 10^{-13}$ and correlation with another observatory can reach $h_0^2\Omega_{GW} \sim 10^{-17}$ [18, 19]. A Japanese mission named DECI-hertz interferometer Gravitational-wave Observatory (DECIGO) has is designed as a 'flying LIGO' which is composed of three satellite and Fabry-Perot interferometers within them (referred to ‘FP-DECIGO’) [101]. An aim of DECIGO missions is to observe IGWB with a sensitivity $h_0\Omega_{GW} \sim 10^{-12\sim13}$, and ultimate version of DECIGO can reach $h_0\Omega_{GW} \sim 10^{-20}$ which is limited by only the quantum mechanical noises.

The correlation analysis is crucial to observe the stochastic gravitational wave background including IGWB.

---

2The original version of DECIGO was designed as a 'downsized LISA', whose arm-length is order of $10^5$ km [17].
Figure 2.1: Schematic diagrams of the evolution histories of GWs in four-dimensional case. The solid line (red) represents the radius of the Hubble horizon, $H^{-1}$. 'Inflation', 'RD', and 'MD' denotes the inflationary epoch, the radiation-dominated epoch and the matter-dominated epoch, respectively.

Figure 2.2: The GWB spectra produced in early time of the universe. The solid line (red) represents the spectrum of IGWB when the energy scale of inflation is $V^{1/4} = 3.4 \times 10^{16}$ GeV constrained by CMB observations [6, 98, 99, 102]. This spectrum is obtained with the amplitude of tensor perturbations on the largest scale [see Eqs. (2.150) and (2.152)] and no tilt $n_T = 0$. The long-dashed line (blue) shows the contribution from Galactic binary systems (binary confusion noise), which disturbs any other GWB observations. 'BBN' indicates the acceptable upper limit of the total energy density of IGWB during the BBN epoch [84]. 'LIGO S4' is current upper limit calculated from the data of 4th LIGO scientific run [103]. 'Pulsar Timing' constrains the amount of low-frequency GWB inhabiting between an observing pulsar and us [104]. Small wavy lines shows the sensitivities of currently proposed future missions of GW observations [17–19, 101].
Chapter 3

Braneworld

Extra-Dimensions have been studied in particle physics since the early 20th century, and many topics associated with them lie in the centre of modern particle physics and gravity theories. We shall devote this chapter to introduce a braneworld scenario based on the Randall–Sundrum single brane model [27]. This chapter is organised as follows. In Sec. 3.1, we shall start with outlining a history of studies on extra-dimensions and motivations of recently proposed braneworld scenarios in this section. In Sec. 3.2, we explain the Randall–Sundrum single brane model which is used in our works explained in the following chapters. After briefly explaining the model, we derive the Friedmann equation with a high-energy correction and solutions of the Einstein equation (3.7) in the bulk in Sec. 3.3. In Sec. 3.4, we demonstrate peculiar cosmic expansion at high-energy regime. Besides we mention the realisation of inflation in the context of the braneworld scenario.

3.1 Motivations and history

In 1920’s, Th. Kaluza and O. Klein have studied an unification of the Einstein gravity field and electro-magnetic field in a five-dimensional spacetime, \( \mathbb{R}^4 \times S^1 \) [105,106]. Since we have never observed the fifth dimension, the extra space has to be compactified to the Planck scale. This novel idea for the unification of gravity with other forces is inherited by recent researches of particle physics, especially, string theories.

In string theories, we consider a string-like object as a most fundamental object. This idea enables us to avoid any singularities near the position of a point particle. The original version of string theories had been developed in nuclear physics to explain some peculiarities of the behaviour of hadrons during 1960’s. After that, during 1970’s, the string theory was re-discovered to describe quantum behaviour of gravity, in which the bosonic string field lives in a 26 dimensional spacetime. Such a large number of dimensions is needed to avoid the quantum anomalies. Supersymmetric versions of string theories (superstring theories) has also been studied, which can be made up in 10 dimensional spacetime.

A concept of braneworld originally appeared as solitonic solutions in a higher-dimensional
spacetime [28, 29]. In 1990’s, it became clear that superstring theories already discovered at that time are related with each other via dualities and those theories are different limits of a unified theory, M-theory. After the so-called second revolution of superstring theories, in 1995, J. Polchinski theoretically discovered a spatially spreading hypersurface in type II superstring theory, which has been called D-branes [107]. This discovery provides us a new picture of our universe, braneworld : we live in a spatial hypersurface confining all matter fields which is embedded in higher-dimensional spacetime [108–111].

Braneworld scenarios have been not only of cosmological interest, but also made a possibility to solve a more fundamental and serious issue, the hierarchy problem of energy scales. In our universe, we know four fundamental interactions, namely, electro-magnetic, weak, strong and gravity fields. The first two interactions were unified in the theory by S. L. Glashow, S. Weinberg and A. Salam in 1960’s. The strong interaction which is observed among hadrons as nuclear forces is unified with the electro-weak field in the framework of grand-unified-theory (GUT). The gravitational interaction, however, is extremely weak in comparison with the others. The energy scale of the unification of electro-weak field is the order of Planck energy, $\sim 10^{19}$ GeV.

Considering a solution of the hierarchy problem, in 1998, N. Arkani-Hamed, S. Dimopoulos and G. Dvali proposed a simple model of braneworld. We usually refer it to ADD model. In this model, a three-dimensional hypersurface, brane, without a tension is embedded in a higher-dimensional flat spacetime [112, 113]. A novel feature of this model is to assume that the matter fields, for instance, electro-magnetic field, fermions, other gauge fields, are confined to the branes, and only the gravity field can travel in the extra-dimensions (bulk spacetime). This setup does not require the extra-dimensions compactified into the Planck scale like the Kaluza–Klein model. It is only required to be consistent with the experiments of Newton’s law of gravity (the inverse square law), whose current upper limit of the length scale of the extra-dimensions is approximately 0.1 nm [114,115].

Let us see how the ADD model resolves the hierarchy problem. The action of $D$-dimensional ADD model is

$$S = \frac{1}{16\pi G_D} \int d^D X \sqrt{-(D)g^{(D)}} R, \quad (3.1)$$

where $G_D$ denotes a $D$-dimensional coupling constant of gravity defined by a fundamental mass scale $M$ as

$$G_D = \frac{1}{M^{D-2}} = \frac{1}{M^{d+2}}, \quad (3.2)$$

where $d = D - 4$ is the number of extra-dimensions. In the ADD picture, the long-distance gravity is mediated by the massless graviton whose wave function is homogeneous in the extra-dimensions. Thus, the integration over the extra spaces whose sizes are $L$ is trivial, and the effective four-dimensional action becomes

$$S_{\text{eff}} = \frac{V_d}{16\pi G_D} \int d^4 x \sqrt{-(^4g)} g^{(4)} R, \quad (3.3)$$
where $V \sim L^d$ is the volume of the extra-dimensions. From this calculation, we can observe the four-dimensional Planck mass as

$$M_{Pl} = M(ML)^{d/2}. \quad (3.4)$$

Suppose the fundamental energy scale is the electro-weak scale, namely, $M \sim 1\text{TeV}$. Then the length scale of the extra-dimensions can be calculated as

$$L \sim \frac{1}{M} \left( \frac{M_{Pl}}{M} \right)^{2/d} \sim 10^{32/d} \cdot 10^{-17} \text{cm}. \quad (3.5)$$

This implies that $R \sim 1 \text{mm}$ if $d = 2$, which is not excluded by the experiments of Newton’s law [114,115].

One year after this model, an alternative solution of the hierarchy problem was introduced by means of an alternative solution of five-dimensional Einstein equations. L. Randall and R. Sundrum proposed a new scenario called *Randall–Sundrum type I (RSI) model*, in which we live in a three-dimensional domain wall ($\partial \mathcal{M}, g_{\mu\nu}$) with a negative tension, $-\lambda$, in the 5-dimensional spacetime ($\mathcal{M}, (5)g_{AB}$) with a negative cosmological constant, $\Lambda_5 < 0$ [26]. Besides there is an extra brane with a positive tension, $+\lambda$, in the bulk. The action in the 5-dimensional spacetime is

$$S = \int_{\mathcal{M}} d^5 X \sqrt{-\langle 5 \rangle g} \left[ \frac{1}{2\kappa_5^2} (5)R - 2\Lambda_5 \right] + \int_{\partial \mathcal{M}} d^4 x \sqrt{-\langle 4 \rangle g} \left[ \mathcal{L}_{\text{matter}} - \lambda \right], \quad (3.6)$$

where $\lambda$ is a tension of the brane and $\mathcal{L}_{\text{matter}}$ is a Lagrangian density of the matter on the brane. The quantity $\kappa_5$ denotes the 5D gravitational constant which can be rewritten using the 5D Planck mass $M_5$ as $\kappa_5^2 = 8\pi/M_5^3$. The Einstein equations for this action is

$$(5)G_{AB} + \Lambda_5 (5)g_{AB} = 0. \quad (3.7)$$

Due to the negative cosmological constant, the bulk spacetime becomes a five-dimensional *anti-de Sitter* spacetime, hereafter shortly denoted by 'AdS$_5$'.

The solution of the Einstein equation (3.7) can be expressed as

$$ds^2 = e^{-2y/\ell}(-dt^2 + dx^2) + dy^2, \quad (3.8)$$

where $y$ denotes a bulk coordinate, and $\ell$ a curvature radius of the AdS$_5$ spacetime. Our brane, a negative-tension brane, is located at $y = L$, and the extra brane is located at $y = 0$. The schematic diagram of the geometrical configuration of RSI model is shown in Fig. 3.1. This model imposes the $\mathbb{Z}_2$ symmetry on the branes, which means that a field living in the bulk, $u(y)$, satisfies

$$u(y) = u(-y), \quad u(L - y) = u(L + y). \quad (3.9)$$

A model without this assumption has been also discussed in, e.g. Ref. [116].

As with the ADD model, all matter fields except for the gravity field are assumed to be confined to the branes in the RSI model. This inequitable treatment can resolve the hierarchy
problem. In fact, the integration of the action (3.6) by the bulk coordinate \( y \) yields a relation between the four-dimensional Planck mass observed on the negative-tension brane and the five-dimensional Planck mass [108]:

\[
M_{Pl}^2 = \ell M_5^3 (e^{2L/\ell} - 1). \tag{3.10}
\]

If we take \( L/\ell \sim 35 \), then the four-dimensional Planck mass becomes \( 10^{19} \) GeV while the five-dimensional one is \( \sim 1 \) TeV. However, the standard Einstein gravity cannot be recovered at a long distance on our brane. Instead, the Brans-Dicke type gravity appears on our brane [117]. Taking a glance at Eq. (3.31), the sign of gravity constant on the brane becomes negative, yielding a repulsive gravity force [118]. According to Ref. [119], however, the authors found that the effective Einstein equation on the negative-tension brane has a positive gravity constant.

### 3.2 Randall–Sundrum single brane model

#### 3.2.1 Schematic description

Alternatively, L. Randall and R. Sundrum introduce a single-brane model called *Randall–Sundrum type II (RSII) model* [27]. In this model, we live on the positive-tension brane, and the negative-tension brane is set to be infinitely far from our brane as shown in Fig. 3.1. In other words, we consider the situation that our brane is embedded in the AdS\(_5\) spacetime with an infinite volume. Also in this case, the gravity is effectively confined to the brane without a second brane because of the curvature of the AdS\(_5\) spacetime, which is a kind of *warped compactification*. Thus the effective compactification scale \( \ell \) is required to be larger than the allowable scale from the experiments of Newton’s law, i.e, \( \ell > 0.1 \) mm [114, 115].

It is noteworthy that, in contrast to the RSI model, the four-dimensional general relativity is recovered on the brane except for a small-scale corrections [117]. Moreover, the tensor structure of the gravitational interactions at a large distance corresponds to the weak field limit of the four-dimensional General Relativity [117, 120]. Instead of these good features, the hierarchy problem cannot be resolved in the RSII model, since the distance between the branes, \( L \), controlling the hierarchy vanished. Actually, the Planck mass on the positive brane becomes [108]

\[
M_{Pl}^2 = \ell M_5^3 (1 - e^{-2L/\ell}) \xrightarrow{L\to\infty} \ell M_5^3,
\]

(3.11)
giving \( M_5 > 10^9 \) GeV if \( \ell < 0.1 \) mm.

Although this model loses the original goal, the simple setup of the RSII model enables us to perform lots of phenomenological calculations, and thus to explore the possibility of the extra-dimensions. As fundamental studies of RSII cosmology, P. Binétruy et al. studied a FLRW brane embedded in the AdS\(_5\) spacetime by solving the Einstein equation (3.7) with a metric ansatz [121, 122]. T. Shiromizu et al. derived the effective Einstein
3.2 Randall–Sundrum single brane model

Equation which is a four-dimensional reduction of the Einstein equation (3.7) in the on-brane observer’s viewpoint [118]. In the following sections and chapters, we consider the braneworld cosmology based on their works.

Figure 3.1: Sketches of geometrical configurations of Randall–Sundrum models. In RSI model, our brane with a negative tension, $-\lambda$, is depicted by a blue plate, and an extra brane with a positive tension, $+\lambda$, depicted by a red plate, is bounded in the AdS$_5$ bulk with a negative cosmological constant. In RSII model, our brane is the positive-tension brane. The negative-tension one is set to be infinitely far from our brane.

3.2.2 Coordinate systems

There are several coordinate systems to cover the bulk spacetime. We devote this subsection to explain the Poincaré coordinate $(\tau, z)$ and the Gaussian–normal coordinates $(t, y)$.

One of the solutions of the Einstein equation (3.7) has been known as a Schwarzschild-anti de Sitter (AdS) solution which is described as [123–125]

\[
ds^2_K = -f(r_K)d\tau^2 + \frac{dr_K^2}{f(r_K)} + r_K^2 d\Sigma^2_K ,
\]

where $d\Sigma^2_K$ is a metric of a unit three-dimensional sphere, plane or hyperboloid for $K = +1, 0, -1$, respectively, and

\[
f(r_K) = K + \frac{r_K^2}{C} - \frac{C}{r_K^2}.
\]

Here $C$ is corresponding to the mass of Schwarzschild black hole in the bulk spacetime. Let us consider a moving brane embedded in the bulk spacetime. The trajectory of the brane with $K = C = 0$ and a given spatial coordinate, $x^i = \text{const.}$, is expressed as $r_K = r_K(\tau)$ shown as a bold curve in the left panel of the conformal diagram, Fig. 3.2. This panel represents how the brane is embedded in the AdS$_5$ spacetime which is depicted as a cylinder. We show
in addition the coordinate with \( K = 1 \) covers the whole AdS\(_5\) spacetime, and its origin is identical to the central axis of the cylinder \( r_+ = 0 \) \[38,125,126\]. White disks represent hypersurfaces of a constant \( \tau \). The brane is embedded at the edge of the disk (dotted line). The short-dashed line is the seam singularity which is introduced in Ref. \[126\].

Figure 3.2: Covered region of AdS\(_5\) spacetime by three coordinate systems. In the left panel, the global chart \( (\tau, r_+) \) covers the whole AdS\(_5\) spacetime depicted by a cylinder. In the middle panel, a cross-section diagram in the Poincaré coordinate and \( (\tau, r_0) \)-chart is shown with the trajectory of the flat brane. In the right panel, the Gaussian–normal coordinates of the flat brane are depicted, which has a coordinate singularity denoted by \( y = y_c \). The long-dashed lines in the middle and right panels represent the regulator branes which we set to fix the computational domain for numerical simulations in Ch. 5 and Ch. 6.

On the other hand, the coordinate with \( K = 0 \) denoted by \( (\tau, r_0) \) covers the triangle region of the cross-section diagram of the cylinder shown in the middle panel in Fig. 3.2 \[38\]. Let us move to another coordinate, called \textit{Poincaré coordinate}, which can be done by

\[
r_0 = \frac{\ell^2}{z}.
\]

This coordinate system is suitable for the numerical simulations since a spatial hypersurface at a given time is definitely spacelike. We will engage in a discussion of tensor perturbations with this coordinate in Ch. 5. The Poincaré coordinate provides a very simple form of the line element:

\[
ds^2 = \left(\frac{\ell}{z}\right)^2 (-d\tau^2 + dx^2 + dz^2).
\]

This coordinate system covers the same region as \( (\tau, r_0) \) coordinate covers. The position of the brane is described as

\[
z = z_b(t),
\]
where $t$ denotes the proper time on the brane. In order to explicitly see the induced metric on the brane, we also change the time coordinate $\tau$ to the proper time $t$:

$$\tau = T(t).$$

(3.17)

Thus the induced metric becomes

$$ds_b^2 = \left( \frac{\ell}{z_b} \right)^2 \left\{ -dt^2 + d\mathbf{x}^2 \right\},$$

(3.18)

where

$$\dot{T} = \sqrt{\frac{z_b^2}{\ell^2} + \dot{z}_b^2} = \frac{\sqrt{1 + H^2 \ell^2}}{a_b},$$

(3.19)

where $H$ denotes the Hubble parameter defined by $H = \dot{a}_b/a_b$. We can identify the factor $(\ell/z_b)^2$ in Eq. (3.15) to the squared scale factor on the brane, namely,

$$a_b(t) = \frac{\ell}{z_b}.$$

(3.20)

Later we derive the five-dimensional version of the Friedmann equation [see Eq. (3.60)]. In this case, the Friedmann equation can be regarded as an equation determining the position of brane.

We use Gaussian–normal coordinates $(t, y)$ to describe the formalisms for metric perturbations in Ch. 4 and numerical simulations of scalar perturbations in Ch. 6. These coordinates are suitable especially for observing the behaviour of gravity near the brane. Slices with $t = \text{const}$ and $y = \text{const}$ in the case of $K = C = 0$ are shown in the right panel of Fig. 3.2. While this coordinates also cover the triangle region shown in Fig. 3.2, a coordinate singularity appears in the bulk at $y = y_c$, which coincides with the past Cauchy horizon shown in the right panel of Fig. 3.2. A spatial hypersurface of a given time becomes null towards the singularity. The latter feature may be awkward sometimes for numerical simulations. In our previous study on the tensor perturbations [33], we must introduce a regulator brane near the singularity, depicted by the long-dashed line, to avoid the influence of it. Hence our study was limited to slightly low-energy scales. This is partially because the effects of the boundary condition imposed on the regulator brane propagates quickly towards the physical brane. In the study on the scalar perturbations which we will discuss in Ch. 6, the numerical simulations with the Gaussian–normal coordinates work well. The details on those calculations will appear in Sec. 6.3.

The following subsections are devoted to derive background equations and those solutions with Gaussian–normal coordinates.

### 3.2.3 Effective Einstein equations

The effective Einstein equation is useful to clarify the difference between the RS model and the standard four-dimensional Einstein gravity [118].
We denote the unit normal vector to the subspace $M$ by $n^A$. The induced metric on the subspace can be written by

$$g_{AB} = (5) g_{AB} - n_A n_B.$$  

(3.21)

Varying the action (3.6) with respect to the metric, we obtain a balance equation between the surface term of the Einstein–Hilbert action and the last term of Eq. (3.6). The junction condition on a domain wall in the bulk spacetime is firstly discussed by W. Israel, which is given by [127, 128]

$$[K^\mu, \nu] = K^{\mu+\nu} - K^{\mu-\nu} = -\kappa_5^2 \left( T^{\mu, \nu} - \frac{1}{3} g^{\mu, \nu} (T - \lambda) \right),$$  

(3.22)

where $K_{\mu\nu}$ is the extrinsic curvature of the brane given by

$$K_{\mu\nu} = g_{D, C} g_{D, D} \nabla_C n_D,$$  

(3.23)

where $^{(5)}\nabla_C$ denotes the five-dimensional covariant derivative. The brane embedded in the bulk is bending due to the existence of matter content on the brane. The junction condition (3.22) indicates the extent of the bending.

Throughout this thesis, we consider only the case that the $Z_2$ symmetry (or mirror symmetry) is imposed on the brane, namely, the brane is located at the edge of an orbifold [129]. This enforces all quantities in the bulk to satisfy $X(y = -0) = X(y = +0)$. This symmetry leads to $[X] = 0$ for all continuous function in the bulk, and for their $y$-derivatives to $[X'] = 2X'(y = +0)$. Hence we obtain

$$K^{\mu+\nu} = -K^{\mu-\nu} = -\frac{\kappa_5^2}{2} \left( T^{\mu, \nu} - \frac{1}{3} g^{\mu, \nu} (T - \lambda) \right).$$  

(3.24)

Here we consider the projection of the Einstein equation (3.7) onto the brane ($4 + 1$ decomposition). According to Ref. [118], we can decompose the Einstein equation into the Gauss–Codazzi equations. The Codazzi equation for the vacuum bulk spacetime [118]

$$\nabla_{\mu} K^{\mu, \nu} - \nabla_{\nu} K = 0,$$  

(3.25)

recovers the local energy–momentum conservation on the brane,

$$\nabla_{\mu} T^{\mu, \nu} = 0,$$  

(3.26)

where $\nabla_{\mu}$ denotes the four-dimensional covariant derivative. On the other hand, the Gauss equation yields the effective Einstein equation,

$$(4) G_{\mu\nu} + \Lambda_{4} g_{\mu\nu} = \kappa_4^2 T_{\mu\nu} + \kappa_4^2 \Pi_{\mu\nu} - E_{\mu\nu},$$  

(3.27)

where $\kappa_4^2 = 8\pi / M_{P1}^2$ and $E_{\mu\nu}$ is the projection of the 5D Weyl tensor,

$$E_{\mu\nu} = (5) C_{ABCD} n_{A} n_{B} g_{C} g_{D},$$  

(3.28)
and $\Pi_{\mu\nu}$ is the quadratic terms of the energy–momentum tensor,

$$
\Pi_{\mu\nu} = -\frac{1}{4} T_{\mu\alpha} T^\alpha_{\nu} + \frac{1}{12} T^\alpha_{\alpha} T_{\mu\nu} + \frac{1}{8} g_{\mu\nu} T_{\alpha\beta} T^{\alpha\beta} - \frac{1}{24} g_{\mu\nu} (T^\alpha_{\alpha})^2. \tag{3.29}
$$

Here we define

$$
\Lambda_4 = \frac{1}{2} \Lambda_5 + \frac{\kappa_4^4}{12} \lambda^2, \tag{3.30}
$$

$$
\kappa_4^2 \equiv \frac{8\pi}{M_{\text{pl}}} = \frac{\kappa_5^4}{6} \lambda. \tag{3.31}
$$

In the effective Einstein equation (3.27), we can identify two extra terms, $\Pi_{\mu\nu}$ and $E_{\mu\nu}$. One can expect that the former term becomes dominant in the high-energy regime on the brane because $\Pi_{\mu\nu}$ is a quadrature of the energy–momentum tensor, $\Pi_{\mu\nu} \propto (T_{\mu\nu})^2$. In fact, the high-energy correction to the Friedmann equation, which we will derive later in Eq. (3.60), comes from this term. The latter extra term, $E_{\mu\nu}$, represents a contribution from the bulk gravity for the on-brane observers. This plays an important role for the cosmological perturbations in the braneworld scenario, which will be discussed in Ch. 4 and Ch. 6. In addition, contracting the effective Einstein equation (3.27) with $\nabla_\mu$ and using the Bianchi identity, $\nabla_\mu (G_{\mu\nu}) = 0$, and the energy conservation law (3.26), we obtain

$$
\kappa_5^3 \nabla_\mu \Pi_{\mu\nu} = \nabla_\mu E_{\mu\nu}. \tag{3.32}
$$

This equation indicates that the matter on the brane can affect the bulk spacetime.

In the Gaussian–normal coordinates, it is easy to explicitly write down all components of the effective Einstein equation (3.27). The induced metric on the brane is given by

$$
ds^2 = -dt^2 + a_b(t)^2 \delta_{ij} dx^i dx^j. \tag{3.33}
$$

From this, the four-dimensional Einstein tensor on the brane, $^{(4)} G_{\mu\nu}$, becomes

$$
^{(4)} G^0_0 = -3 \left( \frac{\dot{a}_b}{a_b} \right), \tag{3.34}
$$

$$
^{(4)} G^i_j = -\frac{2}{a_b^2} \left( \frac{\dot{a}_b}{a_b} \right)^2 \delta_{ij}. \tag{3.35}
$$

The quadratic energy–momentum tensor (3.29) is calculated by means of Eqs. (2.5) and (2.6),

$$
\Pi^0_0 = -\frac{1}{12} \rho^2, \tag{3.36}
$$

$$
\Pi^i_j = \frac{1}{12} \rho (\rho + p) \delta^i_j. \tag{3.37}
$$
In the Gaussian–normal coordinate, the projected Weyl tensor (3.28) can be easily calculated from $E_{\mu\nu} = C^5_{\mu\nu}$, which is explicitly,

$$E^0_0 = \frac{1}{2} \left( \frac{d^n_t}{a_b} + \frac{1}{6} \Lambda_5 \right), \quad (3.38)$$

$$E^i_j = -\frac{1}{6} \left( \frac{d^n_t}{a_b} + \frac{1}{6} \Lambda_5 \right) \delta^i_j. \quad (3.39)$$

This result indicates that the projected Weyl tensor, $-E_{\mu\nu}$, behaves an energy–momentum tensor for radiation. It corresponds to Eqs. (2.5) and (2.6) with $w = 1/3$ in the equation-of-state (2.174). This feature can be seen in the Friedmann equation (3.60) in which the extra term proportional to $C$ evolves as radiation. Following this fact, we usually parameterise the projected Weyl tensor as a perfect fluid of radiation:

$$-E_{00} = -\rho\varepsilon, \quad (3.40)$$

$$-E_{ij} = p\varepsilon, \quad (3.41)$$

where $\rho\varepsilon$ and $p\varepsilon$ satisfy

$$p\varepsilon = \frac{1}{3} \rho\varepsilon. \quad (3.42)$$

This fluid is called the Weyl fluid.

### 3.3 Cosmological setup based on RSII model

#### 3.3.1 Friedmann equation

Next, we introduce a coordinate system based on the position of the brane, namely, the Gaussian–normal coordinates $(t, x, y)$. The metric ansatz in the bulk is given by

$$ds^2 = -n^2(t, y)dt^2 + a^2(t, y)\gamma_{ij}dx^idx^j + b^2(t, y)dy^2, \quad (3.43)$$

where $\gamma_{ij}$ is a maximally symmetric three-dimensional metric and the brane is located at $y = 0$. We choose the normalisation of the lapse function as

$$n(t, 0) = 1, \quad (3.44)$$
without loss of generality. Substituting the ansatz (3.43) into Eq. (3.7), we obtain [121,122]

\begin{equation}
(5)G^0_0 = -\frac{3}{n^2} \left[ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) - n^2 \left\{ \frac{a''}{a} + \frac{a'}{a} \left( \frac{a'}{a} - \frac{\dot{b}}{b} \right) \right\} + K \frac{n^2}{a^2} \right],
\end{equation}

(3.45)

\begin{equation}
(5)G^0_5 = -\frac{3}{n^2} \left( \frac{n'}{n} \frac{a'}{a} + \frac{\dot{b}}{b} - \frac{\dot{a}}{a} \right),
\end{equation}

(3.46)

\begin{equation}
(5)G^5_5 = \frac{3}{b^2} \left[ \frac{a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - \frac{b^2}{n^2} \left\{ \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \right\} - K \frac{b^2}{a^2} \right],
\end{equation}

(3.47)

\begin{equation}
(5)G^i_j = \frac{3}{b^2} \left[ \frac{a^2}{a} \left( \frac{a'}{a} + 2 \frac{n'}{n} \right) - \frac{b^2}{n^2} \left\{ \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{n'}{n} \right) \right\} + 2 \frac{a''}{a} + \frac{n'}{n} \right]
+ \frac{a^2}{n^2} \left( \frac{\ddot{a}}{a} - \frac{\dot{a}}{a} \right) + 2 \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \left( -2 \frac{\dot{a}}{a} + \frac{n'}{n} \right) - \frac{\ddot{b}}{b} \right) - K \right] \delta^i_j,
\end{equation}

(3.48)

where a prime and a dot represent derivatives with respect to \( y \) and \( t \), respectively. \( K \) denotes the parametrised spatial curvature on the brane. Its value \( K = +1, 0, -1 \) corresponds to a spherical, a flat and a hyperbolic universe, respectively.

One can find that a specific combination of \((5)G^0_0\) and \((5)G^0_5\) is written by a \( y \)-derivative of a function \( F \) as

\begin{equation}
\frac{2a'a^3}{3} \left( (5)G^0_0 - \frac{\dot{a}}{a'} (5)G^0_5 \right) = F',
\end{equation}

(3.49)

where \( F(t, y) \) is defined by

\begin{equation}
F(t, y) = \frac{(a'a)^2}{b^2} - \frac{(\dot{a}a)^2}{n^2} - Ka^2.
\end{equation}

(3.50)

From the Einstein equations (3.7),

\begin{equation}
(5)G^0_0 = -\Lambda_5,
\end{equation}

(3.51)

\begin{equation}
(5)G^0_5 = 0,
\end{equation}

(3.52)

the integration of Eq.(3.49) with respect to \( y \) yields a background solution,

\begin{equation}
F + \frac{\Lambda_5}{6} a^4 + C(t) = 0,
\end{equation}

(3.53)

where \( C(t) \) denotes an integration constant. In the same way, one can find

\begin{equation}
F' = \frac{2\ddot{a}a^3}{3} (5)G^5_5 = -\frac{2\ddot{a}a^3}{3} \Lambda_5,
\end{equation}

(3.54)

which implies that \( C(t) \) does not depend on time.

Thus we obtain an explicit form of (3.53),

\begin{equation}
\left( \frac{\dot{a}}{na} \right)^2 = \frac{\Lambda_5}{6} + \left( \frac{a'}{ba} \right)^2 - \frac{K}{a^2} + \frac{C}{a^4},
\end{equation}

(3.55)
The extrinsic curvature (3.23) is calculated from Eq. (3.24):

\[ K^0_0 = \frac{\nu'_b}{n_b} = \frac{\kappa^2_5}{2} \left( \frac{2}{3} \rho + p - \frac{1}{3} \lambda \right), \]  

\[ K^i_j = \frac{\alpha'_b \delta^i_j}{a_b} = -\frac{\kappa^2_5}{6} (\rho + \lambda). \]  

where the subscript ‘b’ means that the value is evaluated at the brane \((y = 0)\). Then, evaluating Eq.(3.53) at \(y = +0\) and using Eq.(3.57), we obtain the Friedmann equation,

\[ H^2 = \frac{\kappa^4_4}{3} \rho \left( 1 + \frac{\rho}{2\lambda} \right) + \frac{\Lambda_4}{3} - \frac{K}{a^2_b} + \frac{C}{a^4_b}. \]  

Here we have used the relation (3.31) to rewrite \(\kappa^4_5\). The last term with the integration constant \(C\) behaves like a radiation behaving as \(a^{-4}\), which is called dark radiation. This term originated with the Weyl tensor in the effective Einstein equation (3.27), and can be interpreted into the mass of the bulk black hole if the bulk spacetime is described by Schwarzschild–AdS spacetime which is a general solution of the Einstein equation (3.7).

Its observational constraint from the Big Bang Nucleosynthesis (BBN) has been studied by K. Ichiki et al. [130]. In this thesis, we focus only on the AdS bulk, assuming no dark radiation, \(C = 0\). Moreover, we consider the spatially flat universe, \(K = 0\), and the four-dimensional cosmological constant \(\Lambda_4\) can be also negligible which follows with Eq.(3.30),

\[ \Lambda_5 = -\frac{\kappa^4_5}{6} \lambda^2. \]  

In this case, the Friedmann equation (3.58) is simplified,

\[ H^2 = \frac{\kappa^4_4}{3} \rho \left( 1 + \frac{\rho}{2\lambda} \right). \]  

The brane tension \(\lambda\) is constrained by abundances of the light elements produced during BBN. If the tension is small enough for the \(\rho^2\)-term to dominate in the Friedmann equations during BBN, the expansion rate of the universe is significantly modified, leading to anomalous abundances. Hence the tension should be \(\lambda \gtrsim (1\text{MeV})^4\). A tighter constraint is obtained from current tests of the Newton’s law [114,115], which yields \(\ell < 0.1\text{ mm}\). Using the relation (3.65), \(\lambda > (100\text{GeV})^4\) [111].

Additionally, the energy conservation on the brane can be derived from 05-component of the Einstein equation (3.52) with the Israel conditions (3.56) and (3.57). The explicit form is the same as four-dimensional one (2.9).

### 3.3.2 Bulk solutions

Next, let us consider solutions of the Einstein equations (3.7) in the bulk. To solve the Einstein equation with the metric ansatz (3.43), we assume that

\[ b(t, y) = 1, \]  

(3.61)
without loss of generality [121,122]. From Eq. (3.44) and (3.52), we find
\[ n(t, y) = \frac{\dot{a}}{a_b}, \]  
(3.62)
where \( a_b = a(t, 0) \) is identified to be the scale factor on the brane. Using Eq.(3.62), the 00-component of the Einstein equation (3.51) can be reduced to
\[ (a^2)'' - \frac{2|\Lambda_5|}{3}a^2 = 2\dot{a}_b^2. \]  
(3.63)
This equation has a solution,
\[ a^2(t, y) = C_1(t) \sinh 2\mu y + C_2(t) \cosh 2\mu y - \frac{\dot{a}_b^2}{2\mu^2}, \]  
(3.64)
where we defined
\[ \mu = \sqrt{\frac{|\Lambda_5|}{6}} = \frac{\kappa_2^2}{6} = \frac{\kappa_4^2}{\kappa_5^2}. \]  
(3.65)
Note that the second equality comes from Eq. (3.59). Differentiating Eq. (3.64) with respect to \( y \) and evaluating it at \( y = 0 \) with the Israel condition (3.57), we find
\[ C_1(t) = -\frac{\kappa_5^2}{6\mu}(\rho + \lambda)a_b^2 = -a_b^2 \left(1 + \frac{\rho}{\lambda}\right), \]  
(3.66)
\[ C_2(t) = \left(1 + \frac{H^2}{2\mu^2}\right)a_b^2. \]  
(3.67)
To compute the square root of Eq.(3.64), we take an ansatz
\[ a(t, y) = a_b(t)(B_1 \cosh \mu|y| + B_2 \sinh \mu|y|). \]  
(3.68)
To get a physically plausible solution, the scale factor should be positive by definition, \( a_b = a(t, 0) > 0 \), and the \( y \)-derivative of the warp factor at the brane be negative by the Israel condition (3.57). From these conditions, we obtain
\[ B_1(t) = 1, \]  
(3.69)
\[ B_2(t) = -\sqrt{1 + \frac{H^2}{\mu^2}}. \]  
(3.70)
In particular, if \( K = C = 0 \), the resultant solution is
\[ a(t, y) = a_b(t) \left\{ \cosh \mu|y| - \left(1 + \frac{\rho}{\lambda}\right) \sinh \mu|y| \right\}. \]  
(3.71)
The lapse function is calculated from Eqs. (3.62) and (2.9) as
\[ n(t, y) = \cosh \mu|y| - \left(1 - \frac{2\rho + 3p}{\lambda}\right) \sinh \mu|y|. \]  
(3.72)
The Gaussian–normal coordinate is suitable to describe the five dimensional phenomena near the brane since the position of the brane is fixed as \( y = 0 \). Note, however, that this coordinate has a coordinate singularity \( y = y_c \) in the bulk,

\[
y_c = \frac{1}{\mu} \coth^{-1} \left( 1 + \frac{\rho}{\lambda} \right),
\]

where the warp factor vanishes, \( a(t, y_c) = 0 \). In the case of de Sitter universe, \( y = y_c \) corresponds to the Cauchy horizon in the bulk spacetime \([32, 131]\). On the other hand, if the universe is described by the conventional Friedmann model, the coordinate singularity is identical to the horizon of \( \text{AdS}_5 \) spacetime \([125]\). Note that, while the warp factor becomes zero at \( y = y_c \), the lapse function remains finite. This is supported by the fact that the spatial hypersurface \((t = \text{const.})\) becomes a null hypersurface towards the coordinate singularity.

Finally, for later convenience, we list useful expressions for derivatives of the warp factor and the lapse function evaluated at the brane:

\[
\begin{align*}
\dot{a} \bigg|_{ab} &= \dot{a}_b \equiv H(t), \\
\dot{a} \bigg|_{ab} &= \dot{H}, \\
\frac{a'}{a} \bigg|_{ab} &= -\mu \sqrt{1 + \frac{H^2}{\mu^2}} \equiv \mathcal{H}(t), \\
\frac{n'}{n} \bigg|_{ab} &= \mathcal{H} + \frac{\dot{H}}{\mathcal{H}}, \\
\frac{a''}{a} \bigg|_{ab} &= \mu^2, \\
\frac{n''}{n} \bigg|_{ab} &= \mu^2,
\end{align*}
\]

### 3.4 Cosmological solutions

#### 3.4.1 Perfect fluid on the brane

In this section, we derive solutions of the Friedmann equation (3.60). To begin with, we consider the perfect fluid confined to the brane. If we assume the simplest form of the equation-of-state,

\[
p = w\rho,
\]

the Friedmann equation (3.60) and the energy conservation law (2.9) yield general solutions of the scale factor and the energy density normalised by the tension,

\[
\epsilon \equiv \rho(t)/\lambda,
\]

\[
\begin{align*}
a_b(t) &\propto e^{-1/3(1+w)}, \\
\epsilon(t) &= \frac{2}{\{3(1+w)t/\ell + 1\}^2 - 1}.
\end{align*}
\]
Particularly, in cases with $w = 0$ (matter dominant), $w = 1/3$ (radiation dominant) and $w = 1$, the energy density becomes

\[
\epsilon(t) = \begin{cases} 
\frac{2\ell^2}{9t^2 + 6t} & \text{for } w = 0, \\
\frac{8t^2 + 4t}{t^2} & \text{for } w = 1/3, \\
\frac{18t^2 + 6t}{t^2} & \text{for } w = 1,
\end{cases}
\]

respectively. The time dependence of the scale factor changes at the critical energy density defined by

\[
\epsilon_{\text{crit}} = \sqrt{2} - 1.
\]

The dependency can be written as

\[
a_b(t) \propto \begin{cases} 
t^{2/3}(\epsilon \ll \epsilon_{\text{crit}}), t^{1/3}(\epsilon \gg \epsilon_{\text{crit}}) & \text{for } w = 0 \\
t^{1/2}, t^{1/4} & \text{for } w = 1/3, \\
t^{1/3}, t^{1/6} & \text{for } w = 1
\end{cases}
\]

### 3.4.2 Slow-roll inflation on the brane

Also in the braneworld picture, inflation can be realised by introducing scalar fields either on the brane or in the bulk. The latter case is called *bulk inflaton models* [132–138], in which the scalar field in the bulk mimics the four-dimensional scalar field driving inflation on the brane. In this thesis, we do not deal with the bulk inflaton model. Instead, we consider inflation driven by scalar fields confined on the brane. We refer to it as *brane inflation model*. In this section, we discuss the inflation dynamics in the case of a single scalar field on the brane.

We consider the situation that the energy on the brane is dominated by a scalar field $\phi$, in which the energy and pressure are described as Eqs. (2.19) and (2.20). In the four-dimensional theory, the scalar field drives the accelerating expansion when the condition (2.16) is satisfied. In the five-dimensional case, the condition is modified to a stronger condition by the existence of the tension of the brane [139]:

\[
p < -\left(\frac{\lambda + 2\rho}{\lambda + \rho}\right)\frac{\rho}{3},
\]

implying that $p < -2\rho/3$ if $\rho \gg \lambda$. This condition can be reduced to

\[
\dot{\phi}^2 - V + \frac{\dot{\phi}^2 + 2V}{8\lambda}(5\dot{\phi}^2 - 2V) < 0,
\]

which corresponds to the condition $\dot{\phi}^2 < V$ in the four-dimensional cases. With this slow-roll condition, the Friedmann equation and the Klein-Goldon equation for the scalar field
become

\begin{align}
H^2 &\simeq \frac{8\pi}{3M_{\text{Pl}}^2} V \left(1 + \frac{V}{2\lambda}\right), \quad (3.86) \\
\dot{\phi} &\approx -\frac{V'}{3H}, \quad (2.28)
\end{align}

where we also assume $|\ddot{\phi}| \ll |H\dot{\phi}|$ [see Eq. (2.37)]. The correction term containing the tension enhances the value of the Hubble parameter relative to the standard four-dimensional model of the same energy. This introduces additional friction on the scalar field, thereby enabling a steeper class of potentials to support inflation [140].

The slow-roll parameters are redefined by [see also Eqs. (2.24) and (2.25)] [139],

\begin{align}
\epsilon_s &\equiv \frac{M_{\text{Pl}}^2}{16\pi} \left(\frac{V'}{V}\right)^2 \left\{\frac{2\lambda(2\lambda + 2V)}{(2\lambda + V)^2}\right\}, \quad (3.87) \\
\eta_s &\equiv \frac{M_{\text{Pl}}^2}{8\pi} \left(\frac{V''}{V}\right) \frac{2\lambda}{2\lambda + V}. \quad (3.88)
\end{align}

The number of e-folds (2.30) turns to

\begin{equation}
N \approx -\frac{M_{\text{Pl}}^2}{8\pi} \int \frac{V}{V'} \left(1 + \frac{V}{2\lambda}\right) d\phi. \quad (3.89)
\end{equation}

Let us reconsider the chaotic inflation discussed in Sec. 2.4.1. The inflaton field at the end of inflation is determined from Eq. (3.87) as $\epsilon_s \approx 1$ which yields in the high energy limit $V \gg \lambda$,

$$\phi_{\text{end}} = \left(\frac{2M_{\text{Pl}}^2\lambda}{\pi m^2}\right)^{1/4}. \quad (3.90)$$

The e-folding number (3.89) yields the inflaton field at the beginning of inflation in the high energy limit,

$$\phi_{\text{init}} \approx \left(\frac{4M_{\text{Pl}}^2\lambda}{\pi m^2 N_{\text{total}}}\right)^{1/4}. \quad (3.91)$$

With the same e-folding number as one used in the four-dimensional case, $N_{\text{total}} = 60$, the ratio of the initial values of four-dimensional case (2.156) and five-dimensional case (3.91) becomes

$$\frac{\phi_{\text{init}}^{\text{5D}}}{\phi_{\text{init}}^{\text{4D}}} \approx 10^{-12} \left(\frac{m}{10^{-6} M_{\text{Pl}}}\right)^{-1/2} \left(\frac{\lambda^{1/4}}{1\text{TeV}}\right)^{1/4} \left(\frac{N_{\text{total}}}{60}\right)^{-1/4}, \quad (3.92)$$

which has possibilities that the energy scale of the inflaton field may be reduced, while the four-dimensional case requires the super-Planckian physics for the initial inflaton field (2.156).
3.4.3 Hawkins–Lidsey model

Instead of using the slow-roll approximation, we can systematically construct the exact background solutions with a single scalar field, which describes the accelerated expansion on the brane. In Ch. 6 in which we will discuss the evolution of inflaton perturbations during the inflation, we use the exact solution as a background solution of inflation instead of solving the background equation numerically.

There exists an analogous formalism to the Hamilton–Jacobi formalism discussed in Sec. 2.2 \[141\]. To proceed, we define a new function \( y(\phi) \):

\[
\rho(\phi) = 2\lambda \frac{y^2}{1 - y^2}, \tag{3.93}
\]

where the restriction \( y^2 < 1 \) must be imposed for the weak energy condition to be satisfied. From Eq.(3.60), the Hubble parameter becomes

\[
H(\phi) = \sqrt{\frac{2\kappa_4^2\lambda}{3}} \frac{y}{1 - y^2}. \tag{3.94}
\]

At low-energy limit, \( y \to 0 \), the right-hand side implies that \( y \) is proportional to \( H(\phi) \). Substituting Eq.(3.94) into Eq.(2.34), we obtain

\[
\dot{\phi} = -\sqrt{\frac{8\lambda}{3\kappa_4^2} \frac{y(\phi)}{1 - y^2}}, \tag{3.95}
\]

and the potential of the scalar field is given by Eq. (2.19) with Eqs. (3.93) and (3.95) :

\[
V(\phi) = 2\lambda \frac{y}{1 - y^2} - \frac{4\lambda}{3\kappa_4^2} \left( \frac{y'}{1 - y^2} \right)^2. \tag{3.96}
\]

Here we rewrite the time-derivative of the scale factor, \( a_b'(t) \), into \( a_b'(\phi) \), then it follows from Eqs.(3.93) and (3.95) that the scale factor, \( a_b(\phi) \), satisfies

\[
y'(\phi)a_b'(\phi) = \frac{\kappa_4^2}{2} ya_b. \tag{3.97}
\]

Integrating this by \( \phi \), the scale factor can be expressed as

\[
a_b(\phi) = \exp \left( -\frac{\kappa_4^2}{2} \int^\phi y \frac{d\phi}{y'} \right). \tag{3.98}
\]

Finally, the cosmic time is evaluated from Eq.(3.95) :

\[
t - t_0 = \sqrt{\frac{3\kappa_4^2}{8\lambda}} \int^\phi y^2 - 1 \frac{d\phi}{y'}, \tag{3.99}
\]

where \( t_0 \) is an arbitrary constant. Thus, we saw that, when the functional form of \( y(\phi) \) is known, the inflation dynamics with a single scalar field is completely specified.
Here we consider the case of

\[ y = \text{sech} \left( \frac{\kappa_4 C}{2} \phi \right), \]  

(3.100)

where \( C \) is an arbitrary constant. From Eqs.(3.93)–(3.99), the physical quantities related to the inflation dynamics can be calculated in an analytic way:

\[ \rho(\phi) = 2\lambda \text{cosech}^2 \left( \frac{\kappa_4 C}{2} \phi \right), \]  

(3.101)

\[ H(\phi) = \sqrt{\frac{2\kappa_4^2 \lambda}{3}} \text{coth} \left( \frac{\kappa_4 C}{2} \phi \right) \text{cosech} \left( \frac{\kappa_4 C}{2} \phi \right), \]  

(3.102)

\[ \dot{\phi} = \sqrt{\frac{2C^2 \lambda}{3}} \text{cosech} \left( \frac{\kappa_4 C}{2} \phi \right), \]  

(3.103)

\[ V(\phi) = \frac{6 - C^2}{3} \lambda \text{cosech}^2 \left( \frac{\kappa_4 C}{2} \phi \right), \]  

(3.104)

\[ a_\theta(\phi) = \left[ \sinh \left( \frac{\kappa_4 C}{2} \phi \right) \right]^{2/C^2}, \]  

(3.105)

\[ t(\phi) - t_0 = \frac{1}{\kappa_4} \sqrt{\frac{6}{\kappa_4^2 \lambda}} \cosh \left( \frac{\kappa_4 C}{2} \phi \right). \]  

(3.106)

The functional form of potential requires \( C^2 < 6 \) to be positive definitive. Furthermore, the pressure is determined from Eq.(2.20) with Eqs. (3.95) and (3.96):

\[ p(\phi) = \frac{2}{3}(C^2 - 3\lambda) \text{ cosech}^2 \left( \frac{\kappa_4 C}{2} \phi \right). \]  

(3.107)

For later convenience, we rewrite these quantities in terms of the cosmic time. First of all, the combination of Eqs.(3.105) and (3.106) yields

\[ a_\theta(t) = \left[ \frac{\kappa_4^2 \lambda C^4}{6} (t - t_0)^2 - 1 \right]^{1/C^2}, \]  

(3.108)

which is the scale factor during a so-called power-law inflation. Without loss of generality, we choose

\[ t_0 = -\sqrt{\frac{6}{\kappa_4^2 \lambda C^4}}, \]  

(3.109)

for the scale factor to be zero at \( t = 0 \). With this treatment, Eqs.(3.100), (3.101)–(3.105)
and (3.107) become

\begin{align}
\dot{y}(t) &= \frac{1}{C^2 \mu t + 1}, \\
\rho(t) &= \frac{2\lambda}{(C^2 \mu t + 1)^2 - 1}, \\
H(t) &= \frac{2\mu (C^2 \mu t + 1)}{(C^2 \mu t + 1)^2 - 1}, \\
\dot{\phi}(t) &= \sqrt{\frac{2C^2 \lambda}{3} \left[ (C^2 \mu t + 1)^2 - 1 \right]^{-1/2}}, \\
V(t) &= \frac{\lambda}{3} (6 - C^2) \left[ (C^2 \mu t + 1)^2 - 1 \right]^{-1}, \\
\alpha(t) &= \left[ (C^2 \mu t + 1)^2 - 1 \right]^{1/2}, \\
\rho(t) &= \frac{2\lambda}{3} (C^2 - 3) \left[ (C^2 \mu t + 1)^2 - 1 \right]^{-1},
\end{align}

where we used Eq.(3.65). We found from Eqs.(3.111) and (3.116) that the scalar field behaves as a perfect fluid with an effective equation of state, \( p = w \rho \), where the index is given by

\begin{equation}
w = \frac{C^2}{3} - 1.
\end{equation}
Chapter 4

Cosmological Perturbations in Braneworld

In this chapter, we shall discuss the five-dimensional cosmological perturbations based on Ref. [77, 142]. We shall first mention the metric perturbations (Sec. 4.1) and their evolution equations rewritten by the master variables (Sec. 4.2). Next, we move on to the topic on the junction conditions, which relates the metric perturbations on the brane with the matter perturbations confined to the brane. In Sec. 4.3, we shall mention the behaviour of the brane bending perturbatively produced by the anisotropic stress on the brane. In Sec. 4.4), from the point of view of the on-brane observers, the junction conditions are derived from the effective Einstein equations. Then we shall devote Sec. 4.5 to discuss the matter perturbations and its evolution equation on the brane. Finally, we mention the difficulty of the quantisation of the scalar perturbations, and demonstrate that of the tensor perturbations in Sec. 4.6.

4.1 Metric perturbations and their gauge transformation

The five-dimensional metric tensor $g_{AB}$ has three scalars ($g_{00}$, $g_{05}$, $g_{55}$), two 3-vectors ($g_0^i$, $g_5^i$), and one 3-tensor ($g_{ij}$). Decomposing them into scalar/vector/tensor variables as we discussed in Sec. 2.3.2, the general form of the metric perturbations in the flat universe ($K = 0$) using the Gaussian–normal coordinates can be written as [77]

$$
(5) g_{AB} = \begin{pmatrix}
-n^2(1 + 2A) & a^2(B_i - S_i) & nA_y \\
\frac{a^2(B_i - S_i)}{nA_y} & a^2[(1 + 2R)\delta_{ij} + 2E_{ij} + F_{i,j} + F_{j,i} + h_{ij}] & a^2(B_{yi} - S_{yi}) \\
\frac{a^2(B_{yi} - S_{yi})}{nA_y} & 1 + 2A_{yy} & nA_y
\end{pmatrix},
$$

where $A, B, R, E, A_y$ and $A_{yy}$ are scalar quantities (each has 1 degree of freedom), $S_i, F_i$ and $S_{yi}$ are divergence-free 3-vectors (each has 2 degrees of freedom), and $h_{ij}$ is a transverse and
traceless 3-tensor (has 2 degrees of freedom). These perturbed quantities have 15 degrees of freedom, which means these have 5 degrees of gauge freedom. In fact, when we transform the coordinates perturbatively, \( x^A \rightarrow \tilde{x}^A = x^A + \xi^A \), or explicitly,

\[
\begin{align*}
t \rightarrow \tilde{t} &= t + \delta t, \\
x^i \rightarrow \tilde{x}^i &= x^i + \delta x^i + \delta x^i, \\
y \rightarrow \tilde{y} &= y + \delta y,
\end{align*}
\]

the metric perturbations become

\[
\tilde{g}_{AB} = g_{AB} - g_{AC} \xi^C ,B - g_{BC} \xi^C ,A g_{AB,C} \xi^C
\]

### 4.1.1 scalar perturbations

The perturbations of the metric are transformed as

\[
\begin{align*}
A \rightarrow \tilde{A} &= A - \dot{\delta t} - \frac{\hat{n}}{n} \delta t - \frac{n'}{n} \delta y, \\
R \rightarrow \tilde{R} &= R - \frac{\dot{a}}{a} \delta t - \frac{a'}{a} \delta y, \\
A_y \rightarrow \tilde{A}_y &= A_y + n \delta t' - \frac{1}{n} \delta y, \\
A_{yy} \rightarrow \tilde{A}_{yy} &= A_{yy} - \delta y', \\
B \rightarrow \tilde{B} &= B + \frac{n^2}{a^2} \delta t - \dot{\delta x}, \\
B_y \rightarrow \tilde{B}_y &= B_y - \delta x' - \frac{1}{a^2} \delta y, \\
E \rightarrow \tilde{E} &= E - \delta x,
\end{align*}
\]

where a dot denotes a derivative with respect to \( t \), and a prime a derivative with respect to \( y \). The true degree of freedom is 7 (perturbed quantities) - 3 (gauge transformation) = 4. Firstly, to eliminate the spatial gauge dependence, \( \delta x \), we introduce the spatially gauge-invariant combinations

\[
\begin{align*}
\sigma &\equiv -B + \dot{\delta x}, \\
\sigma_y &\equiv -B_y + \delta y,
\end{align*}
\]

which can be recognised as the shear of a unit timelike vector projected on the brane, and the shear with respect to a unit spacelike vector orthogonal to the brane, respectively. These variables have temporal and bulk gauge freedom,

\[
\begin{align*}
\sigma \rightarrow \tilde{\sigma} &= \sigma - \frac{n^2}{a^2} \delta t, \\
\sigma_y \rightarrow \tilde{\sigma}_y &= \sigma_y + \frac{1}{a^2} \delta y.
\end{align*}
\]
4.1.2 vector perturbations

The vector perturbations are transformed as

\[ S_i \rightarrow \tilde{S}_i = S_i + \delta x_i , \]
\[ S_{yi} \rightarrow \tilde{S}_{yi} = S_{yi} + \delta x'_i , \]
\[ F_i \rightarrow \tilde{F}_i = F_i - \delta x_i , \]

which have only spatial gauge freedom. Then we introduce the gauge-invariant variables

\[ \tau_i = S_i + \dot{F}_i , \]
\[ \tau_{yi} = S_{yi} + F'_i , \]

and a combination of these quantities is

\[ S = S' - \dot{S}_y = \tau' - \dot{\tau}_y . \]

4.1.3 tensor perturbations

The tensor perturbations \( h_{ij} \) are automatically gauge-invariant:

\[ h_{ij} \rightarrow \tilde{h}_{ij} = h_{ij} . \]

This is because of the absence of tensor components in the gauge transformation.

4.2 Evolution equations of metric perturbations

4.2.1 Scalar part

At this stage, perturbations (4.4)–(4.7) and (2.80)–(4.12) depend on \( \delta y \) and \( \delta t \). In this thesis, we use the 5D-longitudinal gauge for scalar perturbations, which impose additional conditions on the gauge to fully determine the perturbations:

\[ \tilde{\sigma} = \tilde{\sigma}_y = 0. \]

From Eqs. (2.80 and 4.12), we obtain

\[ \delta t = \frac{a^2}{n^2\sigma} , \]
\[ \delta y = -a^2\sigma_y , \]
which define the gauge transformation (4.4)–(4.7). In the 5D-longitudinal gauge, $A, R, A_y$ and $A_{yy}$ given in arbitrary gauges become

\begin{align}
\tilde{A} &= A - \frac{1}{n} \left( \frac{a^2}{n^2 \sigma} \right)' + \frac{n'}{n} (a^2 \sigma_y), \\
\tilde{R} &= R - \frac{a^2}{n^2 \sigma} + \frac{a'}{a} a^2 \sigma_y, \\
\tilde{A}_y &= A_y - n \left( \frac{a^2}{n^2 \sigma} \right)' + \frac{1}{n} (a^2 \sigma_y)' , \\
\tilde{A}_{yy} &= A_{yy} + (a^2 \sigma_y)' .
\end{align}

The perturbed Einstein tensor is given in Eqs. (B.17)–(B.24). From them with the Einstein equation

\begin{equation}
^{(5)} \delta G_{AB} + \Lambda^{(5)} \delta g_{AB} = 0 ,
\end{equation}

we directly obtain the evolution equations for the metric perturbations. In this thesis, we use the formalism in which the bulk metric perturbations are described by a master variable $\Omega$ according to Ref. [143] (also see Ref. [144]). The perturbed variables (4.23)–(4.26) are related by a master variable $\Omega(t,y) :

\begin{align}
\tilde{A} &= -\frac{1}{6a} \left\{ 2\Omega'' - \frac{n'}{n} \Omega' - \mu^2 \Omega + \frac{1}{n^2} \left( \tilde{\Omega} - \frac{n}{n} \tilde{\Omega} \right) \right\} , \\
\tilde{A}_y &= \frac{1}{na} \left( \frac{n'}{n} \Omega' - \frac{n''}{n} \tilde{\Omega} \right) , \\
\tilde{A}_{yy} &= \frac{1}{6a} \left\{ \Omega'' - 2 \frac{n'}{n} \Omega' + \frac{2}{n^2} \left( \tilde{\Omega} - \frac{n}{n} \tilde{\Omega} \right) + \mu^2 \Omega \right\} , \\
\tilde{R} &= \frac{1}{6a} \left\{ \Omega'' + \frac{n'}{n} \Omega' - \frac{1}{n^2} \left( \tilde{\Omega} - \frac{n}{n} \tilde{\Omega} \right) - 2 \mu^2 \Omega \right\} .
\end{align}

From the perturbed Einstein equation (4.27), the master variable obeys a wave equation

\begin{equation}
\left( \frac{1}{na^3} \tilde{\Omega} \right)' - \left( \mu^2 + \frac{\nabla^2}{a^2} \right) \frac{n}{a^3} \Omega - \left( \frac{n}{a^3} \Omega \right)' = 0 .
\end{equation}

Its equivalent form is

\begin{equation}
\tilde{\Omega} - \left( \frac{n'}{n} + 3 \frac{\hat{a}}{a} \right) \Omega - n^2 \left[ \Omega'' + \left( \frac{n'}{n} - 3 \frac{a'}{a} \right) \Omega' + \left( \mu^2 - \frac{k^2}{a^2} \right) \Omega \right] = 0 .
\end{equation}

Note that the spatial off-diagonal components of the perturbed Einstein tensor (B.22) should be zero from the Einstein equation, which yields

\begin{equation}
\tilde{A} + \tilde{R} + \tilde{A}_{yy} = 0 .
\end{equation}

This constraint equation can reduce one degree of freedom for the perturbations, which is due to no matter content in the bulk except for the cosmological constant [see Eq. (4.27)].
4.2.2 Vector part

We decompose the gauge-invariant vector perturbations $\tau_i$ and $\tau_{yi}$ into Fourier modes,

$$\begin{align*}
\tau_i &= \tau(t, y) \hat{e}_i(x^j), \\
\tau_{yi} &= \tau_y(t, y) \hat{e}_i(x^j),
\end{align*}$$

(4.35)

(4.36)

where unit vectors $\hat{e}_i$ satisfies

$$\nabla^2 \hat{e}_i(x^j) = -k^2 \hat{e}_i(x^j).$$

(4.37)

The perturbed Einstein tensor for these perturbations is given in Eqs. (B.25)–(B.28). If the vector perturbations are rewritten by a vector master variable $\Xi(t, y)$:

$$\begin{align*}
\tau &= \frac{n}{a^3} \Xi^2, \\
\tau_y &= \frac{1}{na^3} \Xi,
\end{align*}$$

(4.38)

(4.39)

one can see from the perturbed Einstein equation (4.27) and the perturbed Einstein tensor (B.25)–(B.28) that the master variable is governed by a wave equation:

$$\ddot{\Xi} - \left(\frac{\dot{n} + 3 \frac{\dot{a}}{a}}{n} \right) \frac{\dot{\Xi}}{n} - n^2 \left[ \Xi'' + \left(\frac{n'}{n} - 3 \frac{d^2}{a^2} \right) \Xi' - \frac{k^2}{a^2} \Xi \right] = 0 .$$

(4.40)

4.2.3 Tensor part

In the analogous way for the case of vector perturbations, the tensor perturbations, which are automatically gauge-invariant, can be decomposed into the Fourier modes:

$$h_{ij} = h(t, y) \hat{e}_{ij}(x^m) ,$$

(4.41)

where $\hat{e}_{ij}$ satisfies the transverse-traceless condition and

$$\nabla^2 \hat{e}_{ij}(x^m) = -k^2 \hat{e}_{ij}(x^m).$$

(4.42)

The master equation becomes [131]

$$\ddot{h} + \left(3 \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \frac{\dot{h}}{n} - n^2 \left[ h'' + \left(3 \frac{d^2}{a^2} + \frac{n'}{n} \right) h' - \frac{k^2}{a^2} h \right] = 0 .$$

(4.43)

4.3 Brane bending

In this section, we will see that the anisotropic stress on the brane yields the brane bending. When we numerically treat the tensor perturbations in Ch. 5, however, we will assume no anisotropic stress on the brane for simplicity, and the inflaton perturbations which will be discussed in Ch. 6 do not produce the anisotropic stress. Hence, in this thesis, it is not
necessary to take into account the brane bending. Nevertheless it is important to see how
the brane bending is yielded by the anisotropic stress.

When we deal with the scalar perturbations, we have to consider the location of the
brane being generally perturbed as shown in Fig. 4.1. The location can be written as
\[ y_b = \xi(t, x^i). \tag{4.44} \]
Then the normal vector to the brane can be described as
\[ n_A = (-\xi, 1 + A_{yy}). \tag{4.45} \]
In order to quantify the brane bending in the bulk, let us consider the extrinsic curvature
on the brane defined in Eq. (3.23). The extrinsic curvature with the perturbed energy–
momentum tensor (2.65)–(2.69) yields a junction condition for the bulk metric perturbations
(master variables) through the Israel condition (3.24). The perturbative version of the Israel
condition is given by
\[ \delta K^\mu_\nu = -\frac{\kappa_5^2}{2} \left( \delta T^\mu_\nu - \frac{1}{3} \delta^\mu_\nu \delta T \right), \tag{4.46} \]
where we omit a superscript '+' shown in Eq. (3.24). Each component of the extrinsic
curvature is shown in Appendix B.1. The off-diagonal part of \( \delta K^i_j \) gives
\[ \xi = \frac{\kappa_5^2}{2} a_0^2 \delta \pi^S. \tag{4.47} \]
This result means that the location of the brane is no longer fixed due to the existence of
the anisotropic stress on the brane. Note that we can choose a special gauge in which the
brane location is always fixed, the brane–Gaussian–normal gauge [77, 142]. If we describe
the perturbations in this new gauge, we must remember the transformation between this
and the 5D-longitudinal gauge.

Figure 4.1: Brane bending due to the perturbations in the 5D-longitudinal gauge and the
normal vector to the brane. The solid curve represents the brane located at \( y = \xi(x^i) \). \( n_A \)
denotes the normal vector to the brane.
4.4 Junction conditions

In the section, we discuss the coupling of the matter on the brane to the gravity from the perturbed effective Einstein equation \[118\]

\[
(4.48) \quad \delta G_{\mu\nu} = \kappa_4^2 \delta T_{\mu\nu} + \kappa_3^4 \delta \Pi_{\mu\nu} - \delta E_{\mu\nu},
\]

which yields the junction conditions on the brane. While the junction condition can be more easily derived from the perturbed Israel condition (4.46), the effective Einstein equation gives the explicit contribution from the bulk metric perturbations represented by the projected Weyl tensor, \(E_{\mu\nu}\). We can parameterise the projected Weyl tensor \(\delta E_{\mu\nu}\) as an effective fluid;

\[
\delta E^0_0 = \kappa_4^2 \delta \rho_E, \quad (4.49)
\]

\[
\delta E^0_i = -\kappa_3^2 \delta q_{E,i}, \quad (4.50)
\]

\[
\delta E^i_j = -\kappa_3^4 (\delta p_E \delta^i_j + \delta \pi_E^i_j), \quad (4.51)
\]

where

\[
\delta q_{E,i} = \delta q^S_{E,i} + \delta q^V_{E,i}, \quad (4.52)
\]

\[
\delta \pi^i_j = \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \nabla^2 \right) \delta \pi^S_E + \delta \pi^V_E (\delta^i_j + \delta^j_i) + \delta \pi^T_E \delta^i_j. \quad (4.53)
\]

The Weyl fluid satisfies the radiation-like equation-of-state \(\delta p_E = \delta \rho_E/3\).

4.4.1 Scalar part

We use the 4D-longitudinal gauge on the brane to calculate the scalar perturbations. This gauge has been frequently used in lots of literature \[24,25\]. In this gauge, the perturbation of the induced metric on the brane is given by

\[
(4.54) \quad g_{\mu\nu} = \begin{pmatrix}
-1 & 0 \\
0 & a^2_0 (1 + 2 \bar{\kappa}) \delta_{ij}
\end{pmatrix},
\]

It is important to note that the bulk metric perturbations evaluated on the brane in the 5D-longitudinal gauge, \(\bar{A}_b\) and \(\bar{\mathcal{R}}_b\), do not always coincide with ones in the 4D-longitudinal gauge, \(\bar{A}\) and \(\bar{\mathcal{R}}\). This is because there is a contribution from the perturbatively changed location of the brane. Unless we choose a special gauge, the brane–Gaussian–normal gauge, the location of the brane is generally not fixed by the anisotropic stress on the brane as we discussed in the previous section. The relation between the perturbations defined on the brane and ones defined in the bulk is given by \[145\]

\[
(4.55) \quad g_{AB}^{(4)} = \xi \partial_y^{(5)} g_{AB} + \xi \partial_y^{(5)} g_{AB}.\]
In particular, in the 5D-longitudinal gauge, this relation with (4.47) yields \[77,142,145\]
\[
\bar{A} = \bar{A}_b + \frac{a_b^2}{2} \kappa_5^2 \left( \mathcal{H} + \frac{\dot{H}}{\mathcal{H}} \right) \delta \pi^S, \tag{4.56}
\]
\[
\mathcal{R} = \mathcal{R}_b + \frac{a_b^2}{2} \kappa_5^2 H \delta \pi^S. \tag{4.57}
\]

In Sec. 6, we will treat the scalar field confined to the brane. In which case, the anisotropic stress on the brane is exactly zero [see Eq. (2.108)]. Hence it is not necessary to care about the difference between two gauges as long as the values of the perturbations on the brane are concerned, namely, \(\bar{A} = \bar{A}_b\) and \(\mathcal{R} = \mathcal{R}_b\).

The perturbed Einstein tensor, \(\delta G^\mu_\nu\), the quadratic energy–momentum tensor, \(\delta \Pi^\mu_\nu\), and the projected Weyl tensor, \(\delta \mathcal{E}^\mu_\nu\), are listed in Appendix B.4, B.5, B.6, respectively. Rewriting the metric perturbations, \(\bar{A}, \bar{A}_y, \bar{A}_{yy}, \mathcal{R}\), to the master variable, \(\Omega\), by Eqs. (4.28)–(4.31) and using the master equation (4.33), we can obtain the Weyl fluid components (4.49)–(4.51) \[142\],
\[
\kappa_5^2 \delta \rho_\xi = \left( \frac{k^4 \Omega}{3 \sigma_0^2} \right)_b, \tag{4.58}
\]
\[
\kappa_5^2 \delta q_\xi = - \frac{k^2}{3 \sigma_0^2} \left( H \Omega - \dot{\Omega} \right)_b, \tag{4.59}
\]
\[
\kappa_5^2 \delta \pi_\xi = \frac{1}{6a_b^3} \left\{ 3\dot{\Omega} - 3H\dot{\Omega} + \frac{k^2}{a^2} \Omega - 3 \left( \frac{n'}{n} - \frac{a'}{a} \right) \Omega' \right\}_b. \tag{4.60}
\]

All perturbative quantities can be expanded by scalar harmonic functions \(e^{ikx}\), which enables us to replace the operator \(\nabla^2\) by \(-k^2\). Then we can derive the evolution equations for the perturbative quantities from the effective Einstein equations (4.48) as \[146\]
\[
- \frac{1}{2} \kappa_5^2 \mathcal{H} \delta \rho = 3H(\ddot{\mathcal{R}} - H\bar{A}) + \frac{k^2}{a_b^2} \mathcal{R} - \frac{1}{2} \kappa_5^2 \delta \rho_\xi, \tag{4.61}
\]
\[
- \frac{1}{2} \kappa_5^2 \mathcal{H} \delta q = - \ddot{\mathcal{R}} - H \left( 3 - \frac{\dot{H}}{\mathcal{H}^2} \right) \mathcal{R} + H\ddot{A} + \left( 2H + 3H^2 - \frac{H^2 \dot{H}}{\mathcal{H}^2} \right) \bar{A} \tag{4.62}
\]
\[
- \frac{1}{3} \kappa_5^2 \bar{A} - \frac{1}{3} \left( 1 - \frac{\dot{H}}{\mathcal{H}^2} \right) \frac{k^2}{a_b^2} \mathcal{R} - \frac{1}{6} \left( 1 + \frac{\dot{H}}{\mathcal{H}^2} \right) \kappa_5^2 \delta \rho_\xi,
\]
\[
- \frac{1}{2} \kappa_5^2 \mathcal{H} \delta q = \ddot{\mathcal{R}} - H\bar{A} - \frac{1}{2} \kappa_5^2 \delta q_\xi, \tag{4.63}
\]
\[
\kappa_5^2 \mathcal{H} \left( 1 + \frac{\dot{H}}{2\mathcal{H}^2} \right) \delta \pi^S = \frac{1}{a_b^3}(\bar{A} + \mathcal{R}) + \frac{1}{2} \kappa_5^2 \delta \pi_\xi, \tag{4.64}
\]

where \(\mathcal{H}\) is defined by Eq. (3.76) and used the relation \(p + \rho = 2\dot{H}/\kappa_5^2 \mathcal{H}\) derived from the Friedmann equation Eq. (3.60) and the energy conservation law (2.9).

As we see in Sec. 6.2, Eqs. (4.61)–(4.63) provide a boundary condition for the master variable \(\Omega\), which is described as coupling of gravity to matter on the brane. Note again
that $\mathcal{A}$ and $\mathcal{R}$ are equivalent to $\tilde{\mathcal{A}}_b$ and $\tilde{\mathcal{R}}_b$ as long as $\delta\pi^S = 0$ which is realised by, for example, the perturbations of scalar field.

### 4.4.2 Vector part

The vector perturbations of the induced metric can be written as

$$
^{(4)}g_{AB} = \begin{pmatrix}
-1 & -a^2_b S_i \\
-a^2_b S_i & a^2_b [F_{i,j} + F_{j,i}]
\end{pmatrix}.
$$

The 4D Einstein tensor is given by Eqs. (B.35)–(B.37), the quadratic energy–momentum tensor by Eqs. (B.45)–(B.47) and the projected Weyl tensor by Eqs. (B.55)–(B.57). Rewriting $\mathcal{S}$ defined by Eq. (4.18) and $\tau_y$ to the vector master variable by Eqs. (4.38) and (4.39), we obtain

$$
\kappa^2 \delta q^V = -\frac{k^2}{2a^3_b} (\dot{\Xi} - 2\mathcal{H} \Xi),
$$

$$
\kappa_5^2 \delta \pi^V = \frac{1}{a^3_b} \left\{ \dot{\zeta}^V - \left( 2\mathcal{H} + \frac{\dot{\mathcal{H}}}{\mathcal{H}} \right) \zeta \right\}.
$$

Then we can derive the evolution equations for the perturbative quantities from the effective Einstein equations (4.48) as

$$
-k^2_5 \mathcal{H} \delta q^V = \frac{k^2}{2} \tau - \frac{a^2_b}{2} \left\{ S' + \left( 2\mathcal{H} - \frac{\dot{\mathcal{H}}}{\mathcal{H}} \right) S \right\},
$$

$$
-2\kappa_5^2 \left( \mathcal{H} + \frac{\dot{\mathcal{H}}}{2\mathcal{H}} \right) \delta \pi^V = \dot{\tau} + 3H \tau - \tau'_y - 2\mathcal{H} \tau_y.
$$

### 4.4.3 Tensor part

The tensor perturbations of the induced metric is simply described as

$$
^{(4)}g_{AB} = \begin{pmatrix}
-1 & 0 \\
0 & a^2_b h_{ij}
\end{pmatrix}.
$$

For the tensor perturbations, the 4D Einstein tensor is given by Eqs. (B.38)–(B.39), the quadratic energy–momentum tensor by Eqs. (B.48)–(B.49) and the projected Weyl tensor by Eqs. (B.58)–(B.59). Thus the effective Einstein equation (4.48) yields a non-local$^1$ junction condition on the brane,

$$
\ddot{h} + 3H \dot{h} + \frac{k^2}{a^2_b} h = \kappa^2_4 \left( 1 - \frac{\rho + 3\rho}{2\lambda} \right) \delta \pi^T + 2\kappa_5^2 \delta \pi^T,
$$

$^1$Here we call the junction condition containing time-derivatives 'non-local' junction condition [142,147].
where, if $\delta \pi^T = 0$ and $\lambda \to \infty$, an evolution equation of gravitational waves in the four-dimensional theory is recovered [see Eq. (2.143)]. Using the wave equation for tensor perturbations (4.43), we can rewrite this to another form described by only bulk spatial derivatives:

$$h' = -\frac{\kappa^2}{2} \delta \pi^T,$$

which yields the Neumann condition on the brane when we can neglect the anisotropic stress $\delta \pi^T$.

### 4.5 Evolution equations of matter perturbations

On the brane, the evolution equations for matter perturbations $\delta \rho, \delta q_i$ are derived from the energy-conservation law (2.97) with the induced metric (4.54). Hence the forms of the equations become the same ones derived in the four-dimensional theory (2.98)–(2.99). In the five-dimensional case, however, we have to replace the metric perturbations $A$ and $\mathcal{R}$ by induced metric perturbations on the brane, $\overline{A}$ and $\overline{\mathcal{R}}$. The induce metric perturbations relate with the metric perturbations defined in the 5D-longitudinal gauge, $\overline{A}_b$ and $\overline{\mathcal{R}}_b$ in Eqs. (4.56) and (4.57).

To see explicitly the contribution from the bulk metric perturbations, let us consider the perturbed scalar field on the brane. The energy momentum tensor is given in Eq. (2.18). Its evolution equation is given in Eq. (2.109) in terms of the inflaton perturbation $\delta \phi$ and (induced) metric perturbations $A$ and $\mathcal{R}$. Using the gauge-invariant quantity, called Mukhanov–Sasaki variable (2.110), we can derive the evolution equation for $Q$ in the five-dimensions as

$$\ddot{Q} + 3H \dot{Q} + \frac{k^2}{a_0^2} Q + \left\{ \frac{\dot{H}}{H} - 2 \frac{\dot{H}}{H} \frac{V'(\phi)}{\dot{\phi}} - 2 \left( \frac{\dot{H}}{H} \right)^2 + V''(\phi) \right\} Q = J(\Omega),$$

where

$$J(\Omega) = -\frac{\dot{\phi}}{H} \left[ \left( -\frac{\dot{H}}{H} + \frac{\dddot{H}}{2H^2} \right) \kappa_5^2 \delta \pi^T + \frac{1}{3} \left( 1 - \frac{\dot{H}}{2H^2} \right) \kappa_5^2 \delta \rho^T + \frac{1}{3} \kappa_5^2 \kappa_4^2 \delta \pi^T + \frac{1}{3} \frac{\dot{H}^2}{H^2 a_0^2} \right].$$

Here the left-hand side of Eq. (4.73) coincides with one derive in four-dimensional theory. To eliminate time-derivatives of $A$ and $\mathcal{R}$ in the Mukhanov–Sasaki equation, we used the junction condition (4.61)–(4.64). The $J(\Omega)$-term comes from the results which vanish in the four-dimensional cases. Hence we can state that $J(\Omega)$ describes the the effect of the five-dimensional bulk metric perturbations. Using Eq. (4.33), the perturbed lapse function (4.28) and the curvature perturbation (4.31) evaluated at the brane can be expressed in
terms of $\Omega$ as

$$
\tilde{A}_b = -\frac{1}{6a_b} \left( 3\Omega'' + 3H\dot{\Omega} - 3\mathcal{H}\Omega' - \frac{k^2}{a_b^2} \Omega \right)_b, \quad (4.75)
$$

$$
\tilde{R}_b = \frac{1}{6a_b} \left( 3\mathcal{H}\Omega' - 3H\dot{\Omega} - 3\mu^2\Omega + \frac{k^2}{a_b^2} \Omega \right)_b. \quad (4.76)
$$

## 4.6 Quantisation of perturbations

### 4.6.1 Scalar perturbations

Calculating the amplitude of primordial scalar/tensor-type fluctuations requires the quantisation of those fields. As discussed in Sec. 2.3.8 and Sec. 2.3.9, in the four-dimensional theory, we can quantise the gravitational waves and the inflaton coupled to gravity, yielding their primordial spectra to lowest order of the slow-roll parameters (2.24) and (2.25). In the process of quantisation of the inflaton field, we defined a new variable instead of the original inflaton field $\phi$ as seen in Eq. (2.118). This variable which diagonalise the action (2.116) enables us to quantise these fields and gives a natural definition of the vacuum state called Bunch–Davies vacuum in Eq. (2.129).

In the five-dimensional braneworld, while the second-order action of the inflaton perturbations is obtained by H. Yoshiguchi and K. Koyama [148], we cannot so far define the vacuum state in a natural way. This is partially because, from the viewpoint of the brane, there is an infinite ladder of massive KK-modes, while there is one-degree-of-freedom inflaton perturbation on the brane. For this reason, we cannot quantise the inflaton field and calculate the amplitude of power spectrum of the curvature perturbations in the same way as treated in four-dimensional theory. Nevertheless, it has been reported that the amplitude is enhanced by the high-energy modification of the Friedmann equation. The resultant amplitude in the chaotic inflation model, in which a scalar field is confined to the brane and the potential is given by $V = m^2\phi^2/2$, becomes [139]

$$
A_S = \sqrt{\frac{512\pi}{75M_{Pl}^6}} \frac{V^3}{V'(\phi)^2} \left( 1 + \frac{V}{2\lambda} \right)^3. \quad (4.77)
$$

Comparing with the four-dimensional result (2.151), the last term with parentheses enhances the amplitude, if the inflation occurs at high-energies, $V \gg \lambda$. With this treatment, we obtain the five-dimensional version of the consistency relation between the scalar–tensor ratio, $A_T/A_S$ and the spectral index of primordial gravitational waves, $n_T$, which is the same as one obtained in the four-dimensional case (2.164) [42, 43, 45, 48]. However, this treatment is not fully justified on the point that the bulk metric perturbations are not considered properly. Our work which we will discuss in Ch. 6 focuses on the effects of bulk metric perturbations on the small-scale inflaton perturbations.
4.6.2 Tensor perturbations

As for the tensor perturbations in the braneworld, the gravitational waves decouple to the matter field. This fact enables us to quantise the tensor perturbations. In this subsection, we will see that the zero-mode becomes dominated during the de Sitter inflation, and hence it becomes the initial condition of the tensor perturbations for the numerical analysis in the next chapter.

Let us consider the tensor perturbations during the de Sitter inflation. If the spacetime on the brane is de Sitter one, the energy density $\rho$ becomes constant, and the warp factor (3.71) and the lapse function (3.72) become separable,

$$a(t, y) = a_b(t)N(y), \quad n(t, y) = N(y) = \cosh \mu |y| - \left(1 + \frac{\rho}{\lambda}\right) \sinh \mu |y|, \quad (4.78)$$

where $\rho$ is constant in this case. The separable warp factor enables us to solve the evolution equation of the tensor perturbations (4.43) in an analytic way. In fact, we can divide the variable $h$ into two parts:

$$h(t, y) = \int \phi_m(t)u_m(y) \, dm. \quad (4.79)$$

Then, the evolution equation (4.43) becomes

$$\frac{d^2\phi_m}{dt^2} + 3H \frac{d\phi_m}{dt} + \left(m^2 + \frac{k^2}{a_b^2}\right) \phi_m = 0, \quad (4.80)$$

$$\frac{d^2u_m}{dy^2} + 4\frac{N'}{N} \frac{du_m}{dy} + \frac{m^2}{N^2} u_m = 0, \quad (4.81)$$

where $m$ is effectively a mass of a mode function (graviton). The solution for $m = 0$ is called zero-mode, and solutions for $m > 0$ are called Kaluza–Klein modes, or KK-modes. Using a conformal time $\eta = -1/a_b H$, the time-dependent wave equation (4.80) becomes [30,32]

$$\frac{d^2\phi_m}{d\eta^2} - \frac{2}{\eta} \frac{d\phi_m}{d\eta} + \left(k^2 + \frac{\nu^2 + 9/4}{\eta^2}\right) \phi_m = 0, \quad (4.82)$$

whose general solution is given by

$$\phi_0(\eta) = C_1^{(0)}(-k\eta)^{3/2}Z_{3/2}(-k\eta), \quad (4.83)$$

$$\phi_m(\eta) = C_1^{(m)}(-k\eta)^{3/2}Z_{\nu}(-k\eta), \quad \nu = \sqrt{\frac{m^2}{H^2} - \frac{9}{4}}, \quad (4.84)$$

where $Z_\nu$ denotes a linear combination of the Bessel functions of order $\nu$, and $C_1^{(m)}$ is a normalisation factor determined from the quantum theory as discussed later part of this section. This result indicates that there is a mass gap between the lightest KK-mode and the zero-mode:

$$m^2 = \left(\nu^2 + \frac{9}{4}\right)H^2 \geq \frac{9}{4}H^2. \quad (4.85)$$
4.6 Quantisation of perturbations

Figure 4.2: Behaviour of zero-mode (red solid line) [Eq. (4.83)] and massive KK-modes with $m = 3H/2$ (green short-dashed line) and $m = \sqrt{13}H/2$ (blue long-dashed line) on super-horizon scales [Eq. (4.84)]. These modes become larger than the Hubble horizon when $k\eta = -1$.

As we plotted in Fig. 4.2, these massive modes are sufficiently damped in the super-horizon scale $|k\eta| < 1$. Therefore, during inflation, the zero-mode may dominate on the brane [131].

The equation of spatial mode functions (4.81) can be rewritten as a Schrödinger-type equation. Here we introduce the conformal bulk-coordinate,

$$z = \int \frac{dy}{N(y)} = \frac{\text{sign}(y)}{H} \log \left[ \coth \frac{\mu}{2}(y_c - |y|) \right],$$

(4.86)

where $y_c$ is a coordinate singularity defined by Eq. (3.73). Then the brane is located at $z = \pm z_b$ with

$$z_b = \frac{1}{H} \sinh^{-1} \frac{H}{\mu},$$

(4.87)

and the warp factor (4.78) becomes

$$N(z) = \frac{H}{\mu \sinh(H|z|)}.$$  

(4.88)

Introducing a dimension-less spatial coordinate $\xi \equiv Hz$, Eq. (4.81) becomes

$$\frac{d^2 u_m}{d\xi^2} - 3 \coth \xi \frac{d u_m}{d\xi} + \left( \nu^2 + \frac{9}{4} \right) u_m = 0.$$  

(4.89)

Defining $\chi_m = N^{3/2} u_m$, we obtain the Schrödinger-like equation of motion for $\chi_m(z)$:

$$\frac{d^2 \chi_m}{dz^2} - V(z)\chi_m = -m^2 \chi_m,$$

(4.90)

where

$$V(z) = \frac{15H^2}{4 \sinh^2(Hz)} + \frac{9}{4} H^2 - 3\mu \left( 1 + \frac{\rho}{\lambda} \right) \delta(z - z_b).$$

(4.91)

This volcano potential has one bound state due to the delta function originated from the matter confined to the brane.
Returning to Eq. (4.89), one finds that the solution can be described by a linear combination of the associated Legendre functions, \( P_n^m(x) \) and \( Q_n^m(x) \) [30, 32]:

\[
\begin{align*}
    u_0(\xi) &= C_2^{(0)} + C_3^{(0)} (\cosh 3\xi - 9 \cosh \xi), \\
    u_m(\xi) &= C_2^{(m)} \sinh^2 \xi \left[ P_{-1/2+im}^{-2}(\cosh \xi) - C_3^{(m)} Q_{-1/2+im}^{-2}(\cosh \xi) \right],
\end{align*}
\]  

(4.92) (4.93)

where \( C_2^{(m)} \) and \( C_3^{(m)} \) are normalisation constants.

The next task is to determine normalisation factors, \( C_1^{(m)} \), \( C_2^{(m)} \) and \( C_3^{(m)} \), from the quantum theory. Treating the tensor perturbation \( h \) as a field operator, it can be expanded by use of the creation and annihilation operators:

\[
\hat{h}(\eta, \xi) = \int (\hat{a}_m h_m + \hat{a}_m^\dagger h_m^*) \, dm,
\]

(4.94)

where \( h_m \equiv \phi_m(\eta) u_m(\xi) \). According to a standard text of the quantum theory [81], we can define a scalar product between two scalar fields \( \phi_1, \phi_2 \):

\[
\begin{align*}
    (\phi_1, \phi_2) &= -i \int_{\Sigma} (\phi_1 \partial_{\mu} \phi_2^* - \phi_2^* \partial_{\mu} \phi_1) \sqrt{-g} \, d\Sigma^\mu \\
    &= -2i \int_{\xi_b}^{\infty} \frac{d\xi}{\eta^2 \sinh^3 \xi} (\phi_1 \partial_\eta \phi_2^* - \phi_2^* \partial_\eta \phi_1),
\end{align*}
\]

(4.95)

where \( d\Sigma^\mu \) is a volume element of a spacelike hypersurface \( \Sigma \), and \( d\Sigma^0 = d\xi \) in the present case. This scalar product is called Wronskian, which does not depend on time. We impose some conditions to the Wronskian in order that the quantum field \( \hat{h} \) is canonically normalised:

\[
\begin{align*}
    (h_0, h_0) &= -(h_0^*, h_0^*) = 1, \\
    (h_m, h_m) &= -(h_m^*, h_m^*) = \delta(m - m'),
\end{align*}
\]

(4.96)

Rewriting \( h_m = \phi_m u_m \), we obtain normalisation conditions for \( \phi_m(\eta) \) and \( u_m(\xi) \):

\[
\begin{align*}
    -\frac{i}{\mu \eta^2} (\phi_m \partial_\eta \phi_m^* - \phi_m^* \partial_\eta \phi_m) &= 1, \\
    2 \int_{\xi_b}^{\infty} \frac{d\xi}{\sinh^3 \xi} u_0^* u_0 &= 1, \\
    2 \int_{\xi_b}^{\infty} \frac{d\xi}{\sinh^3 \xi} u_m^* u_m &= \delta(m - m').
\end{align*}
\]

(4.97) (4.98) (4.99)

From Eq. (4.97), we can determine the normalisation constants \( C_1^{(0)} \) and \( C_1^{(m)} \) for the time-dependent mode function (4.83) (4.84) [32],

\[
\phi_m(\eta) = \frac{\sqrt{\pi}}{2} \mu^3/2 e^{-\pi \eta/2} (-\eta)^{3/2} H_{1/2}^{(1)}(-k\eta).
\]

(4.100)

Here we chose the functional form of \( Z_\nu \) for the mode function \( \phi_m \) to coincide the asymptotic form of Eq. (2.129) at \(-k\eta \to 0\).
As for the spatial mode functions $u_m$, let us firstly consider the zero-mode ($m = 0$). The junction condition (4.72) is imposed on the zero-mode solution (4.92). Then we obtain

$$C_3^{(0)} = 0,$$

(4.101)

where we assumed no anisotropic stress on the brane ($\delta\pi^T = 0$). From the normalisation condition for the zero-mode (4.98), we can calculate the constant $C_2^{(0)}$ [32,131],

$$C_2^{(0)} = \frac{H}{\mu} F \left( \frac{H}{\mu} \right),$$

(4.102)

where

$$F(x) \equiv \left( \sqrt{1 + x^2} - x^2 \log \frac{\sqrt{1 + x^2} + 1}{x} \right)^{-1/2}.$$  

(4.103)

This factor in low energy ($x \ll 1$) and high energy ($x \gg 1$) limits becomes

$$F(x) \rightarrow \begin{cases} 
\left( \frac{27M_{Pl}^2}{16\pi\lambda} \right)^{1/4} H^{1/2}, & \text{for } x \gg 1 \\
1, & \text{for } x \ll 1,
\end{cases}$$

(4.104)

indicating the observed spectrum of gravitational waves is enhanced by the factor $F(x)$ if the inflation occurs at high-energies ($H/\mu \gg 1$).

For the massive KK-modes ($m \geq 3H/2$), the constant $C_2^{(m)}$ is determined from Eq. (4.99) as

$$C_2^{(m)} = \left[ \left| \frac{\Gamma(i\nu)}{\Gamma(5/2 + i\nu)} \right|^2 + \left| \frac{\Gamma(-i\nu)}{\Gamma(5/2 - i\nu)} - \pi C_3 \frac{\Gamma(i\nu - 3/2)}{\Gamma(1 + i\nu)} \right|^2 \right]^{-1/2}.$$

(4.105)

$C_3^{(m)}$ is determined from the junction condition (4.72) [30,32,133],

$$C_3^{(m)} = \frac{P_{2/2 - i\nu}^{-1}}{Q_{2/2 - i\nu}^{-1}} \frac{\cosh \xi_b}{\cosh \xi_b}.$$  

(4.106)

As mentioned above, we can determine the normalisation and phase of the tensor perturbations in the de Sitter inflation without ambiguities.

The KK-modes with a mass $0 < m < 3H/2$ diverge at the Cauchy horizon. Above the mass $m > 3H/2$, KK-modes rapidly decay during the inflation, and go towards zero in the super-horizon scales shown in Fig. 4.2, which indicates that the massive states remain their vacuum states. Therefore, only the zero-mode ($m = 0$) can contribute to the observed primordial spectrum of gravitational waves. Combining Eqs. (4.83) and (4.92) with normalisation factors (4.101) and (4.102), we obtain the asymptotic form of the zero-mode

$$|h_0| \rightarrow \frac{H^2 \mu}{2k^2} F \left( \frac{H}{\mu} \right).$$

(4.107)
Hence the primordial spectrum of gravitational waves is calculated as [32]

\[ P_h^{5D \ 1/2} = 2 \left( \frac{k^3 \ 16\pi}{2\pi^2 \ M_5^3} \left| h_0 \right|^2 \right)^{1/2} \]

\[ = \frac{4}{\sqrt{\pi} \ M_{Pl}} \ F \left( \frac{H}{\mu} \right), \]  

(4.108)

where the factor ‘2’ appeared in the first line comes from the existence of two polarisations, and the factor \( 16\pi/M_5^3 \) is required for the quantum field \( h_m \) to be canonically normalised [see Eq. (2.137) in the four-dimensional case]. When we take a limit \( \mu \to \infty \), the four-dimensional result (2.148) is recovered including the factor.
Chapter 5

Numerical Studies on Tensor Perturbations – Gravitational Wave Background

5.1 Introduction

As an ultimate cosmological tool, the IGWB may also be useful to probe the presence of extra-dimensional spaces. Recent developments in particle physics suggest that we live in a higher-dimensional spacetime as we discussed in Sec. 3.1. In particular, braneworld scenarios have recently attracted much attention theoretically and observationally. According to them, we live in a three-dimensional hypersurface (brane) embedded in the higher-dimensional spacetime (bulk). While gravity can propagate in the bulk with the curvature scale $\ell \equiv \mu^{-1}$, the standard model particles are confined to the three-dimensional brane. In the low-energy regime of the universe ($H\ell \ll 1$), four-dimensional general relativity is successfully recovered and the extra-dimensional effects should be fairly small. On the other hand, in the high-energy regime ($H\ell \gg 1$), the localisation of gravity is not always guaranteed and a significant deviation of the time evolution of GWs from the standard four-dimensional theory is expected. If this scenario is true, the spectrum of the IGWB may be significantly modified by the high-energy effects, which can provide a direct probe of the extra-dimensions.

The goal of this chapter is to investigate the high-energy effects on the evolution of the IGWB after inflation and quantify the observed energy spectrum. During the inflationary epoch, the wavelength of the GWs exceeds the Hubble horizon scale due to the exponential expansion of the universe and the amplitude of GWs becomes frozen. After the end of inflation, the universe enters the decelerated expansion phase and wavelengths of GWs soon become shorter than the Hubble scale. When the Hubble horizon scale becomes comparable or smaller than the characteristic size of the extra-dimension, $H\ell \gtrsim 1$, the high-energy effects may significantly affect the evolution of GWs. There are two main high-energy effects: i) peculiar cosmological expansion due to the high-energy correction of the Friedmann equation
(3.60), which enhances the spectrum in the high frequency region and ii) excitation of Kaluza–Klein modes (KK-modes) freely escaping from the brane to the bulk spacetime, which may suppress the amplitude of the GWs on the brane. While the former effect is simply estimated from the expansion rate of the universe, the amount of the latter effect requires the knowledge of the wave propagation in the bulk. Focusing on the Randall–Sundrum single-brane model which has been discussed in Sec. 3.2 [27], many authors have tried to estimate these effects in an analytic way. However, these analyses were restricted to the idealistic situations [30,32,34] or low-energy cases in the Friedmann universe [35,36], since the analytic study of wave equation is generally intractable due to the complicated form of equation as well as boundary condition. Thus, in this chapter, we numerically solve the wave equation and try to estimate the observed IGWB spectrum.

Concerning numerical studies, several authors have used different numerical techniques and coordinate systems to solve the wave equation of GWs [33,37–41,149]. In our previous studies [33, 37], numerical simulations were carried out in the two types of coordinate systems. One is the Gaussian–normal coordinate system in which we found the suppression of the amplitude of the IGWB on the brane. Unfortunately, the coordinate singularity appears in the bulk [see Eq. (3.73)] and this restricts our analyses to a relatively low-energy scale [33] [see the discussion in Sec. 3.2.2]. On the other hand, another coordinate system we used is the Poincaré coordinates (3.15) in which we observed that the two high-energy effects compensate each other and the spectrum became same one as predicted in the four-dimensional theory [37].

This chapter is organised as follows : In Sec. 5.2, we discuss how the spectrum of the IGWB is affected by the presence of the extra-dimensions. In Sec. 5.3, we introduce the Poincaré coordinate system and derive the evolution equation of GWs. After briefly discussing the details of the numerical simulations and initial conditions in Sec. 5.4, we check our numerical scheme by solving a simple case in Sec. 5.5. In Sec. 5.6, we present the numerical results in the case that the IGWB re-enters the Hubble horizon during the radiation dominated (RD) universe. Further, we show that the spectrum of the IGWB estimated here is robust against several numerical artifacts, such as the dependence of the regulator brane and initial time. Also, in Sec. 5.6, the results in cases with other equation of state (EOS) are shown. Finally, Sec. 5.7 is devoted to the summary and conclusions.

### 5.2 High-energy effects in the braneworld cosmology

In a braneworld scenario, the propagation of gravity may be modified by the presence of extra-dimensional spaces, which affects the observed spectrum of IGWB. Here we consider RS single brane model and the flat FLRW universe with the equation-of-state (2.174) on the brane. In general, the RS model may possess a bulk black hole whose mass is described as constant $C$ in the Friedmann equation (3.58), but we do not consider it.

As we mentioned in Sec. 5.1, there are two important effects on the spectrum in the
5.2 High-energy effects in the braneworld cosmology

high-energy regime. Let us first consider the non-standard cosmological expansion in the presence of $\rho^2$-term. From Eqs. (3.79) and (3.80), the power-law index of the scale factor in the high-energy regime ('H' in Fig. 5.1 is related to the EOS parameter $w$ as

$$n = \frac{1}{3(1+w)} \quad \text{for} \quad t \ll \ell. \quad (5.1)$$

On the other hand, at the late-time phase ($t \ll t_{\text{crit}}$), the power-law index just coincides with the four-dimensional result (2.175). As a result, the energy spectrum (2.173) is modified to

$$\Omega_{\text{GW}} = \begin{cases} 
\left(\frac{f}{f_{\text{crit}}^3}\right)^{\frac{3}{5}w} & \text{for} \quad f > f_{\text{crit}}, \\
\left(\frac{f}{f_{\text{crit}}^3}\right)^{\frac{1}{5}w} & \text{for} \quad f < f_{\text{crit}},
\end{cases} \quad (5.2)$$

where $f_{\text{crit}}$ denotes the critical frequency given by

$$f_{\text{crit}} = \frac{1}{2\pi \ell} \frac{a_{\text{crit}} a_{\text{eq}}}{a_0}$$

$$= 5.6 \times 10^{-5} \, \text{Hz} \left(\frac{\ell}{0.1 \, \text{mm}}\right)^{-1/2} \left(\frac{H_0}{72 \, \text{km/s} \cdot \text{Mpc}}\right)^{1/2} \left(\frac{1 + z_{\text{eq}}}{3200}\right)^{-1/4}. \quad (5.3)$$

That is, the wavelength at the horizon re-entry time just coincides with the curvature scale, namely, $H_s = \ell^{-1}$ (cf. [150]). Note that the curvature radius $\ell$ is constrained to $\ell < 0.1$ mm by table-top experiments on Newton’s law of gravity [114,115]. For the frequency $f > f_{\text{crit}}$, GWs re-enter the horizon during the RD phase of the $\rho^2$-term dominated epoch as shown in Fig. 5.1 [cf. Fig. 2.1]. In the high-energy RD phase, the spectrum of the IGWB is modified to

$$\Omega_{\text{GW}} \propto f^{4/3} \quad (f_{\text{crit}} < f < f_{\text{inf}}), \quad (5.4)$$

which is shown in short-dashed lines in Fig 5.2. Additionally, the inflationary cut-off frequency (2.168) is modified to

$$f_{\text{inf}} \approx \frac{1}{2\pi} \frac{a_{\text{inf}} a_{\text{crit}}}{a_{\text{eq}} a_0} H_{\text{inf}}$$

$$= 4.7 \times 10^6 \, \text{GHz} \left(\frac{H_{\text{inf}}}{6 \times 10^{-5} M_{\text{pl}}}\right)^{3/4} \left(\frac{H_0}{72 \, \text{km/s} \cdot \text{Mpc}}\right)^{1/2} \left(\frac{\ell}{0.1 \, \text{mm}}\right)^{1/4} \left(\frac{1 + z_{\text{eq}}}{3200}\right)^{-1/4}. \quad (5.5)$$

Notice that the above estimate neglects another remarkable high-energy effect caused by the excitation of Kaluza–Klein modes (KK-modes). The KK-modes can propagate into the bulk and may be observed on the brane as massive gravitons. By contrast, the GW propagating on the brane is called the 'zero-mode', and it behaves like a massless graviton on the brane. Strictly speaking, the KK-modes and zero-mode are coordinate-dependent concepts and are mathematically well-defined only in the case of the Minkowski brane and the de Sitter brane. Nevertheless, we keep to use these terms in the FLRW case to distinguish these propagation features.
As we will see in the next section, the brane is generally moving in the bulk spacetime. From the analogy of the moving mirror problem [81], even if only the zero-mode is generated in the inflationary epoch, the zero-mode is partially transferred to KK-modes with arbitrary masses. If this is true, the amplitude of the zero-mode observed on the brane may be suppressed in comparison with the result neglecting the KK-modes, i.e., (5.4). The envisaged spectrum involving the two effects is schematically shown as solid lines in Fig. 5.2. Unfortunately, we cannot fully estimate the KK-mode effects in an analytic way. Thus, in order to construct the IGWB spectrum including the KK-mode effects, we must directly solve the evolution equations of GWs in the five-dimensional cosmology.

In the next section, we present the formalism to solve the evolution equations for GWs numerically.

![Figure 5.1: Schematic diagrams of the evolution histories of GWs in five-dimensional case.](image)

In the high-energy RD regime of the latter case (denoted by “H”), the $\rho^2$-term in the Friedmann equation changes the time dependence of the Hubble parameter. This fact yields a new characteristic frequency $f_{\text{crit}}$. See also Fig. 2.1.

### 5.3 Evolution equation and initial conditions

#### 5.3.1 Evolution equation of GWs

Here we revisit the formulations for the tensor perturbations in the $\text{AdS}_5$ spacetime. While we have discussed them in the Gaussian–normal coordinates in Chap. 4, we use the Poincaré coordinate $(\tau, x, z)$ which is discussed in Sec. 3.2.2. The tensor perturbations defined in this new coordinate coincide with ones defined in Gaussian–normal coordinates thanks to no tensors in the coordinate transformation.
5.3 Evolution equation and initial conditions

\[
\Omega_{GW}(f) \approx 10^{-5} \text{ Hz}
\]

\[
\Omega_{GW} \propto f^{4/3}
\]

Figure 5.2: A schematic diagram of the spectrum of IGWB. The four-dimensional prediction is shown in long-dashed lines. Considering the modification due to the non-standard cosmological expansion, the spectrum behaves as \( \Omega_{GW} \propto f^{4/3} \) shown in short-dashed line, which may appear upon the detection limit of advanced LIGO (dot-dashed line). Moreover, KK-mode excitations modify the spectrum as solid lines. The main issue in this chapter is to clarify whether the resultant spectrum (solid) exceeds the four-dimensional prediction (long-dashed).

The trajectory of the brane is given by Eqs. (3.16) and (3.17). In Fig. 5.3, we show the trajectory denoted by 'Friedmann brane' in the conformal chart where the surfaces of \( \tau = \text{const.} \) and \( z = \text{const.} \) are plotted as dashed lines. One can check that this trajectory induces the metric of the four-dimensional flat FLRW model on the brane:

\[
ds^2_b = -dt^2 + a^2(t)\delta_{ij}dx^idx^j.
\]

Note that, in the case of de Sitter brane, the scale factor becomes \( a(t) = e^{Ht} \) and the Hubble parameter \( H \) is constant. Hence the trajectory becomes

\[
z_b^{\text{dS}} = -\sqrt{1 + (Ht)^2} e^{-Ht}, \quad z_b^{\text{dS}} = \ell e^{-Ht}, \quad (5.6)
\]

which yields the straight trajectory in the bulk; that is, \( \tau_b^{\text{dS}} \) becomes proportional to \( z_b^{\text{dS}} \).

In the Poincaré coordinate, the perturbed metric of the AdS₅ spacetime is given by

\[
ds^2 = \left(\frac{\ell}{z}\right)^2 \left\{ -d\tau^2 + (\delta_{ij} + h_{ij})dx^idx^j + dz^2 \right\}, \quad (i, j = 1, 2, 3), \quad (5.7)
\]

where \( h_{ij} \) satisfies the transverse-traceless (TT) condition,

\[
\partial_ih_{ij} = h_i^i = 0. \quad (5.8)
\]

Let us focus on the evolution of the tensor perturbations \( h_{ij} \). Here we decomposed the perturbative quantity as Eq. (4.41) in the Poincaré coordinate \((\tau, z)\). In terms of the Fourier
modes, the evolution equation for the perturbations is derived from the perturbed Einstein equation (4.27) [146],
\[
\frac{\partial^2 h}{\partial \tau^2} - \frac{\partial^2 h}{\partial z^2} + \frac{3}{z} \frac{\partial h}{\partial z} + k^2 h = 0, \tag{5.9}
\]
The equation (5.9) must be solved with the junction condition imposed on the brane. In the Poincaré coordinate, the explicit form of the junction condition becomes [146]
\[
\frac{\partial h}{\partial n}\bigg|_{\text{brane}} = \left( \frac{\partial h}{\partial \tau} - \frac{\sqrt{1 + H^2 \ell^2} \frac{\partial h}{\partial z}}{H \ell} \right)_{z = z_n(t)} = 0, \tag{5.10}
\]
where \( \partial / \partial n \) denotes the normal vector of the brane. These Eqs. (5.9) and (5.10) are identical to ones calculated in the Gaussian–normal coordinates (4.43) (4.72). While there is generally a contribution from the tensor part of anisotropic stress tensor on the brane \( \pi^T_{ij} \) as seen in the junction condition (4.72), we neglect it for simplicity and set \( \pi^T_{ij} = 0 \) hereafter.

It is important to note that the evolution equation (5.9) can be written in a separable form and by using this fact, one obtains the general solutions (e.g. [30, 146])
\[
h(\tau, z) = \int_0^\infty dm \ \left\{ \tilde{h}_1(m) \ z^2 H_2^{(1)}(mz) e^{i \omega \tau} + \tilde{h}_2(m) \ z^2 H_2^{(2)}(mz) e^{-i \omega \tau} \right\}, \tag{5.11}
\]
where \( \omega^2 = m^2 + k^2 \). The functions \( H_2^{(1)} \) and \( H_2^{(2)} \) denote respectively the Hankel functions of first and second kind, and \( \tilde{h}_1(m) \) and \( \tilde{h}_2(m) \) represent arbitrary coefficients. The above expression implies that the GWs propagating in the bulk are described as a superposition of the zero mode \( (m = 0) \) and the KK-modes \( (m > 0) \). Solving the wave equation (5.9) with the junction condition (5.10) is the equivalent task to determining the coefficients \( \tilde{h}_{1,2}(m) \) that satisfy the junction condition [146]. In the very high-energy case, technical difficulties hinder efforts to calculate these analytically because of the significant contribution from the massive modes \( (m \ell \gg 1) \). For this reason, we solve numerically the wave equation (5.9) with the junction condition (5.10).

### 5.3.2 Initial conditions

In order to correctly estimate the effects of the KK-modes, we must specify the initial conditions for the perturbed quantity \( h \) after inflation. In this chapter, we specifically consider a brane inflation model in which the exponential expansion takes place on the brane. As we have discussed in Sec. 4.6, the zero-mode tensor perturbation, which is homogeneous in the bulk, becomes dominant during the inflation. Solving the equations (4.80) and (4.81) with \( m = 0 \), the zero-mode is described explicitly as
\[
h(\eta, y) = C(-k \eta)^{-3/2} H_{3/2}^{(1)}(-k \eta), \tag{5.12}
\]
where \( C \) is a normalisation constant and \( \eta \) is the conformal time \( \eta = -1/aH \). We do not pay attention to the normalisation factor since we will treat the perturbed field as a classical field in this chapter.

Our previous work [33] has been done with these formulae in the Gaussian–normal coordinates.
To see how the solution (5.12) looks like in the Poincaré coordinates, we rewrite the general solution (5.11) in the GN coordinates. In the de Sitter case, the coordinate transformation between the GN coordinates and the Poincaré coordinates is explicitly given as \[30,32\]

\[ z = -\eta \sinh(y/\ell), \quad \tau = \eta \cosh(y/\ell), \] (5.13)

Substituting them into the general solution (5.11), we obtain

\[ h(\tau, z) = \int_0^\infty dm \left\{ \eta \sinh(y/\ell)^2 \left\{ \tilde{h}_1(m) H_2^{(1)}(m\eta \sinh(y/\ell)) e^{-i\omega \eta \cosh(y/\ell)} + \tilde{h}_2(m) H_2^{(2)}(m\eta \sinh(y/\ell)) e^{i\omega \eta \cosh(y/\ell)} \right\} \right. \]

Comparing (5.12) with (5.14), we see that the zero-mode solution given in the inflationary epoch cannot be simply expressed in terms of the zero-mode solution in the Poincaré coordinates, which indicates that a mixture of KK-modes is required to express the zero-mode solution in the inflationary epoch. Nevertheless, in the long-wavelength limit \( k \to 0 \), both the zero-mode solutions become constant over the time and the bulk space, and they coincide with each other. Since we are specifically concerned with the evolution of long-wavelength GWs after inflation, the constant mode, i.e.,

\[ h = \text{const.}, \quad \frac{dh}{d\tau} = 0, \] (5.15)
seems a natural and a physically plausible initial condition for our numerical calculation in
the Poincaré coordinate. Strictly speaking, however, this is valid only in the long-wavelength
limit \( k \rightarrow 0 \). This point will be discussed in details in Sec 5.5.2.

## 5.4 Setup and parameters for numerical simulations

On the basis of the formalism presented in the previous section, we now discuss the numerical
treatment used to solve the wave equation (5.9) with the junction condition (5.10). First of
all, the computational domain should be finite. We introduce an artificial cutoff (regulator)
boundary in the bulk at \( z = z_{\text{reg}} \) (shown in Fig. 5.3 and, in the middle panel of Fig. 3.2 as
a long-dashed line) and impose the Neumann condition at the regulator boundary, i.e.,

\[
\left( \frac{\partial h}{\partial z} \right)_{z=z_{\text{reg}}} = 0. \tag{5.16}
\]

We are especially concerned with the late-time evolution of GWs after the inflation. For
this purpose, we focus on the evolution equation (5.9) in the RD phase. In order to quantify
high-energy effects, we define a useful parameter \( \epsilon_{\text{a}} \) which represents the normalised energy
density at the horizon re-entry time \( t_{\text{a}} \) of the GWs concerned, namely, \( \epsilon_{\text{a}} = \epsilon(t_{\text{a}}) \). From the
Friedmann equation (3.60) and the definition of the scale factor (3.79), the comoving wave
number of GWs is rewritten in terms of the parameter \( \epsilon_{\text{a}} \) as

\[
k_{\text{a}} = a_{\text{a}} H_{\text{a}} = \left( \frac{\epsilon_{\text{crit}}}{\epsilon_{\text{a}}} \right)^{1/3(1+w)} \sqrt{\epsilon_{\text{a}}^2 + 2\epsilon_{\text{a}}}. \tag{5.17}
\]

For higher frequency GWs, \( \epsilon_{\text{a}} \) becomes larger. One thus expects that the high-frequency
GW modes tend to be significantly affected by the high-energy effects.

Notice that the location of the regulator brane is another important parameter. Here,
the location of the boundary is set to \( z_{\text{reg}} = 25\text{–}200\ell \), which is far enough away from the
physical brane to avoid artificial suppression of light KK-modes. Further, we must stop the
numerical calculations before the influence of the boundary condition at \( z = z_{\text{reg}} \) reaches the
physical brane \( z_{\text{b}} \). The arrival time of the influences of the regulator brane can be estimated
by drawing a null line from the initial position of the regulator boundary \( (\tau_0, z_{\text{reg}}) \) towards
the physical brane. With these treatments, we have checked that the amplitude of GWs on
the brane is fairly insensitive to the location of regulator boundaries. Thus, all the results
presented in Sec. 5.5 and Sec. 5.6 are free from the effect of regulator boundary.

In the situation considered here, the initial time \( t_{\text{init}} \) is also an important parameter,
which turns out to have an important effect on the GWs in the bulk [37]. We parameterise
the initial time as

\[
s_{\text{init}} = \frac{a(t_{\text{init}}) H(t_{\text{init}})}{k}, \tag{5.18}
\]

which represents the wavelength of GWs normalised by the Hubble horizon scale at the
initial time \( t_{\text{init}} \). In order to get a reliable estimate, we set \( s_{\text{init}} \gg 1 \) and run the simulations
until \( \epsilon(t) \ll 1 \), when the high-energy effects on the GWs become negligible.
Finally, we adopt the constant mode \( h(t_{\text{init}}, \xi) = 1 \) as an initial condition according to the discussion in Sec. 5.3.2. Although it is plausible, the validity of the constancy of the super-horizon modes must be checked. This point will be carefully discussed in Sec. 5.5.2.

## 5.5 Code check and qualitative behaviour of GWs

### 5.5.1 Code check

We employ the spectral collocation method for our numerical simulation [52, 53] with the predictor–corrector method as a temporal method. Details on this method are described in appendix C. In order to check our numerical code, we first consider the simplest case in which the location of the brane is given by \( z = z_b = \text{constant} \). This is the so-called Minkowski brane embedded in the AdS\(_5\) bulk. The solution of the evolution equation (5.9) is given as [27,30,151]

\[
h_{\text{exact}}(\tau, z) = z^2 Z_2(mz) E(\omega \tau),
\]

where \( \omega \equiv \sqrt{k^2 + m^2} \). The function \( Z_2 \) and \( E \) respectively denote linear combinations of the Bessel functions of order 2 and the sinusoidal functions. Imposing the junction condition (5.10) at \( z_b = \ell \), we obtain

\[
h_{\text{exact}}(\tau, z) = \left( \frac{z}{\ell} \right)^2 \{ Y_1(m)J_2(mz) - J_1(m)Y_2(mz) \} \cos(\omega \tau).
\]

According to the analytic solution (5.20), we set \( h = h_{\text{exact}}(0, z) \) and \( \dot{h} = \partial h_{\text{exact}}/\partial \tau \rVert_{\tau=0} = 0 \) for the initial condition of numerical simulation and compare the numerical results with the analytic solution.

Fig. 5.4 shows the behaviour of GWs in the AdS\(_5\) bulk. The right panel of the figure is the projection of the left panel. In this simulation, we chose parameters as \( z_{\text{reg}} = 20\ell \), \( m\ell = k\ell = 2 \) and the number of the collocation points, \( N = 1024 \). The left panel of Fig. 5.5 shows the snapshot of the waveform at \( \tau = \tau_1 = 10\ell \), which illustrates that the numerical result accurately recovers the exact solution (5.20) in the interval between \( z_b \leq z \leq z_{\text{reg}} - \tau_1 \).

Outside this region, the numerical simulation is contaminated by the boundary condition of the regulator brane. In the right panel of Fig. 5.5, the fractional error of the amplitude \( \left| (h_{\text{num}} - h_{\text{exact}})/h_{\text{exact}} \right| \) evaluated at the time \( \tau = \tau_1 \) is plotted as a function of bulk coordinate. We found that the error is suppressed to the order of \( 10^{-3} \) near the physical brane. Note that there appear several sharp spikes, whose locations roughly match those of the zero-point \( h_{\text{exact}}(\tau, z) = 0 \). Thus, the cancellation of significant digits occurs. These numerical errors can be reduced when the number of collocation points \( N \) is increased.
Numerical Studies on Tensor Perturbations – Gravitational Wave Background

Figure 5.4: The behaviour of a test wave with $m\ell = k\ell = 2$ in the bulk. The Minkowski brane is located at $z = \ell$. The right panel depicts the projection of the three-dimensional waves of the left panel.

Figure 5.5: The left panel illustrates that the numerical solution at $\tau = 10$ is consistent with the analytical solution (5.20) in the region $z_b \leq z \leq 10$. The outer region $10 \leq z \leq z_{\text{reg}}$ in the bulk is contaminated by the boundary condition on the regulator brane at $z = z_{\text{reg}}$. The right panel shows the numerical errors $|(h_{\text{num}} - h_{\text{exact}})/h_{\text{exact}}|$ estimated at that time, which is suppressed by $10^{-3}$ near the physical brane. The spike-shapes reflects the cancellation of significant digits because of $h_{\text{exact}}(10, z) \approx 0$.

5.5.2 Behaviour of GWs in the bulk and the validity check of the initial condition

Having checked the reliability of our numerical scheme, we now focus on the cosmological evolution of GWs. As we mentioned in Sec 5.3.2, we must first clarify the validity and the
sensitivity of the initial condition (5.15), namely, the constancy of the super-horizon modes. It should be stressed that, only in the long-wavelength limit $k \to 0$ during the inflationary phase, the constant mode coincides with the zero-mode solution [see Eq.(5.14)]. Therefore, the constancy of GW amplitudes after inflation cannot be guaranteed even on super-horizon scales. Depending on the choice of the parameters $s_{\text{init}}$ and $\epsilon_*$, the mode $h = \text{const.}$ may not be a good approximation to the initial condition for numerical simulations in the RD epoch.

Fig. 5.6 shows the time evolution of GWs in the Poincaré coordinate system in the de Sitter case. In this simulation, we set $s_{\text{init}} = 100$ and $H\ell = \sqrt{3}$. The universe on the brane experiences accelerated cosmological expansion and the wavelength of GWs becomes longer than the Hubble horizon. Fig. 5.6 indicates that the GWs on initially super-horizon-scale remains constant not only on the brane but also in the bulk. The right panel of this figure, which depicts the projection of the left panel, shows that a very slight change of the amplitude is observed (a fraction of the original amplitude of $\leq 1\%$) and the amplitude finally converged to a fixed value. In this sense, the constant mode $h = \text{const.}$ is suitable for the initial condition of the super-horizon-scale GWs in the RD phase if we impose the initial condition just after the inflation.

Adopting this initial condition, we then performed simulations in the radiation-dominated FLRW case ($w = 1/3$) with the same parameters, $\epsilon_* = 1.0$ and $s_{\text{init}} = 100$. This result is shown in Fig 5.7. We found that the constancy of the GW amplitude no longer holds in the bulk even before $\tau \approx 0$, where the wavelength of GW on brane just re-enters the Hubble horizon. In particular, GWs emanating from the physical brane are observed, which propagate into the bulk spacetime almost along a null line. This indicates that the excitation of KK-modes occurs near the brane even if the wavelength of GWs is still outside the Hubble horizon.

It is noteworthy that the different behaviours of GWs in the AdS$_5$ bulk may be caused by the difference in the motion of the brane [see Eqs.(3.20) and (5.6)]. In the moving mirror problem in an electromagnetic field, the acceleration or deceleration of the mirror yields the creation of photons due to vacuum polarisation (e.g., Sec. 4.4 of Ref. [81]). A similar phenomenon may occur in the AdS$_5$ bulk, that is, the KK-modes (massive gravitons) are excited by the deceleration of the brane which is depicted by an arc with a non-zero curvature $d^2z_b/d\tau^2 < 0$.

Figs. 5.6 and 5.7 reveal that the constant mode $h(t_{\text{init}}, \xi) = \text{const.}$ can be used as the initial condition if we set this just after the end of inflation, but, the constancy of the long-wavelength mode would not be guaranteed in the RD epoch even on super-horizon scales. This implies that the choice of the initial time $t_{\text{init}}$ (or $s_{\text{init}}$) defined in (5.18) is crucial when setting the initial condition at the RD epoch.

In Figs. 5.8 and 5.9, the dependence of the evolution of GWs on the initial time is shown by varying the parameter $s_{\text{init}}$ in low-energy ($\epsilon_* = 0.1$) and high-energy ($\epsilon_* = 10$) cases. Fig. 5.8 plots the snapshots of the amplitude $h(\tau, z)$ in the bulk when the wavelength of GWs becomes five times larger than the Hubble horizon, i.e., $aH/k = 5$. Clearly, in
Figure 5.6: The evolution of a GW in the bulk in the case of a de Sitter brane. We set the Hubble parameter to $H \ell = \sqrt{3}$ with $(s_{\text{init}}, z_{\text{reg}}) = (10, 20)$. The right panel depicts the projection of the three-dimensional waves of the left panel, zooming in the image in $0 \leq z/\ell \leq 3$. The empty corner in the surface represents the motion of the brane [see Eq.(5.6)].

Figure 5.7: The evolution of a GW in the bulk in the case of a Friedmann brane. We set the comoving wave number to $k = \sqrt{3}/\ell$ or $\epsilon_* = 1.0$ with $(s_{\text{init}}, z_{\text{reg}}) = (200, 80)$. The right panel depicts the projection of the three-dimensional waves of the left panel. The empty corner in the surface represents the motion of the brane [see Eq.(3.20)].

In the bulk, the amplitude of GWs is very sensitive to the choice of the parameter $s_{\text{init}}$, or equivalently, the initial time $t_{\text{init}}$. The resultant wave-form away from the physical brane does not show any convergence even in the low-energy case ($\epsilon_* = 0.1$). This behaviour may be caused by the fact that the constant mode with the comoving wave number $k$ in the RD epoch immediately starts to oscillate as $h(\tau, z) \propto e^{ik\tau}$, which is the massless ($m \to 0$) limit of Eq. (5.11).

On the other hand, in Fig. 5.9, the GW amplitudes tend to converge on the brane if we set the initial time $t_{\text{init}}$ early enough. This convergence property might be due to the
presence of the junction condition (5.10). Therefore, as far as we choose \( s_{\text{init}} \gtrsim 50 \) for our interest of the energy scale \( 0.01 \lesssim \epsilon_a \lesssim 100 \), we do not need to care about the initial time, when we estimate the IGWB spectra on the brane. In Appendix 5.6.3, quantitative aspects of the convergence properties of the amplitude are discussed. Moreover, Seahra addressed these points in an analytic way in Ref. [41].

![Figure 5.8: Snapshots of the GW amplitudes in the bulk for various choices of initial time. The snapshots were taken when the wavelength of GWs becomes five times longer than the Hubble horizon, i.e., \( aH/k = 5 \).](image1)

![Figure 5.9: Evolved results of GWs projected on the brane starting with the various initial times.](image2)
5.6 IGWB spectra

5.6.1 Comparison with reference waves

Keeping the results in Sec. 5.5 in mind, let us now quantitatively estimate the high-energy effects of the GWs and evaluate the energy spectra of the IGWB on the brane. To quantify these, it is helpful to discriminate the influence of KK-mode excitation in the bulk from the non-standard cosmological expansion caused by the $\rho^2$-term in the Friedmann equation (3.60). For this purpose, we introduce the reference wave $h_{\text{ref}}$, which is a solution of the four-dimensional evolution equation of the amplitude obtained by replacing the scale factor and the Hubble parameter derived from the standard Friedmann equation with those from the modified Friedmann equation (3.60). The resultant equation is given by

$$\ddot{h}_{\text{ref}} + 3H\dot{h}_{\text{ref}} + \left(\frac{k}{a}\right)^2 h_{\text{ref}} = 0,$$

(5.21)

which is just the Klein-Goldon equation for a scalar field in the FLRW background (e.g. [84, 97]) and is same as (4.80) for $m = 0$ [see also Eqs. (2.143) and (4.71)]. Comparing the numerical simulations with the solution of the wave equation (5.21), the effect of the KK-mode excitation can be quantified separately.

![Figure 5.10: Squared amplitude of GWs on the brane in low-energy (left) and the high-energy (right) regimes. In both panels, solid lines represent the numerical solutions of wave equation (5.9). The dashed lines are the amplitudes of reference wave $h_{\text{ref}}$ obtained from equation (5.21).](image)

Fig. 5.10 shows the squared amplitude of the GWs, $h_{5D}^2$ and $h_{\text{ref}}^2$ as functions of the scale factor $a$. The left panel shows the low-energy case ($\epsilon_* = 0.1$), while the right panel
5.6 IGWB spectra

depicts the result in the high-energy regime ($\epsilon_a = 50$). As we increase the energy scale at the horizon re-entry time, the GW amplitude $h_{5D}$ becomes considerably reduced compared to the reference wave, $h_{\text{ref}}$. Since the late-time evolution of GWs simply scales as $h \propto 1/a$ in both $h_{5D}$ and $h_{\text{ref}}$, the suppression of the amplitude $h_{5D}$ is caused by the excitation of KK-modes around the horizon re-entry time. Notice that the normalised energy density at the horizon re-entry time $\epsilon_a$ is related to the observed proper frequency $2\pi f = k/a_0 = (a_s/a_0) H_a$ as

$$\frac{f}{f_{\text{crit}}} = \left(\frac{a_s}{a_{\text{crit}}}\right) \ell H_a = \left(\frac{\epsilon_{\text{crit}}}{\epsilon_a}\right)^{1/3(1+w)} \sqrt{\epsilon_a^2 + 2\epsilon_a},$$

(5.22)

where the critical frequency $f_{\text{crit}}$ is defined in (5.3) as $2\pi f_{\text{crit}} = (a_{\text{crit}}/a_0) \ell^{-1}$, and $w = 1/3$ in this case. From this relation, one expects that the KK-mode excitation is essential in the high-energy regime and the deviation from the standard four-dimensional prediction for the spectrum of IGWB becomes more prominent above the critical frequency, $f > f_{\text{crit}}$.

5.6.2 IGWB spectrum in the five-dimensional cosmology

We are in position to estimate the influence of KK-mode excitation on the shape of the spectrum. To do so, we ran simulations for the parameters listed in Table 5.1 ($w = 1/3$ case) and estimated the ratio of amplitudes $|h_{5D}/h_{\text{ref}}|$ for a different set of parameters. Note that in the simulations with $\epsilon_a \lesssim 1$, the location of regulator brane $z_{\text{reg}}$ should be set far away from the physical brane. This is because the long-term evolution is needed to follow the oscillatory behaviour.

We show the frequency dependence of the ratio in Fig. 5.11. The ratio is evaluated at the low-energy regime long after the horizon re-entry time and is plotted as a function of the frequency $f/f_{\text{crit}}$. Clearly, the ratio $|h_{5D}/h_{\text{ref}}|$ monotonically decreases with the frequency and the suppression of amplitude $h_{5D}$ becomes significant above the critical frequency $f_{\text{crit}}$. Using the data points in the asymptotic region $\epsilon_a \gg 5$, we try to fit the ratio of amplitudes with $s_{\text{init}} = 200$ to a power-law function. Employing the least-squares method, the result becomes

$$\left|\frac{h_{5D}}{h_{\text{ref}}}\right| = \alpha \left(\frac{f}{f_{\text{crit}}}\right)^{-\beta}$$

(5.23)

with $\alpha = 0.76 \pm 0.01$ and $\beta = 0.67 \pm 0.01$ (dashed line in Fig. 5.11). In Appendix 5.6.3, we calculate the ratios for various $s_{\text{init}}$ for each combination ($\epsilon_a, z_{\text{reg}}$) to check the robustness of this result.

The power-law fit (5.23) can be immediately translated to the energy spectrum of IGWB, $\Omega_{GW}$. The spectrum taking account of the KK-mode excitations is calculated as

$$\Omega_{GW} = \left|\frac{h_{5D}}{h_{\text{ref}}}\right|^2 \Omega_{\text{ref}},$$

(5.24)

where we used the fact $\Omega_{GW} \propto h^2 f^2$. As discussed in Sec. 5.2, if we neglect the effect of the KK-mode excitation, the spectrum becomes $\Omega_{\text{ref}} \propto f^{4/3}$ [See Eq.5.4]. Then, combining it
Figure 5.11: Frequency dependence of the ratio of amplitudes $|h_{5D}/h_{\text{ref}}|$ between the numerical simulation of wave equation (5.9) and the reference wave (5.21). The vertical solid line represents the critical frequency. The dashed line indicates the fitting result (5.23), where fitting was examined using the data with $s_{\text{init}} = 200$ at the asymptotic region $\epsilon_* \geq 5$.

Table 5.1: Numerical parameters used for the simulations to estimate the frequency dependence of the ratio of amplitudes $|h_{5D}/h_{\text{ref}}|$.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$(\epsilon_*, z_{\text{reg}})$</th>
<th>$s_{\text{init}}$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(0.1, 200\ell)$ $\sim (100, 50\ell)$</td>
<td>50, 100, 200, 400</td>
<td>2048 or 4096</td>
</tr>
<tr>
<td>1/3</td>
<td>$(0.01, 200\ell)$ $\sim (100, 25\ell)$</td>
<td>50, 100, 200, 400, 800</td>
<td>2048 or 4096</td>
</tr>
<tr>
<td>1</td>
<td>$(0.1, 500\ell)$ $\sim (50, 50\ell)$</td>
<td>200, 400, 800, 3200, 12800</td>
<td>2048 or 4096</td>
</tr>
</tbody>
</table>

with the result (5.23), the IGWB spectrum becomes nearly flat above the critical frequency:

$$\Omega_{GW} \propto f^0,$$

which is shown in filled squares in Fig. 5.12. In this figure, the spectrum calculated from the results of the reference waves $\Omega_{\text{ref}}$ is also shown in filled circles. Note that the normalisation factor of the spectrum is determined so as to be $\Omega_{GW} = 10^{-14}$ according to the constraint from the CMB observation. The short-dashed line and the solid line represent each asymptotic behaviour in the high-frequency region. The spectrum taking account of the two high-energy effects seems almost indistinguishable from the standard four-dimensional prediction shown in long-dashed line in the figure. In other words, while the effect due to the non-standard cosmological expansion lifts up the spectrum, the KK-mode effect reduces the GW amplitude, which results in the same spectrum as the one predicted in the four-dimensional theory. Additionally notice that the amplitude taking account of the two effects...
near $f \approx f_{\text{crit}}$ is slightly decreasing, which agrees with the results in our previous study for $\epsilon_\ast \lesssim 0.3$ using the GN coordinates [33]. At this point, however, it is unclear whether the results obtained here is generic or accidental for certain range of the model parameters. To clarify the cosmological dependence of the KK-mode excitations in more quantitative way, we next study the cases with different EOS parameter $w$.

![Image](image.png)

Figure 5.12: The energy spectrum of the IGWB around the critical frequency. The filled circles represent the spectrum caused by the non-standard cosmological expansion of the universe. Taking account of the KK-mode excitations, the spectrum becomes the one plotted as filled squares. Particularly, in the asymptotic region depicted in the solid line, the frequency dependence becomes almost same as the one predicted in the four-dimensional theory (long-dashed line).

### 5.6.3 Initial time dependence

As seen in 5.5.2, the amplitudes on the brane becomes insensitive to the choice of the initial time as long as $s_{\text{init}}$ is large enough. In this appendix, we show that the ratios of amplitudes discussed in 5.6.2 tend to converge to a fixed value in the low-energy and high-energy cases. This validates the estimation of the power spectrum of the IGWB using the results obtained from the simulations with $s_{\text{init}} = 200$.

Fig. 5.13 shows the dependence of the ratios $h_{5D}/h_{\text{ref}}$ on the parameter $s_{\text{init}}$. The left and right panels show the high-energy ($\epsilon_\ast = 50$) and the low-energy ($\epsilon_\ast = 0.1$) cases, respectively. In both cases, the ratios clearly converge to certain asymptotic values as increasing $s_{\text{init}}$. Using the nonlinear least-square method, we tried to fit these values to the function

$$\frac{h_{5D}}{h_{\text{ref}}} = A s_{\text{init}}^B + C,$$

(5.26)
where $A, B, C$ are fitting parameters depending on $\epsilon_\ast$. Then we obtained

$$
(A, B, C) = \begin{cases} 
(0.566, -0.573, 0.772) & \text{for } \epsilon_\ast = 0.1, \\
(0.456, -0.880, 0.118) & \text{for } \epsilon_\ast = 50,
\end{cases} \tag{5.27}
$$

which are shown in solid curves in Fig. 5.13. Note that $C$ represents the asymptotic values shown in long-dashed lines in each panel.

Figure 5.13: Convergence of the ratio of amplitudes $h_{5D}/h_{\text{ref}}$. Solid lines represent the fitting curve calculated by employing the nonlinear least-square method. Long-dashed lines denote the asymptotic value of the ratio.

Picking up the values at $s_{\text{init}} = 200$ in both cases, one can see that the deviations from the asymptotic values $C$ keep to be less than a few percent. From this fact, we use the ratios with $s_{\text{init}} = 200$ to construct the power spectra of the IGWB without deriving the asymptotic values for each $\epsilon_\ast$.

### 5.6.4 Dependence on equation of state

To quantify the EOS dependence, we ran simulations for the MD case $w = 0$ and the somewhat stiff matter case $w = 1$, which might be realised by introducing the kinetically driven scalar field (e.g. the quintessential inflation [152, 153]). Varying the EOS parameter changes the acceleration of the brane. One naively expects that the different motion of the brane may suppress or enhance the KK-mode excitation.

With the same procedure as in the previous subsection, we calculated the ratio of amplitudes $|h_{5D}/h_{\text{ref}}|$ for various $\epsilon_\ast$ in the case of $w = 0$ and $w = 1$. The results are summarised in Fig. 5.14, where the horizontal axis represents $\sqrt{1 + H^2 \xi^2}$. Fitting the power-law function
to all cases shown in the figure, we found that the ratios universally scale as
\[
\frac{|h_{3D}|}{h_{\text{ref}}} = \tilde{\alpha}(1 + H_s^2 \ell^2)^{-0.24} \approx \tilde{\alpha}(1 + H_0^2 \ell^2)^{-1/4},
\]
(5.28)
where \(\tilde{\alpha} = 0.75, 0.81\) and 0.83 for \(w = 0, 1/3\) and 1, respectively, which indicates that the quantity \(\tilde{\alpha}\) may be related to \(w\) as \(\tilde{\alpha} = -0.155w^2 + 0.241w + 0.748\) by simply fitting the quadratic function. An important point to emphasise is that this scaling property does not depend on the parameter \(w\) or the acceleration of the brane. Combining the scaling relation with (5.24), the IGWB spectra can be estimated as
\[
\Omega_{\text{GW}} = \tilde{\alpha}^2(1 + H_s^2 \ell^2)^{-1/2} \Omega_{\text{ref}}.
\]
(5.29)
In particular, in the high-frequency region \(f \gg f_{\text{crit}}\), the prefactor of the right-hand side behaves as
\[
(1 + H_s^2 \ell^2)^{-1/2} \approx (H_s \ell)^{-1} \propto f^{-\frac{3(w+1)}{3w+2}}
\]
(5.30)
from equations (2.171),(2.172) and (5.1). Combining the upper case of Eq. (5.2), the energy spectrum (5.29) behaves as
\[
\Omega_{\text{GW}} \propto f^{\frac{3w-1}{3w+2}} \text{ for } f \gg f_{\text{crit}}.
\]
(5.31)
Owing to these calculations (5.2) and (5.31), for \(w = 0\) (MD), we can observe the KK-mode effects on the spectra in the higher frequency region:
\[
\Omega_{\text{ref}} \propto \begin{cases} f^1, & \text{for } f \gg f_{\text{crit}}, \\ f^{-2}, & \text{for } f \ll f_{\text{crit}}, \end{cases}
\]
(5.32)
and for \(w = 1\) case,
\[
\Omega_{\text{ref}} \propto \begin{cases} f^{8/5}, & \text{for } f \gg f_{\text{crit}}, \\ f^1, & \text{for } f \ll f_{\text{crit}}. \end{cases}
\]
(5.33)
These results imply that the spectrum generally changes from the four-dimensional prediction. Indeed, transforming the numerical results (5.28) to the energy spectra in the cases with \(w = 0\) and \(w = 1\), the frequency dependence of the spectra including the two high-energy effects (solid lines) clearly differ from each four-dimensional prediction (long-dashed lines).

Taking these results into account, one may conclude that the cancellation of the high-energy effects in the RD epoch is accidental and the KK-mode excitation dominates over the non-standard cosmological expansion when \(w > 1/3\).

5.7 Conclusion

We have investigated the power spectrum of the IGWB in the five-dimensional cosmology based on the Randall–Sundrum model. In the braneworld scenario, the two high-energy
Figure 5.14: EOS dependence on the amplitude ratio. The filled circles, squares and triangles show the cases with $w = 0, 1/3$ and 1, respectively. The solid line, short-dashed line and long-dashed line show each fitting result above the critical frequency corresponding to $(1 + H_{\text{crit}}^2 l^2)^{1/2} = \sqrt{2}$ depicted in the vertical dot-dashed line.

Figure 5.15: The energy spectrum of the IGWB in the background EOS with $w = 0$ (left), and with $w = 1$ (right). The amplitude is normalised by the value of the four-dimensional prediction at the critical frequency.

effects affect the shape of the spectrum above the critical frequency $f_{\text{crit}}$ defined in (5.3). One is the non-standard cosmological expansion on the brane caused by the high-energy correction of the Friedmann equation. The analytical estimate taking account of this effect
reveals that the effect makes the spectrum steeply blue [see Eq. (5.4)]. By contrast, another important effect is the excitations of KK-modes which escapes from our brane into the five-dimensional bulk, leading to the suppression of the spectrum. In order to quantify these two effects, we solved the wave equation of each Fourier mode of GWs numerically for various EOS parameters $w$.

The systematic survey of numerical simulations with various parameter sets reveals that there may exist the universal scaling law for the KK-mode excitation in the high-energy regime [Eq. (5.28)]:

$$\frac{h_{5D}}{h_{\text{ref}}} \propto (1 + H^2 t^2)^{-1/4}.$$  \hfill (5.34)

Using the universal scaling law, we constructed the power spectrum of the IGWB in the cases with $w = 0$ (MD universe), $w = 1/3$ (RD universe) and $w = 1$ (stiff matter dominant universe). From the results (2.176) and (5.31), the frequency dependence of the spectrum in the high-frequency region $f \gg f_{\text{crit}}$ becomes

$$\Omega_{\text{GW}} \propto \begin{cases} f^{-2} & \text{for } w = 0, \\ f^{0} & \text{for } w = 1/3, \\ f^{1} & \text{for } w = 1. \end{cases}$$  \hfill (5.35)

Particularly, in the RD case, the accidental cancellation of the two high-energy effects occurs, which yields the same spectrum as one predicted in the four-dimensional (4D) theory. This scaling law might be understood in a context of moving mirror problems. The discussion about the analytic derivation of the scaling law is work in progress.

Finally, we briefly comment on the other numerical works using different numerical schemes.

The numerical calculation based on the quantum theory has been performed by T. Kobayashi and T. Tanaka [40]. They reported the same spectrum in the RD case as ours, even if KK modes are taken into account in the initial de Sitter phase. After that, Kobayashi also calculated the cases for $w = 0, 2/3, 1$ and obtained a result which coincides with ours shown in Eq. (5.31) [149].

Moreover, S. S. Seahra recently solved numerically the wave equation in the null coordinate system based on the Poincaré coordinates using a sophisticated numerical scheme [41]. He observed the agreement between his numerical results and ours for $w = 1/3$ with the same initial conditions as ours. In addition, it is observed that, even if a constant initial condition on the initial null hypersurface is chosen, the amplitude of GWs on the brane is almost identical with our results. He also showed in analytical way why the final amplitude of GWs is insensitive to a choice of the initial time, $s_{\text{init}}$, as long as $s_{\text{init}}$ is enough large as we discussed in Sec. 5.6.3.

On the other hand, K. Ichiki and K. Nakamura have obtained a tilted spectrum $\Omega_{\text{GW}} \propto f^{-0.46}$ [38]. While their early results have included errors associated with numerical accuracy, the new calculation using the revised code did not converge to the flat spectrum either. Currently, we do not know the reason why the result by K. Ichiki and K. Nakamura is different.
from ours. In order to understand these numerical results well, the analytical study of the scaling relation (5.28) is essential, which is definitely our next task.
Chapter 6
Numerical Studies on Scalar Perturbations – Inflaton Perturbations

6.1 Introduction

We dealt with IGWB in the context of a braneworld scenario in the previous chapter. We go on to consider the inflaton perturbations. A simple way to study the effects of the higher-dimensional gravity on inflation models is to consider a 4D inflaton field confined to the brane in the context of the Randall–Sundrum model [139]. Work to date has concentrated mainly on deriving corrections to the standard 4D result by assuming that 5D metric perturbations generated by the inflaton perturbations can be neglected. Then the modification of spectrum is coming from the modified Friedmann equation (3.60). Although a lot of interesting results have been obtained in this direction [42–48], it remains to see the consistency of neglecting the bulk metric perturbations. The analysis of the evolution of gravitational waves in this model has revealed that the creation of massive graviton modes in the bulk is as important as the modification of the Friedmann equation [33, 37–41, 149]. In fact, these two higher-dimensional effects cancel out to give a nearly identical spectrum with the conventional 4D models for the stochastic background of gravitational waves in the radiation dominated universe [54]. This highlights the importance of taking into account the higher-dimensional effects consistently [see Eq. (5.25) and Fig. 5.12].

The study of the effects of bulk metric perturbations has been initiated in Ref. [49, 50] (see [51] for a different approach). Using the approximation that the geometry of the brane is approximated by de Sitter spacetime for a slow-roll inflation, the bulk metric perturbations sourced by inflaton fluctuations on the brane were calculated. It turns out that the inflaton fluctuations excite an infinite ladder of massive modes of bulk metric perturbations. Then Ref. [50] tried to estimate the effects of the backreaction of bulk metric perturbations. Assuming that the inflaton field behaves in the same ways as 4D models on sufficiently
small scales at the 0-th order of slow-roll parameters, the corrections to the evolution of inflaton perturbations at the first order of slow-roll parameters were calculated. Unlike in 4D models where the slow-roll corrections can be completely ignored on sub-horizon scales, the slow-roll corrections play a role even on sub-horizon scales, yielding the change of the amplitude at a horizon crossing. This result may not be so surprising as the bulk metric perturbations are strongly coupled to inflaton perturbations on small scales where gravity becomes 5D. However, this indicates that we should carefully check the validity of using the 4D formula just by modifying the background Friedmann equations.

This chapter focuses on the classical evolution of inflaton perturbations coupled to bulk metric perturbations (cf. the four-dimensional case in Sec. 2.3.7). We do not assume de Sitter geometry on the brane and we take into account the backreaction of inflaton dynamics consistently. This entails a numerical analysis to solve the coupled system directly. We investigate whether the inflaton perturbations behave as free massless fields on small scales.

The structure of this chapter is as follows. In Sec. 6.2, we summarise basic equations to describe the evolution of inflaton perturbations coupled to metric perturbations, which has been already reviewed in Ch. 4. We use the Hawkins–Lidsey model of the inflation on the brane where the background solution is derived analytically [141] which has been discussed in Sec. 3.4.3. In Sec. 6.3, we explain initial conditions and boundary conditions to integrate the evolution equations by means of a scheme discussed in Appendix C. Then we study the evolution of the comoving curvature perturbations defined in Eq. (2.95), which are directly related to observables.

### 6.2 Summary of basic equations

#### 6.2.1 Background spacetime

In this section, we briefly revisit the formulations for the scalar perturbations discussed in Ch. 4.

We use the Gaussian–normal (GN) coordinates for the bulk spacetime. The warp factor and the lapse function are given by Eqs. (3.71) and (3.72). The Friedmann equation is given by

\[ H^2 = \frac{\kappa^2}{3} \rho \left( 1 + \frac{\rho}{2\lambda} \right), \tag{3.60} \]

in which we assume again that the cosmological constant \( \Lambda_4 \) and the dark radiation \( C \) are neglected in a flat \( (K = 0) \) universe. We consider a case that a scalar field driving the inflation becomes dominant component of the energy density on the brane, whose energy–momentum tensor is given in Eq. (2.18). In this situation, the energy density and pressure on the brane are given by Eqs. (2.19) and (2.20). In order to specify the inflaton potential, we use a brane inflation models studied by R. M. Hawkins and J. E. Lidsey [141], which enables us to avoid numerical integrations of the background equations as we discussed in
Sec.3.4.3. One of their proposed models is characterised by a potential energy given by
\[ V(\phi) = \frac{6 - C^2}{3} \lambda \cosh^2 \left( \frac{\kappa_4 C}{2} \phi \right). \] (3.104)

With this model potential, a power-law inflation controlled by a parameter \( C \) is realised on the brane. The scale factor grows as
\[ a_b(t) = \left[ (C^2 \mu t + 1)^2 - 1 \right]^{1/C^2}. \] (3.115)

### 6.2.2 Bulk metric perturbations

We choose the 5D-longitudinal gauge for the scalar perturbations [see Eqs. (4.21) and (4.22)]. In this gauge, the metric with scalar perturbations (4.1) is simplified as
\[
^{(5)}g_{AB} = \begin{pmatrix}
-n^2(1 + 2\tilde{A}) & 0 & n\tilde{A}_y \\
0 & a^2(1 + 2\tilde{R})\delta_{ij} & 0 \\
n\tilde{A}_y & 0 & 1 + 2\tilde{A}_{yy}
\end{pmatrix}.
\] (6.1)

The perturbative quantities, \( \tilde{A}, \tilde{R}, \tilde{A}_y \) and \( \tilde{A}_{yy} \), can be rewritten in terms of a master variable \( \Omega \), which are given by Eqs. (4.28)–(4.31). The master variable obeys an evolution equation:
\[
\ddot{\Omega} - \left( \frac{\dot{H}}{n} + 3\frac{\dot{a}}{a} \right) \dot{\Omega} - n^2 \left[ \Omega'' + \left( \frac{n'}{n} - 3\frac{a'}{a} \right) \Omega' + \left( \frac{\mu^2}{a^2} - \frac{k^2}{a^2} \right) \Omega \right] = 0.
\] (4.33)

### 6.2.3 Inflaton perturbations

The inflaton perturbation on the brane is defined by Eq. (2.100). The perturbation contributes the energy density and pressure on the brane as shown in Eqs. (2.105)–(2.107). Introducing a gauge-invariant variable \( Q \), Mukhanov–Sasaki variable, defined in Eq. (2.110), the evolution equation becomes one derived in four-dimensional theory with an additional term \( J(\Omega) \):
\[
\ddot{Q} + 3H\dot{Q} + \frac{k^2}{a_b^2} Q + \left\{ \frac{\ddot{H}}{H} - 2 \frac{\dot{H}}{H} \frac{V'(\phi)}{\phi} - 2 \left( \frac{\dot{H}}{H} \right)^2 + V''(\phi) \right\} Q = J(\Omega),
\] (4.73)

where
\[
J(\Omega) = -\frac{\dot{\phi}}{H} \left[ \left( \frac{\dot{H}}{H} + \frac{\dot{H}}{H} \right) \kappa_4^2 \delta q_\xi + \frac{1}{3} \left( 1 - \frac{\dot{H}}{2H^2} \right) \kappa_4^2 \delta \rho_\xi + \frac{1}{3} k^2 \kappa_4^2 \delta \pi_\xi + \frac{1}{3} \frac{\dot{H}}{H} \frac{k^2}{a_b^2} \tilde{R} \right].
\] (4.74)

Thus, in the five-dimensional case, the Mukhanov–Sasaki equation has a source term \( J(\Omega) \) induced by the bulk metric perturbations. We well see that this term plays an important role to affect the evolution of inflaton perturbations.
Note that, according to Eq. (2.108), the inflaton perturbation does not produce anisotropic stress, $\delta \pi^S = 0$. This means that each component of the induced metric perturbations in 4D-longitudinal gauge (4.54) is equivalent to the corresponding component of the bulk metric perturbation evaluated at the brane in the 5D-longitudinal gauge as we mentioned in Eqs. (4.56) and (4.57).

### 6.2.4 Junction conditions

From the (0, i) component of the effective Einstein equation (4.63) with Eqs. (2.107), (4.75) and (4.76), we can relate $Q$ and $\Omega$ as

$$Q = \frac{1}{\kappa_5^2 \phi a_h} \left[ \dot{\Omega} - \mathcal{H} \dot{\Omega} - \frac{\dot{H}}{\mathcal{H}} \left( \mathcal{H} \ddot{\Omega} - \mu^2 \Omega + \frac{k^2}{3a_h^2} \Omega \right) \right],$$

which gives a non-local boundary condition for $\Omega$. Alternatively, we can combine Eqs. (4.61) and (4.63) to get

$$a_h \kappa_5^2 (\delta \rho - 3H \delta q) = -\frac{k^2}{a_h^2} \left( \Omega - \frac{\dot{\Omega}}{\mathcal{H}} + \frac{\Omega}{\mathcal{H}} \right).$$

Then using Eqs. (2.105) and (2.107) and rewriting $\delta \phi$ by $Q$, we obtain

$$\kappa_5^2 a_h \phi^2 \left( \frac{H}{\phi} Q \right) = \frac{k^2}{a_h^2} \left( \frac{\phi^2}{6} \left( \dot{\Omega} - \mathcal{H} \Omega \right) + H \left( \Omega' - \frac{\dot{\Omega}}{a} \right) \right).$$

Note that, combining Eqs. (4.61) - (4.63), one can derive a boundary condition for $\Omega$ which does not include $Q$ [147]. This boundary condition is equivalent to choose (4.73) and (6.2) as boundary conditions.

In summary, the master equation (4.33), the Mukhanov–Sasaki equation (4.73) with a source (4.74) and the junction condition (6.2) or (6.4) form a closed system for the coupled inflaton and metric perturbations.

### 6.3 Evolution of curvature perturbations

#### 6.3.1 Initial conditions and boundary conditions

First of all, in order to avoid the effect from the coordinate singularity at $y = y_c$ in the GN coordinate [see Eq. (3.73)], we introduce an artificial cutoff (regulator) boundary in the bulk at $y = y_{\text{reg}} = \gamma y_c$ shown in the right panel of Fig. 3.2 as a long-dashed line. The constant $\gamma$ controls the position of the regulator brane. Since the regulator brane should be far from the physical brane and $\gamma < 1$, we set $\gamma = 0.99$ in our numerical simulations. We discuss the boundary condition at the regulator brane in the next section.

The boundary condition at the brane is given by the junction condition Eq. (6.2) or Eq. (6.4). At the regulator brane, we use a Dirichlet boundary condition

$$\Omega|_{y = y_{\text{reg}}} = 0.$$
This is because the analysis in de Sitter brane background [49,50] suggests that the inflaton fluctuation excites the bound states that decays quickly away from the brane. If we focus on this bound state the result should not be sensitive to the location of the regulator brane. We will check this fact numerically later. Different boundary conditions from Eq. (6.5) may produce reflection waves from a regulator brane, which affect the evolution of \( Q \). A proper boundary condition should be imposed on the past Cauchy horizon. This requires a quantisation of 5D metric perturbations in the bulk, which will turn out to be a highly non-trivial problem due to the coupling to the inflaton perturbations.

In 4D cases, \( Q \) satisfies
\[
\ddot{Q} + 3H\dot{Q} + \frac{k^2}{a_b^2}Q = 0, \tag{6.6}
\]
on small scales since \( k^2 \) term is dominant over the mass term induced by the dynamics of the inflaton [cf. see (2.121) and (4.73)]. Thus the solution on small scales is given by simple plane waves and we can choose the usual 4D Bunch–Davis vacuum state, which specifies an appropriate phase and normalisation. On the other hand, in the present 5D case, it is not guaranteed that the perturbed inflaton field obeys the usual 4D Klein-Gordon equation due to the coupling to the bulk metric perturbations, which is described by \( J(\Omega) \) in the right-hand side of Eq. (4.73). In order to obtain the appropriate initial conditions for \( Q(t) \), we again need a quantisation of the coupled system of the inflaton perturbations on a brane and 5D metric perturbations in the bulk.

The aim of this chapter is to check whether we can neglect \( J(\Omega) \) on small scales or not. If this term could be neglected on the small scales, the quantity \( a_bQ \) would behave just as plane waves with a constant amplitude as in the standard 4D cases. We call the solution of Eq. (4.73) with \( J(\Omega) = 0 \) or, equivalently, Eq. (6.6) on small scales 4D solutions. If the solutions for the coupled equations agree with 4D solutions, we can assume that the inflaton perturbation behaves as a free massless field without taking into account the effect of 5D metric perturbations, which is the assumption made in lots of literature.

For this purpose, we take the simplest possible initial conditions for \( \Omega(y, t) \) and then specify initial conditions for \( Q(t) \) so that they are consistent with the junction condition (6.4). We set
\[
\begin{align*}
\Omega(y, t_{\text{init}}) &= 0, \quad \dot{\Omega}(y, t_{\text{init}}) = 0, \quad Q(t_{\text{init}}) = 1. \tag{6.7}
\end{align*}
\]
Then the time-derivative of \( Q(t) \) is determined from the junction condition (6.4). Note that our initial conditions do not give standard quantum vacuum state for 4D solutions. They are merely references to see the effects of \( J(\Omega) \) on a classical evolution of \( Q(t) \).

We employ the spectral collocation method for the integration of the master equation (4.33), discussed in Appendix C. The number of the collocation points is set to be \( N = 64 \) or \( N = 128 \). We adopt the predictor–corrector method as a temporal evolution method [see Sec. C.2], and evolve simultaneously the Mukhanov–Sasaki equation (4.73) and the master equation.
6.3.2 Evolution of curvature perturbations

We solve numerically the wave equation for the bulk metric perturbations (4.33) in the GN coordinates with the junction condition (6.4) supplemented by the evolution equation for inflaton perturbations (4.73). We set the parameter of the Hawkins–Lidsey model as \( C/10 \) and the initial time for the simulations as \( \mu t_{\text{init}} = 40 \). At the initial time, the Hubble scale is given by

\[
\frac{H(t_{\text{init}})}{\mu} \approx 2.92.
\]  

(6.8)

Thus we are considering high-energy region \( H/\mu > 1 \). The numerical simulations were performed by means of the spectral collocation method discussed in Appendix C.

The observable is the comoving curvature perturbation defined by

\[
R_c = -\frac{H}{\phi_0} Q,
\]  

(6.9)

which coincides with Eq. (2.95). We focus on the dynamics of this quantity. Fig. 6.1 shows the behaviour of curvature perturbations for \( k/\mu = 5 \) which corresponds to \( k/a_b(t_{\text{init}})\mu \approx 296 \). The curvature perturbations \( R_c \) becomes constant on super-horizon scales, which has been shown to be valid even in brane world models [76]. On sub-horizon scales, while \( R_c \propto 1/a_b \) in the 4D cosmology, a suppression of the amplitude is observed in the braneworld model, which is due to the coupling to the bulk metric perturbations. In Fig. 6.2, we compare the amplitude of \( R_c \) obtained in numerical simulations with the one obtained by neglecting the coupling to the bulk metric perturbations, i.e. \( J(\Omega) = 0 \) in Eq. (4.73). While the difference is very small for the long-wavelength modes (left panel; \( k/a_b(t_{\text{init}})\mu = 296 \)), the suppression becomes significant for the short-wavelength modes (right panel; \( k/a_b(t_{\text{init}})\mu = 2960 \)). Due to this, the spectrum of \( R_c \) just before the horizon crossing acquires a scale dependence as is shown in Fig. 6.3. Perturbations with larger wavenumber stay on sub-horizon scales for a longer time than those with smaller wavenumber, so they receive more suppression.

The suppression of the amplitude \( R_c \) under horizon may be understood as the excitation of metric perturbations. The suppression of the amplitude of \( Q \) is transferred into the enhancement of the metric perturbations \( \Omega \) in the bulk, which is shown in the left panel of Fig. 6.4.

Here we comment on the reason why we evaluated \( R_c \) just before the horizon crossing. In the 4D case, we can choose the Bunch–Davis vacuum as an initial condition. Hence we can determine the amplitude and phase of the inflaton perturbations and obtain the spectrum of \( R_c \) without any ambiguities. On the contrary, in the present 5D case, the initial conditions cannot be determined without quantising the coupled system. Even in the classical level, the junction condition (6.2) or (6.4) restricts our choice of the initial conditions. On super-horizon scales, the initial phase of the perturbations essentially determines its final amplitude after the horizon crossing because the phase determines the amplitude of the growing mode solution on large scales. Then we get an unphysical \( k \)-dependence of the
curvature perturbations evaluated on super-horizon scales due to the restricted choice of $dQ/dt$ for given $k$. On the other hand, the initial phase is not important when we evaluate the amplitude of the curvature perturbations on small scales. Our aim is to see how $J(\Omega)$ term changes the amplitude of $Q$ in comparison with the 4D solution for various $k$ on small scales. Therefore we chose to calculate the spectrum below the horizon scales.

6.3.3 Check of numerical calculations

In this section, we first check the assumption of introducing a regulator. As expected from the previous work, $\Omega$ decays fast away from the brane as is shown in the right panel of Fig. 6.4. This justifies the use of boundary condition $\Omega(y_{\text{reg}}, t) = 0$. In addition, thanks to this behaviour of $\Omega$, the evolution of the curvature perturbations does not depend on the location of the regulator. In fact, even at late times when the information from the regulator brane comes into the brane, the behaviour of the curvature perturbations does not change. Thus our result is not obscured by the regulator brane nor the existence of the coordinate singularity of the GN coordinate.

Finally, we check the accuracy of our numerical simulations by checking the constraint equations derived from the Einstein equations. These equations can be generally expressed as

$$\mathcal{L}(t) = \sum_{k} C_k(t) f_k(t) = 0,$$

(6.10)

where

$$f_k(t) = \{ \dot{\mathcal{R}}_b, \dot{\mathcal{R}}_b, \mathcal{R}_b, \dot{A}_b, A_b, \delta \rho_\Sigma, \delta q_\Sigma, \delta \pi_\Sigma \},$$

(6.11)
Numerical Studies on Scalar Perturbations – Inflaton Perturbations

Figure 6.2: The curvature perturbations (multiplied by the scale factor) in the inflationary epoch for a long-wavelength mode (left; \( k/a_b(t_{\text{init}}) \mu \approx 296 \)) and for a short-wavelength mode (right; \( k/a_b(t_{\text{init}}) \mu \approx 2960 \)). The solid lines represent numerical results and the dashed lines show the 4D predictions obtained by neglecting the coupling to the bulk metric perturbations, that is, by solving Eq. (4.73) with \( J(\Omega) = 0 \). We set \( C = 0.1 \) and \( \mu t_{\text{init}} = 40 \).

and \( C_k(t) \) is a set of their coefficients. We calculated the relative error for the constraint equations (4.61) (4.63) and (4.64) as

\[
E(t) = \frac{|\mathcal{L}(t)|}{\max_k |C_k(t)f_k(t)|}.
\]  

(6.12)

Note that the constraint equation (4.62) contains higher derivative terms of \( \Omega \) like \( \dddot{\Omega} \), which are not solved in our simulations. Hence it is impossible to evaluate the numerical accuracy of (4.62) from our numerical simulation. In fact, if we rewrite the higher-derivative terms of \( \Omega \) by \( Q \) using Eq. (6.2), Eq. (4.62) becomes an identity.

Fig. 6.5 shows that the relative error of Eq. (4.61) is suppressed to less than \( 10^{-5} \) and Eqs. (4.63) and (4.64) are very precisely satisfied, indicating that our numerical scheme for solving the coupling system (4.33) and (4.73) is reliable. Note that the constraint equation (4.64) has a different character from the others, which measures the numerical accuracy of the master equation (4.33) on the brane.

6.4 Conclusion and discussion

In this chapter, we studied the evolution of the curvature perturbations in a braneworld inflation model where an inflaton is living on a single brane in a 5D AdS spacetime. We used the Hawkins–Lidsey model of the inflaton potential which enables us to derive the
background solutions analytically. We solved the full coupled evolution equations for the inflaton perturbations described by the Mukhanov–Sasaki variable and the bulk metric perturbations described by the master variable. This is the first numerical result for the evolution of scalar curvature perturbations in a braneworld that consistently takes into account the backreaction of the metric perturbations.

We used the GN coordinate to describe the bulk spacetime that has a coordinate singularity. Then we are forced to introduce a regulator brane to cut the spacetime. We have checked that the evolution of curvature perturbations is insensitive to the location of the regulator. This is because the bulk metric perturbations decay fast away from the brane. We adopted the simplest possible initial condition for $\Omega$ by taking $\Omega = 0$ and $\dot{\Omega} = 0$ at an initial time. Then we fix $Q$ so that the boundary condition is consistent. In order to check whether we can consistently neglect the coupling to bulk metric perturbations described by $J(\Omega)$ in Mukhanov–Sasaki equation (4.73) or not, we followed the subsequent evolution of curvature perturbations numerically.

The evolution of curvature perturbations showed a suppression of the amplitude compared with the conventional 4D model even on sub-horizon scales. This means that it is impossible to neglect a coupling to gravity even on small scales due to the coupling to the higher-dimensional gravity through $J(\Omega)$. The suppression of the amplitude may be understood as a loss of energy due to the excitation of the bulk metric perturbations as we took the initial condition that $\Omega(y, t_{\text{init}}) = 0$ and $\dot{\Omega}(y, t_{\text{init}}) = 0$. On super-horizon scales,
the curvature perturbations become constant, which confirms the fact that the constancy of the curvature perturbations is independent of gravitational theory for adiabatic perturbations [76].

Our result suggests that an usual assumption that the inflaton perturbations (the Mukhanov–Sasaki variable) approach to a free massless field on small scales cannot be applied in a brane world models on small scales at high energies. Then the spectrum becomes sensitive to initial conditions not only for the inflaton perturbations but also for bulk metric perturbations. In this sense, we should not take the result Fig. 6.3 at face value because this result is based on a special choice of initial conditions. In order to determine the initial conditions without ambiguity, it is required to quantise the coupled brane (inflaton)-bulk (metric perturbations) system in a consistent manner along the line of [148, 154]. This is an outstanding open question to be addressed in order to derive the spectrum of the inflaton perturbations in a braneworld.

Nevertheless, before closing this chapter, we evaluate the running of the spectrum of the curvature perturbations observed today to quantify the bulk effects. Assuming that the inflation begins when the scale factor $a_b = a_{\text{init}}$, and lasts until the e-folding number becomes $N \sim 60$, the present physical wave number of the perturbation with a given physical scale
Figure 6.5: The numerical errors of constraint equations (4.61)[upper-left], (4.63)[upper-right] and (4.64)[lower]. These plots are results of calculations for $k/a(t_{\text{ini}})\mu = 296$.

at the beginning of inflation, $k_\ast/\mu a_{\text{ini}}$, becomes

$$
\frac{k_\ast}{a_0} = \frac{k_\ast}{a_{\text{ini}}} \frac{a_{\text{end}} a_{\text{eq}}}{a_{\text{ini}} a_{\text{end}} a_{\text{eq}}} a_0, \quad
= 2.0 \times 10^{-3} \text{ Mpc}^{-1}
\times \left( \frac{k_\ast/\mu a_{\text{ini}}}{10^2} \right) \left( \frac{H_{\text{end}}}{\mu} \right)^{-1/2} \left( \frac{T_{\text{rh}}}{10^6 \text{ GeV}} \right) \left( \frac{T_0}{2.3 \times 10^{-4} \text{ eV}} \right)^{-1} \times \left( \frac{0.72}{h_0} \right) \left( \frac{z_{\text{eq}}}{3300} \right)^{-1/4} \left( \frac{e^{-N}}{e^{-60}} \right), \quad (6.13)
$$

where $H_{\text{end}}$ the Hubble parameter at the end of inflation, $T_{\text{rh}}$ the reheating temperature, $T_0$ the present temperature of the universe, $h_0$ the present reduced Hubble parameter, $z_{\text{eq}}$ the redshift of matter-radiation equality, and $N$ the e-folding number.
Let us consider to fit the spectrum, Fig. 6.3, to a functional form

\[
\left| \frac{\mathcal{R}_{\text{D}}}{\mathcal{R}_{\text{E}}^{\text{D}}} \right| = \left( \frac{k}{k_\ast} \right)^{n_S - 1 + \frac{dn_S}{d\log k}} \left| \frac{d\log k}{k - k_\ast} \right| \log \frac{k}{k_\ast}. \tag{6.14}
\]

Assuming a scale-invariant spectrum with a running spectral index, \(n_S = 1\) and \(dn_S/d\log k \neq 0\), then we obtain

\[
\frac{dn_S}{d\log k} \bigg|_{k = k_\ast} = -0.0094 \pm 0.0001, \tag{6.15}
\]

where we set \(k_\ast/\mu a_{\text{init}} = 10^2\) corresponding to its physical scale shown in Eq. (6.13). Although our numerical results strongly depend on a choice of inflation models and initial time, they indicate that the braneworld scenario may explain the negative running spectral index [12].
Chapter 7

Conclusion

7.1 Summary

In this thesis, we have discussed the scalar and tensor perturbations in the cosmological setup based on the Randall–Sundrum single brane model (RSII). In particular, we have focused on the high-energy effects on the behaviour of IGWB (tensor perturbations) and inflaton perturbations coupled to the bulk metric perturbations (scalar perturbations). One of the effects is derived from the high-energy correction to the Friedmann equation, leading to the peculiar expansion of the universe (hereafter called 'effect I' and see Eq. (3.83)). The other is due to the existence of the bulk metric perturbations, which can be interpreted as the energy loss of the perturbations on the brane (hereafter called 'effect II').

In Ch. 5, we focused on the evolution of IGWB after the inflation and estimated an observed spectrum of the IGWB. The universe after the inflation was assumed to be filled by a perfect fluid whose equation-of-state (EOS) is described by \( p = w \rho \). We performed numerical simulations with the EOS parameters, \( w = 0, 1/3, 1 \), corresponding to matter-dominated, radiation-dominated and stiff-matter-dominated universe. Then we found the following results:

- The two effects cancel each other, resulting that the spectrum of IGWB becomes almost the same as the one predicted in the standard four-dimensional theory [see Fig. 5.12].
- This cancellation occurs only in the radiation dominated case. In cases with \( w = 0, 1 \), the spectra becomes different from the four-dimensional ones [see Fig. 5.15].
- There is a universal relation between the ratio of amplitudes, \( h_{3D}/h_{\text{ref}} \), and a function \( \sqrt{1 + H^2_\ell} \) shown in Eq. (5.28) where \( H_\ell \) denotes the Hubble parameter at the horizon crossing and \( h_{\text{ref}} \) is the final amplitude of the solution of Eq. (5.21). Using the relation, we concluded that the spectrum of IGWB in the RS single brane model becomes Eq. (5.31).
These results imply that, if we assume that the standard history of the universe (inflation–radiation dominant–matter dominant), it remains difficult for us to observe the IGWB, which is constrained to $\Omega_{GW} < 10^{-14}$ by the COBE observations. Even though future observational technologies can reach such tiny signals, our present results using the RS model indicate that there may be no evidences of existence of extra-dimensions in the IGWB observations.

In Ch. 6, we have investigated the evolution of the scalar perturbations in the context of the brane inflation scenario. For this purpose, we used the model proposed by R. M. Hawkins and J. E. Lidsey, in which a power-law inflation takes place by a single scalar field confined to the brane as we discussed in Sec. 3.4.3. In order to correctly treat the backreaction of the metric perturbations, i.e., effect II, we employed a full numerical integration of the inflaton perturbations coupled with the bulk metric perturbations. The inflaton perturbation and the bulk metric perturbations are expressed by the Mukhanov–Sasaki variable, $Q(t)$, and the master variable, $\Omega(t, y)$, respectively. We adopted a simplified initial condition (6.7), in which no bulk metric perturbation is present at the initial time, to observe the effect II represented by the right-hand side of the Mukhanov–Sasaki equation (4.74), $J(\Omega) \neq 0$.

We found the following results:

• the comoving curvature perturbation (6.9) becomes constant on the super-horizon scales, shown in Fig. 6.1, which is the same feature as one observed in the standard four-dimensional theory.

• Comparing with the solution of Eq. (4.73) with $J(\Omega) = 0$, the amplitude on the sub-horizon scales is suppressed, shown in Fig. 6.2.

The latter result implies that one may not neglect a coupling to the bulk gravity on small scales. Therefore, unlike the four-dimensional case, one cannot naively treat the initial inflaton field as a free massless field (cf. Eq. 2.129). This fact means that the observed spectra of the curvature perturbations or density perturbations become sensitive to the bulk metric perturbations at the initial time.

### 7.2 Discussion and future prospects

As we mentioned in the last paragraph of the previous section, investigating the braneworld scenario from a phenomenological viewpoint requires to specify the vacuum state, namely, the initial conditions for the inflaton perturbations coupling to the bulk metric perturbations. In the brane inflation scenario, the difficulty to specify the vacuum state comes from the fact that there is an infinite ladder of massive KK-modes, while there is one-degree-of-freedom inflaton perturbation on the brane. Overcoming the difficulty is crucial to predict the spectrum of the curvature perturbations in the context of braneworld.

Here we introduce two special cases that we can specify the vacuum state. G. Dvali, G. Gabadadze and M. Porrati proposed a braneworld model with an induced Einstein–Hilbert term on the brane, called \textit{DGP model} [155]. Considering a de Sitter brane embed-
ded in a five-dimensional Minkowski bulk spacetime in the \( - \) branch of the DGP model. Ref. [156] confirmed that small scale perturbations are described by the four-dimensional Brans-Dicke theory. This result implies that we can specify the vacuum state without ambiguity based on the four-dimensional Brans-Dicke theory on sufficiently small scales. Another case is based on the recent work by K. Koyama et al [157]. They investigated the quantisation of the coupled system in the case of \( \dot{\phi} = \text{const.} \) In this limited case, they found that one can choose the vacuum state satisfying the junction condition (6.4), whose mode functions are quite different from four-dimensional ones (2.129).

We got rid of technical difficulties to perform the simulations of the scalar perturbations, which enables us to start phenomenological studies on the braneworld scenarios if we can specify the initial conditions. For example, our works can be applied to recent studies by Y. Sendouda et al. discussing the primordial blackholes in the RS model [158–163].

As for the tensor perturbations, the numerical simulations for those evolutions in the \( \text{AdS}_5 \) bulk spacetime showed that the brane motion excites the KK-modes. Moreover, there is a universal relation between the ratio \( |h_{5D}/h_{\text{ref}}| \) and \( \sqrt{1 + \frac{H_s^2 \ell^2}{H^2}} \) where \( H_s \) is the Hubble parameter at the horizon crossing. It is noteworthy that the relation depends on the Lorentz factor of the brane motion at the horizon crossing, \( \gamma_s = \sqrt{1 + \frac{H_s^2 \ell^2}{H^2}} \) [see Eqs. (D.9) and (D.10)], which means that the universal relation can be expressed as \( |h_{5D}/h_{\text{ref}}| \propto \gamma_s^{-1/2} \). This relation is insensitive to the choice of EOS parameter, \( w \).

These numerical results indicate that the amount of KK-mode excitations is related only to the motion of brane. Thus, in order to understand the universal relation (5.28), it is important to investigate the effects of the brane motion with an arbitrary trajectory on the KK-mode excitations. For simplicity, let us consider the Minkowski bulk instead of the \( \text{AdS}_5 \). The details of the test calculation are shown in Appendix D. Note that, in the Minkowski bulk, the zero-mode is not localised at the brane and there is no mass gap between the zero-mode and the lightest KK-mode. Even in this simple setup, we can recover the universal relation (5.28) as shown in the right panel of Fig (D.3). This result indicates that the KK-mode excitations are controlled by the brane motion like the Doppler effect. Especially, the Lorentz factor of the brane, \( \gamma_s \), plays a role to determine the final zero-mode amplitude. Moreover this simulation indicates that the background metric in the bulk is not essential for the KK-mode excitations.

The final task of this topic is to draw the universal relation in an analytic way. It is a big step towards understanding the relation that we revealed the moving brane effect on the IGWB in this thesis.
acknowledgements

First of all, I would like to express my special gratitude to Dr. Atsushi Taruya, who is my supervisor and my collaborator. He recommended me to study gravitational waves at the beginning of my research life in the laboratory, and has continuously taught me how to research and what a researcher has to be.

I wish to express my gratitude to Prof. Yasushi Suto who accepted me into his laboratory and granted my self-centered wish to study a different research field from his. His continuous encouragement and fabulously strong support have been

Thanks are due to Dr. Kazuya Koyama who introduced me the research field of braneworld. He is my collaborator and has given me many oppotunities to visit the Institute of Cosmology and Gravitation in United Kingdom and to have greatly useful discussion.

Further I should express my sincere appreciation to Prof. Katsuhiro Sato and Prof. Shigehiro Nagataki for warm support of life in the laboratory. For a comprehensible instruction of string theories and braneworld scenarios, I would like to thank Dr. Shinji Mukohyama. I appriciate very much Prof. Roy Maartens and Dr. Sanjeev S. Seahra for very kind hospitality extended to me during my stay at Institute of Cosmology and Gravitation in United Kingdom, and for carefully reading and correcting my poor English in my articles.

For helpful discussion on the observations of GWB and GWB of astrophysical origin, I would like to highlight the contribution of my collaborators, Dr. Kei Kotake, Dr. Hideaki Kudoh and Dr. Yoshiaki Himemoto, and I am sincerely greatful to them. I would like to thank Dr. Kiyotomo Ichiki, Dr. Shuntaro Mizuno and Dr. Tsutomu Kobayashi for useful and helpful discussion on the cosmological perturbations in braneworld scenarios. I wish to thank Dr. Shinpei Kobayashi and Dr. Shunji Matsuura for helpful instruction about string theory. I would like to thank Dr. Yuuiti Sendouda and Dr. Shunichiro Kinoshita for useful discussions on the general relativity and early universe cosmology, and for giving me a lot of technical information about Mathematica and LaTeX. I also deeply thank Dr. Kohji Yoshikawa, Dr. Mamoru Shimizu, Dr. Takeshi Kuwabara, Dr. Issha Kayo, Dr. Masamune Oguri, Dr. Kazuhiro Yahata, Dr. Tomoya Takiwaki, and Dr. Hidetaka Sonoda for discussion about numerical simulations and supporting my computer environment. I wish to thank Dr. Norio Narita, Dr. Akihito Shirata, Dr. Shunsaku Horiuchi, Dr. Hajime Takami, Dr. Yudai Suwa, Dr. Takahiro Nishimichi, Dr. Shun Saito and all other members of the Theoretical Astrophysics Group in the University of Tokyo.
Finally, I greatly appreciate Prof. Shigenori Hiramatsu and his family for their invaluable support for my life.
Appendix A

Notations

In this chapter, we summarise notations and definitions of quantities used in this thesis.

Tensor indices

In this thesis, we use

- Greek alphabets (\(\mu, \nu, \alpha, \cdots\)) for 0, 1, 2, 3,
- small Roman alphabets (\(i, j, k, \cdots\)) for 1, 2, 3,
- capital Roman alphabets (\(A, B, C, \cdots\)) for 0, 1, 2, 3, 5.

Definition of curvature tensors

A covariant derivative is given by

\[
\nabla_\mu T^\nu_\lambda = \partial_\mu T^\nu_\lambda + \Gamma^\nu_{\mu\alpha} T^\alpha_\lambda - \Gamma^\lambda_{\alpha\mu} T^\nu_\alpha.
\]

A commutation of the covariant derivative operator yields the curvature of the spacetime:

\[
(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\alpha = R^\alpha_{\beta\mu\nu} V^\beta,
\]

where the Riemann curvature tensor is defined as

\[
R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\nu\beta,\mu} - \Gamma^\alpha_{\mu\beta,\nu} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\beta}.
\]

In a \(D\)-dimensional spacetime, the Weyl tensor is defined by

\[
C^\alpha_{\beta\mu\nu} = R^\alpha_{\beta\mu\nu} - \frac{2}{D-2} \left( \delta_\mu^{[\alpha} R^{\beta]_{\nu]} + R_{\mu[\alpha} g_{\beta]\nu]} \right) + \frac{2}{(D-1)(D-2)} \delta_\mu^{[\alpha} g_{\nu]\beta]} R,
\]

where a square bracket \([,]\) in subscripts means

\[
X_{[\alpha\beta]} = \frac{1}{2}(X_{\alpha\beta} - X_{\beta\alpha}).
\]
Unit

We set the fundamental constants $\hbar = c = k_B = 1$, where they denote Planck’s constant, the speed of light and Boltzmann’s constant, respectively. Thus, there is a fundamental dimension, energy, so that,

\[
\begin{align*}
1 \text{ GeV} & = 1.6022 \times 10^{-3} \text{ erg} \\
1 \text{ GeV} & = 1.1605 \times 10^{13} \text{ K} \\
1 \text{ GeV} & = 1.7827 \times 10^{-24} \text{ g} \\
1 \text{ GeV}^{-1} & = 1.9733 \times 10^{-14} \text{ cm} \\
1 \text{ GeV}^{-1} & = 6.5821 \times 10^{-25} \text{ sec} \\
1 \text{ GeV}^4 & = 2.3201 \times 10^{17} \text{ g/cm}^3
\end{align*}
\]

1 erg = $6.2414 \times 10^2$ GeV
1 K = $8.6170 \times 10^{-14}$ GeV
1 g = $5.6095 \times 10^{23}$ GeV
1 cm = $5.0677 \times 10^{13}$ GeV$^{-1}$
1 sec = $1.5193 \times 10^{24}$ GeV$^{-1}$
1 g/cm$^3$ = $4.3102 \times 10^{-18}$ GeV$^4$

Fundamental Constants

The values of the fundamental constants are provided by the CODATA website [164].

Planck’s constant : \[ \hbar = 6.6260693(11) \times 10^{-34} \text{ J} \cdot \text{sec} \]
reduced Planck’s constant : \[ \hbar = 1.05457168(18) \times 10^{-34} \text{ J} \cdot \text{sec} \]
speed of light : \[ c = 2.99792458 \times 10^8 \text{ m/sec} \ (\text{exact}) \]
Newtonian constant of gravity : \[ G = 6.6742(10) \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{sec}^2 \]
Boltzmann’s constant : \[ k_B = 1.3806505(24) \times 10^{-23} \text{ J/K} \]
elementary charge : \[ e = 1.60217653(14) \times 10^{-19} \text{ C} \]
Planck length : \[ \ell_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^3}} = 1.61624(12) \times 10^{-35} \text{ m} \]
Planck time : \[ t_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^5}} = 5.39121(40) \times 10^{-44} \text{ sec} \]
Planck mass : \[ M_{\text{Pl}} = \sqrt{\frac{\hbar c^5}{G}} = 2.17645(16) \times 10^{-8} \text{ kg} \]

Here the number in brackets is the standard uncertainty (e.g., $1.234(56) = 1.234 \pm 0.056$).
**Alphabet**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>00-component of the metric perturbation, $\delta g_{00}$ (scalar)</td>
</tr>
<tr>
<td>$A^{(m)}_j$</td>
<td>coefficients of $(m+1)$th-order predictor</td>
</tr>
<tr>
<td>$A_m$</td>
<td>coefficients of integration of $p_m(t)$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>05-component of the metric perturbation, $\delta g_{05}$ (scalar)</td>
</tr>
<tr>
<td>$A_{yy}$</td>
<td>55-component of the metric perturbation, $\delta g_{55}$ (scalar)</td>
</tr>
<tr>
<td>$A_S$, $A_T$</td>
<td>amplitude of spectrum of scalar/tensor perturbations</td>
</tr>
<tr>
<td>$a(t)$</td>
<td>scale factor of the universe</td>
</tr>
<tr>
<td>$a_k$, $a_k^l$</td>
<td>creation and annihilation operators for $\hat{u}$</td>
</tr>
<tr>
<td>$\hat{a}(t, y)$</td>
<td>warp factor</td>
</tr>
<tr>
<td>$a_{crit}$</td>
<td>scale factor evaluated at $\epsilon = \epsilon_{crit}$</td>
</tr>
<tr>
<td>$B$</td>
<td>0i-component of the metric perturbation, $\delta g_{0i}$ (scalar)</td>
</tr>
<tr>
<td>$B_j^{(m)}$</td>
<td>coefficients of $(m+1)$th-order corrector</td>
</tr>
<tr>
<td>$B_j$</td>
<td>coefficients of integration of $\hat{p}_m(t)$</td>
</tr>
<tr>
<td>$B_g$</td>
<td>5i-component of the metric perturbation, $\delta g_{5i}$ (scalar)</td>
</tr>
<tr>
<td>$B^{(2)}_X$, $B^{(2)}_Y$</td>
<td>coefficients of general boundary conditions ($X = t\xi, \xi, \xi, t, \xi$)</td>
</tr>
<tr>
<td>$b_k^l$, $b_k^\lambda$</td>
<td>creation and annihilation operators for $\hat{v}_k^\lambda$</td>
</tr>
<tr>
<td>$C$</td>
<td>amplitude of dark radiation</td>
</tr>
<tr>
<td>$C_{ABCD}$</td>
<td>5D Weyl tensor</td>
</tr>
<tr>
<td>$c_n$</td>
<td>correction factor to norm of $T_n(\xi)$</td>
</tr>
<tr>
<td>$D$</td>
<td>number of dimension of spacetime (D = d + 1)</td>
</tr>
<tr>
<td>$d$</td>
<td>number of spatial dimension (d = D - 1)</td>
</tr>
<tr>
<td>$E$</td>
<td>ij-component of the metric perturbation, $\delta g_{ij}$ (scalar)</td>
</tr>
<tr>
<td>$E_{\mu\nu}$</td>
<td>Weyl tensor projected onto the brane</td>
</tr>
<tr>
<td>$\hat{e}<em>i$, $\hat{e}</em>{ij}$</td>
<td>vector/tensor harmonics</td>
</tr>
<tr>
<td>$F$</td>
<td>1) enhancement factor for spectra of tensor perturbations</td>
</tr>
<tr>
<td>$f_{hi}$, $f_{eq}$, $f_{inf}$</td>
<td>GW frequency crossing the horizon at the present time, at the matter–radiation equality, at the end of inflation,</td>
</tr>
<tr>
<td>$f_{crit}$</td>
<td>at $\epsilon = \epsilon_{crit}$</td>
</tr>
<tr>
<td>$G$</td>
<td>1) 4D gravity constant</td>
</tr>
<tr>
<td>$\hat{G}_D$</td>
<td>D-dimensional gravity constant</td>
</tr>
<tr>
<td>$[D]G_{AB}$</td>
<td>D-dimensional Einstein tensor</td>
</tr>
<tr>
<td>$[D]g_{AB}$</td>
<td>D-dimensional metric tensor</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$g_{AB}$</td>
<td>metric tensor induced on the brane</td>
</tr>
<tr>
<td>$g_{\mu\nu}$</td>
<td>4D metric tensor</td>
</tr>
<tr>
<td>$(D)g$</td>
<td>determinant of $D$-dimensional metric</td>
</tr>
<tr>
<td>$H(t)$</td>
<td>Hubble parameter</td>
</tr>
<tr>
<td>$\mathcal{H}(t)$</td>
<td>spatial gradient of warp factor</td>
</tr>
<tr>
<td>$h$</td>
<td>1) reduced Hubble parameter</td>
</tr>
<tr>
<td></td>
<td>2) amplitude of IGWB</td>
</tr>
<tr>
<td>$h_{ij}$</td>
<td>$ij$-component of the metric perturbation, $\delta g_{ij}$ (tensor)</td>
</tr>
<tr>
<td>$h_{\text{ref}}(t)$</td>
<td>reference solutions</td>
</tr>
<tr>
<td>$J(\Omega)$</td>
<td>source term in 5D Mukhanov–Sasaki equation</td>
</tr>
<tr>
<td>$K$</td>
<td>3D spatial curvature</td>
</tr>
<tr>
<td>$K_X, K_X^{(2)}$</td>
<td>coefficients of a general hyperbolic equation ($X = t\xi, \xi, t, \xi$)</td>
</tr>
<tr>
<td>$K_{\mu\nu}$</td>
<td>extrinsic curvature of the brane</td>
</tr>
<tr>
<td>$k_c$</td>
<td>critical comoving wave number</td>
</tr>
<tr>
<td>$L$</td>
<td>1) compactification scale of extra-dimensions</td>
</tr>
<tr>
<td></td>
<td>2) distance between two branes</td>
</tr>
<tr>
<td>$\ell$</td>
<td>curvature radius of AdS$_5$ spacetime</td>
</tr>
<tr>
<td>$M$</td>
<td>fundamental scale in the ADD model</td>
</tr>
<tr>
<td>$M_{\text{Pl}}$</td>
<td>4D Planck mass ($= 1.2 \times 10^{19}$ GeV)</td>
</tr>
<tr>
<td>$M_5$</td>
<td>5D fundamental scale in RS model</td>
</tr>
<tr>
<td>$N$</td>
<td>1) e-folding number</td>
</tr>
<tr>
<td></td>
<td>2) lapse function in a de Sitter brane</td>
</tr>
<tr>
<td></td>
<td>3) number of collocation points</td>
</tr>
<tr>
<td>$n(t, y)$</td>
<td>lapse function</td>
</tr>
<tr>
<td>$n_A$</td>
<td>normal vector to a brane</td>
</tr>
<tr>
<td>$n_S, n_T$</td>
<td>spectral index for scalar/tensor perturbations</td>
</tr>
<tr>
<td>$P(k)$</td>
<td>the observed spectrum of curvature perturbations</td>
</tr>
<tr>
<td>$\mathcal{P}_R, \mathcal{P}_h$</td>
<td>power spectra of $R, h_{ij}$</td>
</tr>
<tr>
<td>$\mathcal{P}_h^{(5)}$</td>
<td>5D version of $\mathcal{P}_h$</td>
</tr>
<tr>
<td>$p, p_\xi$</td>
<td>pressure of the universe/Weyl fluid</td>
</tr>
<tr>
<td>$p_m(t), \hat{p}_m(t)$</td>
<td>approximation polynomial</td>
</tr>
<tr>
<td>$Q$</td>
<td>Mukhanov–Sasaki variable</td>
</tr>
<tr>
<td>$R$</td>
<td>1) 4D Ricci scalar</td>
</tr>
<tr>
<td></td>
<td>2) right-hand side of general boundary conditions</td>
</tr>
<tr>
<td>$(D)R$</td>
<td>$D$-dimensional Ricci scalar</td>
</tr>
<tr>
<td>$\mathcal{R}$</td>
<td>$ij$-component of the metric perturbation, $\delta g_{ij}$ (scalar)</td>
</tr>
<tr>
<td>$\mathcal{R}_c$</td>
<td>comoving curvature perturbation (curvature perturbation in the comoving gauge)</td>
</tr>
<tr>
<td>$\mathcal{R}_k$</td>
<td>Fourier components of $\mathcal{R}_c$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$^{(3)}R_{ijkl}$</td>
<td>3D Riemann tensor</td>
</tr>
<tr>
<td>$r$</td>
<td>scalar–tensor ratio</td>
</tr>
<tr>
<td>$r_K$</td>
<td>radial coordinate ($K = +1, 0, -1$) in AdS$_5$</td>
</tr>
<tr>
<td>$S$</td>
<td>gauge-invariant variable defined by $S_i$ and $S_{yi}$</td>
</tr>
<tr>
<td>$S_i$</td>
<td>$0i$-component of the metric perturbation, $\delta g_{0i}$ (vector)</td>
</tr>
<tr>
<td>$S_{yi}$</td>
<td>$5i$-component of the metric perturbation, $\delta g_{5i}$ (vector)</td>
</tr>
<tr>
<td>$s_{\text{init}}$</td>
<td>GW wavelength normalised by the Hubble horizon scale at $t_{\text{init}}$</td>
</tr>
<tr>
<td>$T(t)$</td>
<td>brane position in the Poincaré coordinate (time coordinate)</td>
</tr>
<tr>
<td>$T_n(\xi)$</td>
<td>Tchebychev polynomials</td>
</tr>
<tr>
<td>$T_{\mu\nu}$</td>
<td>energy–momentum tensor</td>
</tr>
<tr>
<td>$T_{\text{R}}(k)$</td>
<td>the transfer function for curvature perturbations</td>
</tr>
<tr>
<td>$t$</td>
<td>cosmic time</td>
</tr>
<tr>
<td>$t_{\text{init}}$</td>
<td>cosmic time at the beginning of simulations</td>
</tr>
<tr>
<td>$\tilde{u}^{(k)}(t)$</td>
<td>spectral coefficients for $\delta^k u / \delta \xi^k$ in the Tchebychev space</td>
</tr>
<tr>
<td>$u(\eta, x)$</td>
<td>canonically normalised $R_c$</td>
</tr>
<tr>
<td>$u_k(\eta)$</td>
<td>mode functions for $\tilde{u}(\eta, x)$</td>
</tr>
<tr>
<td>$u_m(y)$</td>
<td>spatial part of tensor perturbations (4.79)</td>
</tr>
<tr>
<td>$u^\mu$</td>
<td>velocity of fluid</td>
</tr>
<tr>
<td>$V(\phi)$</td>
<td>inflaton potential</td>
</tr>
<tr>
<td>$v_k^c$</td>
<td>canonically normalised tensor perturbations</td>
</tr>
<tr>
<td>$v^i$</td>
<td>velocity perturbation of fluid</td>
</tr>
<tr>
<td>$w$</td>
<td>parameter of equation-of-state (EOS) for perfect fluid</td>
</tr>
<tr>
<td>$y$</td>
<td>1) spatial coordinate in Gaussian–normal coordinates</td>
</tr>
<tr>
<td></td>
<td>2) intermediate variable in Hawkins–Lidsey model</td>
</tr>
<tr>
<td>$y_b$</td>
<td>perturbed position of the brane in Gaussian–normal coordinates</td>
</tr>
<tr>
<td>$y_c$</td>
<td>coordinate singularity in Gaussian–normal coordinates</td>
</tr>
<tr>
<td>$y_{\text{reg}}$</td>
<td>position of the regulator brane in Gaussian–normal coordinates</td>
</tr>
<tr>
<td>$Z_\nu(z)$</td>
<td>linear combination of the Bessel functions</td>
</tr>
<tr>
<td>$z$</td>
<td>1) canonical normalisation factor of $R_c$</td>
</tr>
<tr>
<td></td>
<td>2) redshift</td>
</tr>
<tr>
<td></td>
<td>3) Poincaré coordinate in AdS$_5$</td>
</tr>
<tr>
<td></td>
<td>4) conformal Gaussian–normal coordinates (spatial coordinate)</td>
</tr>
<tr>
<td>$z_b$</td>
<td>1) position of the brane in the Poincaré coordinate</td>
</tr>
<tr>
<td></td>
<td>2) position of the brane in conformal Gaussian–normal coordinates</td>
</tr>
<tr>
<td>$z_{\text{reg}}$</td>
<td>position of the regulator brane in the Poincaré coordinate</td>
</tr>
</tbody>
</table>
### Notations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma^{\lambda}_{\mu\nu}$</td>
<td>0th-order 4D Christoffel symbols</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>parameter controlling the position of the regulator brane</td>
</tr>
<tr>
<td>$\gamma_{ij}$</td>
<td>3D spatial metric with a spatial curvature $K$</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>time-step</td>
</tr>
<tr>
<td>$\delta q_i, \delta \xi_i$</td>
<td>momentum perturbation of fluid/Weyl fluid</td>
</tr>
<tr>
<td>$\delta \pi, \delta \xi$</td>
<td>anisotropic stress of fluid/Weyl fluid</td>
</tr>
<tr>
<td>$\varepsilon_{ij}(k, \lambda)$</td>
<td>polarisation tensors for tensor perturbations</td>
</tr>
<tr>
<td>$\epsilon(t)$</td>
<td>normalised energy density of the universe</td>
</tr>
<tr>
<td>$\varepsilon_H, \varepsilon_s$</td>
<td>slow-roll parameter defined by $H(\phi)/V(\phi)$</td>
</tr>
<tr>
<td>$\epsilon_{\text{crit}}$</td>
<td>critical energy density</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>curvature perturbation on uniform density surface</td>
</tr>
<tr>
<td>$\eta$</td>
<td>conformal time</td>
</tr>
<tr>
<td>$\eta_H, \eta_s$</td>
<td>slow-roll parameter defined by $H(\phi)/V(\phi)$</td>
</tr>
<tr>
<td>$\kappa_4, \kappa_5$</td>
<td>4D ($\kappa_4^2 = 8\pi/M_{Pl}^2$) and 5D ($\kappa_5^2 = 8\pi/M_5^2$) gravity constant</td>
</tr>
<tr>
<td>$\Lambda_D$</td>
<td>D-dimensional cosmological constant</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>brane tension</td>
</tr>
<tr>
<td>$\mu$</td>
<td>1) index of Hankel functions</td>
</tr>
<tr>
<td></td>
<td>2) curvature scale of AdS$_5$ ($\equiv \ell^{-1}$)</td>
</tr>
<tr>
<td>$\nu$</td>
<td>index of Hankel functions</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>master variable for vector perturbations</td>
</tr>
<tr>
<td>$\xi$</td>
<td>1) perturbed position of the brane</td>
</tr>
<tr>
<td></td>
<td>2) dimension-less conformal Gaussian–normal coordinate</td>
</tr>
<tr>
<td></td>
<td>3) spatial coordinate where $T_n(\xi)$ are defined ($-1 \leq \xi \leq 1$)</td>
</tr>
<tr>
<td>$\xi^A$</td>
<td>changes of coordinates induced by gauge-transformation</td>
</tr>
<tr>
<td>$\xi_H$</td>
<td>slow-roll parameter defined by $H(\phi)$</td>
</tr>
<tr>
<td>$\xi_n$</td>
<td>Gauss–Lobatto collocation points</td>
</tr>
<tr>
<td>$\Pi_{\mu\nu}$</td>
<td>quadratic energy–momentum tensor</td>
</tr>
<tr>
<td>$\rho, \rho_\xi$</td>
<td>energy density of the universe/Weyl fluid</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>critical density</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>shear of fluid</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>shear of fluid defined by $B_y$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>time coordinate in AdS$_5$</td>
</tr>
<tr>
<td>$\tau_i$</td>
<td>gauge-invariant variable defined by $S_i$ and $F_i$</td>
</tr>
<tr>
<td>$\tau_{yi}$</td>
<td>gauge-invariant variable defined by $S_{yi}$ and $F_i$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>scalar field (inflaton)</td>
</tr>
<tr>
<td>$\phi_m(t)$</td>
<td>temporal part of tensor perturbations</td>
</tr>
</tbody>
</table>
\( \phi_0 \) | background inflaton field | (2.100)  
\( \chi \) | auxiliary variable | (C.4)  
\( \Omega \) | 1) density parameter | (2.13)  
| | 2) master variable for scalar perturbations | (4.28)  
\( \Omega_{GW}(f) \) | present energy spectrum of gravitational wave background | (2.149)  
\( \Omega_{\text{ref}}(f) \) | \( \Omega_{GW}(f) \) evaluated for \( h_{\text{ref}} \) | (5.24)  

**Symbols**

\( (r)_q \) | the Pochhammer symbol | (C.25)  
\( (\cdots, \cdots) \) | scalar product (Klein–Gordon inner product) | (4.95)  
\( \hat{x} \) (hat) | field operator | (2.119)  
\( \bar{x} \) (bar) | perturbed quantities in 4D-longitudinal gauge | (2.88) (4.54)  
\( \tilde{x} \) (tilde) | 1) gauge-transformed quantities | (2.41)  
| | 2) perturbed quantities in 5D-longitudinal gauge | (4.20)  
| | 3) quantity transformed into Tchebychev space | (C.8)  
subscript 's' | evaluated at the horizon crossing (for \( a, H, h, t \)) | (2.167)  
subscript '0' | evaluated at the present time (for \( a, H, h \)) | (2.167)  
subscript 'b' | evaluated at the brane (for \( a, n \)) | (3.19) (3.56)  
subscript 'eq' | evaluated at matter–radiation equality (for \( a, H, z \)) | (2.167)  
subscript 'inf', 'end' | evaluated at the end of inflation (for \( a, H \)) | (2.168) (6.13)  
subscript 'T', 'TF' | trace part/trace-free part | (B.4)  
superscript 'S', 'V', 'T' | scalar/vector/tensor type perturbations | (2.49) (2.52)  
\( \nabla^q \) | backward difference operator | (C.25)  
\( \nabla_\mu, \nabla_\nu \) | covariant derivative defined by \( g_{AB} / \gamma_{ij} \) | (3.25) (2.71)  
\( \nabla_A \) | covariant derivative defined by \( (5) g_{AB} \) | (3.23)  
\( \nabla^2 \) | Laplace–Beltrami operator defined by \( \gamma_{ij} \) | (2.71) (4.32)
Appendix B

Calculations of Some Tensor Quantities

B.1 Extrinsic curvature

The extrinsic curvature of the brane is defined in Eq. (3.23). In the 5D-longitudinal gauge, the extrinsic curvature perturbed by the scalar perturbations (4.23)–(4.26) is calculated as

\[
\delta K^0_0 = \ddot{A} - \left( \mathcal{H} + \frac{\dot{H}}{\mathcal{H}} \right) \dddot{A}_{yy} + \dddot{A}_y + \dddot{\xi} - \left( H^2 + 2\dot{H} + \frac{\dot{H}^2}{\mathcal{H}^2} \right) \xi, \quad (B.1)
\]

\[
\delta K^0_i = \left( \frac{1}{2} \dddot{A}_y + \dddot{\xi} - H\xi \right) , \quad (B.2)
\]

\[
\delta K^i_j = \delta^i_j \delta K_T + \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \nabla^2 \right) \delta K_{TF}, \quad (B.3)
\]

\[
\delta K_T = \dddot{\xi} + H \left( \dddot{\xi} - H\xi + \dddot{A}_y \right) - \mathcal{H} \dddot{A}_{yy}, \quad (B.4)
\]

\[
\delta K_{TF} = -\frac{1}{a_b^2} \xi, \quad (B.5)
\]

where \( \xi \) is the position of the brane (4.44), and the dot and the prime denote derivatives with respect to \( t \) and \( y \), respectively. Note that Eqs. (B.1) and (B.4) contains terms proportional to \( \xi \) which comes from the last term of the right-hand side of Eq. (4.55).

The vector perturbations contribute as

\[
\delta K^0_0 = 0, \quad (B.6)
\]

\[
\delta K^0_i = \frac{a_b^2}{2} \mathcal{S} \dddot{e}_i, \quad (B.7)
\]

\[
\delta K^i_j = \frac{1}{2} \mathcal{F} ( \dddot{e}^i_j + \dddot{e}_j^i ), \quad (B.8)
\]

where \( \dddot{e}_i \) denotes a vector harmonic function (2.51).
Finally, the extrinsic curvature induced by the tensor perturbations is

\begin{align*}
\delta K^0_0 &= \delta K^0_i = 0, \\
\delta K^i_j &= \frac{1}{2} \hat{\kappa} \hat{e}^i_j,
\end{align*}

(B.9)\(\text{(B.10)}\)

where \(\hat{e}^i_j\) denotes a tensor harmonic function (2.54).

### B.2 4D Christoffel symbol

In this section, we calculate the Christoffel symbol from the induced metric on the brane:

\[ \Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} \left( (4) g_{\alpha\lambda,\beta} + (4) g_{\beta\lambda,\alpha} - (4) g_{\alpha\beta,\lambda} \right). \]

(B.11)

The induced metric with its scalar perturbations in 4D-longitudinal gauge is given by Eq. (4.54). The 0th-order components of the Christoffel symbols are calculated as

\begin{align*}
\Gamma^{(0)}_{000} &= \Gamma^{(0)}_{0i0} = \Gamma^{(0)}_{k00} = \Gamma^{(0)}_{kij} = 0, \\
\Gamma^{(0)}_{0ij} &= a_b^2 H \delta_{ij}, \quad \Gamma^{(0)}_{k0i} = H \delta^k_i.
\end{align*}

(B.12)\(\text{(B.13)}\)

The scalar perturbations yield

\begin{align*}
\delta \Gamma^{(0)}_{000} &= \delta \Gamma^{(0)}_{0i0} = \delta \Gamma^{(0)}_{k00} = \frac{\delta \Gamma^{(0)}_{kij}}{a_b^2}, \\
\delta \Gamma^{(0)}_{0ij} &= a_b^2 \left\{ \hat{\mathcal{R}} + 2H(\mathcal{R} - \bar{A}) \right\} \delta_{ij}, \\
\delta \Gamma^{(0)}_{k0i} &= \frac{\delta \mathcal{R}}{a_b^2} \delta_{kij}, \quad \delta \Gamma^{(0)}_{kij} = \mathcal{R} \delta_{kij} + \bar{\mathcal{R}} \delta_{kij} - \mathcal{R} \delta_{kij}.
\end{align*}

(B.14)\(\text{(B.15)}\)\(\text{(B.16)}\)

### B.3 5D Perturbed Einstein tensor

Taking the 5D-longitudinal gauge, the scalar perturbations of the metric are given in Eqs. (4.23)–(4.26). They yield the perturbed Einstein tensor [77] :

\begin{align*}
^{(5)}\delta G^0_0 &= \frac{6}{n^2} \left\{ \left( \frac{\dot{a}}{a} \right)^2 \bar{A} - \frac{\dot{\mathcal{R}}}{a} \right\} + 3\bar{\mathcal{R}}'' + 12\frac{\dot{a}'}{a} \bar{\mathcal{R}}' + \frac{2}{a^2} \nabla^2 \bar{\mathcal{R}} + \frac{1}{a^2} \mathcal{A}_{yy} \\
&\quad - 3 \left[ \frac{\dot{a}'}{a} \mathcal{A}_{yy} + \frac{\dot{a}}{an^2} \mathcal{A}_{yy} + 2 \left\{ \left( \frac{\dot{a}'}{a} \right)^2 + \frac{a''}{a} \right\} \mathcal{A}_{yy} - \frac{1}{n} \left\{ \frac{\dot{a}}{a} \mathcal{A}_y + \left( \frac{2\dot{a}'}{a^2} + \frac{\dot{a}'}{a} \right) \mathcal{A}_y \right\} \right],
\end{align*}

(B.17)
The spatial part of the perturbed Einstein tensor is decomposed into its trace part and trace-free part:

\[ \delta G^0_0 = \left[ \frac{2}{n^2} \left( \frac{\dot{a}}{a} \tilde{A} + \dot{\mathcal{R}} \right) - \frac{1}{2n} \left\{ \tilde{A}_y' + \left( \frac{2n'}{n} + \frac{a'}{a} \right) \tilde{A}_y \right\} + \frac{1}{n^2} \left( \tilde{A}_{yy} - \frac{\dot{a}}{a} \tilde{A}_{yy} \right) \right], \quad (B.18) \]

\[ \delta G^0_5 = \frac{3}{a^2} \left\{ \left( \frac{\dot{a}}{a} + \frac{n'}{n} \right) \tilde{A} - 2 \tilde{\mathcal{R}}' - \frac{1}{2n} \left( \tilde{A}_y' + \frac{\dot{a}}{a} \tilde{A}_y \right) + \left( \frac{n'}{n} + \frac{2a'}{a} \right) \tilde{A}_{yy} \right\}, \quad (B.19) \]

\[ \delta G^5_i = \left\{ -\tilde{A}' + \left( \frac{a'}{a} - \frac{n'}{n} \right) \tilde{A} - 2 \tilde{\mathcal{R}}' - \frac{1}{2n} \left( \tilde{A}_y' + \frac{\dot{a}}{a} \tilde{A}_y \right) + \left( \frac{n'}{n} + \frac{2a'}{a} \right) \tilde{A}_{yy} \right\}, \quad (B.20) \]

\[ \delta G^5_5 = \frac{1}{a^2} \nabla^2 \tilde{A} + \frac{3d'}{a} \tilde{A} + \frac{3}{n^2} \left[ \left( \frac{\dot{a}}{a} + \frac{a'}{a} \right)^2 \tilde{A} - \tilde{\mathcal{R}} + \left( \frac{n'}{n} - 4 \frac{a'}{a} \right) \tilde{\mathcal{R}} \right] + \left[ \frac{\dot{a}}{a} + \left( \frac{a'}{a} \right)^2 - \frac{\dot{a}n}{an} \right] \tilde{A} - \tilde{\mathcal{R}} + \left( \frac{n'}{n} - 4 \frac{a'}{a} \right) \tilde{\mathcal{R}} \right\} + \frac{3}{n^2} \left\{ \tilde{A}_y' + \frac{\dot{a}}{a} \tilde{A}_y \right\} - 6 \left\{ \frac{a'}{an} + \frac{\dot{a}n}{an} \right\} \tilde{A}_{yy} \quad (B.21) \]

\[ \delta G^i_k = \delta^i_j (B.25) \delta G_T + \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \nabla^2 \right) (B.27) \delta G_{TF}, \quad (B.22) \]

The spatial part of the perturbed Einstein tensor is decomposed into its trace part and trace-free part:

\[ \delta G^0 = \frac{2}{3a^2} \nabla^2 (\tilde{A} + \dot{\mathcal{R}}) + \frac{2}{n^2} \left\{ \left( \frac{n'}{n} - \frac{\dot{a}}{a} \right) \dot{\mathcal{R}} - \frac{\dot{a}}{a} \tilde{A} + \left( \frac{2\dot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{\dot{a}n}{an} \right) \tilde{A} \right\} + \frac{1}{n} \left\{ \dot{\tilde{A}} + \left( \frac{n'}{n} + \frac{2a'}{a} \right) \tilde{A}_y + \frac{\dot{a}}{a} \dot{\tilde{A}} + \left( \frac{n'}{n} - \frac{\dot{a}n}{an} + 2 \frac{\dot{a}a'}{a^2} + 4 \frac{\dot{a}}{a} \right) \tilde{A}_y \right\} + \frac{1}{n^2} \left\{ \tilde{A}_{yy} - \left( \frac{n'}{n} - 2 \frac{\dot{a}}{a} \right) \tilde{A}_{yy} \right\} + \frac{2}{3a^2} \nabla^2 \tilde{A}_{yy} - \left( \frac{n'}{n} + 2 \frac{d'a'}{a^2} \right) \tilde{A}_{yy} \quad (B.23) \]

\[ \delta G_{TF} = -\frac{1}{a^2} (\tilde{A} + \tilde{\mathcal{R}} + \tilde{A}_{yy}). \quad (B.24) \]

For the vector perturbations (4.16)–(4.18), the perturbed Einstein tensor becomes

\[ \delta G^0_0 = \delta G^0_5 = \delta G^5_5 = 0, \quad (B.25) \]

\[ \delta G^0_i = -\frac{1}{2n^2} \left[ \nabla^2 \tau + a^2 \left\{ \left( \frac{5a'}{a} - \frac{n'}{n} \right) S + S' \right\} \right] \tilde{e}_i, \quad (B.26) \]

\[ \delta G^5_i = \frac{1}{2} \left[ \nabla^2 \tau_y + \frac{2}{a^2} \left\{ - \left( \frac{n'}{n} - \frac{5}{a} \right) S + S' \right\} \right] \tilde{e}_i, \quad (B.27) \]

\[ \delta G^i_j = -\frac{1}{2} \left\{ \left( \frac{n'}{n} + \frac{3a'}{a} \right) \tau_y + \tau_y' + \frac{1}{n^2} \left\{ \left( \frac{n'}{n} - \frac{3}{a} \right) \tau - \tau' \right\} \right\} (\tilde{e}^i_j + \tilde{e}^j_i), \quad (B.28) \]
Calculations of Some Tensor Quantities

For tensor perturbations (4.41), only $ij$-component has a non-zero value:

$$^{(5)}\delta G^i_j = -\frac{1}{2} \left\{ \left( \frac{n' + 3 a'}{n} \right) h' + \frac{1}{n} \left( \frac{\dot{h}}{n} - 3 \frac{\dot{a}}{a} \right) \dot{h} + \frac{1}{a^2} \nabla^2 h + h'' - \frac{1}{n^2} h \right\} \hat{e}^i_j. \quad (B.29)$$

### B.4 4D perturbed Einstein tensor

For the scalar perturbations in the 4D-longitudinal gauge (4.54), the perturbed Einstein tensor is calculated as [77] :

$$^{(4)}\delta G^0_0 = 6H \left( -\dot{\mathbf{\cal R}} + H\mathbf{\cal A} \right) \frac{2}{a^6} \nabla^2 \mathbf{\cal R}, \quad (B.30)$$

$$^{(4)}\delta G^0_i = -2(H\mathbf{\cal A} - \dot{\mathbf{\cal R}})_i, \quad (B.31)$$

$$^{(4)}\delta G^i_j = \delta^i_j(4\delta G_T + \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \nabla^2 \right) \delta G_{TF}) \quad (B.32)$$

where

$$^{(4)}\delta G_T = 2 \left\{ (3H^2 + 2\dot{H})\mathbf{\cal A} + H\dot{\mathbf{\cal A}} - \dot{\mathbf{\cal R}} - 3H\dot{\mathbf{\cal R}} + \frac{1}{3a^2} \nabla^2 (\mathbf{\cal R} + \mathbf{\cal A}) \right\}, \quad (B.33)$$

$$^{(4)}\delta G_{TF} = -\frac{1}{a^2} (\mathbf{\cal R} + \mathbf{\cal A}). \quad (B.34)$$

For the vector perturbations, we obtain the perturbed Einstein tensor in terms of the gauge-invariant variables defined in Eqs. (4.16)–(4.18) [77] :

$$^{(4)}\delta G^0_0 = 0 , \quad (B.35)$$

$$^{(4)}\delta G^0_i = \frac{k^2}{2} \tau \hat{e}_i , \quad (B.36)$$

$$^{(4)}\delta G^i_j = \frac{1}{2} \left( \dot{\tau} + 3H\tau \right) (\hat{e}^i_j + \hat{e}^j_i) , \quad (B.37)$$

Finally, for the tensor perturbations, the perturbed Einstein tensor is given by [77]

$$^{(4)}\delta G^0_0 = ^{(4)}\delta G^0_i = 0 , \quad (B.38)$$

$$^{(4)}\delta G^i_j = \frac{1}{2} \left( \ddot{\mathbf{\cal R}} + 3H\dot{\mathbf{\cal R}} + \frac{k^2}{a^2} h \right) \hat{e}^i_j. \quad (B.39)$$

### B.5 Quadratic energy–momentum tensor

The quadratic energy–momentum tensor $\delta \Pi_{\mu\nu}$ is calculated from its definition, Eq. (3.29), with Eqs. (2.5)–(2.6) and Eqs. (2.65)–(2.69). For the scalar perturbations, it becomes [77]

$$\delta \Pi^0_0 = -\frac{1}{6} \rho \delta \rho , \quad (B.40)$$

$$\delta \Pi^0_i = \frac{1}{6} \rho \delta q_{i,i} , \quad (B.41)$$

$$\delta \Pi^i_j = \delta^i_j \delta \Pi_T + \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \nabla^2 \right) \delta \Pi_{TF} \quad (B.42)$$
where
\[\delta \Pi_T = \frac{1}{6} \{(\rho + p)\delta \rho + \rho \delta p \}, \quad (B.43)\]
\[\delta \Pi_{TF} = -\frac{1}{12} (\rho + 3p) \delta \pi, \quad (B.44)\]

For the vector perturbations, the quadratic energy–momentum tensor \(\delta \Pi_{\mu \nu}\) is calculated as [77]
\[\delta \Pi^0_0 = 0 \quad (B.45)\]
\[\delta \Pi^0_0 = \frac{\rho}{6} \delta q^V \dot{e}_i, \quad (B.46)\]
\[\delta \Pi^i_j = -\frac{\rho + 3p}{12} \delta \pi^V (\dot{e}^i_j + \dot{e}^j_i). \quad (B.47)\]

Finally, for the tensor perturbations, the quadratic energy–momentum tensor \(\delta \Pi_{\mu \nu}\) is calculated as [77]
\[\delta \Pi^0_0 = \delta \Pi^0_0 = 0, \quad (B.48)\]
\[\delta \Pi^i_j = -\frac{\rho + 3p}{12} \delta \pi^V \dot{e}^i_j. \quad (B.49)\]

## B.6 Projected Weyl tensor

The Weyl fluid components can be obtained from a direct calculation of the Weyl tensor (3.28) and its evaluation on the brane. In the 5D-longitudinal gauge (4.23)–(4.26), the Weyl fluid components are
\[\delta E^0_0 = \dot{H} (\dot{R} + 2\dot{A}) + \frac{H \dot{H}}{2 \dot{H}} - \frac{\mathcal{H}^2}{2 \mathcal{H}} \dot{A}_y + \frac{\mathcal{H}}{2} \left( \dot{\mathcal{R}} - \mathcal{H}^2 + 3 \mathcal{H} \dot{\mathcal{R}} \right) + \frac{H}{2} \mathcal{A}_x, \quad (B.50)\]
\[\delta E^0_i = \frac{4}{3} \left\{ H \left( \mathcal{A} + \dot{\mathcal{R}} \right) - \frac{\dot{H}}{2 \mathcal{H}} \mathcal{A}_y - \frac{1}{2} \mathcal{A}_y - \frac{1}{2} \mathcal{R} - \mathcal{A}_y \right\}_i, \quad (B.51)\]
\[\delta E^i_j = \delta^i_j \delta E_T + \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \nabla^2 \right) \delta E_{TF}, \quad (B.52)\]
\[\delta E_T = -\frac{\dot{H}}{3} (\dot{R} + 2\dot{A}) - \frac{1}{6 \mathcal{H}} \left\{ H \dot{H} \left( 1 + \frac{\dot{H}}{\mathcal{H}^2} \right) + \ddot{H} \right\} \mathcal{A}_y + \frac{\mathcal{H}^2 + 3 \dot{H} \mathcal{A}_x}{6 \mathcal{H}} \mathcal{A}_y - \frac{1}{6 \mathcal{H}} \mathcal{A}_y - \frac{\dot{A}_y - \mathcal{A}_y - 2 \mathcal{R}}{6 \mathcal{H}}, \quad (B.53)\]
\[\delta E_{TF} = \frac{1}{a_0^2} (\mathcal{A} + \dot{\mathcal{R}}), \quad (B.54)\]
Here we used the trace-free condition (4.34) to eliminate $\tilde{A}_{yy}$.

For the vector perturbations, the projected Weyl tensor (3.28) becomes [77]

$$\delta E^0_{0} = 0,$$

$$\delta E^0_{i} = -\frac{a_{b}^{2}}{3}\left\{ S' + \left( \mathcal{H} - \frac{\dot{H}}{\mathcal{H}} \right) S + \frac{k^2}{2a_{b}^{2}} \tau \right\} \hat{e}_i,$$

$$\delta E^i_{j} = -\frac{1}{6}\left\{ \dot{\tau} + 3H\tau + 2\tau' + \left( 2\mathcal{H} - \frac{\dot{H}}{\mathcal{H}} \right) \tau_y \right\} (\hat{e}_i^j + \hat{e}_j^i).$$

Finally, for the tensor perturbations, the projected Weyl tensor becomes [77]

$$\delta E^0_{0} = \delta E^0_{i} = 0,$$

$$\delta E^i_{j} = -\left( \frac{1}{2} h'' + \mathcal{H}h' \right) \hat{e}_j^i.$$
Appendix C

The Spectral Collocation Method

In this appendix, we briefly describe the implementation of the spectral collocation method in our numerical scheme.

C.1 Reduction of evolution equations

Throughout the thesis, the numerical calculations of this system were carried out by employing the spectral collocation method \[52,53\]. We focus on the variables to be integrated, \(\Omega(t, y)\) for scalar perturbations (4.33) and \(h(\tau, z)\) for tensor perturbations (5.9). They are firstly decomposed by the Tchebychev polynomials defined as

\[ T_n(\xi) = \cos(n \cos^{-1} \xi) \quad \text{for} \quad -1 \leq \xi \leq 1, \quad (C.1) \]

which is a polynomial function of \(\xi\) of order \(n\). Here the variable \(\xi\) is related to the GN coordinate \(y\). To implement the spectral collocation method, we must transform the original spatial coordinate \(y\) in the GN coordinates or the Poincaré coordinate (in this case, denoted by \(z\)) by

\[ y = \frac{1}{2} \left( (y_{\text{reg}} - y_0) \xi + (y_{\text{reg}} + y_0) \right), \quad (C.2) \]

so that the locations of both the physical and the regulator branes are kept fixed and the spatial coordinate \(y\) is projected to the compact domain \(-1 \leq \xi \leq 1\).

From the coordinate transformation (C.2), the original equations are generally expressed as

\[ \frac{\partial^2 u}{\partial t^2} + K_{t\xi} \frac{\partial^2 u}{\partial t \partial \xi} + K_{\xi \xi} \frac{\partial^2 u}{\partial \xi^2} + K_t \frac{\partial u}{\partial t} + K_\xi \frac{\partial u}{\partial \xi} + Ku = 0, \quad (C.3) \]

where the coefficients \(K_{t\xi}, K_{\xi \xi}, K_t, K_\xi\) and \(K\) are functions of \(t\) and \(\xi\). We use the predictor–corrector method for the temporal evolution, which will be discussed in Sec. C.2. To implement this, we introduce an auxiliary variable \(\chi(t, \xi)\) satisfying the equation

\[ \frac{\partial u}{\partial t} = \chi - K_{t\xi} \frac{\partial u}{\partial \xi} = F(\chi, u'; t, \xi_n), \quad (C.4) \]

\[ 121 \]
where the prime denotes the derivative with respect to $\xi$ [see Eq.(C.2)]. With this definition, the time evolution of $\chi$ satisfying the original equations is rewritten as

$$\frac{\partial \chi}{\partial t} = -K_{\xi} \frac{\partial^2 u}{\partial \xi^2} - K_{\xi}^{(2)} \frac{\partial u}{\partial \xi} - K_{t}^{(2)} \frac{\partial u}{\partial t} - K u = G(\chi, u, u', u''; t, \xi_n),$$

(C.5)

where

$$K_{\xi}^{(2)} = K_{\xi} - K_{t}^{(2)} - K_{t} K_{\xi}.$$  

(C.6)

Notice that the function $G$ does not contain the derivative $\partial \chi / \partial \xi$. Empirically, the presence of this derivative causes numerical instability. The functions $F$ and $G$ are evaluated at each discretised point $\xi_n$. Here we use an inhomogeneous grid for the spatial coordinate $\xi$, which is called Gauss–Lobatto collocation points defined as

$$\xi_n = \cos \frac{n\pi}{N},$$

(C.7)

where $n$ runs from 0 to $N$, so we take $N + 1$ collocation points. An advantage of this grid is that we can use the Fast-Fourier-Transformation (FFT) algorithm when we transform a quantity into the spectral space. The quantity $u$ is transformed as

$$u(t, \xi_i) = \sum_{n=0}^{N} \tilde{u}(t) T_n(\xi_i).$$

(C.8)

A advantage of the spectral method is that we can precisely evaluate the spatial differentiations of $u$ from the Fourier modes of $u$. A spatially differential operator $\partial / \partial \xi$ changes $T_n(\xi_i)$ to $T_n'(\xi_i)$ in the right-hand side of Eq.(C.8). We can re-expand the differentiated right-hand side by $T_n(\xi)$ :

$$\frac{\partial u}{\partial \xi}(t, \xi_i) = \sum_{n=0}^{N} \tilde{u}(t) T_n'(\xi_i) = \sum_{n=0}^{N} \tilde{u}^{(1)}(t) T_n(\xi_i),$$

(C.9)

where $\tilde{u}^{(1)}$ is computed in decreasing order by the recurrence relations [52, 53]

$$c_n \tilde{u}^{(1)}_n = \tilde{u}^{(1)}_{n+1} + 2(n + 1) \tilde{u}_{n+1}, \quad \tilde{u}^{(1)}_{n \geq N} = 0.$$  

(C.10)

The normalisation constant $c_n$ is defined by

$$c_n = \begin{cases} 
2 & \text{for } n = 0, N \\
1 & \text{for } 1 \leq n \leq N - 1
\end{cases}.$$  

(C.11)

The second-order differentiations are calculated in the same way :

$$\frac{\partial^2 u}{\partial \xi^2}(t, \xi_i) = \sum_{n=0}^{N} \tilde{u}(t) T_n''(\xi_i) = \sum_{n=0}^{N} \tilde{u}^{(2)}(t) T_n(\xi_i),$$

(C.12)

where $\tilde{u}^{(1)}$ is determined from

$$c_n \tilde{u}^{(2)}_n = \tilde{u}^{(2)}_{n+2} + 2(n + 1) \tilde{u}^{(1)}_{n+1}, \quad \tilde{u}^{(2)}_{n \geq N} = 0.$$  

(C.13)
C.1 Reduction of evolution equations

Then, transforming to the Tchebychev space by Eq. (C.8), we obtain a set of ordinary differential equations:

\[
\frac{d\tilde{u}_n}{dt} = \tilde{F}_n(t), \quad \frac{d\tilde{\chi}_n}{dt} = \tilde{G}_n(t), \quad \text{for } 0 \leq n \leq N-2. \tag{C.14}
\]

We use boundary conditions to determine the Fourier modes with \( n = N - 1 \) and \( N \). With the reduced equations (C.14), the predictor–corrector method based on the Adams–Bashforth–Moulton scheme can be used to obtain the time evolution of \( \tilde{u}_n(t) \) and \( \tilde{\chi}_n(t) \).

At each time step, the boundary conditions (5.10) and (5.16) for the tensor case, (6.2) or (6.4) and (6.5) for scalar cases are used to evaluate \( \tilde{u}_{N-1} \) and \( \tilde{u}_N \). A general form of boundary conditions imposed on the brane is given as

\[
B_t \frac{\partial u}{\partial t} + B_{t\xi} \frac{\partial u}{\partial \xi} + B_{\xi\xi} \frac{\partial^2 u}{\partial \xi^2} + B_{\xi} \frac{\partial u}{\partial \xi} + Bu = R, \tag{C.15}
\]

where \( B_t, B_{t\xi}, B_{\xi\xi}, B_{\xi}, B \) and \( R \) are time-dependent coefficients. We impose two sets of the boundary conditions on each boundary, \( \xi = \pm 1 \). Introducing the auxiliary variable \( \chi \) by Eq.(C.4), the boundary condition becomes

\[
B^{(2)}_{\xi\xi} \frac{\partial u^2}{\partial \xi^2} + B^{(2)}_{\xi} \frac{\partial u}{\partial \xi} + Bu + B_{t\xi} \frac{\partial \chi}{\partial \xi} + B_{t} \chi = R, \tag{C.16}
\]

where

\[
B^{(2)}_{\xi\xi} = B_{\xi\xi} - B_{t\xi} K_{t\xi}, \tag{C.17}
\]
\[
B^{(2)}_{\xi} = B_{\xi} - B_{t\xi} K'_{t\xi} - B_{t} K_{t\xi}. \tag{C.18}
\]

Note that \( B^{(2)}_{\xi\xi} \) is vanished because \( K_{t\xi} = B_{\xi\xi} = 0 \) for tensor cases and \( B_{t\xi} = B_{\xi} = 0 \) for scalar cases in this thesis. Furthermore, we use the definition of \( \chi \), Eq.(C.4), evaluated at \( \xi = \pm 1 \) as a boundary condition for \( \chi \). In this condition, \( \partial u/\partial t \) is evaluated by a backward approximation formula,

\[
\frac{\partial u}{\partial t}(t_i, y) = \frac{25u(i) - 48u(i-1) + 36u(i-2) - 16u(i-3) + 3u(i-4)}{12\Delta t}, \quad u(k) \equiv u(t_k, y), \tag{C.19}
\]

which has an error of order of \((\Delta t)^4\).

Also on the regulator brane, we take boundary conditions (C.3) for \( u \), and Eq.(C.4) for \( \chi \). \( u \) and its derivatives are decomposed by Tchebychev polynomials at each boundary as

\[
u(t_i, \pm 1) = \sum_{n=0}^{N-2} (\pm 1)^n \tilde{u}_n \pm \tilde{u}_{N-1} + \tilde{u}_N, \tag{C.20}\]

\[
\frac{\partial u}{\partial \xi}(t_i, \pm 1) = \sum_{n=0}^{N-2} (\pm 1)^n n^2 \tilde{u}_n \mp (N-1)^2 \tilde{u}_{N-1} - N^2 \tilde{u}_N. \tag{C.21}\]
The auxiliary quantity $\chi$ is also decomposed in the same way. These four conditions Eqs.(C.3) and (C.4) for $\tilde{u}_N, \tilde{u}_{N-1}, \tilde{\chi}_N$ and $\tilde{\chi}_{N-1}$ are solved simultaneously.

In summary, we firstly evaluate the functions $F$ and $G$ in the physical space at the time $t$. Then, transforming them into the Tchebychev space by (C.8), we obtain $\tilde{u}_n$ and $\tilde{\chi}_n$ at the next time step $t + \Delta t$ by solving the reduced equations (C.14) for $0 \leq n \leq N - 2$, and imposing the boundary conditions and the additional conditions (C.16) into $\tilde{u}_n$ and $\tilde{\chi}_n$ for $n = N - 1, N$.

In the case of scalar perturbations, there is a more degree of freedom on the brane, namely, an inflaton perturbation. We have to simultaneously evolve the ordinary differential equation of the perturbation (4.73) as well as the evolution equations of the master equation (4.33) in the bulk.

### C.2 Temporal methods

#### C.2.1 The predictor–corrector method with Adams–Bashforth and Adams–Moulton formulae

As a preparation for numerical simulations, we discuss on the temporal methods provided by J. C. Adams, F. Bashforth and F. R. Moulton, which are used to solve a first-order differential equation. These methods form the foundation of the predictor–corrector method.

Let us consider a first-order differential equation

$$\frac{dy}{dt} = f(y, t). \quad (C.22)$$

Integrating this equation from $t = t_n$ to $t = t_{n+1} = t_n + h$, we obtain

$$y_{n+1} - y_n \equiv y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t)) \, dt. \quad (C.23)$$

To proceed the integration of the right-hand side, we replace an interpolating polynomial $p(t)$ by $f(t, y(t))$. While we can choose many types of the interpolating polynomial $p(t)$, two of them are explained here to use as the predictor–corrector method.

Defining

$$f_{n-k} = f(t_{n-k}, y_{n-k}), \quad 0 \leq k \in \mathbb{Z}, \quad (C.24)$$

the polynomial $p(x)$ can be obtained from $m$th-order Newton’s backward difference formula :

$$p_m(t) = \sum_{q=0}^{m} \frac{(r)_q}{q!} \nabla^q f_n, \quad (C.25)$$

where $r = (t - t_n)/h$ and

$$(r)_q \equiv r(r + 1)(r + 2) \cdots (r + q - 1), \quad (C.26)$$
is the Pochhammer symbol. The difference operator $\nabla$ yields difference formula for $f_n$ of any order:

$$\nabla^k f_n = \nabla^{k-1} f_n - \nabla^{k-1} f_{n-1}, \quad \nabla f_n = f_n - f_{n-1}. \quad (C.27)$$

This recursive relation can be reduced to

$$\nabla^k f_n = \sum_{j=0}^k C_j f_{n-j}. \quad (C.28)$$

The integration (C.23) can be reduced to the integration with respect to $r$:

$$\int_{t_n}^{t_{n+1}} p_m \, dt = \Delta t \int_0^1 p_m \, dr = h \sum_{q=0}^m A_q \nabla^q f_n, \quad (C.29)$$

where $A_q$ is the integration of the coefficients of Eq. (C.25):

$$A_q = \int_0^1 \frac{(r)_q}{q!} \, dr, \quad (C.30)$$

whose values is tabulated in Table C.1. Then, we obtain the $m$th-order Adams–Bashforth formula:

$$y_{n+1} = y_n + \Delta t \sum_{j=0}^m A_j^{(m)} f_{n-j}. \quad (C.31)$$

Here we rearranged the order of indices of sums in Eqs. (C.25) and (C.28), $\sum_{q}$ and $\sum_{j}$, and obtained

$$A_j^{(m)} = \sum_{q=j}^m C_j A_q. \quad (C.32)$$

The values of $A_j^{(m)}$ are showed in Table C.2.

Alternatively, the polynomial (C.25) can be interpolated with the points $t_{n+1}, t_n, t_{n-1}, \ldots$. In this case,

$$\hat{p}_m(t) = \sum_{q=0}^m \frac{(r)_q}{q!} \nabla^q f_{n+1}, \quad (C.33)$$

The integration of the polynomial corresponds to that with respect to $r$ between $-1$ and $0$ [see Eq. (C.29)]:

$$\int_{t_n}^{t_{n+1}} \hat{p}_m \, dt = \Delta t \int_{-1}^0 \hat{p}_m \, dr = \Delta t \sum_{q=0}^m B_q \nabla^q f_n, \quad (C.34)$$

where $B_q$ is

$$B_q = \int_{-1}^0 \frac{(r)_q}{q!} \, dr, \quad (C.35)$$

which is tabulated in Table C.1. As a result, we obtain the $m$th-order Adams–Moulton formula:

$$y_{n+1} = y_n + \Delta t \sum_{j=0}^q \tilde{A}_j^{(m)} f_{n-j}, \quad (C.36)$$
where $\tilde{B}_j^{(m)}$ is also obtained from rearranging the summations:

$$\tilde{B}_j^{(m)} = \sum_{q=j}^{m} qC_j B_q.$$  \hspace{1cm} (C.37)

The values of $\tilde{B}_j^{(m)}$ can be seen in Table C.2.

<table>
<thead>
<tr>
<th>$q$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_q$</td>
<td>1</td>
<td>1/2</td>
<td>5/12</td>
<td>3/8</td>
<td>251/720</td>
<td>95/288</td>
<td>19087/60480</td>
</tr>
<tr>
<td>$B_q$</td>
<td>1</td>
<td>-1/2</td>
<td>-1/12</td>
<td>-1/24</td>
<td>-19/720</td>
<td>-3/160</td>
<td>-863/60480</td>
</tr>
</tbody>
</table>

Table C.1: Integrations of the coefficients of polynomials $p_m(t)$ and $\hat{p}_m(t)$

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_j^{(1)}$</td>
<td>3/2</td>
<td>-1/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{B}_j^{(1)}$</td>
<td>1/2</td>
<td>1/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_j^{(2)}$</td>
<td>23/12</td>
<td>-4/3</td>
<td>5/12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{B}_j^{(2)}$</td>
<td>5/12</td>
<td>2/3</td>
<td>-1/12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_j^{(3)}$</td>
<td>55/24</td>
<td>-59/24</td>
<td>37/24</td>
<td>-3/8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{B}_j^{(3)}$</td>
<td>3/8</td>
<td>19/24</td>
<td>-5/24</td>
<td>1/24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_j^{(4)}$</td>
<td>1901/720</td>
<td>-1387/360</td>
<td>109/30</td>
<td>-637/360</td>
<td>251/720</td>
<td></td>
</tr>
<tr>
<td>$\tilde{B}_j^{(4)}$</td>
<td>251/720</td>
<td>323/360</td>
<td>-11/30</td>
<td>53/360</td>
<td>-19/720</td>
<td></td>
</tr>
<tr>
<td>$A_j^{(5)}$</td>
<td>4277/1440</td>
<td>-2641/480</td>
<td>4991/720</td>
<td>-3649/720</td>
<td>959/480</td>
<td>-95/288</td>
</tr>
<tr>
<td>$\tilde{B}_j^{(5)}$</td>
<td>95/288</td>
<td>1427/1440</td>
<td>-133/240</td>
<td>241/720</td>
<td>-173/1440</td>
<td>3/160</td>
</tr>
</tbody>
</table>

Table C.2: The coefficients of Adams–Bashforth and Adams–Moulton formula.

Using the Adams–Bashforth and Adams–Moulton formulae, we can develop the predictor–corrector method to integrate a differential equation. For example, adopting the 2nd-order method ($m = 2$), the Adams–Bashforth and Adams–Moulton formulae are given by

$$y_{n+1}^* = y_n + \frac{\Delta t}{12} (23f_n - 16f_{n-1} + 5f_{n-2}),$$  \hspace{1cm} (C.38)

$$y_{n+1} = y_n + \frac{\Delta t}{12} (5f_{n+1}^* + 8f_n - f_{n-1}).$$  \hspace{1cm} (C.39)

Here we use the Adams–Bashforth formula to predict the value of $y$ at the time $t = t_{n+1}$, which is written by $y_{n+1}^*$. Using this predicted value, we next estimate the function $f_{n+1}^* = f(t_{n+1}, y_{n+1}^*)$. Finally, we obtain the corrected value of $y_{n+1}$ from the Adams–Moulton formula (C.39).

In our numerical simulations, we take $m = 4$ formulae. Furthermore, in order to enhance the numerical accuracy, we check that the difference between the predicted value $y_{n+1}^*$ and the corrected value $y_{n+1}$ is suppressed to a given tolerance, $|y_{n+1}^* - y_{n+1}| < \epsilon$ at each time step.
C.2.2 Runge–Kutta method

Given an initial condition \( y_0 \), we have to estimate \( y_1, y_2, y_3, y_4 \) in advance to implement the \( m = 4 \) predictor–corrector method. For this purpose, we use the 4th-order Runge–Kutta method developed by C. D. T. Runge and M. W. Kutta.

Let us consider again a first-order differential equation (C.22) and solve it with an initial condition \( y_0 \) at a given time \( t = t_0 \). Defining \( y_n = y(t_n) \), the value of \( y \) at \( t = t_{n+1} \) can be described in the Runge–Kutta method by a following general form:

\[
y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i k_i,
\]

where \( b_i \) and \( c_i \) satisfy

\[
\sum_{i=1}^{s} b_i = 1, \quad c_i = \sum_{j=1}^{s} a_{ij}.
\]

To develop an explicit formula, we assume \( a_{ij} = 0 \) for \( j \geq i \). The \( s = 1 \) case is well-known as the Euler method:

\[
y_{n+1} = y_n + \Delta t f(t_n, y_n).
\]

In the case of \( s = 2 \), we can develop two type of methods. One of them is called the modified Euler method:

\[
y_{n+1} = y_n + \Delta t f(t_n, y_n),
\]

\[
\begin{align*}
  k_1 &= f(t_n, y_n), \\
  k_2 &= f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_1),
\end{align*}
\]

and the other is called the improved Euler method or the 2nd-order Heun method:

\[
y_{n+1} = y_n + \frac{\Delta t}{2} (k_1 + k_2)
\]

\[
\begin{align*}
  k_1 &= f(t_n, y_n), \\
  k_2 &= f(t_n + \Delta t, y_n + \Delta t k_1).
\end{align*}
\]

The most popular formula is known as the classical 4th-order explicit method:

\[
y_{n+1} = y_n + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4)
\]

\[
\begin{align*}
  k_1 &= f(t_n, y_n), \\
  k_2 &= f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_1), \\
  k_3 &= f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_2), \\
  k_4 &= f(t_n + \Delta t, y_n + \Delta t k_3).
\end{align*}
\]

In our numerical code, we use this 4th-order method to estimate \( y_1, y_2, y_3, y_4 \) for the following predictor–corrector method.
Appendix D

Moving Brane in Minkowski Bulk

We assume that a moving brane is embedded in the Minkowski bulk whose metric with its tensor perturbations is given by

\[ ds^2 = -d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j + dz^2. \]  

(D.1)

The trajectory of the moving brane is arbitrary, but, in order to mimic the Friedmann brane, here we consider a limited case that the velocity \( \frac{dz}{d\tau} \) varies from \(-1\) to \(0\). We therefore apply a test function for the velocity of the brane,

\[ \beta_b \equiv \frac{dz_b}{d\tau} = \frac{1}{2} \left( \tanh \frac{\tau - \tau_c}{\Delta} - 1 \right), \]  

(D.2)

whose behaviour is shown in Fig. D.1. Here we set \( \tau_c = 3.0 \) and \( \Delta = 0.2 \). Then the induced metric on the moving brane is calculated as

\[ ds^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j, \]  

(D.3)

where we defined the proper time on the brane, \( t \), as

\[ t = \int_0^\tau \sqrt{1 - \beta_b^2} \, d\tau. \]  

(D.4)

In the Minkowski bulk, the evolution equation of GWs is given by eliminating the third term of Eq. (5.9),

\[ \frac{\partial^2 h}{\partial \tau^2} - \frac{\partial^2 h}{\partial z^2} + k^2 h = 0. \]  

(D.5)

This equation is equivalent to an wave equation for a massive field with a mass \( k^2 \). Hence the general solution is given as a linear combination of sinusoidal functions :

\[ h(\tau, z) = \int_{0}^{\infty} \{ C_1(m)e^{i(mz+\omega \tau)} + C_2(m)e^{-i(mz+\omega \tau)} \} \, dm, \]  

(D.6)

where

\[ \omega = \sqrt{m^2 + k^2}. \]  

(D.7)
Figure D.1: The functional form of the velocity of the test brane. We set $\tau_c = 3.0$ and $\Delta = 0.2$.

The initial condition for the simulations is taken as the same form as Eq. (5.15), namely, $h(0, z) = 1$ and $dh/d\tau(0, z) = 0$. To fix the computational domain in the bulk, we again introduce the regulator brane at $z = z_{\text{reg}}$ where we fix $z_{\text{reg}} = 50$. The boundary conditions are also same as ones imposed in the five-dimensional cases (5.10) and (5.16), but the functional form of former one represented with $\partial / \partial \tau$ and $\partial / \partial z$ is different from the five-dimensional one due to the difference of the metric in the bulk. In the present case, we obtain

$$\frac{\partial h}{\partial n} = \beta_b \gamma_b \frac{\partial h}{\partial \tau} + \gamma_b \frac{\partial h}{\partial z} = 0,$$  \hspace{1cm} (D.8)

where $\gamma_b$ is the Lorentz factor of the moving brane, given by

$$\gamma_b \equiv \frac{1}{\sqrt{1 - \beta_b^2}}.$$  \hspace{1cm} (D.9)

In the present setup, there exists no 'Hubble horizon', that is, we have to define a concept of 'horizon crossing' like the Friedmann cases. In the case of Friedmann brane or four-dimensional setup, the time at the horizon crossing, $aH = k$, corresponds to $|k\eta| = 1$ where $\eta$ is the conformal time. The counterpart of the present case is $k\tau = 1$ in the sense that the wave on the brane represented by Eq. (D.6) starts to oscillate at that time. Hence the quantities with a subscript '*' are defined by their values evaluated at $\tau = \tau_* = 1/k$.

Then, in both present and the Friedmann cases, the Lorentz factor of the brane becomes

$$\gamma_b = \sqrt{1 + H_*^2 \ell^2}.$$  \hspace{1cm} (D.10)

In addition, we have to specify the terminologies, 'zero-mode' and 'KK-modes'. In the present case, the zero-mode is defined as a wave with $m = 0$ and the KK-modes with $m > 0$. Thus the initial condition imposed here is definitely the zero-mode.
We perform the simulations starting from $\tau = 0$ to $\tau = 100$ for GWs with wave numbers, $k = 10, 20, 50, 75, 100, 200, 500$. The resultant waveforms with $k = 10$ in the bulk and on the brane are shown in Fig. (D.2). To avoid the contamination on the brane from the reflected waves by the regulator brane, we stopped the simulations at $\tau = 100$. We use the Fourier analysis to estimate the remained zero-mode. The Fourier component of the resultant amplitude is calculated as

$$\tilde{h}(\omega) = \mathcal{N} \int_{\tau_0}^{\tau_1} h(\tau)e^{i\omega\tau}d\tau,$$  \hspace{1cm} (D.11)$$

where the normalisation factor $\mathcal{N}$ is determined for $\tilde{h}(\omega)$ to be unity when the integrand is set to the wave corresponding to the present initial condition with $\beta_0 = 0$, namely, $h(\tau) = \cos k\tau$. Note that, since the integral interval is not infinite, we have to take care of the sideband waves generated by the finite Fourier transformation. The left panel of Fig. D.3 shows that the amplitude of zero-mode with $\omega = k = 100$ is reduced from the initial amplitude, $\tilde{h}(k) = 1$, and instead lots of KK-modes are excited. The final amplitudes for $k = 10, 20, 50, 75, 100, 200, 500$ are summarised in the right panel of Fig. D.3, which mimics the universal relation found in the five-dimensional cases [see Eq. (5.34) and Fig. 5.14].

Figure D.2: The waveform with $k = 10$ on the brane (left) and in the bulk (right).
Figure D.3: The spectrum with $k = 100$ (left) and the $\gamma_*$ dependence of the amplitude of $h$ in the toy model (right).
References


[18] [http://universe.nasa.gov/program/bbo.html](http://universe.nasa.gov/program/bbo.html).


REFERENCES


