Monte Carlo studies of the spontaneous rotational symmetry breaking in a matrix model with the complex action

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1. Introduction

Matrix models as a constructive definition of superstring theory

iKKT model (IIB matrix model)

$\Rightarrow$ Promising candidate for constructive definition of superstring theory.

N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115.

$$S = N\left(-\frac{1}{4}\mathrm{tr}\,[A_\mu, A_\nu]^2 + \frac{1}{2}\mathrm{tr}\,\bar{\psi}_\alpha(\Gamma_\mu)_{\alpha\beta}[A_\mu, \psi_\beta]\right).$$

- Dimensional reduction of $\mathcal{N}=1$ 10d Super-Yang-Mills (SYM) theory to 0d.
  A_μ (10d vector) and ψ_α (10d Majorana-Weyl spinor) $\Rightarrow$ $N \times N$ matrices .
- Evidences for spontaneous breakdown of SO(10) $\to$ SO(4).
  J. Nishimura and F. Sugino, hep-th/0111102,
  H. Kawai, et. al. hep-th/0204240,0211272,0602044,0603146.
- Complex fermion determinant:
  * Crucial for rotational symmetry breaking.
    J. Nishimura and G. Vernizzi, hep-th/0003223.
  * Difficulty of Monte Carlo simulation.

2. Simplified IKKT matrix model

Simplified model with spontaneous rotational symmetry breakdown,

J. Nishimura, hep-th/0108070.

$$S = \underbrace{\frac{N}{2}\mathrm{tr}\,A_\mu^2}_{=S_b} \underbrace{-\bar{\psi}^f_\alpha(\Gamma_\mu)_{\alpha\beta}A_\mu\psi^f_\beta}_{=S_f}$$

- A_μ: $N \times N$ hermitian matrices ($\mu = 1, \cdots, 4$)
  $\bar{\psi}^f_\alpha, \psi^f_\alpha$: N-dim vector ($\alpha = 1, 2, f = 1, \cdots, N_f$),
  $\Rightarrow$ CPU cost $O(N^3)$ (instead of $O(N^6)$ in IKKT)
  N_f = (number of flavors).

$$\Gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\Gamma_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma_4 = i\sigma_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

- $SU(N)$ symmetry and $SO(4)$ rotational symmetry.
- No supersymmetry.
- Partition function:

$$Z = \int dA e^{-S_B}(\det \mathcal{D})^{N_f} = \int dA e^{-S_0} e^{i\Gamma}, \quad \text{where}$$
$$\mathcal{D} = \Gamma_\mu A_\mu = \begin{pmatrix} iA_3 + A_4 & iA_1 + A_2 \\ iA_1 - A_2 & -iA_3 + A_4 \end{pmatrix} = (2N \times 2N \text{ matrices}),$$

Phase-quenched partition function

$$Z_0 = \int dA e^{-S_0} = \int dA e^{-S_B} |\det \mathcal{D}|^{N_f}.$$

$(\Gamma_4)^\dagger = \Gamma_4$, $(\Gamma_i)^\dagger = -\Gamma_i$ $(i = 1, 2, 3)$
$\Rightarrow$ $\mathcal{D}$ becomes complex conjugate under

$$A_i^P = A_i (i = 1, 2, 3), \quad A_4^P = -A_4$$

Analytical studies of the model

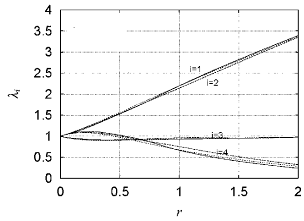
Gaussian expansion analysis up to 9th order:

T. Okubo, J. Nishimura and F. Sugino, hep-th/0412194.

Observable for probing dimensionality : $T_{\mu\nu} = \frac{1}{N}\mathrm{tr}\,(A_\mu A_\nu)$.

λ_i $(i = 1, 2, 3, 4)$: eigenvalues of $T_{\mu\nu}$ $(\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \lambda_4)$

Spontaneous breakdown of $SO(4)$ to $SO(2)$ at finite r $\left(= \frac{N_f}{N}\right)$.



3. Monte Carlo simulation

Factorization method

Numerical approach to the complex action problem.

K. N. Anagnostopoulos and J. Nishimura, hep-th/0108041,

J.Ambjorn, K.N.Anagnostopoulos, J.Nishimura and J.J.M.Verbaarschot, hep-lat/0208025.

Overlap problem: Discrepancy of a distribution function between the phase-quenched model Z_0 and the full model Z.

Standard reweighting method:

$$\langle \lambda_i \rangle = \frac{\langle \lambda_i \cos\Gamma \rangle_0}{\langle \cos\Gamma \rangle_0},$$

where $\langle * \rangle_0$ = (V.E.V. for the phase-quenched model Z_0).

($\sharp$ of configurations required) $\simeq e^{O(N^2)}$. $\Rightarrow$ complex-action problem.

Distribution function

$$\rho_i(x) \stackrel{\text{def}}{=} \langle \delta(x - \tilde{\lambda}_i) \rangle = \frac{1}{C}\rho_i^{(0)}(x) w_i(x),$$

where

$$\tilde{\lambda}_i = \lambda_i / \langle \lambda_i \rangle_0, \quad C = \langle \cos\Gamma \rangle_0,$$
$$\rho_i^{(0)}(x) = \langle \delta(x - \tilde{\lambda}_i) \rangle_0, \quad w_i(x) = \langle \cos\Gamma \rangle_{i,x},$$
$$\langle * \rangle_{i,x} = [\text{V.E.V. for the partition function } Z_{i,x}]$$
$$Z_{i,x} = \int dA e^{-S_0} \delta(x - \tilde{\lambda}_i)].$$

Resolution of the overlap problem:

$\Rightarrow$ Visit the configurations where $\rho_i(x)$ is important.

Monte Carlo evaluation of $\rho_i^{(0)}(x)$ and $w_i(x)$

Approximation of the partition function $Z_{i,x}$:

$$Z_{i,V} = \int dA e^{-S_0} \underbrace{e^{-V(\lambda_i)}}_{\simeq \delta(x - \tilde{\lambda}_i)}, \quad \text{where}$$
$$V(x) = \frac{\gamma}{2}(x - \xi)^2, \quad \gamma, \xi = \text{(parameters)}.$$

Monte Carlo evaluation of $\rho_i^{(0)}(x)$ and $w_i(x)$:

$$\rho_{i,V}(x) \stackrel{\text{def}}{=} \langle \delta(x - \tilde{\lambda}_i) \rangle_{i,V} \propto \rho_i^{(0)}(x) \exp(-V(\langle \lambda_i \rangle_0 x)).$$

The position x_p of the peak for $\rho_{i,V}(x)$:

$$0 = \frac{\partial}{\partial x}\log \rho_{i,V}(x) = f_i^{(0)}(x) - \langle \lambda_i \rangle_0 V'(\langle \lambda_i \rangle_0 x),$$
$$f_i^{(0)}(x) \stackrel{\text{def}}{=} \frac{\partial}{\partial x}\log \rho_i^{(0)}(x).$$

- Determination of x_p: Approximated as $x_p \simeq \langle \tilde{\lambda}_i \rangle_{i,V}$.
- Determination of $\rho_i^{(0)}(x)$:
  1. Vary ξ.
  2. Calculate $f_i^{(0)}(x_p)$ for different x_p (and ξ).
  3. Evaluate $\rho_i^{(0)}(x) = \exp\left\{\int_0^x dz f_i^{(0)}(z) + \text{const.}\right\}$.

Why such a roundabout way?

$\Rightarrow$ To capture the skirt of $\rho_i^{(0)}(x)$.

Monte Carlo evaluation of $\langle\tilde{\lambda}_i\rangle$

$w_i(x) > 0 \Rightarrow \langle\tilde{\lambda}_i\rangle$ is the minimum of $\mathcal{F}_i(x)$:

$$\mathcal{F}_i(x) = (\text{free energy density}) = -\frac{1}{N^2}\log\rho_i(x).$$

We solve $\mathcal{F}_i'(x) = 0$, namely $\frac{1}{N^2}f_i^{(0)}(x) = -\frac{d}{dx}\left\{\frac{1}{N^2}\log w_i(x)\right\}$.

Both $\frac{1}{N^2}\log w_i(x)$ and $\frac{1}{N^2}f_i^{(0)}(x)$ scale at large N as

$$\frac{1}{N^2}\log w_i(x) \to \Phi_i(x), \qquad \frac{1}{N^2}f_i^{(0)}(x) \to F_i(x).$$

Behavior of $\Phi_i(x)$

Analytical behavior of $\Phi_i(x) = \frac{1}{N^2}\log w_i(x)$ at $x \ll 1$ and $x \gg 1$.

When we fix the i-th largest eigenvalue $\to$

- $x \ll 1$ $(i = 2,3,4)$: $(5-i)$ directions are shrunk
  $\Rightarrow$ $(i-1)$-dimensional configuration
- $x \gg 1$ $(i = 1,2,3)$: $(4-i)$ directions are shrunk
  $\Rightarrow$ i-dimensional configuration

Fermion determinant $\mathcal{D}$ is complex conjugate under

$$A_i^P = -A_i (i = 1,2,3), \quad A_4^P = A_4$$

$\Omega_d = \{\{A_\mu\}; n_\mu^{(i)}A_\mu = 0 \text{ for } \exists n_\mu^{(i)} (i = 1,\cdots 4-d)\}$

3-dimensional configuration $\Omega_3 \Rightarrow$ Fermion determinant is real.

J. Nishimura and G. Vernizzi, hep-th/0003223.

For d-dimensional configuration Ω_d,

$$\frac{\partial^n\Gamma}{\partial A_{\mu_1}^{a_1}\cdots\partial A_{\mu_n}^{a_n}} = 0 \text{ for } n = 1,\cdots 3-d$$

(Up to $(3-d)$-order perturbation $\Rightarrow$ configuration $\in \Omega_3$)

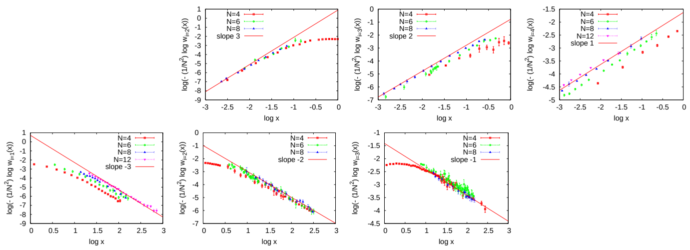
Expected power behaviors:

$$\Phi_i(x) \propto \begin{cases} c_{i,0}x^{5-i} + c_{i,1}x^{\frac{11}{2}-i} + \cdots & (x \ll 1, i = 2,3,4) \\ \frac{d_{i,0}}{x^{4-i}} + \frac{d_{i,1}}{x^{\frac{9}{2}-i}} + \cdots & (x \gg 1, i = 1,2,3) \end{cases}$$

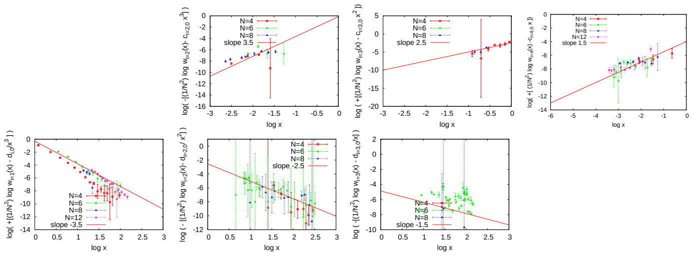
(*) x has the order of the eigenvalues of $T_{\mu\nu} = \frac{1}{N}\mathrm{tr}(A_\mu A_\nu)$.

Simulation for $r = 1$

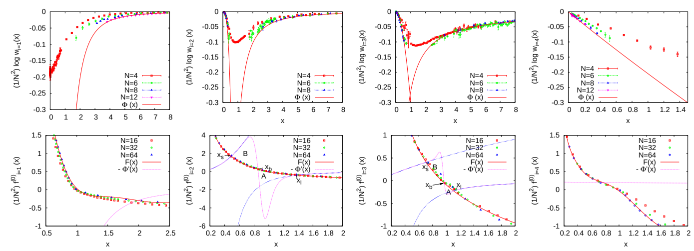
Contribution of the leading order



Contribution of the next-leading order



Evaluation of $\langle\tilde{\lambda}_i\rangle$



$i = 2,3 \Rightarrow$ double-peak structure of $\rho_i(x)$.

Extrapolation of $\Phi_i(x)$:

$$\Phi_i(x) \simeq \begin{cases} \phi_{i,s}(x) = c_{i,0}x^{5-i} + c_{i,1}x^{\frac{11}{2}-i} + \cdots, & (x \ll 1), \\ \phi_{i,l}(x) = \frac{d_{i,0}}{x^{4-i}} + \frac{d_{i,1}}{x^{\frac{9}{2}-i}} + \cdots, & (x \gg 1), \\ \frac{\phi_{i,s}(x)e^{-\mathcal{C}(x-\alpha)} + \phi_{i,l}(x)e^{\mathcal{C}(x-\alpha)}}{e^{-\mathcal{C}(x-\alpha)} + e^{\mathcal{C}(x-\alpha)}}, & (\text{intermediate } x). \end{cases}$$

At $x = \alpha$, $\phi_{i,s}(x) = \phi_{i,l}(x)$.

Three solutions of $\mathcal{F}_i'(x) = 0$ $(x_s < x_b < x_l)$.

Which peak is higher?

- $\frac{1}{N^2}(\log\rho_i(x_l) - \log\rho_i(x_b)) = \int_{x_b}^{x_l} dx (F_i(x) + \Phi_i'(x)).$
  $=$ (A's area).
- $\frac{1}{N^2}(\log\rho_i(x_s) - \log\rho_i(x_b)) = -\int_{x_s}^{x_b} dx (F_i(x) + \Phi_i'(x)).$
  $=$ (B's area).

Difference of the height:

$$\begin{aligned} \Delta_i &= \frac{1}{N^2}(\log\rho_i(x_l) - \log\rho_i(x_s)) \\ &= (\Phi_i(x_l) - \Phi_i(x_s)) + \int_{x_s}^{x_l} dx F_i(x) \\ &= (\text{A's area})\text{-}(\text{B's area}) \\ &\simeq \begin{cases} +0.28\cdots > 0, & (i = 2), \\ -0.10\cdots < 0, & (i = 3). \end{cases} \end{aligned}$$

$\langle\tilde{\lambda}_{i=2}\rangle = 1.4$, $\langle\tilde{\lambda}_{i=3}\rangle = 0.7$

$\Rightarrow$ Rotational symmetry breaking SO(4) $\to$ SO(2).

Result of 9th-order Gaussian expansion:

$$\tilde{\lambda}_{i=1} \simeq 1.4, \ \tilde{\lambda}_{i=2} \simeq 1.4, \ \tilde{\lambda}_{i=3} \simeq 0.7, \ \tilde{\lambda}_{i=4} \simeq 0.5.$$

Behavior of $\frac{1}{N^2}f_i^{(0)}(x)$

Small x $(x \ll 1) \to (5-i)$ directions are shrunk.

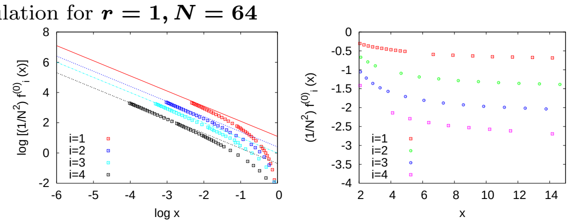
- $i = 2,3,4$: $\rho_i^{(0)}(x) \simeq (\sqrt{x})^{N^2(5-i)}$
  $\Rightarrow \frac{1}{N^2}f_i^{(0)}(x) \simeq \left(\frac{5-i}{2x}\right)$
- $i = 1$: Eigenvalues of A_μ are collapsed to zero.
  $\Rightarrow$ Add the effect of fermionic determinant
  (polynomial of A_μ with degree $2N^2r$).
  $\Rightarrow \rho_{i=1}^{(0)}(x) = (\sqrt{x})^{2N^2(1+r)} \Rightarrow \frac{1}{N^2}f_i^{(0)}(x) \simeq \left(\frac{2+r}{x}\right)$

$$\log\left(\frac{1}{N^2}f_i^{(0)}(x)\right) = \begin{cases} -\log x + \log(2+r), & i = 1, \\ -\log x + \log\left(\frac{5-i}{2}\right), & i = 2,3,4. \end{cases}$$

Large x $(x \gg 1)$:

$$\frac{1}{N^2}f_i^{(0)}(x) \overset{x\to+\infty}{\to} (\text{costant})$$

Simulation for $r = 1, N = 64$



4. Simulation of IKKT model

6d version of IKKT model

- Gaussian expansion $\Rightarrow$ Symetry breaking SO(6) $\to$ SO(3)
  T .Aoyama, J. Nishimura, T. Okubo and S. Takeuchi
- Monte Carlo simulation
  K.N. Anagnostopoulos, T. Aoyama, T.A., M .Hanada and J. Nishimura

Fermion is not vector but adjoint representation.

$\Rightarrow$ More CPU cost $O(N^6)$

Supersymmetry $\Rightarrow$ Solution of $\mathcal{F}_i'(x) = 0$ at $x \ll 1$ and $x \gg 1$.

Asymptotic behaviors of $\frac{1}{N^2}\log w_i(x)$ and $\frac{1}{N^2}f_i^{(0)}(x)$ are important.