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Monte Carlo Studies of Dynamical Compactification of Extra Dimensions in a Model of Non-perturbative String Theory (arXiv:1509.05079)

Takehiro Azuma (Setsunan Univ.)

with Konstantinos N. Anagnostopoulos and Jun Nishimura

Tea Duality seminar at TIFR,

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1. Introduction

Difficulties in putting complex partition functions on computers.

$$Z = \int dA \exp(-S_0 + i\Gamma), \;\; Z_0 = \int dA e^{-S_0}$$

e.g. lattice QCD, matrix models for superstring theory

1. Sign problem:
   The reweighting $\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\Gamma} \rangle_0}{\langle e^{i\Gamma} \rangle_0}$ requires configs. $\exp[O(N^2)]$
   $\langle * \rangle_0$ = (V.E.V. for the phase-quenched partition function Z_0)

2. Overlap problem:
   Discrepancy of important configs. between Z_0 and Z.

2. Factorization method

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Method to sample important configurations for Z.

[J. Nishimura and K.N. Anagnostopoulos, hep-th/0108041
K.N. Anagnostopoulos, T.A. and J. Nishimura, arXiv:1009.4504]

We constrain the observables $\Sigma = \{\mathcal{O}_k | k = 1, 2, \cdots, n\}$ correlated with the phase Γ.

They are normalized as $\tilde{\mathcal{O}}_k = \mathcal{O}_k / \langle \mathcal{O}_k \rangle_0$

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The distribution function factorizes as

$$\rho(x_1,\cdots,x_n) \stackrel{\text{def}}{=} \left\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\rangle \stackrel{\text{reweighting}}{=} \frac{\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) e^{i\Gamma} \rangle_0}{\langle e^{i\Gamma} \rangle_0}$$

$$= \frac{1}{\langle e^{i\Gamma}\rangle_0} \times \left\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\rangle_0 \times \frac{\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) e^{i\Gamma} \rangle_0}{\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \rangle_0}$$

$$= \frac{1}{\langle e^{i\Gamma}\rangle_0} \times \left\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\rangle_0 \times \frac{\int dA e^{-S_0} \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) e^{i\Gamma}}{\int dA e^{-S_0}} \div \frac{\int dA e^{-S_0} \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k)}{\int dA e^{-S_0}}$$

$$= \underbrace{\frac{1}{\langle e^{i\Gamma}\rangle_0}}_{=1/C} \times \underbrace{\left\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\rangle_0}_{\stackrel{\text{def}}{=}\rho^{(0)}(x_1,\cdots,x_n)} \times \underbrace{\frac{\int dA e^{-S_0} \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) e^{i\Gamma}}{\int dA e^{-S_0} \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k)}}_{\stackrel{\text{def}}{=} w(x_1,\cdots,x_n)}$$

$$w(x_1,\cdots,x_n) = \langle e^{i\Gamma} \rangle_x$$

$$\left(\langle * \rangle_x = \left\{ \text{V.E.V. for } Z_x = \int dA e^{-S_0} \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\} \right)$$

Partition function in the constrained system.

Simulation of Z_x with a proper choice of the set Σ
⇒sample the important region for Z.

Evaluation of the observables $\langle \tilde{\mathcal{O}}_k \rangle$

Peak of the distribution function ρ at V=(system size)→∞.
= Minimum of the free energy $\mathscr{F} = -\frac{1}{N^2} \log \rho$

⇒Solve the saddle-point equation $\frac{1}{N^2} \frac{\partial}{\partial x_n} \log \rho^{(0)} = -\frac{\partial}{\partial x_n} \frac{1}{N^2} \log w$

Applicable to general systems with sign problem.

3. The model

IKKT model (or the IIB matrix model)

⇒Promising candidate for nonperturbative string theory

[N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115]

$$S = \underbrace{-\frac{N}{4}\mathrm{tr}[A_\mu, A_\nu]^2}_{=S_B} + \underbrace{\frac{N}{2}\mathrm{tr}\bar{\psi}_\alpha(\Gamma^\mu)_{\alpha\beta}[A_\mu, \psi_\beta]}_{=S_F}$$

Euclidean case after the Wick rotation $A_0 \to iA_{10}$, $\Gamma^0 \to -i\Gamma_{10}$.

⇒Path integral is finite without cutoff.

[W. Krauth, H. Nicolai and M. Staudacher, hep-th/9803117, P. Austing and J.F. Wheater, hep-th/0103059]

A_μ, Ψ_α ⇒N×N Hemitian matrices (μ=1,2,…,d=10, α,β=1,2,…,16)

Ψ_α : Majorana-Weyl fermion

Matrix regularization of the type IIB string action:

$$S_{\mathrm{Sh}} = \int d^2\sigma \left\{ \sqrt{g}\alpha \left(\frac{1}{4}\{X_\mu, X_\nu\}^2 - \frac{i}{2}\bar{\psi}\Gamma^\mu\{X_\mu, \psi\} \right) + \beta\sqrt{g} \right\}.$$

$$-i[,] \leftrightarrow \{,\}, \quad \mathrm{tr} \leftrightarrow \int d^2\sigma\sqrt{g}.$$

N=2 supersymmetry

$$\tilde{\delta}_\varepsilon^{(1)} = \delta_\varepsilon^{(1)} + \delta_\varepsilon^{(2)} \qquad \tilde{\delta}_\varepsilon^{(2)} = i(\delta_\varepsilon^{(1)} - \delta_\varepsilon^{(2)})$$ where

$$\delta_\varepsilon^{(1)} A_\mu = i\varepsilon(\mathscr{C}\Gamma_\mu)\psi, \;\; \delta_\varepsilon^{(1)}\psi = \frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\varepsilon, \;\; \delta_\varepsilon^{(2)} A_\mu = 0, \;\; \delta_\varepsilon^{(2)}\psi = \varepsilon.$$

$$[\tilde{\delta}_\varepsilon^{(a)}, \tilde{\delta}_\xi^{(b)}]A_\mu = -2i\delta^{ab}\varepsilon(\mathscr{C}\Gamma_\mu)\xi, \;\; [\tilde{\delta}_\varepsilon^{(a)}, \tilde{\delta}_\xi^{(b)}]\psi = 0, \;\; (a, b = 1, 2).$$

Eigenvalues of A_μ ⇒spacetime coordinate.

Dynamical emergence of the spacetime due to the Spontaneous Symmetry Breaking (SSB) of SO(10).

Result of Gaussian Expansion Method (GEM)

Order parameter of the SO(10) rotational symmetry breaking

$$\lambda_n(\lambda_1 \geq \cdots \geq \lambda_{10}) : \text{eigenvalues of } T_{\mu\nu} = \frac{1}{N}\text{tr}(A_\mu A_\nu)$$

$$\langle\lambda_1\rangle = \cdots = \langle\lambda_d\rangle (= R^2) \gg \langle\lambda_{d+1}\rangle = \cdots = \langle\lambda_{10}\rangle (= r^2)$$

Extended d-dim. and shrunken (10-d) dim. at N→∞

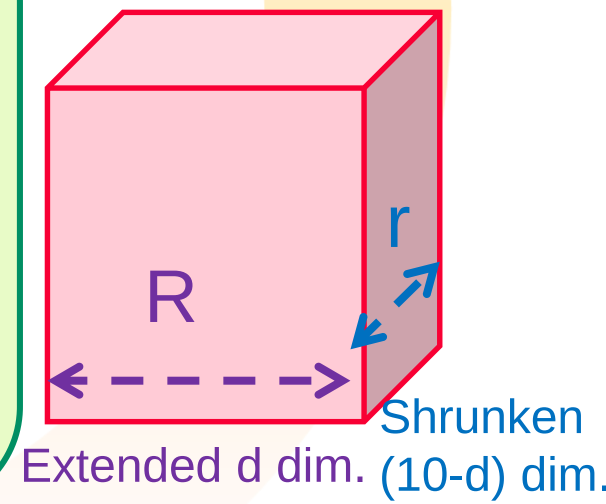
SSB SO(10)→SO(d)

Main Results of GEM

[J. Nishimura, T. Okubo and F. Sugino, arXiv:1108.1293]

- Universal compactification scale $r^2 \cong 0.15$ for SO(d) ansatz (d=2,3,…7).
- Constant volume property except d=2 $V = R^d \times r^{10-d} = l^{10}$, $l^2 \cong 0.38$
- SSB SO(10)→SO(3).

10 dim. volume $V = R^d \times r^{10-d}$



Mechanism of SSB in Euclidean case

Partition function of the model:

$$Z = \int dA \; e^{-S_B} \underbrace{\left(\int d\psi e^{-S_F} \right)}_{= \mathrm{Pf}\mathcal{M} = |\mathrm{Pf}\mathcal{M}| e^{i\Gamma}} = \int dA \; \underbrace{e^{-S_0}}_{= e^{-S_B} |\mathrm{Pf}\mathcal{M}|} \; e^{i\Gamma}$$

$$\mathcal{M}_{a\alpha, b\beta} = -i f_{abc} (\mathcal{C}\Gamma)_{\alpha\beta} (A_\mu)^c \quad (a,b,c = 1,2,\cdots,N^2-1, \;\; \alpha,\beta = 1,2,\cdots,16)$$

$16(N^2-1) \times 16(N^2-1)$ anti-symmetric matrix

The Pfaffian PfM is complex in the Euclidean case

⇒Complex phase Γ is crucial for the SSB of SO(10).

[J. Nishimura and G. Vernizzi hep-th/0003223]

Under the parity transformation $A_{10} \Rightarrow -A_{10}$,
PfM is complex conjugate
⇒ PfM is real for $A_{10}=0$ (hence 9-dim config.).

For the d-dim config, $$\frac{\partial^m \Gamma}{\partial A_{\mu_1} \cdots \partial A_{\mu_m}} = 0$$

Up to m=9-d, the config. is at most 9-dim.

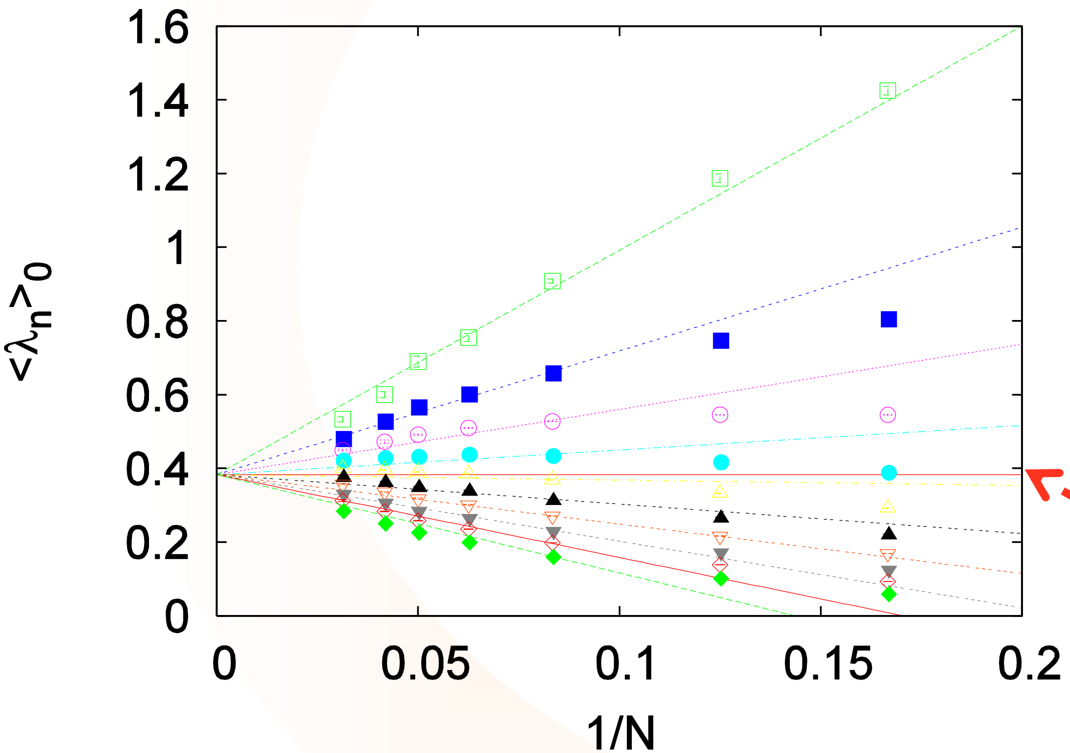
The phase is more stationary for lower d.

No SSB with the phase-quenched partition function.

$$Z_0 = \int dA e^{-S_0}$$

$<*>_0$=V.E.V. for Z_0

[J. Ambjorn, K.N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, hep-th/0003208,0005147]



Result of the Euclidean IKKT model for N≦32.

The N→∞ limit is consistent with the constant volume property l^2=0.38.

4. Results of the Monte Carlo simulation

It turns out sufficient to constrain only one eigenvalue λ_{d+1}

$\Sigma = \{\lambda_{d+1} \text{ only}\}$ Corresponds to the SO(d) vacuum

$$\langle\lambda_1\rangle = \cdots = \langle\lambda_d\rangle (= R^2) \gg \langle\lambda_{d+1}\rangle = \cdots = \langle\lambda_{10}\rangle (= r^2)$$

$\tilde{\lambda}_n \stackrel{\text{def}}{=} \lambda_n / \langle\lambda_n\rangle_0$ corresponds to $(r/l)^2$[≃0.15/0.38=0.40 (GEM)]

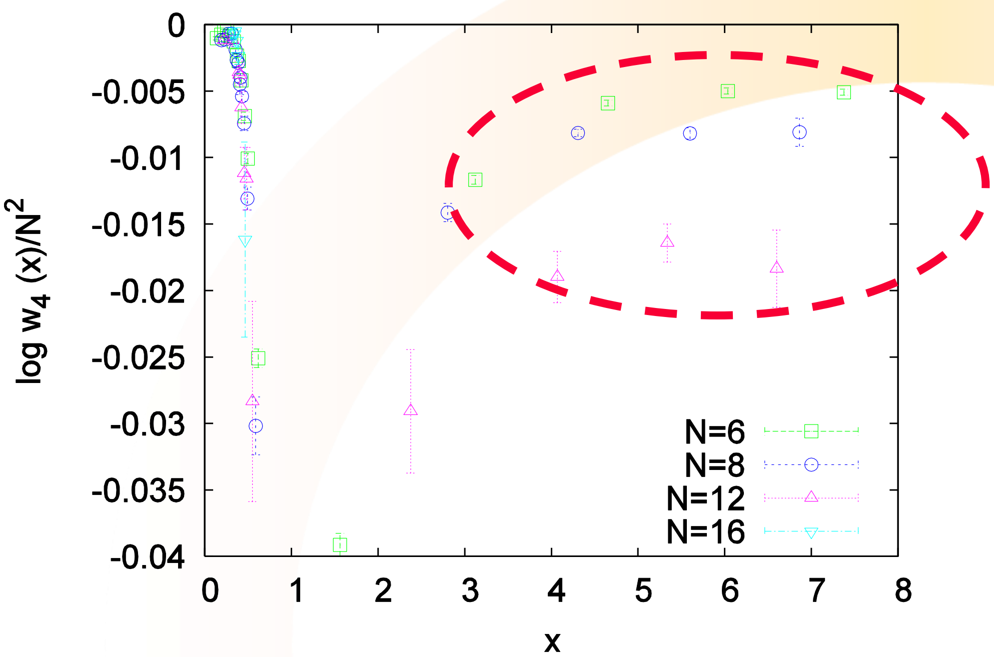
$$\frac{1}{N^2} f_n^{(0)}(x) = -\frac{d}{dx}\frac{1}{N^2}\log w_n(x) \text{ where n=d+1}$$

$$f_n^{(0)}(x) \stackrel{\text{def}}{=} \frac{d}{dx}\log\langle\delta(x-\tilde{\lambda}_n)\rangle_0, \quad w_n(x) \stackrel{\text{def}}{=} \langle e^{i\Gamma}\rangle_{n,x} = \langle\cos\Gamma\rangle_{n,x}$$

$$\langle * \rangle_{n,x} = \left\{ \text{V.E.V. for } Z_{n,x} = \int dA e^{-S_0}\delta(x-\tilde{\lambda}_n) \right\}$$

S_0 and $T_{\mu\nu} = \frac{1}{N}\text{tr}(A_\mu A_\nu)$(hence λ_n) are invariant under $A_{10} \Rightarrow -A_{10}$.

The solution $\bar{x}_n$ corresponds to $\bar{x}_n = \langle\tilde{\lambda}_{d+1}\rangle_{\text{SO(d)}}$ in the SO(d) vacuum.



$\frac{1}{N^2}\log w_n(x)$ is almost constant at large x.

No need to constrain the larger eigenvalues $\lambda_{1\sim d}$.

The phase $w_n(x)$ scales at large N as

$$\Phi_n(x) = \lim_{N\to\infty} \frac{1}{N^2} \log w_n(x) \simeq -a_n x^{10-(n-1)} - b_n \quad (x<1)$$

For the d=(n-1) dim. config, $\frac{\partial^m \Gamma}{\partial A_{\mu_1} \cdots \partial A_{\mu_m}} = 0$ up to m=9-d.

⇒The fluctuation of the phase is $\delta\Gamma \propto (\delta A/|A|)^{10-d} \propto (\sqrt{x})^{10-d}$

(x corresponds to the eigenvalues of $T_{\mu\nu} = \frac{1}{N}\mathrm{tr}(A_\mu A_\nu) = \mathrm{O}(A^2)$)

Assume that Γ's distribution is Gaussian:

$$\langle e^{i\Gamma}\rangle = \int d\Gamma \frac{1}{\sqrt{2\pi}\sigma} e^{-\Gamma^2/(2\sigma^2)} e^{i\Gamma} = e^{-\sigma^2/2}$$

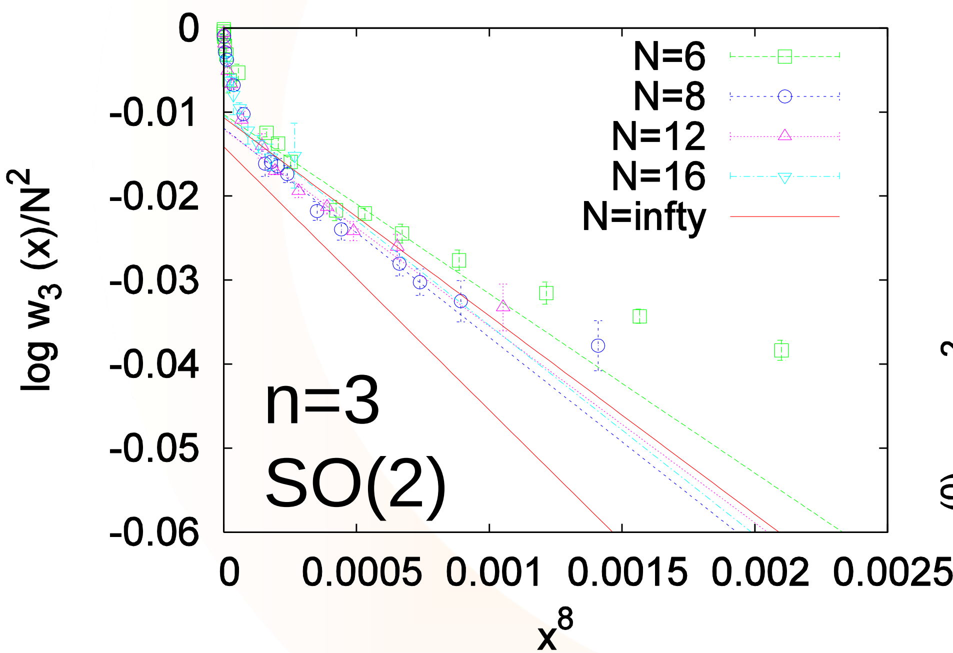
We have $\log w_n(x) = -\frac{\sigma^2}{2} = -\mathrm{O}(x^{10-d}) = -\mathrm{O}(x^{10-(n-1)})$

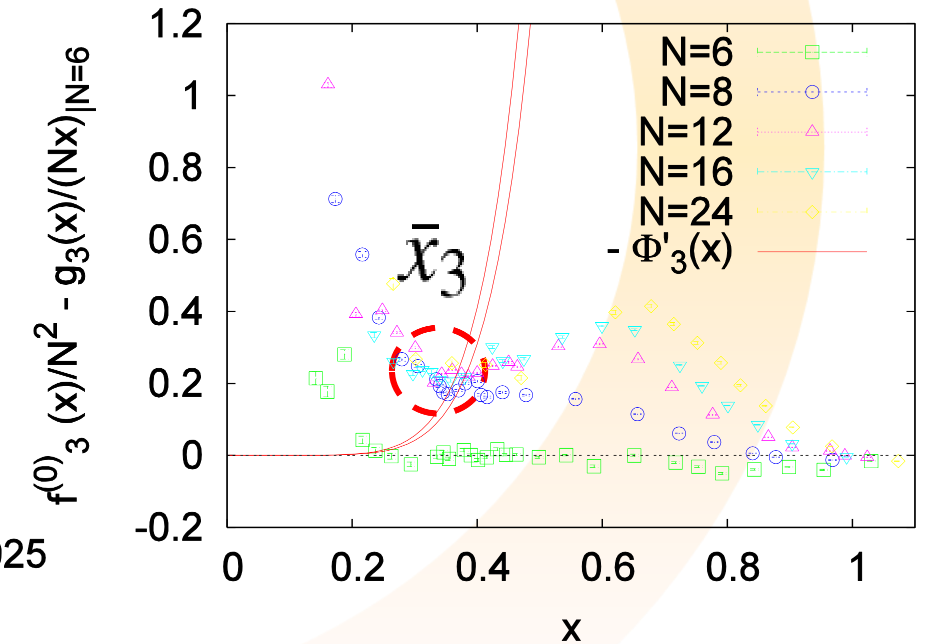
Around $x \cong 1$: $f_n^{(0)}(x)/N$ scales at large N:

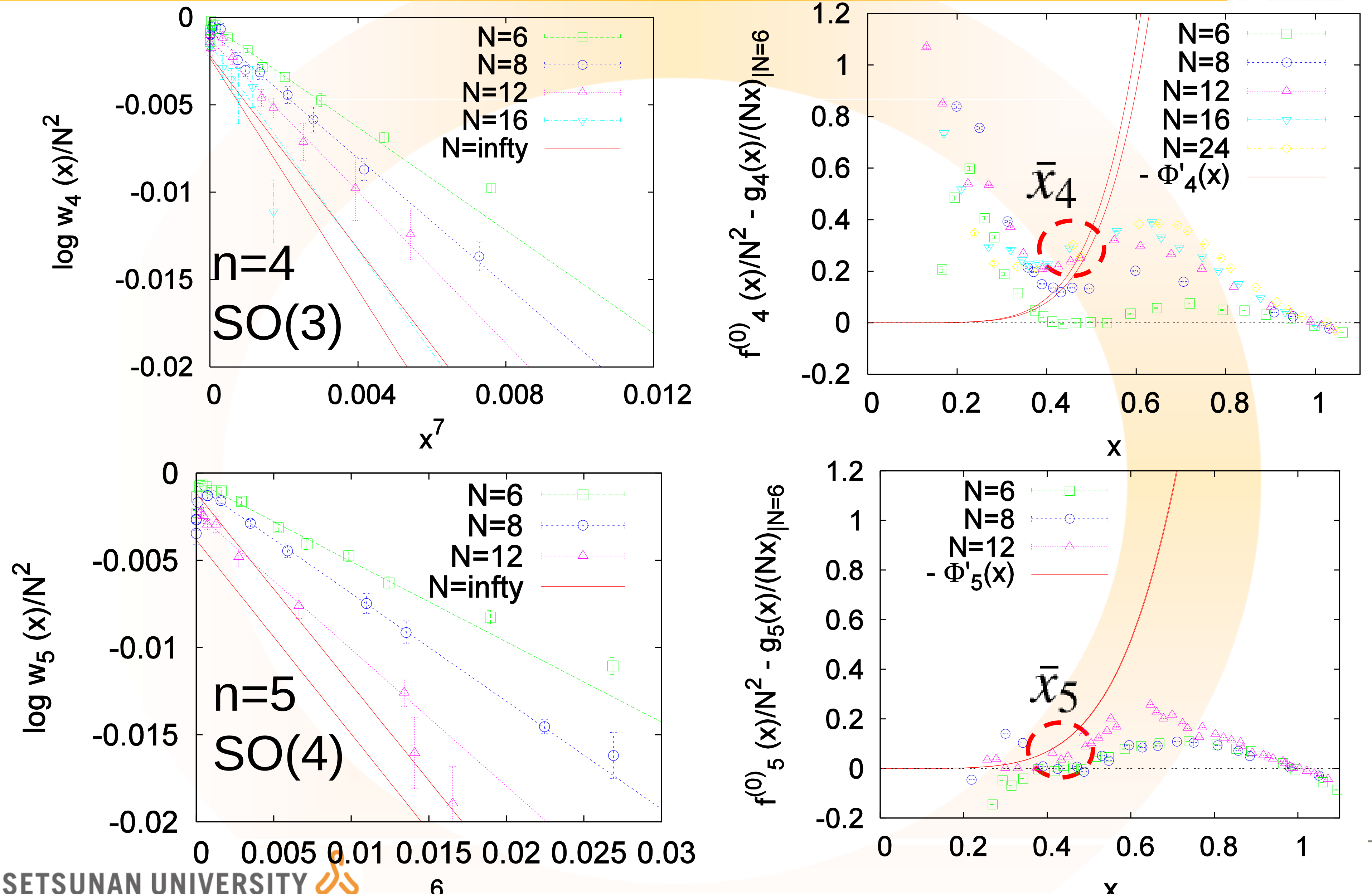
$$\frac{x}{N} f_n^{(0)}(x) \simeq g_n(x) = c_{1,n}(x-1) + c_{2,n}(x-1)^2$$

Around $x<0.4$: $f_n^{(0)}(x)/N^2$ scales at large N
→ existence of the hardcore potential.

GEM suggests
$\bar{x}_n = \langle \tilde{\lambda}_{d+1} \rangle_{\mathrm{SO(d)}} \simeq 0.40$







log w4 (x)/N^2
x^7
N=6
N=8
N=12
N=16
N=infty
n=4
SO(3)
f^(0)4 (x)/N^2 - g4(x)/(Nx)|N=6
x
N=24
- Φ'4(x)
x̄4
log w5 (x)/N^2
x^6
n=5
SO(4)
f^(0)5 (x)/N^2 - g5(x)/(Nx)|N=6
- Φ'5(x)
x̄5

Comparison of the free energy

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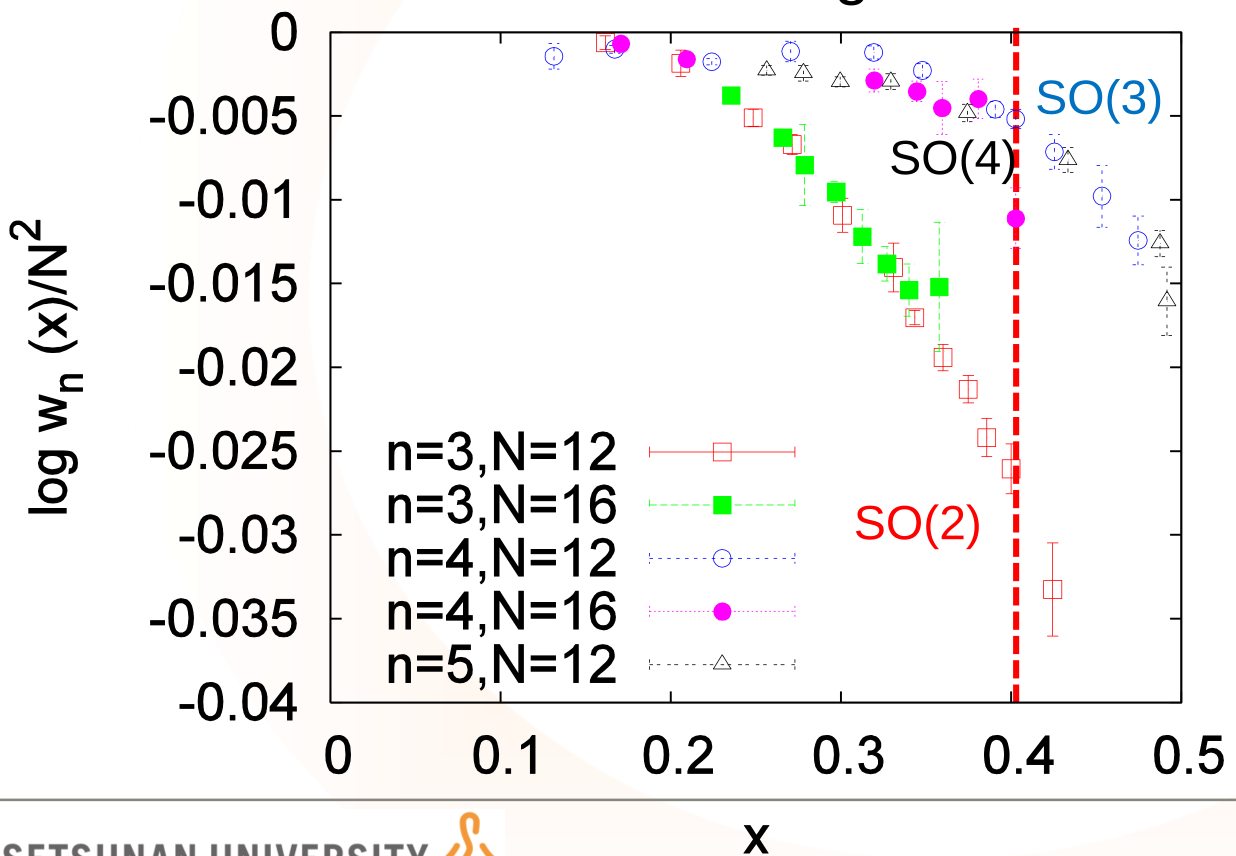
Free energy for the SO(d) vacuum:

$$\mathscr{F}_{SO(d)} = \int_{\bar{x}_n}^{1} \frac{1}{N^2} f_n^{(0)}(x)dx - \frac{1}{N^2}\log w_n(\bar{x}_n), \text{ where } n = d+1$$

→0 at large N



The SO(2) vacuum is disfavored.

$$\mathscr{F}_{SO(3,4)} \ll \mathscr{F}_{SO(2)}$$

5. Similar toy models

(1) Gaussian toy model

$$S = \frac{N}{2}\mathrm{tr}(A_\mu)^2 - \bar{\psi}^f_\alpha (\Gamma_\mu)_{\alpha\beta} A_\mu \psi^f_\beta$$

(μ=1,2,3,4, α,β=1,2, f=1,2,…,N_f , Euclidean)

[J. Nishimura, hep-th/0108070, K.N. Anagnostopoulos, T.A. and J. Nishimura arXiv:1009.4504,1108.1534]

- Fermion Ψ : N-dim. vector (not adjoint)
  → CPU cost of det M is $O(N^3)$.
- No supersymmetry

Severe overlap problem
→ constrain all eigenvalues $\Sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$

Effect of including other observables $\mathcal{O} = -\frac{1}{N}\mathrm{tr}[A_\mu, A_\nu]^2$ in Σ
⇒Without O, the remaining overlap problem is small.

(2) 6-dim Euclidean IKKT model with supersymmetry

det M is complex in 6d (real in 4d).

dynamics similar to that of 10d IKKT model.

$\rightarrow$ constrain only one eigenvalue $\Sigma = \{\lambda_{d+1} \text{ only}\}$

- One-loop effective action: CPU power $O(N^3)$

[K.N. Anagnostopoulos and J. Nishimura hep-th/0108041]

- full model : CPU power $O(N^5)$ with RHMC

[K.N. Anagnostopoulos, T.A. and J. Nishimura arXiv:1306.6135]

$\Rightarrow$captures the short-distance effect

crucial for qualitative agreement with GEM

In both (1)(2), the free energy was difficult to evaluate.

6. Summary

We have studied the dynamical compactification of the spacetime in the Euclidean IKKT model.

Monte Carlo simulation via factorization method
⇒We have obtained the results consistent with GEM:

- Universal compactification scale for SO(2,3,4) vacuum.
- SO(2) vacuum is disfavored.

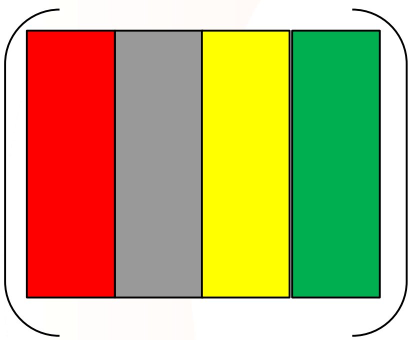
Future prospects

Euclidean IKKT model:

- In 10d model, the finite-N effect seems severer.

The volume is $N^2=L^6$ (6d) while $N^2=L^{10}$ (10d)

⇒it is important to pursue large-N simulation.



Parallelization by Message Passing Interface (MPI).
Each node works on each block in matrix multiplication.

Better preconditioning for the CG method
⇒reduce the iteration

Future prospects

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Factorization method is applicable to general systems with sign problem.

- Random matrix model

[J. Ambjorn, K.N. Anagnostopoulos, J. Nishimura and J.J.M. Verbaarschot, hep-lat/0208025]

- Application to various other systems

Analysis with Complex Langevin Method
⇒ works well for the Gaussian Toy Model.

[Y. Ito and J. Nishimura, arXiv:1609.04501]

backup: RHMC

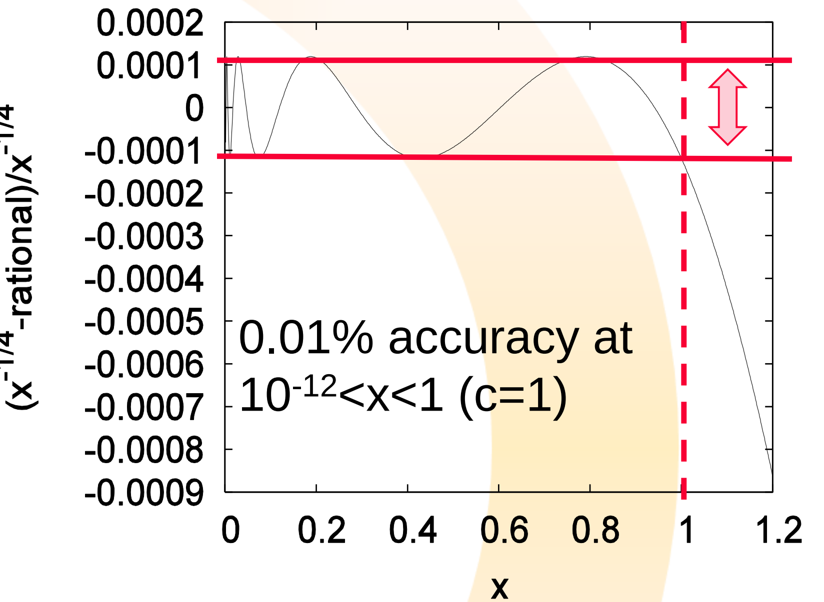
Simulation of Z_0 via Rational Hybrid Monte Carlo (RHMC) algorithm. [Chap 6,7 of B.Ydri, arXiv:1506.02567, for a review]

We exploit the rational approximation

$$x^{-1/4} \simeq a_0 + \sum_{k=1}^{Q} \frac{a_k}{x+b_k}$$

after a proper rescaling.
(typically Q=15⇒valid at $10^{-12}c<x<c$)
a_k, b_k come from Remez algorithm.

[M. A. Clark and A. D. Kennedy, https://github.com/mikeaclark/AlgRemez]



$$S_0 = S_B - \log|\mathrm{Pf}\mathscr{M}|$$

$$|\mathrm{Pf}\mathscr{M}| = |\det\mathscr{M}|^{1/2} = (\det\mathscr{D})^{1/4} \simeq \int dF dF^* \exp\left(-F^*\mathscr{D}^{-1/4}F\right) \simeq \int dF dF^* e^{-S_{\mathrm{PF}}}$$

$$S_{\mathrm{PF}} = a_0 F^* F + \sum_{k=1}^{Q} a_k F^* (\mathscr{D}+b_k)^{-1} F, \quad (\text{where } \mathscr{D} = \mathscr{M}^\dagger \mathscr{M})$$

F: *bosonic* $16(N^2-1)$-dim vector (called *pseudofermion*)

backup: RHMC

Hot spot (most time-consuming part) of RHMC:
⇒Solving $(\mathscr{D} + b_k)\chi_k = F \quad (k = 1, 2, \cdots, Q)$
by conjugate gradient (CG) method.

Multiplication $\mathscr{M}\chi_k$ ⇒ use the expression $\Gamma^\mu[A_\mu, \chi_k]$
($\mathscr{M}$ is a very sparse matrix. No need to build $\mathscr{M}$ explicitly.)
⇒CPU cost is $O(N^3)$

The iteration for CG method is $O(N^2)$ *in the IKKT model*.

In total, the CPU cost is $O(N^5)$
(while direct calculation of $\mathscr{M}^{-1}$ costs $O(N^6)$.)

Multimass CG solver: [B. Jegerlehner, hep-lat/9612014]
Solve $(\mathscr{D} + b_k)\chi_k = F$ only for the smallest b_k
⇒The rest can be obtained as a byproduct,
which saves O(Q) CPU cost.

backup: RHMC

Conjugate Gradient (CG) method:
Iterative algorithm to solve the linear equation Ax=b
(A: symmetric, positive-definite n × n matrix)

Initial config. $\mathbf{x}_0 = 0 \quad \mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0 \quad \mathbf{p}_0 = \mathbf{r}_0$

(for brevity, no preconditioning on x_0 here)

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k \quad \mathbf{r}_{k+1} = \mathbf{r}_k - \alpha_k A \mathbf{p}_k \quad \alpha_k = \frac{(r_k, r_k)}{(p_k, Ap_k)}$$

$$\mathbf{p}_{k+1} = \mathbf{r}_{k+1} + \frac{(\mathbf{r}_{k+1}, \mathbf{r}_{k+1})}{(\mathbf{r}_k, \mathbf{r}_k)} \mathbf{p}_k$$

Iterate this until $\sqrt{\frac{(\mathbf{r}_{k+1}, \mathbf{r}_{k+1})}{(\mathbf{r}_0, \mathbf{r}_0)}} < (\text{tolerance}) \simeq 10^{-4}$

The approximate answer of Ax=b is $x = x_{k+1}$.