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Complex Langevin analysis of the spontaneous rotational symmetry breaking in the dimensionally-reduced super-Yang-Mills models (arXiv:1712.07562)

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1. Introduction

Difficulties in simulating complex partition functions.

$$Z=\int dA\exp(-S_0+i\Gamma),\ \ Z_0=\int dA e^{-S_0}$$

Sign problem:

The reweighting $\langle \mathcal{O}\rangle = \frac{\langle \mathcal{O} e^{i\Gamma}\rangle_0}{\langle e^{i\Gamma}\rangle_0}$ requires configs. exp[O(N^2)]

$\langle * \rangle_0$ = (V.E.V. for the phase-quenched partition function Z_0)

Various methods to address the sign problem:

(Complex Langevin Method (CLM), factorization method, Lefschetz-thimble method…)

In the following, we discuss CLM.

2. The Euclidean IKKT model

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IKKT model [N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115]

⇒Promising candidate for nonperturbative string theory

$$Z = \int_N dA d\psi e^{-(S_b + S_f)}$$

$$S_b = -\frac{N}{4}\mathrm{tr}[A_\mu, A_\nu]^2, \quad S_f = N\mathrm{tr}\bar{\psi}_\alpha(\Gamma^\mu)_{\alpha\beta}[A_\mu, \psi_\beta]$$

Euclidean case after Wick rotation $A_0 \to iA_D, \Gamma^0 \to -i\Gamma_D$.

⇒Path integral is finite without cutoff.

[W. Krauth, H. Nicolai and M. Staudacher, hep-th/9803117, P. Austing and J.F. Wheater, hep-th/0103059]

- A_μ, Ψ_α ⇒N×N Hermitian traceless matrices.

$$\mu = 1, 2, \cdots, D, \quad \alpha, \beta = \begin{cases} 1,2,3,4 & (D=6) \\ 1,2,\cdots,16 & (D=10) \end{cases}$$

- Originally defined in **D=10** (ψ: Majonara-Weyl)

  We consider the ***simplified D=6 case as well***

  (ψ: Weyl, not Majorana $d\psi \to d\psi d\bar{\psi}$)

2. The Euclidean IKKT model

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- Matrix regularization of the type IIB string action:

$$S_{\rm Sh} = \int d^2\sigma \left\{ \sqrt{g}\alpha \left(\frac{1}{4}\{X_\mu, X_\nu\}^2 - \frac{i}{2}\bar{\psi}\Gamma^\mu\{X_\mu, \psi\} \right) + \beta\sqrt{g} \right\}.$$

$$-i[X,Y] \leftrightarrow \{X,Y\} = \frac{1}{\sqrt{g}}\varepsilon^{ab}\partial_a X \partial_b Y, \quad \mathrm{tr} \leftrightarrow \int d^2\sigma\sqrt{g}.$$

- Eigenvalues of A_μ : spacetime coordinate⇒$\mathcal{N}$ =2 SUSY

$$\tilde{\delta}^{(1)}_\varepsilon = \delta^{(1)}_\varepsilon + \delta^{(2)}_\varepsilon \qquad \tilde{\delta}^{(2)}_\varepsilon = i(\delta^{(1)}_\varepsilon - \delta^{(2)}_\varepsilon)$$ where

$$\delta^{(1)}_\varepsilon A_\mu = i\varepsilon(\mathscr{C}\Gamma_\mu)\psi, \quad \delta^{(1)}_\varepsilon \psi = \frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\varepsilon, \quad \delta^{(2)}_\varepsilon A_\mu = 0, \quad \delta^{(2)}_\varepsilon \psi = \varepsilon.$$

$$[\tilde{\delta}^{(a)}_\varepsilon, \tilde{\delta}^{(b)}_\xi]A_\mu = -2i\delta^{ab}\varepsilon(\mathscr{C}\Gamma_\mu)\xi, \quad [\tilde{\delta}^{(a)}_\varepsilon, \tilde{\delta}^{(b)}_\xi]\psi = 0, \quad (a,b = 1,2).$$

2. The Euclidean IKKT model

- Result of Gaussian Expansion Method (GEM)

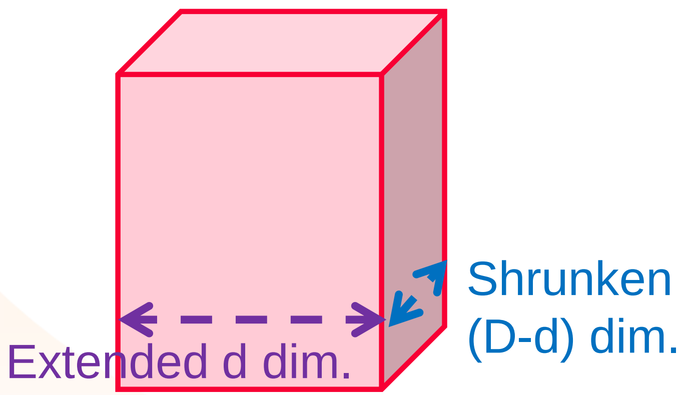
[T.Aoyama, J.Nishimura, and T.Okubo, arXiv:1007.0883、J.Nishimura, T.Okubo and F.Sugino, arXiv:1108.1293]

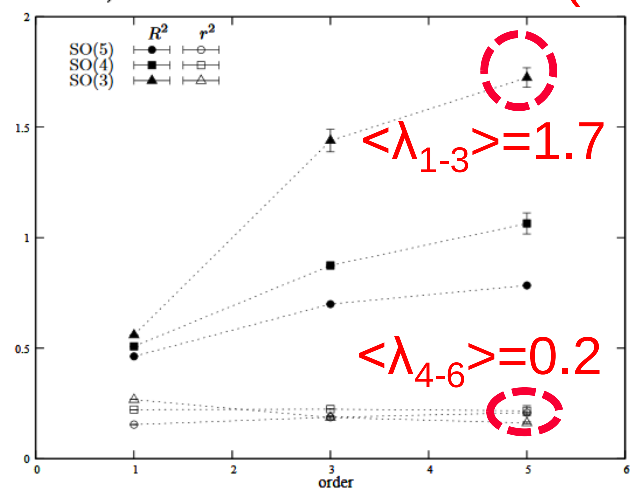
SSB **SO(6)→SO(3)** (In D=10, too, SO(10)→SO(3))
Dynamical compactification to 3-dim spacetime.

$$\lambda_n(\lambda_1 \geq \cdots \geq \lambda_D) : \text{eigenvalues of } T_{\mu\nu} = \frac{1}{N}\mathrm{tr}(A_\mu A_\nu)$$

$$\rho_\mu = \frac{\langle\lambda_\mu\rangle}{\sum_{\nu=1}^{D}\langle\lambda_\nu\rangle} = \begin{cases} 0.30 & (\mu = 1,2,3) \\ 0.035 & (\mu = 4,5,6) \end{cases} \quad (D=6)$$

arXiv:1007.0883 (D=6)





2. The Euclidean IKKT model

$$Z = \int dA e^{-S_b} \underbrace{\left(\int d\psi e^{-S_f}\right)}_{=\det/\mathrm{Pf}\mathcal{M} = |\det/\mathrm{Pf}\mathcal{M}| e^{i\Gamma}} = \int dA \underbrace{e^{-S}}_{e^{-\{S_b - \log(\det/\mathrm{Pf}\mathcal{M})\}}}$$

- Integrating out ψ yields det$\mathcal{M}$ in D=6 (Pf$\mathcal{M}$ in D=10)
- det/Pf $\mathcal{M}$'s *complex phase* contributes to the Spontaneous Symmetry Breaking (SSB) of SO(D).

Under the parity transformation $A_D \Rightarrow -A_D$,
det/PfM is complex conjugate
⇒ det/PfM is real for $A_D=0$ (hence (D-1)-dim config.).

For the d-dim config, $\frac{\partial^m \Gamma}{\partial A_{\mu_1} \cdots \partial A_{\mu_m}} = 0$ (m=1,2,…,(D-1)-d)

The phase is more stationary for lower d.

[J. Nishimura and G. Vernizzi hep-th/0003223]

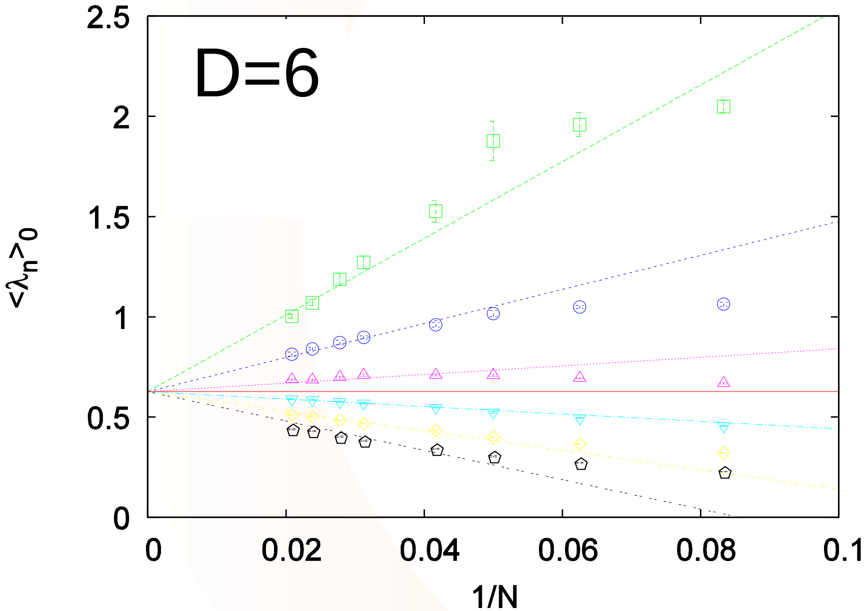
2. The Euclidean IKKT model

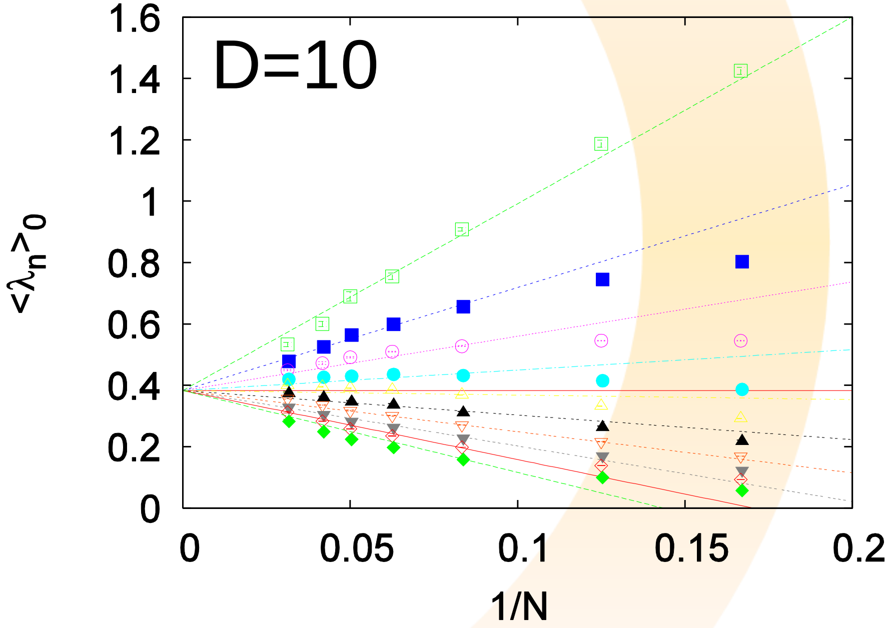
No SSB with the phase-quenched partition function.

$$Z_0 = \int dA e^{-S_0} = \int dA e^{-S_b} |\det/\text{Pf}\mathcal{M}|$$

$\langle * \rangle_0$=V.E.V. for Z_0

[J. Ambjorn, K.N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, hep-th/0003208,0005147, K.N. Anagnostopoulos, T. Azuma, J.Nishimura arXiv:1306.6135, 1509.05079]





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Complex Langevin Method (CLM)

⇒Solve the complex version of the Langevin equation.

[Parisi, Phys.Lett. 131B (1983) 393, Klauder, Phys.Rev. A29 (1984) 2036]

"Real" case

$x_k(t)$ and the action S are real (t: fictitious time)

$$\dot{x}_k^{(\eta)}(t) = \underbrace{-\frac{\partial S}{\partial x_k(t)}}_{\text{drift term}} + \eta_k(t)$$

・η_μ: White noise obeying the probability distribution

$$\exp\left(-\frac{1}{4}\int \eta_k^2(t)dt\right)$$

3. Complex Langevin Method

Probability distribution of $x^{(\eta)}_k(t)$

$$P(x;t) = \left\langle \prod_k \delta(x_k - x_k^{(\eta)}(t)) \right\rangle_\eta$$ where

$$\langle \cdots \rangle_\eta = \frac{\int \mathscr{D}\eta \cdots \exp\left(-\frac{1}{4}\int \eta_k^2(t)dt\right)}{\int \mathscr{D}\eta \exp\left(-\frac{1}{4}\int \eta_k^2(t)dt\right)} \quad \langle \eta_k(t_1)\eta_l(t_2)\rangle_\eta = 2\delta_{kl}\delta(t_1 - t_2)$$

This obeys the Fokker-Planck (FP) equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x_k}\left(\frac{\partial S}{\partial x_k} + \frac{\partial}{\partial x_k}\right) P$$

Time-independent solution $P_{\text{time-indep.}}(x) \propto e^{-S}$

Equivalent to the path integral.

3. Complex Langevin Method

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Putting the real Langevin equation on a computer
⇒discretized version

$$x_k^{(\eta)}(t+\Delta t) = x_k^{(\eta)}(t) - (\Delta t)\frac{\partial S}{\partial x_k} + \overbrace{(\Delta t)\eta_k(t)}^{=\sqrt{\Delta t}\,\tilde{\eta}_k(t)}$$

The white noise obeys $\exp\left(-\frac{1}{4}\int \eta_k^2(t)dt\right) \to \exp\left(-\frac{1}{4}\sum_t \overbrace{(\Delta t)\eta_k^2(t)}^{=\tilde{\eta}_k^2(t)}\right)$

Derivation of the Fokker-Planck equation

$$\left\langle f\left(x^{(\eta)}(t+\Delta t)\right)\right\rangle_\eta - \left\langle f\left(x^{(\eta)}(t)\right)\right\rangle_\eta = \int f(x)\left(P(x;t+\Delta t) - P(x;t)\right)dx$$

$$= -(\Delta t)\left\langle \frac{\partial f}{\partial x_k}\frac{\partial S}{\partial x_k}\right\rangle_\eta + \frac{1}{2}\sqrt{(\Delta t)}^2\left\langle \frac{\partial^2 f}{\partial x_k \partial x_l}\tilde{\eta}_k(t)\tilde{\eta}_l(t)\right\rangle_\eta + O((\Delta t)^2)$$

$$= (\Delta t)\left\{-\left\langle \frac{\partial f}{\partial x_k}\frac{\partial S}{\partial x_k}\right\rangle_\eta + \frac{1}{2}\left\langle \underbrace{\frac{\partial^2 f}{\partial x_k \partial x_l}}_{\text{depends on } \eta(0),\cdots,\eta(t-\Delta t)}\right\rangle_\eta \underbrace{\langle \tilde{\eta}_k(t)\tilde{\eta}_l(t)\rangle_\eta}_{=2\delta_{kl}}\right\} + O((\Delta t)^2)$$

$$= (\Delta t)\left\{\int \left(-\frac{\partial f}{\partial x_k}\frac{\partial S}{\partial x_k} + \frac{\partial^2 f}{\partial x_k^2}\right)P(x;t)dx\right\} + O((\Delta t)^2) = (\Delta t)\left\{\int f(x)\left\{\frac{\partial}{\partial x_k}\left(\frac{\partial S}{\partial x_k} + \frac{\partial}{\partial x_k}\right)P(x;t)\right\}dx\right\} + O((\Delta t)^2)$$

Integration by part w.r.t. real x only

3. Complex Langevin Method

Extension to complex actions

The action S(x) is complex for real x.
x(t) is complexified as x⇒z=x+iy
(S(z) is holomorphic by analytic continuation)

$$\dot{z}_k^{(\eta)}(t) = -\underbrace{\frac{\partial S}{\partial z_k(t)}}_{\text{drift term}} + \eta_k(t)$$

・η_μ: real white noise obeying $\exp\left(-\frac{1}{4}\int \eta_k^2(t)dt\right)$

Probability distribution $P(x,y;t) = \left\langle \prod_k \delta(x_k - x_k^{(\eta)}(t))\delta(y_k - y_k^{(\eta)}(t)) \right\rangle_\eta$

$$\langle\cdots\rangle_\eta = \frac{\int \mathscr{D}\eta \cdots \exp\left(-\frac{1}{4}\int \eta_k^2(t)dt\right)}{\int \mathscr{D}\eta \exp\left(-\frac{1}{4}\int \eta_k^2(t)dt\right)} \quad \langle \eta_k(t_1)\eta_l(t_2)\rangle_\eta = 2\delta_{kl}\delta(t_1 - t_2)$$

3. Complex Langevin Method

P(x,y;t) satisfies $\frac{\partial P}{\partial t} = L^\top P$

When the boundary term vanishes,

$$\int (Lf(x,y))g(x,y)dxdy = \int f(x,y)(L^\top g(x,y))dxdy$$

$$L^\top = \frac{\partial}{\partial x_k}\left\{\mathrm{Re}\left(\frac{\partial S}{\partial z_k}\right) + \frac{\partial}{\partial x_k}\right\} + \frac{\partial}{\partial y_k}\left\{\mathrm{Im}\left(\frac{\partial S}{\partial z_k}\right)\right\}$$

$$L = \left\{-\mathrm{Re}\left(\frac{\partial S}{\partial z_k}\right) + \frac{\partial}{\partial x_k}\right\}\frac{\partial}{\partial x_k} + \left\{-\mathrm{Im}\left(\frac{\partial S}{\partial z_k}\right)\right\}\frac{\partial}{\partial y_k}$$

To justify the CLM, does the following actually hold?

$$\int \underbrace{\mathcal{O}(x+iy)}_{\text{holomorphic}} P(x,y;t)dxdy \stackrel{?}{=} \int \mathcal{O}(x)\rho(x;t)dx$$

$$\frac{\partial \rho(x;t)}{\partial t} = \overbrace{\frac{\partial}{\partial x_k}\left(\frac{\partial S}{\partial x_k} + \frac{\partial}{\partial x_k}\right)}^{=L_0^\top}\rho(x;t) \Rightarrow \rho_{\text{time-indep.}}(x) \propto e^{-S}$$

3. Complex Langevin Method

At t=0, we choose $P(x,y;t=0)=\rho(x;t=0)\delta(y)$

Time evolution at t>0: we define an observable O(z;t)

$$\frac{\partial}{\partial t}\mathscr{O}(z;t)=\underbrace{\left(\frac{\partial}{\partial z_k}-\frac{\partial S}{\partial z_k}\right)\frac{\partial}{\partial z_k}}_{=\tilde{L}}\mathscr{O}(z;t) \quad [\text{initial condition } \mathscr{O}(z;t=0)=\mathscr{O}(z)]$$

Setting y=0, $$\frac{\partial}{\partial t}\mathscr{O}(x;t)=\overbrace{\left(\frac{\partial}{\partial x_k}-\frac{\partial S}{\partial x_k}\right)\frac{\partial}{\partial x_k}}^{=L_0}\mathscr{O}(x;t)$$

$$[\mathscr{O}(x;t=0)=\mathscr{O}(x)]$$

$$\int (L_0 f(x))g(x)dx=\int f(x)(L_0^{\top}g(x))dx$$

S(z) is holomorphic ⇒ O(z;t) remains holomorphic.

$$Lf(z) = \left\{-\mathrm{Re}\left(\frac{\partial S}{\partial z_k}\right)+\underbrace{\frac{\partial}{\partial x_k}}_{=\partial/\partial z_k}\right\}\underbrace{\frac{\partial f(z)}{\partial x_k}}_{=\partial f(z)/\partial z_k}+\left\{-\mathrm{Im}\left(\frac{\partial S}{\partial z_k}\right)\right\}\underbrace{\frac{\partial f(z)}{\partial y_k}}_{=i\partial f(z)/\partial z_k}$$

$$= \left\{-\left(\frac{\partial S}{\partial z_k}\right)+\frac{\partial}{\partial z_k}\right\}\frac{\partial f(z)}{\partial z_k}=\tilde{L}f(z)$$

f(z)'s holomorphy

3. Complex Langevin Method

Interpolating function $F(t,\tau) = \int dxdy \mathcal{O}(x+iy;\tau)P(x,y;t-\tau)$

$$\frac{\partial F(t,\tau)}{\partial \tau} = \int dxdy \left\{ \frac{\partial \mathcal{O}(x+iy;\tau)}{\partial \tau} P(x,y;t-\tau) + \mathcal{O}(x+iy;\tau) \frac{\partial P(x,y;t-\tau)}{\partial \tau} \right\}$$

$$= \int dxdy (\tilde{L}\mathcal{O}(x+iy;\tau))P(x,y;t-\tau) - \int dxdy \mathcal{O}(x+iy;\tau) L^{\top} P(x,y;t-\tau)$$

integration by part

$$\overset{\nabla}{=} \int dxdy \overbrace{\{(\tilde{L}-L)\mathcal{O}(x+iy;\tau)\}}^{=0} P(x,y;t-\tau) = 0$$

Similarly, $\frac{\partial}{\partial \tau} \int dx \mathcal{O}(x;\tau)\rho(x;t-\tau) \overset{\nabla}{=} 0$ Integration by part w.r.t. real x only.

Integration by part is justified when P(x,y;t) damps rapidly

- in the imaginary direction
- around the singularity of the drift term

[G. Aarts, F.A. James, E. Seiler and O. Stamatescu, arXiv:1101.3270,
K. Nagata, J. Nishimura and S. Shimasaki, arXiv:1508.02377]

3. Complex Langevin Method

$$F(t,0)=\int dxdy\,\underbrace{\mathcal{O}(x+iy;0)}_{=\mathcal{O}(x+iy)}P(x,y;t)\overset{\nabla}{=}F(t,t)=\int dxdy\,\underbrace{\mathcal{O}(x+iy;t)}_{=e^{t\tilde{L}}\mathcal{O}(x+iy)}\underbrace{P(x,y;0)}_{=\rho(x,0)\delta(y)}$$

$$=\int dx\mathcal{O}(x;t)\rho(x;0)\overset{\nabla}{=}\int dx\overbrace{\mathcal{O}(x;0)}^{=\mathcal{O}(x)}\rho(x;t)$$

Is this well-defined at large t?

$$\frac{\partial\mathcal{O}(z;t)}{\partial t}=\tilde{L}\mathcal{O}(z;t)\Rightarrow\mathcal{O}(z;t)=e^{t\tilde{L}}\mathcal{O}(z)$$

$$\int dxdy\{e^{\tau\tilde{L}}\mathcal{O}(z)\}P(x,y;t)=\sum_{n=0}^{+\infty}\frac{\tau^n}{n!}\int dxdy\{\tilde{L}^n\mathcal{O}(z)\}P(x,y;t)$$

This series should have a finite convergence radius

⇒Probability of the drift term should fall exponentially.

[K. Nagata, J. Nishimura and S. Shimasaki, arXiv:1606.07627]

Look at the drift term ⇒ Get the drift of CLM!!

3. Complex Langevin Method

Complex Langevin equation for the IKKT model:

$$\frac{d(A_\mu)_{ij}}{dt} = -\frac{\partial S}{\partial (A_\mu)_{ji}} + \eta_{\mu,ij}(t)$$

$$\frac{\partial S}{\partial (A_\mu)_{ji}} = \frac{\partial S_{\mathrm{b}}}{\partial (A_\mu)_{ji}} - c_d \mathrm{Tr}\left(\frac{\partial \mathscr{M}}{\partial (A_\mu)_{ji}} \mathscr{M}^{-1}\right)$$

$$c_d = \begin{cases} 1 & (D=6 \to \det \mathscr{M}) \\ \frac{1}{2} & (D=10 \to \mathrm{Pf}\mathscr{M}) \end{cases}$$

- A_μ : Hermitian→general comlex traceless matrices.
- η_μ: Hermitian white noise obeying $\exp\left(-\frac{1}{4}\int \mathrm{tr}\eta^2(t)dt\right)$

3. Complex Langevin Method

CLM does not work when it encounters these problems:

(1) Excursion problem: A_μ is too far from Hermitian

⇒Gauge Cooling minimizes the Hermitian norm

$$\mathcal{N} = \frac{-1}{DN} \sum_{\mu=1}^{D} \mathrm{tr}[(A_\mu - (A_\mu)^\dagger)^2]$$

[K. Nagata, J. Nishimura and S. Shimasaki, arXiv:1604.07717]

A_μ : Hermitian→general complex traceless matrices.

⇒We make use of this extra symmetry:

After each step of discretized Langevin equation,

$$A_\mu \to g A_\mu g^{-1}, \ \ g = e^{\alpha H}, \ \ H = \frac{-1}{N} \sum_{\mu=1}^{D} [A_\mu, A_\mu^\dagger]$$

α: real parameter, such that $\mathcal{N}$ is minimized.

3. Complex Langevin Method

(2) Singular drift problem:

The drift term $dS/d(A_\mu)_{ji}$ diverges due to $\mathcal{M}$'s near-zero eigenvalues.

We trust CLM when the distribution p(u) of the drift norm

$$u = \sqrt{\frac{1}{DN^3} \sum_{\mu=1}^{D} \sum_{i,j=1}^{N} \left| \frac{\partial S}{\partial (A_\mu)_{ji}} \right|^2}$$

falls exponentially as $p(u) \propto e^{-au}$.

[K. Nagata, J. Nishimura and S. Shimasaki, arXiv:1606.07627]

3. Complex Langevin Method

SETSUDAI

Mass deformation [Y. Ito and J. Nishimura, arXiv:1609.04501]

- SO(D) symmetry breaking term $\Delta S_{\rm b} = \frac{1}{2} N \varepsilon \sum_{\mu=1}^{D} m_\mu \mathrm{tr}(A_\mu)^2$

Order parameters for SSB of SO(D): $\lambda_\mu = \mathrm{Re}\left\{ \frac{1}{N} \mathrm{tr}(A_\mu)^2 \right\}$

- Fermionic mass term:

$$\Delta S_{\rm f} = N m_{\rm f} \mathrm{tr}\left(\bar{\psi}_\alpha \gamma_{\alpha\beta} \psi_\beta\right), \quad \gamma = \begin{cases} \Gamma_6 & (D=6) \\ i\Gamma_8 \Gamma_9^\dagger \Gamma_{10} & (D=10) \end{cases}$$

Avoids the singular eigenvalue distribution of $\mathscr{M}$.

This breaks SO(6)→SO(5) (SO(10)→SO(7))

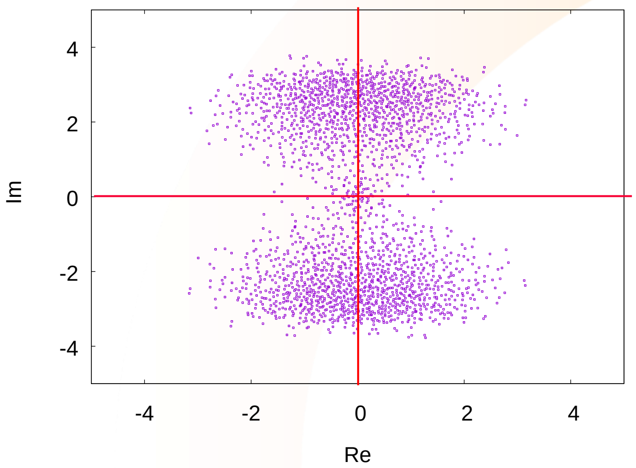
We study the SSB of the remaining symmetry.

Extrapolation (i) N→∞ ⇒(ii)ε→0 ⇒ (iii) $m_{\rm f}$→0.

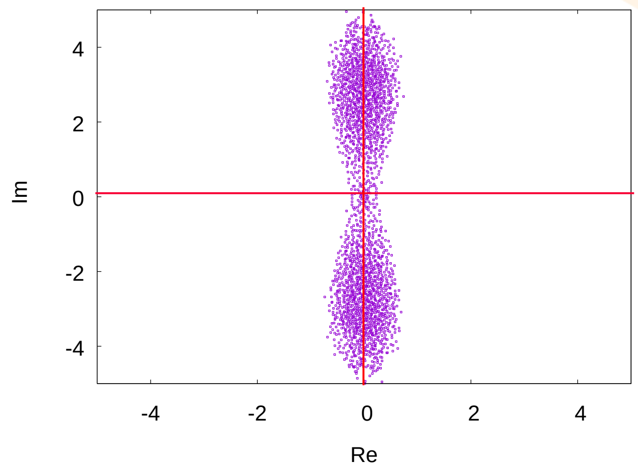
4. Result for D=6

The effect of adding these mass terms

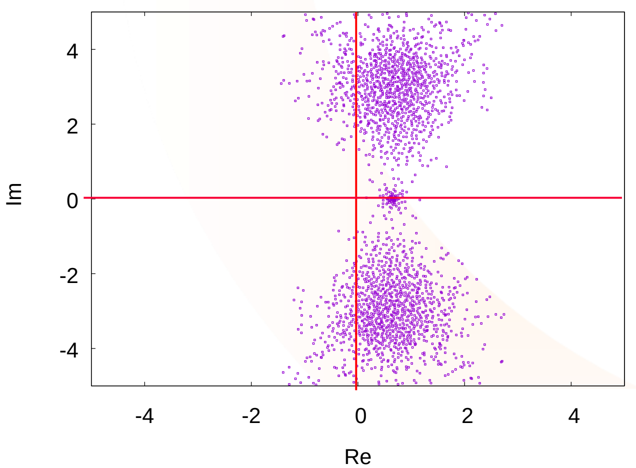
$(\varepsilon, m_f)=(0.00,0.00)$



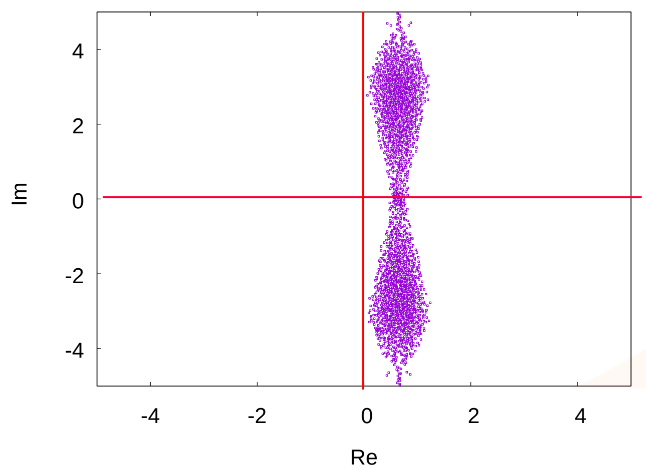
$(\varepsilon, m_f)=(0.25,0.00)$



$(\varepsilon, m_f)=(0.00,0.65)$



$(\varepsilon, m_f)=(0.25,0.65)$



Scattering plots of the eigenvalues of the $4(N^2-1)\times 4(N^2-1)$ matrix $\mathscr{M}$ for D=6, N=24.

ΔS_b narrows the eigenvalue distribution.

ΔS_f shifts the eigenvalues, to evade the origin.

4. Result for D=6

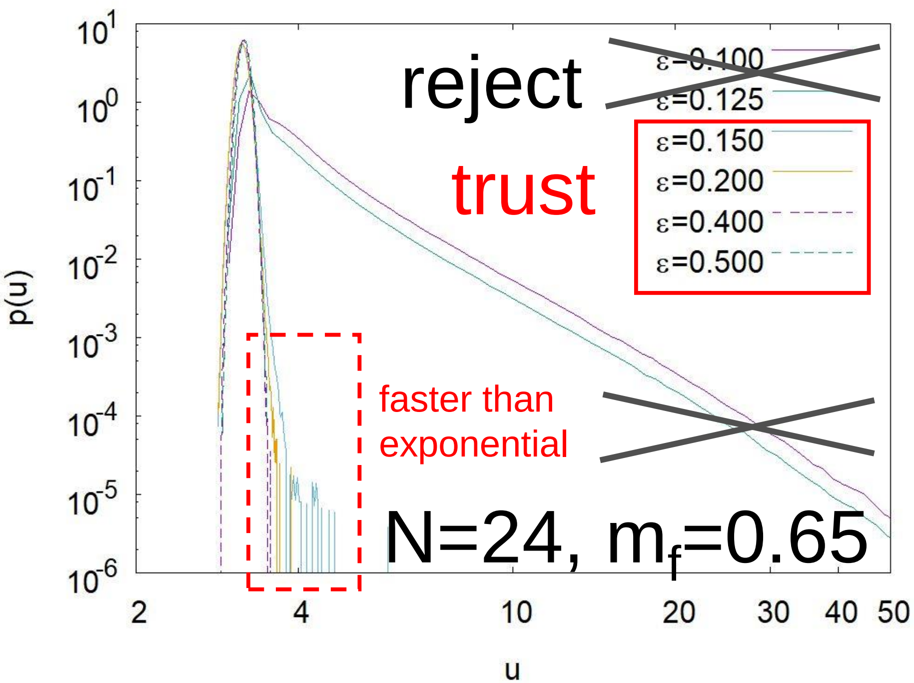
$$\Delta S_b = \frac{1}{2} N \varepsilon \sum_{\mu=1}^{D} m_\mu \mathrm{tr}(A_\mu)^2$$

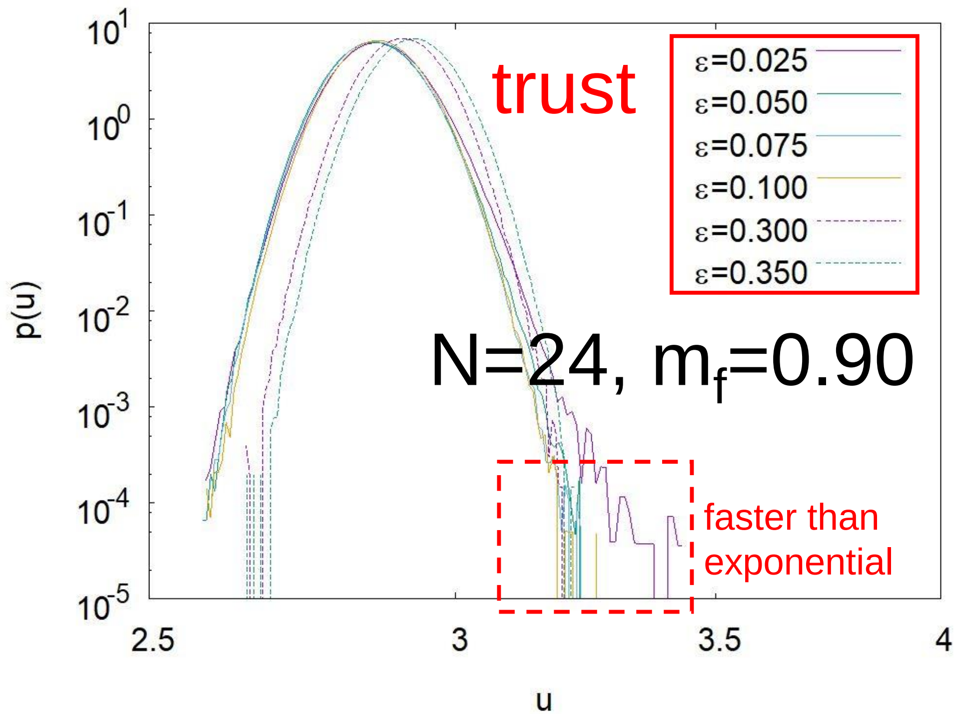
$$\Delta S_f = N m_f \mathrm{tr}(\bar{\psi}_\alpha (\Gamma_D)_{\alpha\beta} \psi_\beta) \quad (D=6)$$

$m_\mu = (0.5, 0.5, 1, 2, 4, 8)$

$$u = \sqrt{\frac{1}{DN^3} \sum_{\mu=1}^{D} \sum_{i,j=1}^{N} \left| \frac{\partial S}{\partial (A_\mu)_{ji}} \right|^2}$$

's distribution p(u) (log-log)



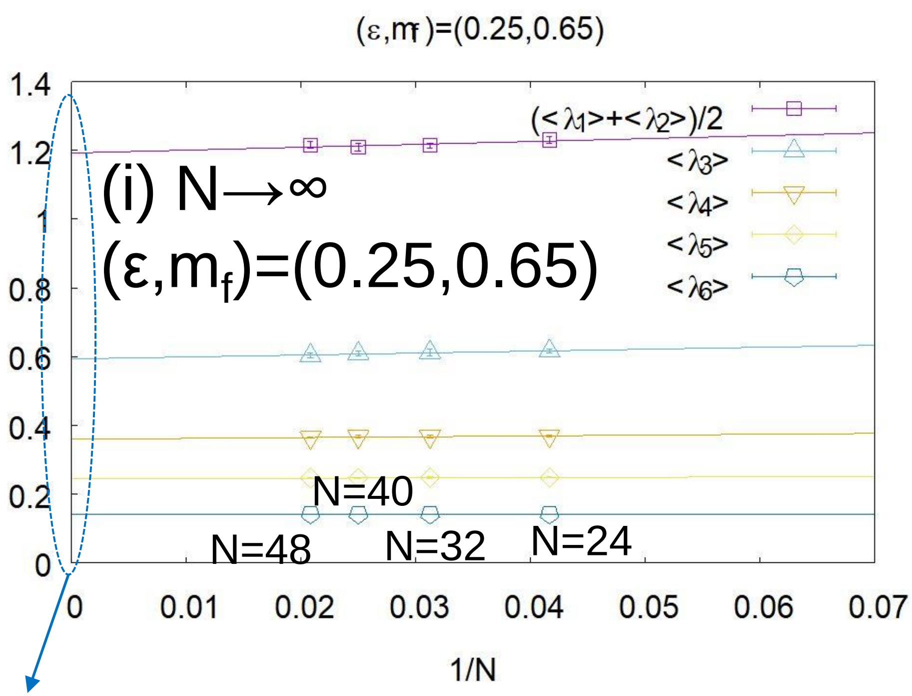


4. Result for D=6

$$\Delta S_{\mathrm{b}} = \frac{1}{2} N \varepsilon \sum_{\mu=1}^{D} m_\mu \mathrm{tr}(A_\mu)^2 \qquad \Delta S_{\mathrm{f}} = N m_{\mathrm{f}} \mathrm{tr}(\bar{\psi}_\alpha (\Gamma_D)_{\alpha\beta} \psi_\beta) \ (D=6)$$

m_μ=(0.5, 0.5, 1, 2, 4, 8)

(i) N→∞ limit for fixed (ε, m_f)



$\langle \lambda_\mu \rangle_{\varepsilon, m_f}$ at large N

$(\varepsilon, m_f) \to (0,0)$ extrapolation for finite N

⇒We cannot observe SSB of SO(D).

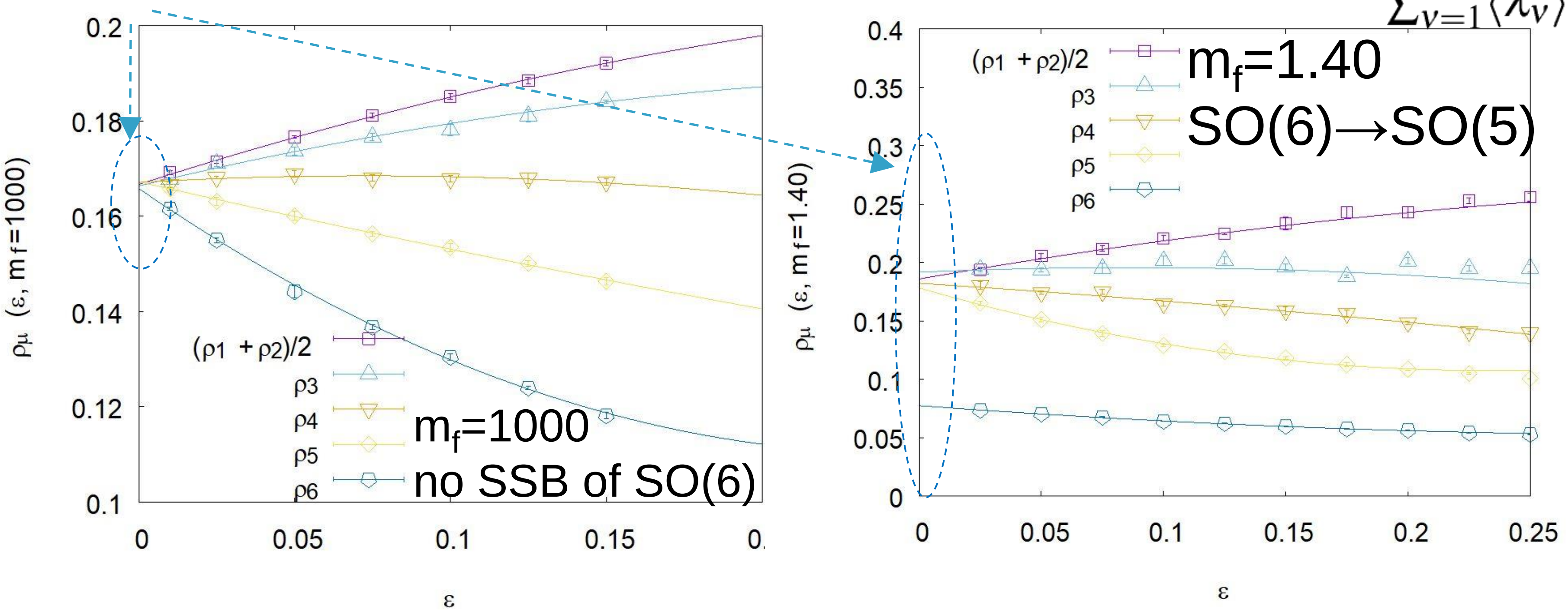
4. Result for D=6

$$\Delta S_b = \frac{1}{2} N \varepsilon \sum_{\mu=1}^{D} m_\mu \mathrm{tr}(A_\mu)^2 \qquad \Delta S_f = N m_f \mathrm{tr}(\bar{\psi}_\alpha (\Gamma_D)_{\alpha\beta} \psi_\beta) \ (D=6)$$

m_μ=(0.5, 0.5, 1, 2, 4, 8)

(ii) ε→0 after N→∞

$$\rho_\mu(\varepsilon, m_f) = \frac{\langle \lambda_\mu \rangle_{\varepsilon, m_f}}{\sum_{\nu=1}^{D} \langle \lambda_\nu \rangle_{\varepsilon, m_f}}$$



- $m_f \to \infty$: Ψ decouples from A_μ and reduces to the bosonic IKKT.
- The bosonic IKKT S_b does not break SO(D).

[T. Hotta, J. Nishimura and A. Tsuchiya, hep-th/9811220]

- The SSB of SO(D) is not an artifact of ε→0 but a physical effect.

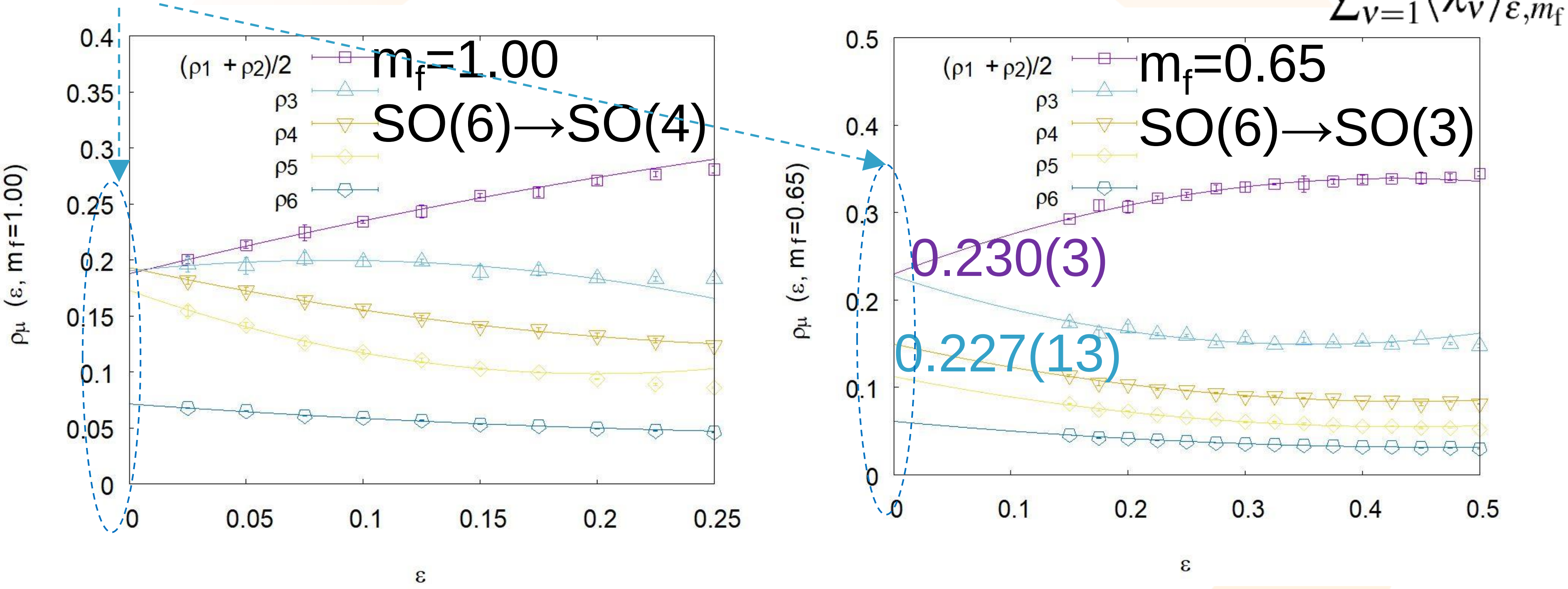
4. Result for D=6

$$\Delta S_{\mathrm{b}} = \frac{1}{2} N \varepsilon \sum_{\mu=1}^{D} m_\mu \mathrm{tr}(A_\mu)^2 \qquad \Delta S_{\mathrm{f}} = N m_{\mathrm{f}} \mathrm{tr}(\bar{\psi}_\alpha (\Gamma_D)_{\alpha\beta} \psi_\beta) \quad (D=6)$$

m_μ=(0.5, 0.5, 1, 2, 4, 8)

(ii) ε→0 after N→∞

$$\rho_\mu(\varepsilon, m_{\mathrm{f}}) = \frac{\langle \lambda_\mu \rangle_{\varepsilon, m_{\mathrm{f}}}}{\sum_{\nu=1}^{D} \langle \lambda_\nu \rangle_{\varepsilon, m_{\mathrm{f}}}}$$



4. Result for D=6

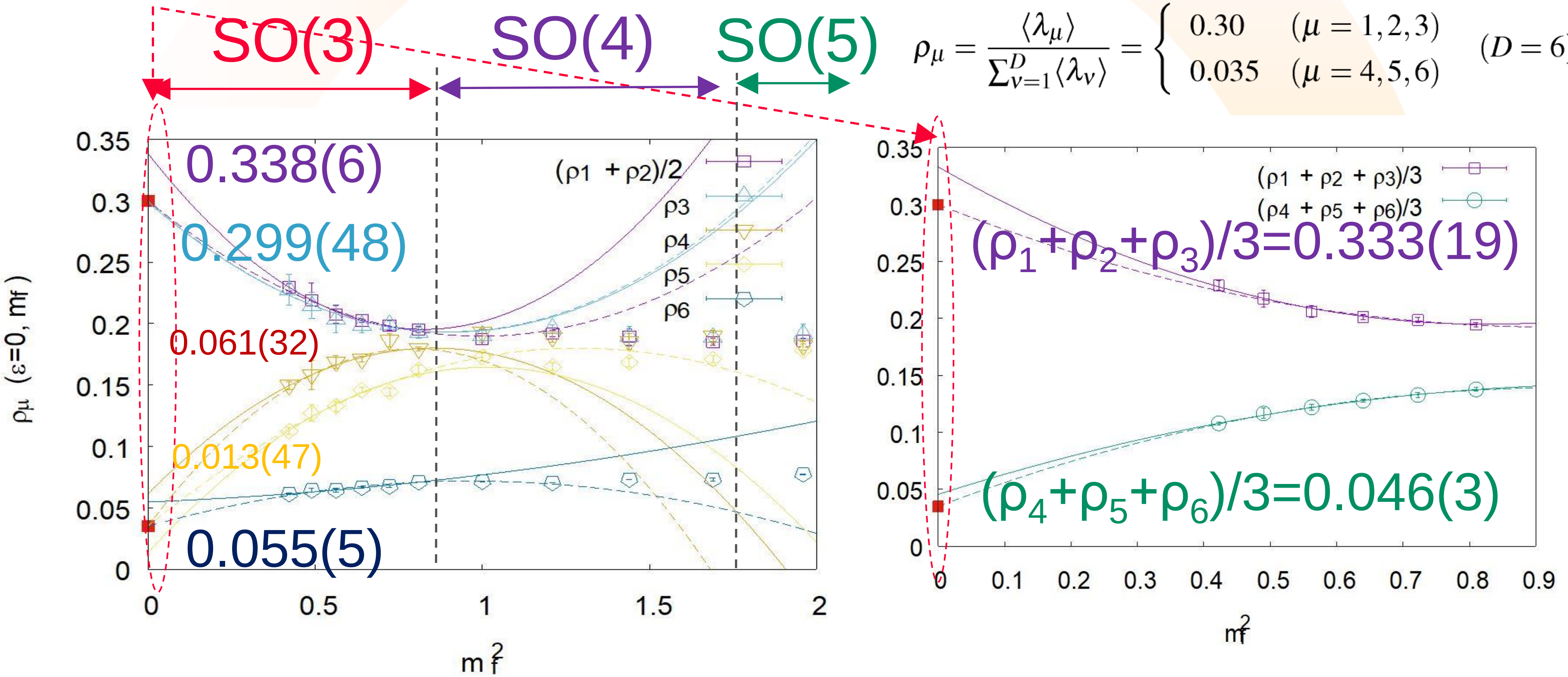
$$\Delta S_b = \frac{1}{2} N\varepsilon \sum_{\mu=1}^{D} m_\mu \mathrm{tr}(A_\mu)^2$$

$$\Delta S_f = N m_f \mathrm{tr}(\bar{\psi}_\alpha (\Gamma_D)_{\alpha\beta} \psi_\beta) \quad (D=6)$$

m_μ=(0.5, 0.5, 1, 2, 4, 8)

(iii) $m_f \to 0$ after $\varepsilon \to 0$

$$\rho_\mu = \frac{\langle \lambda_\mu \rangle}{\sum_{\nu=1}^{D} \langle \lambda_\nu \rangle} = \begin{cases} 0.30 & (\mu = 1,2,3) \\ 0.035 & (\mu = 4,5,6) \end{cases} \quad (D=6)$$



(dotted line: $m_f \to 0$ limit fixed to GEM results)

SSB SO(6) → at most SO(3) Consistent with GEM.

5. Result for D=10 (preliminary)

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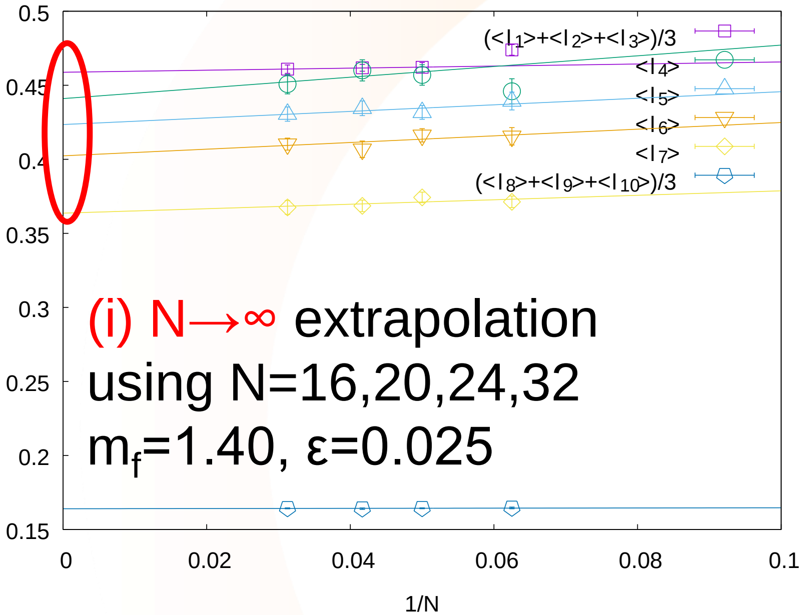
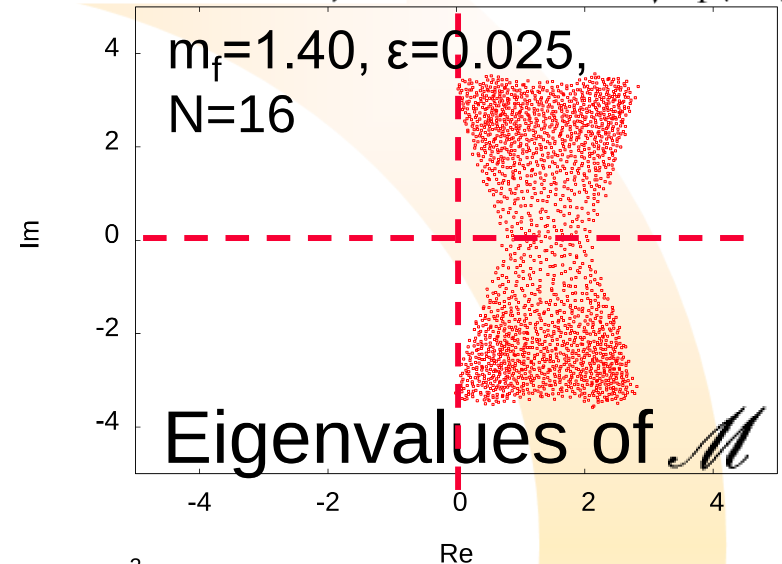
$$\Delta S_{\mathrm{b}} = \frac{1}{2} N\varepsilon \sum_{\mu=1}^{D} m_\mu \mathrm{tr}(A_\mu)^2 \qquad \Delta S_{\mathrm{f}} = N m_{\mathrm{f}} \mathrm{tr}\left(\bar{\psi}_\alpha (i\Gamma_8 \Gamma_9^\dagger \Gamma_{10})_{\alpha\beta} \psi_\beta\right) \qquad \rho_\mu(\varepsilon, m_{\mathrm{f}}) = \frac{\langle \lambda_\mu \rangle_{\varepsilon, m_{\mathrm{f}}}}{\sum_{\nu=1}^{D} \langle \lambda_\nu \rangle_{\varepsilon, m_{\mathrm{f}}}}$$

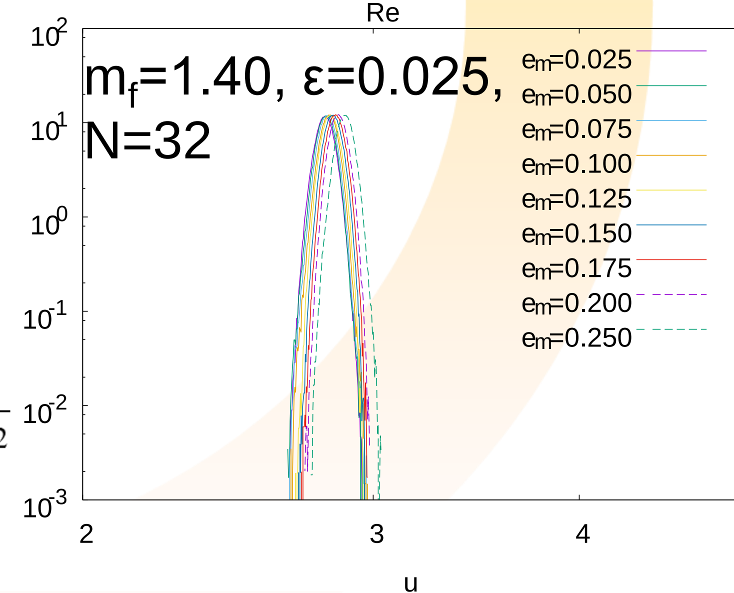
$m_\mu = (0.5, 0.5, 0.5, 1, 2, 4, 8, 8, 8, 8)$





Distribution of

$$u = \sqrt{\frac{1}{DN^3} \sum_{\mu=1}^{D} \sum_{i,j=1}^{N} \left| \frac{\partial S}{\partial (A_\mu)_{ji}} \right|^2}$$



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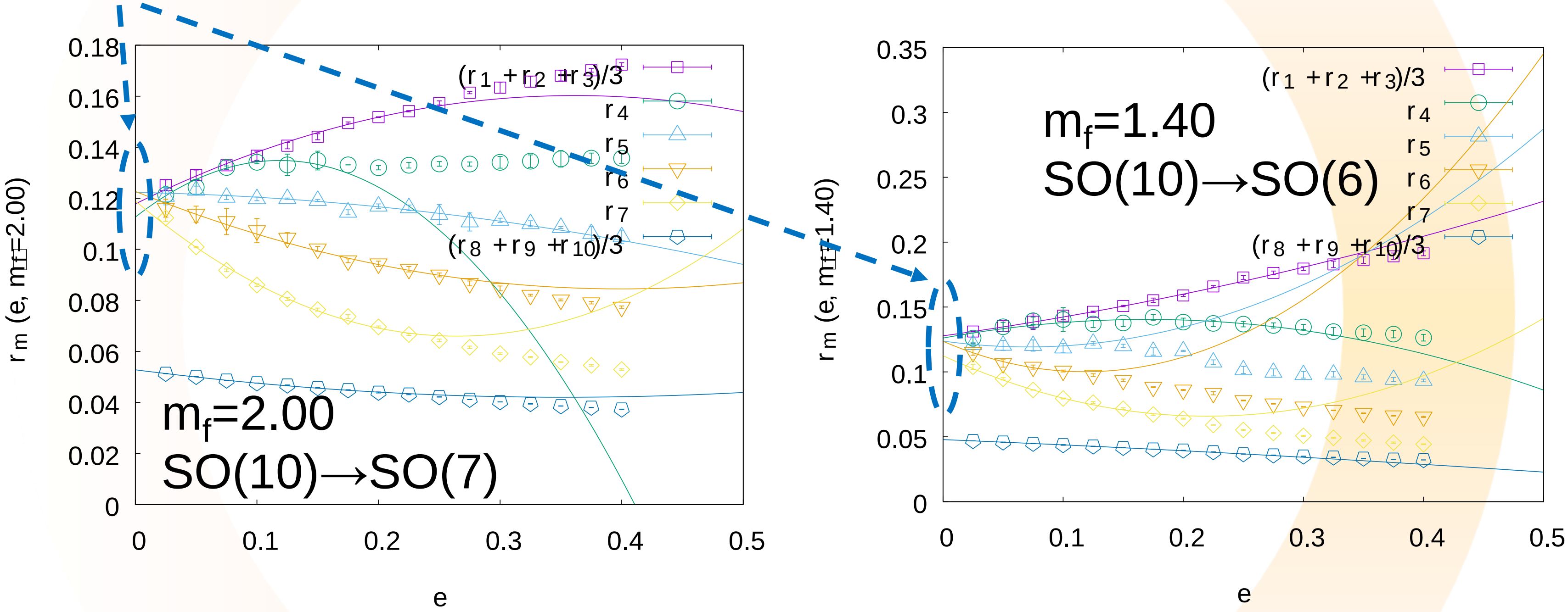
$$\Delta S_{\mathrm{b}} = \frac{1}{2} N\varepsilon \sum_{\mu=1}^{D} m_\mu \mathrm{tr}(A_\mu)^2 \qquad \Delta S_{\mathrm{f}} = N m_{\mathrm{f}} \mathrm{tr}\left(\bar{\psi}_\alpha (i\Gamma_8 \Gamma_9^\dagger \Gamma_{10})_{\alpha\beta} \psi_\beta\right) \qquad \rho_\mu(\varepsilon, m_{\mathrm{f}}) = \frac{\langle \lambda_\mu \rangle_{\varepsilon, m_{\mathrm{f}}}}{\sum_{\nu=1}^{D} \langle \lambda_\nu \rangle_{\varepsilon, m_{\mathrm{f}}}}$$

m_μ=(0.5,0.5,0.5,1,2,4,8,8,8,8)

(ii) ε→0 after N→∞



Transition from SO(7) to SO(6) at m_f<2.0

6. Summary

Dynamical compactification of the spacetime in the simplified Euclidean IKKT model.
"Complex Langevin Method" ⇒trend of SO(D)→SO(3).

Future works

Test various ideas

- Reweighting method [J. Bloch, arXiv:1701.00986]
- Other deformations than the mass deformation

(z=1:original Euclidean、 pure imaginary z: fermion det/Pf is real)

$$N\mathrm{tr}\left(\bar{\psi}(z\Gamma_D)[A_D,\psi]+\sum_{k=1}^{D-1}\bar{\psi}\Gamma_k[A_k,\psi]\right)$$

[Y. Ito, J. Nishimura, arXiv:1710.07929]

6. Summary

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Future works

Application of CLM to other cases

Lorentzian version of the IKKT model

generalization to Gross-Witten-Wadia model

$$S_{\mathrm{g}} = N(a\mathrm{tr}U + b\mathrm{tr}U^{\dagger})$$

[P. Basu, K. Jaswin and A. Joseph arXiv:1802.10381]

BFSS model $S=S_b+S_f$ (D=5,9 ⇒ det/Pf M is complex)

$$S_{\mathrm{b}} = N\int_0^{\beta} \mathrm{tr}\left\{\frac{1}{2}\sum_{\mu=1}^{D}(D_t X_\mu(t))^2 - \frac{1}{4}\sum_{\mu,\nu=1}^{D}[X_\mu(t),X_\nu(t)]^2\right\}dt$$

$$S_{\mathrm{f}} = N\int_0^{\beta} \mathrm{tr}\left\{\sum_{\alpha=1}^{p}\bar{\psi}_\alpha(t)D_t\psi_\alpha(t) - \sum_{\mu=1}^{D}\sum_{\alpha,\eta=1}^{p}\bar{\psi}_\alpha(t)(\Gamma_\mu)_{\alpha\eta}[X_\mu(t),\psi_\eta(t)]\right\}dt$$

backup: example of CLM

Example [G. Aarts, arXiv:1512.05145]

$$S(x) = \frac{1}{2}\underbrace{(a+ib)}_{=\sigma}x^2, \ (a, b \in \mathbf{R}, \ a > 0)$$

S(x) is complex for real x.
Complexify to z=x+iy.

$$S(z) = \frac{1}{2}\sigma z^2 = \frac{1}{2}(a+ib)\overbrace{(x+iy)^2}^{=z^2} = \frac{a(x^2-y^2)}{2} + ibxy, \quad \frac{\partial S}{\partial z} = \sigma z = (a+ib)(x+iy)$$

Complex Langevin equation for this action

$$\dot{x}(t) = -\mathrm{Re}\left(\frac{\partial S}{\partial z}\right) + \eta(t) = (-ax+by) + \eta(t) \qquad \dot{y}(t) = -\mathrm{Im}\left(\frac{\partial S}{\partial z}\right) = (-ay-bx)$$

The real white noise satisfies

$$\langle \eta(t_1)\eta(t_2)\rangle = 2\delta(t_1 - t_2) \qquad \langle \cdots \rangle = \frac{\int \mathscr{D}\eta \cdots \exp\left(-\frac{1}{4}\int \eta^2(t)dt\right)}{\int \mathscr{D}\eta \exp\left(-\frac{1}{4}\int \eta^2(t)dt\right)}$$

backup: example of CLM

Solution of the Langevin equation

$$x(t) = e^{-at} \underbrace{[x(0)\cos bt + y(0)\sin bt]}_{=A(t)} + \int_0^t \eta(s) e^{-a(t-s)} \cos[b(t-s)] ds$$

$$y(t) = e^{-at} [y(0)\cos bt - x(0)\sin bt] - \int_0^t \eta(s) e^{-a(t-s)} \sin[b(t-s)] ds$$

$$\begin{aligned}\langle x^2 \rangle &= \lim_{t\to+\infty} \langle x^2(t) \rangle = \lim_{t\to+\infty} \Big\{ \underbrace{e^{-2at} A(t)^2}_{\to 0} + 2e^{-at} A(t) \int_0^t \underbrace{\langle \eta(s) \rangle}_{=0} e^{-a(t-s)} \cos[b(t-s)] ds \\ &\quad + \int_0^t \int_0^t \underbrace{\langle \eta(s)\eta(s') \rangle}_{=2\delta(s-s')} e^{-a(2t-s-s')} \cos[b(t-s)] \cos[b(t-s')] ds ds' \Big\} \\ &= \lim_{t\to+\infty} \Big\{ 2\int_0^t e^{-2a(t-s)} \cos^2[b(t-s)] \Big\} ds = \frac{2a^2+b^2}{2a(a^2+b^2)}\end{aligned}$$

Similarly, $\langle y^2 \rangle = \dfrac{b^2}{2a(a^2+b^2)}, \quad \langle xy \rangle = \dfrac{-b}{2(a^2+b^2)}$

This replicates $\langle z^2 \rangle = \langle x^2 \rangle - \langle y^2 \rangle + 2i\langle xy \rangle = \dfrac{a-ib}{a^2+b^2} = \dfrac{1}{\sigma}$

backup: example of CLM

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Fokker-Planck equation

$$\frac{\partial P}{\partial t} = L^\top P \quad \text{where} \quad L^\top = \frac{\partial}{\partial x}\left\{\underbrace{\mathrm{Re}\left(\frac{\partial S}{\partial z}\right)}_{=ax-by} + \frac{\partial}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\underbrace{\mathrm{Im}\left(\frac{\partial S}{\partial z}\right)}_{=ay+bx}\right\}$$

Ansatz for its static solution:

$$P(x,y) = N\exp\left(-\alpha x^2 - \beta y^2 - 2\gamma xy\right) = N\exp\left(-\beta\left(y + \frac{\gamma x}{\beta}\right)^2 - \overbrace{\left(\alpha - \frac{\gamma^2}{\beta}\right)}^{=a(a^2+b^2)/(2a^2+b^2)} x^2\right)$$

$$0 = \partial_t P = L^\top P = [\underbrace{(2a-2\alpha)}_{=0\to a=\alpha} + x^2\underbrace{(4\alpha^2 - 2a\alpha - 2b\gamma)}_{=0\to\gamma=a^2/b} + y^2\underbrace{(4\gamma^2 + 2b\gamma - 2a\beta)}_{=0\to\beta=a(1+2a^2/b^2)} + xy\underbrace{(4(2\alpha-a)\gamma + 2b(\alpha-\beta))}_{=0}]P$$

Using $\dfrac{\int_{-\infty}^{+\infty} t^2 e^{-At^2}dt}{\int_{-\infty}^{+\infty} e^{-At^2}dt} = \dfrac{1}{2A} \quad (A>0)$ we have

$$\langle x^2\rangle = \frac{\iint x^2 P(x,y)dxdy}{\iint P(x,y)dxdy} = \frac{1}{2} \div \frac{a(a^2+b^2)}{2a^2+b^2} = \frac{2a^2+b^2}{2a(a^2+b^2)}$$

backup: noisy estimator

Noisy estimator: method to calculate Tr A using Gaussian random numbers (A: n × n matrix)

X_k, Y_k independently obey the standard normal distribution N(0,1).

$$\chi_k = \frac{X_k + iY_k}{\sqrt{2}} \Rightarrow \langle \chi_j^* \chi_k \rangle = \delta_{jk} \quad (j,k = 1,2,\cdots,n)$$

$$\sum_{j,k=1}^{n} \langle \chi_j^* A_{jk} \chi_k \rangle = \sum_{j,k=1}^{n} A_{jk} \langle \chi_j^* \chi_k \rangle = \sum_{j,k=1}^{n} A_{jk} \delta_{jk} = \mathrm{Tr} A$$

backup: noisy estimator

Integrate out $\Psi \Rightarrow$ $\int d\bar\psi d\psi e^{-(S_f+\Delta S_f)} = \det \mathcal{M}$ (D= 6⇒p= 4)

$\int d\psi e^{-(S_f+\Delta S_f)} = \mathrm{Pf}\mathcal{M}$ (D=10⇒p=16)

$$\gamma = \begin{cases} \Gamma_6 & (D=6) \\ i\Gamma_8\Gamma_9^\dagger\Gamma_{10} & (D=10) \end{cases}$$

$$S_f + \Delta S_f = N\left\{\mathrm{tr}\left(\bar\psi_\alpha(\Gamma_\mu)_{\alpha\beta}[A_\mu,\psi_\beta]\right) + m_f \mathrm{tr}\left(\bar\psi_\alpha(\gamma_{\alpha\beta})\psi_\beta\right)\right\}$$

Tracelessness of $\Psi \Rightarrow \mathcal{M}$ is a $p(N^2-1) \times p(N^2-1)$ matrix

$$\mathcal{M}_{a_1a_2\alpha,b_1b_2\beta} = [(a_1+(a_2-1)N+(\alpha-1)(N^2-1), b_1+(b_2-1)N+(\beta-1)(N^2-1))\ \text{element}]$$

$$= \mathcal{M}'_{a_1a_2\alpha,b_1b_2\beta} - \mathcal{M}'_{NN\alpha,b_1b_2\beta}\delta_{a_1a_2} - \mathcal{M}'_{a_1a_2\alpha,NN\beta}\delta_{b_1b_2} + \mathcal{M}'_{NN\alpha,NN\beta}\delta_{a_1a_2}\delta_{b_1b_2}$$

$a_1,a_2,b_1,b_2 = 1,2,\cdots,N$, except for $(a_1,a_2)=(N,N)$, $(b_1,b_2)=(N,N)$ $\quad \alpha,\beta = 1,2,\cdots,p$

$$\mathcal{M}'_{a_1a_2\alpha,b_1b_2\beta} = (\Gamma_\mu)_{\alpha\beta}\{(A_\mu)_{a_2b_1}\delta_{a_1b_2} - (A_\mu)_{b_2a_1}\delta_{a_2b_1}\} + m_f\gamma_{\alpha\beta}\delta_{a_1b_2}\delta_{a_2b_1}.$$

$a_1,a_2,b_1,b_2 = 1,2,\cdots,N$, including $(a_1,a_2)=(N,N)$, $(b_1,b_2)=(N,N)$ $\quad \alpha,\beta = 1,2,\cdots,p$

$\mathcal{M}$ in the scattering plots

(without altering det/Pf $\mathcal{M}$ up to a constant)

unit matrix

$$\mathcal{M}'_{a_1a_2\alpha,b_1b_2\beta} = (\Gamma_\mu\gamma^{-1})_{\alpha\beta}\{(A_\mu)_{a_2b_2}\delta_{a_1b_1} - (A_\mu)_{b_1a_1}\delta_{a_2b_2}\} + m_f\delta_{\alpha\beta}\delta_{a_1b_1}\delta_{a_2b_2}.$$

backup: noisy estimator

$\mathcal{M}$ is a p(N^2-1) × p(N^2-1) matrix (p=4 for D=6 and p=16 for D=10)

⇒Naively calculating $\mathrm{Tr}\left(\frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1}\right)$ takes CPU cost O(N^6).

Instead, we use the noisy estimator

χ=(random number vector)

$$\mathrm{Tr}\left(\frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1}\right) = \left\langle \chi^* \frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \underbrace{\mathcal{M}^{-1}\chi}_{=\zeta} \right\rangle$$

$\mathcal{M}^\dagger \mathcal{M} \zeta = \mathcal{M}^\dagger \chi$ by conjugate gradient (CG) method.

- $\mathcal{M}^\dagger \mathcal{M}$ is symmetric and positive definite.
- $\mathcal{M}$ is sparse ⇒ CPU cost O(N^3) per CG iteration.
- In solving Langevin eq., we use one noisy estimator $\chi^* \frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1}\chi$ instead of the average $\left\langle \chi^* \frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1}\chi \right\rangle$

backup: noisy estimator

Conjugate Gradient (CG) method:
Iterative algorithm to solve the linear equation Ax=b
(A: symmetric, positive-definite n × n matrix)

Initial config. $\mathbf{x}_0 = 0 \quad \mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0 \quad \mathbf{p}_0 = \mathbf{r}_0$

(for brevity, no preconditioning on x_0 here)

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k \quad \mathbf{r}_{k+1} = \mathbf{r}_k - \alpha_k A\mathbf{p}_k \quad \alpha_k = \frac{(r_k, r_k)}{(p_k, Ap_k)}$$

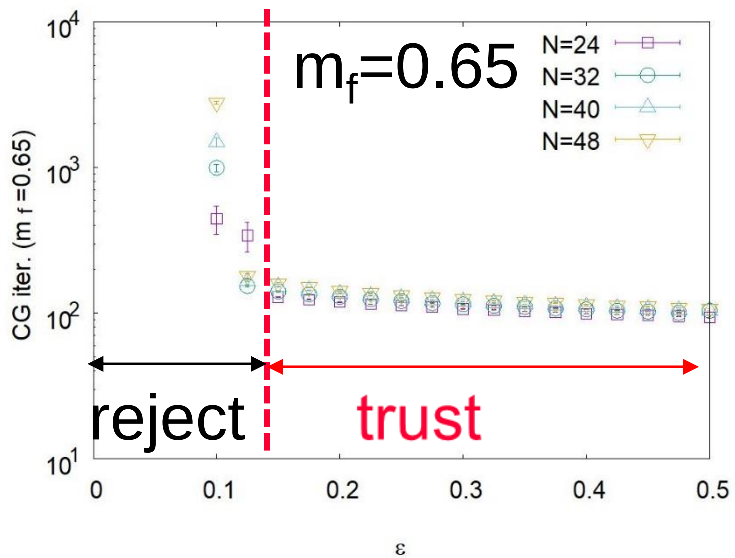
$$\mathbf{p}_{k+1} = \mathbf{r}_{k+1} + \frac{(\mathbf{r}_{k+1}, \mathbf{r}_{k+1})}{(\mathbf{r}_k, \mathbf{r}_k)} \mathbf{p}_k$$

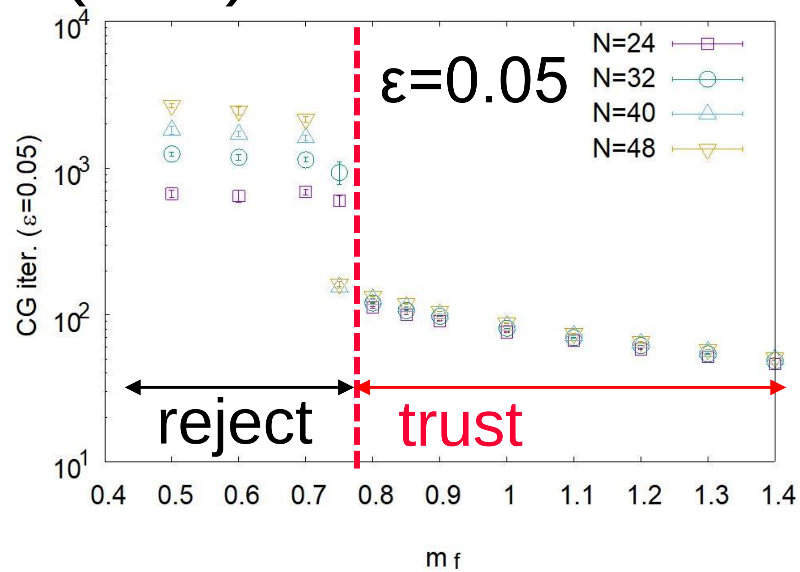
Iterate this until $\sqrt{\frac{(\mathbf{r}_{k+1}, \mathbf{r}_{k+1})}{(\mathbf{r}_0, \mathbf{r}_0)}} < (\text{tolerance}) \simeq 10^{-4}$

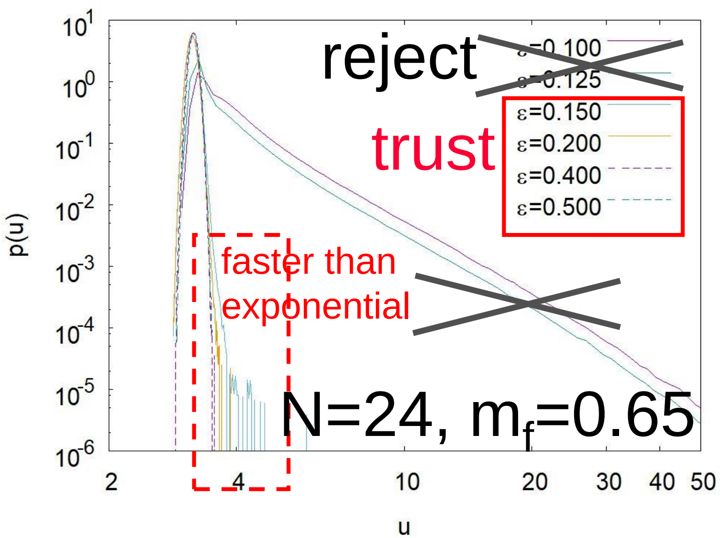
The approximate answer of Ax=b is $x = x_{k+1}$.

backup: noisy estimator

Required CG iteration time (D=6)







When we can trust CLM, there is small dependence of CG iter. on N.

In total, the CPU cost for

$$\mathrm{Tr}\left(\frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1}\right)$$

is $O(N^3)$.