Monte Carlo studies of the rotational symmetry breaking in dimensionally reduced matrix models

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with K.N. Anagnostopoulos and J. Nishimura, `arXiv:1009.4504,1108.1534`

Contents

1 Introduction

Matrix models as a constructive definition of superstring theory

IKKT model (IIB matrix model)

$\Rightarrow$ **Promising candidate for the constructive definition of superstring theory.**
Ishibashi, Kawai, Kitazawa and Tsuchiya, hep-th/9612115.

$$S = N \left(-\frac{1}{4} \mathrm{tr}\, [A_\mu, A_\nu]^2 + \frac{1}{2} \mathrm{tr}\, \bar{\psi}_\alpha (\Gamma_\mu)_{\alpha\beta} [A_\mu, \psi_\beta] \right).$$

- A_μ (10d vector) and ψ_α (10d Majorana-Weyl spinor) $\Rightarrow$ $N \times N$ matrices .
- Euclidean model after Wick rotation $\Rightarrow$ SO(10) rotational symmetry.

  Path integral is finite without cutoff.
  W. Krauth, H. Nicolai and M. Staudacher, hep-th/9803117, P. Austing and J.F. Wheater hep-th/0103159.
- Recent observation from Gaussian Expansion Method (GEM):
  J. Nishimura, T. Okubo and F. Sugino, arXiv:1108.1293.

  Rotational symmetry breaking SO(10) $\to$ SO(3).

- **After integrating out fermion,** $\int d\psi e^{-S_f} = \mathrm{Pf}\mathcal{M}$**, where** $S_f = \frac{N}{2}\mathrm{tr}\,\bar{\psi}_\alpha(\Gamma_\mu)_{\alpha\beta}[A_\mu, \psi_\beta]$**,**
  $(\mathcal{M})_{a\alpha,b\beta} = -if_{abc}(\mathcal{C}\Gamma_\mu)_{\alpha\beta}A^c_\mu = (16(N^2-1) \times 16(N^2-1)\mathrm{matrix})$**.**

  For the Euclidean model, this Pfaffian is complex in general.

  * **Crucial for rotational symmetry breaking.**
    Nishimura and Vernizzi, hep-th/0003223.

  * **Difficulty of Monte Carlo simulation.**

2 Gaussian toy model

We want to understand the rotational symmetry breaking in the Euclidean IKKT model.

Toy model with similarity to the Euclidean IKKT model.
Nishimura, hep-th/0108070.

$$S = \underbrace{\frac{N}{2}\mathrm{tr}\, A_\mu^2}_{=S_b} \underbrace{- \bar{\psi}_\alpha^f (\Gamma_\mu)_{\alpha\beta} A_\mu \psi_\beta^f}_{=S_f}$$

$$\Gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Gamma_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \Gamma_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Gamma_4 = i\sigma_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

- A_μ: $N \times N$ **hermitian matrices** ($\mu = 1, \cdots, 4$)
  $\bar{\psi}_\alpha^f, \psi_\alpha^f$: **$N$-dim vector** ($\alpha = 1, 2, f = 1, \cdots, N_f =$ **(number of flavors)**)
  $\Rightarrow$ **CPU cost is $O(N^3)$ (instead of $O(N^6)$ in the IKKT model)**
- **SO(4) rotational symmetry.**
- **No supersymmetry.**

- **Partition function:**

$$Z = \int dA e^{-S_B} (\det \mathcal{D})^{N_f} = \int dA e^{-S_0} e^{i\Gamma}, \text{ where}$$

$$\mathcal{D} = \Gamma_\mu A_\mu = \begin{pmatrix} A_3 + iA_4 & A_1 - iA_2 \\ A_1 + iA_2 & -A_3 + iA_4 \end{pmatrix} = (2N \times 2N \text{ matrices}),$$

Phase-quenched partition function

$$Z_0 = \int dA e^{-S_0} = \int dA e^{-S_B} |\det \mathcal{D}|^{N_f}.$$

$\det \mathcal{D}$ becomes complex conjugate under

$$A_n^P = A_n (n = 1, 2, 3), \; A_4^P = -A_4.$$

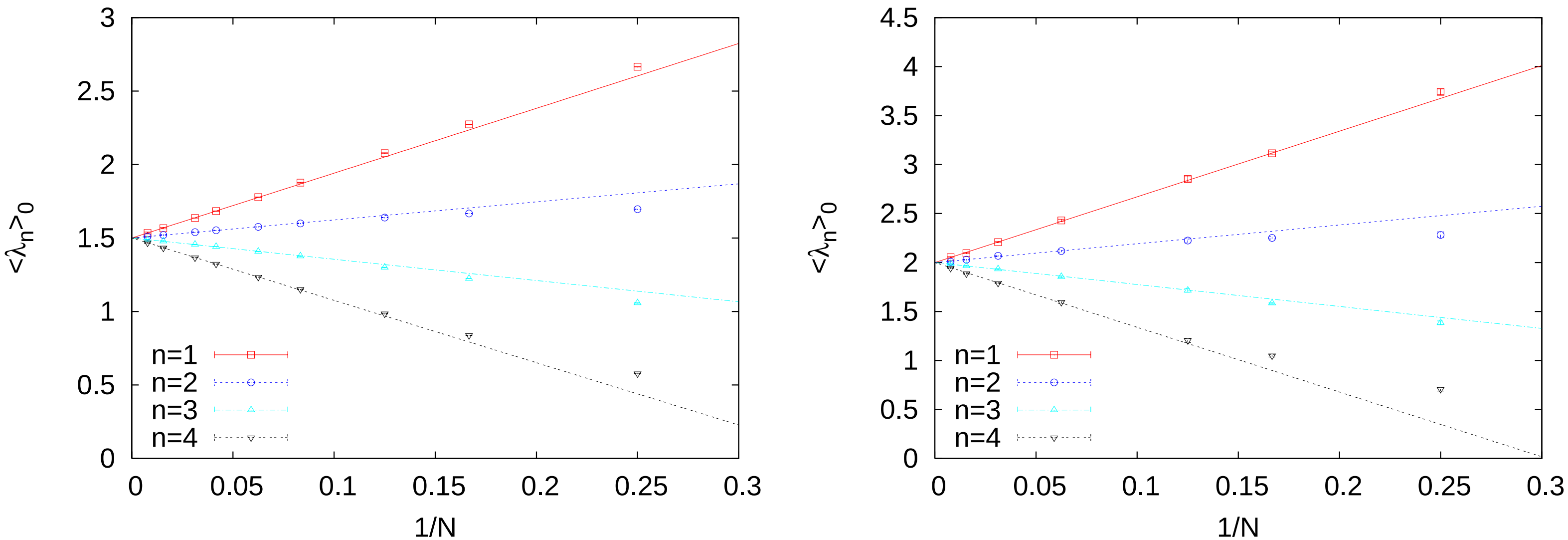
In general, $\det \mathcal{D}$ is complex, while $\det \mathcal{D}$ is real when $A_4 = 0$.

Monte Carlo simulation of the phase-quenched model

Simulation of the partition function Z_0 with the phase omitted.

Observable for probing dimensionality : $T_{\mu\nu} = \frac{1}{N}\mathrm{tr}\,(A_\mu A_\nu)$.

λ_n $(n = 1, 2, 3, 4)$ **: eigenvalues of** $T_{\mu\nu}$ $(\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \lambda_4)$



Results for $r = 1$ (left) and $r = 2$ (right).

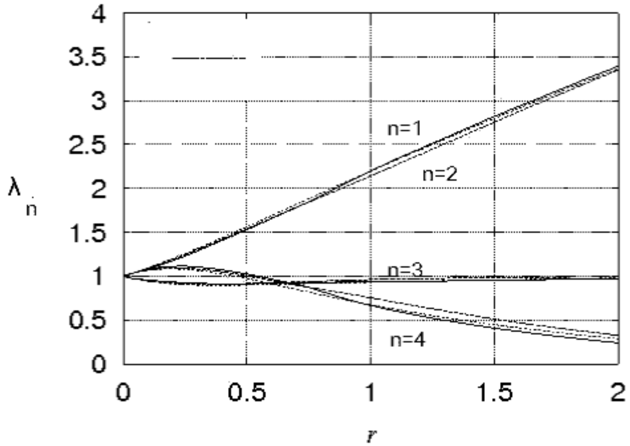
$$\langle \lambda_1 \rangle_0 = \cdots = \langle \lambda_4 \rangle_0 \to 1 + \frac{r}{2} \ (\text{as } N \to \infty),$$

$\langle * \rangle_0 =$ **(V.E.V. for the phase-quenched model Z_0).**

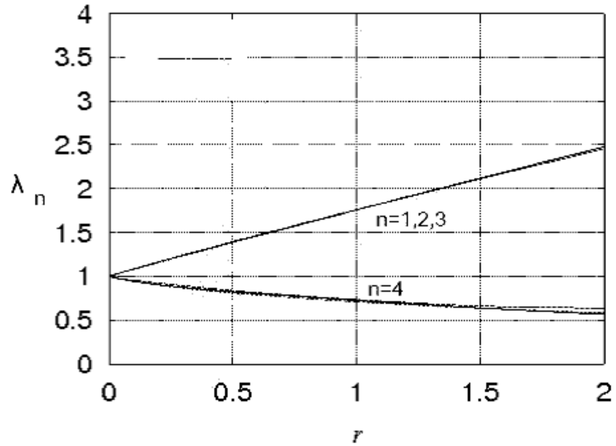
The effect of the phase is crucial for the spontaneous rotational symmetry breaking.

Gaussian expansion analysis up to 9th order:

Okubo, Nishimura and Sugino, hep-th/0412194.



SO(2) ansatz



SO(3) ansatz

Results for $r = 1$ $\left(\tilde{\lambda}_n = \dfrac{\lambda_n}{\langle \lambda_n \rangle_0} = \dfrac{\lambda_n}{1 + \frac{r}{2}}\right)$:

- **SO(2) ansatz:** $\langle \tilde{\lambda}_1 \rangle = \langle \tilde{\lambda}_2 \rangle = 1.4$, $\langle \tilde{\lambda}_3 \rangle = 0.7$, $\langle \tilde{\lambda}_4 \rangle = 0.5$.
- **SO(3) ansatz:** $\langle \tilde{\lambda}_1 \rangle = \langle \tilde{\lambda}_2 \rangle = \langle \tilde{\lambda}_3 \rangle = 1.17$, $\langle \tilde{\lambda}_4 \rangle = 0.5$.
- **No constant volume property :** $\displaystyle\prod_{n=1}^{4} \langle \tilde{\lambda}_n \rangle \simeq \begin{cases} 0.69 & (\mathrm{SO}(2)\ \text{ansatz}) \\ 0.79 & (\mathrm{SO}(3)\ \text{ansatz}) \end{cases} \neq 1.$
- **Comparison of free energy:** $F_{\mathrm{SO}(2)} = -1.8 < F_{\mathrm{SO}(3)} = -1.5 \Rightarrow$ **SO(2) vacuum is favored.**

Spontaneous breakdown of SO(4) to SO(2) at finite $r \left(= \dfrac{N_f}{N}\right)$.

3 Single-variable factorization method

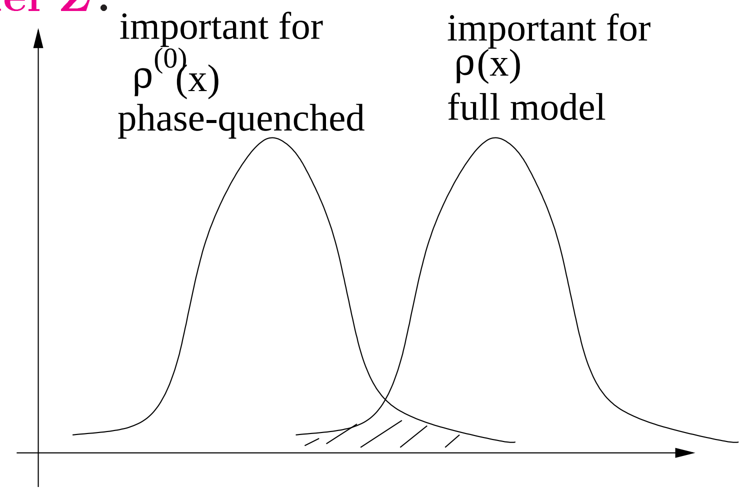
Difficulty in simulating complex-action systems:

- **Sign problem: Standard reweighting method:**

  $$\langle \lambda_n \rangle = \frac{\langle \lambda_n e^{i\Gamma} \rangle_0}{\langle e^{i\Gamma} \rangle_0} = \frac{\langle \lambda_n e^{i\Gamma} \rangle_0}{\langle e^{i\Gamma} \rangle_0},$$ **where** $\langle * \rangle_0 =$ **(V.E.V. for the phase-quenched model** Z_0**).**

  (Number of configurations required) $\simeq e^{O(N^2)}$.

- **Overlap problem: Discrepancy of a distribution function between the phase-quenched model** Z_0 **and the full model** Z**.**



Factorization method: An approach to overcome the overlap problem in Monte Carlo simulation.

K. N. Anagnostopoulos and J. Nishimura, hep-th/0108041,
J. Ambjorn, K. N. Anagnostopoulos, J. Nishimura and J. J. M. Verbaarschot, hep-lat/0208025.

Factorization property of the distribution function

($\tilde{\lambda}_n \stackrel{\text{def}}{=} \lambda_n / \langle \lambda_n \rangle_0$: deviation from 1 ⇒ effect of the phase)

$$\begin{aligned}
\rho_n(x) &\stackrel{\text{def}}{=} \langle \delta(x-\tilde{\lambda}_n)\rangle \stackrel{\text{reweighting}}{=} \frac{\langle \delta(x-\tilde{\lambda}_n) e^{i\Gamma}\rangle_0}{\langle e^{i\Gamma}\rangle_0} \\
&= \frac{1}{\langle e^{i\Gamma}\rangle_0} \times \langle \delta(x-\tilde{\lambda}_n)\rangle_0 \times \frac{\langle \delta(x-\tilde{\lambda}_n) e^{i\Gamma}\rangle_0}{\langle \delta(x-\tilde{\lambda}_n)\rangle_0} \\
&= \frac{1}{\langle e^{i\Gamma}\rangle_0} \times \langle \delta(x-\tilde{\lambda}_n)\rangle_0 \times \frac{\int e^{-S_0}\delta(x-\tilde{\lambda}_n) e^{i\Gamma} dA}{\int e^{-S_0} dA} \div \frac{\int e^{-S_0}\delta(x-\tilde{\lambda}_n) dA}{\int e^{-S_0} dA} \\
&= \underbrace{\frac{1}{\langle e^{i\Gamma}\rangle_0}}_{=\frac{1}{C}} \times \underbrace{\langle \delta(x-\tilde{\lambda}_n)\rangle_0}_{=\rho_n^{(0)}(x)} \times \underbrace{\frac{\int e^{-S_0}\delta(x-\tilde{\lambda}_n) e^{i\Gamma} dA}{\int e^{-S_0}\delta(x-\tilde{\lambda}_n) dA}}_{=w_n(x)} = \frac{1}{C}\rho_n^{(0)}(x) w_n(x)
\end{aligned}$$

where

$$C = \langle e^{i\Gamma}\rangle_0, \quad \rho_n^{(0)}(x) = \langle \delta(x-\tilde{\lambda}_n)\rangle_0, \quad w_n(x) = \langle e^{i\Gamma}\rangle_{n,x},$$

$$\langle * \rangle_{n,x} = [\text{V.E.V. for the partition function } Z_{n,x} = \int dA e^{-S_0}\delta(x-\tilde{\lambda}_n)].$$

In fact, $w_n(x) = \langle e^{i\Gamma} \rangle_{n,x} = \langle \cos \Gamma \rangle_{n,x}$ in our model.

Under parity transformation $A_n^P = A_n (n = 1, 2, 3)$, $A_4^P = -A_4 \Rightarrow$

- **Partition function Z is complex conjugate.**
- **However, the action S_0 is invariant $\rightarrow C = \langle e^{i\Gamma} \rangle_0 = \langle \cos \Gamma \rangle_0$**
- **λ_n (eigenvalues of $T_{\mu\nu} = \frac{1}{N} \mathrm{tr}\, A_\mu A_\nu$) are also invariant $\rightarrow w_n(x) = \langle e^{i\Gamma} \rangle_{n,x} = \langle \cos \Gamma \rangle_{n,x}$.**

Simulation of partition function $Z_{n,x} \Rightarrow x$ is trapped at $\tilde{\lambda}_n$.

The system visits the configurations important for full partition function Z.

Resolution of overlap problem.

In practice, we approximate the partition function $Z_{n,x}$ by

$$Z_{n,V} = \int dA e^{-S_0} e^{-V(\lambda_n)}, \text{ where } V(x) = \frac{\gamma}{2}(x-\xi)^2, \quad \gamma, \xi = (\text{parameters}).$$

Monte Carlo evaluation of $\rho_n^{(0)}(x)$ and $w_n(x)$:

$$\rho_{n,V}(x) \stackrel{\text{def}}{=} \langle \delta(x - \tilde{\lambda}_n) \rangle_{n,V} \propto \rho_n^{(0)}(x) \exp(-V(\langle \lambda_n \rangle_0 x)).$$

The position of the peak x_p for the distribution function $\rho_{n,V}(x)$:

$$0 = \frac{\partial}{\partial x} \log \rho_{n,V}(x) = f_n^{(0)}(x) - \langle \lambda_n \rangle_0 V'(\langle \lambda_n \rangle_0 x), \text{ where } f_n^{(0)}(x) \stackrel{\text{def}}{=} \frac{\partial}{\partial x} \log \rho_n^{(0)}(x).$$

- **Determination of x_p: $\rho_{n,V}(x)$ has a sharp peak for large γ**
  $\Rightarrow$ x_p is approximated as $x_p \simeq \langle \tilde{\lambda}_n \rangle_{n,V}$.

- **Determination of $\rho_n^{(0)}(x)$: Vary ξ, and calculate $f_n^{(0)}(x_p)$ for different x_p.**
  Then, evaluate $\rho_n^{(0)}(x) = \exp\left\{ \int_0^x dz f_n^{(0)}(z) + \text{const.} \right\}$.

Monte Carlo evaluation of $\langle\tilde{\lambda}_n\rangle$

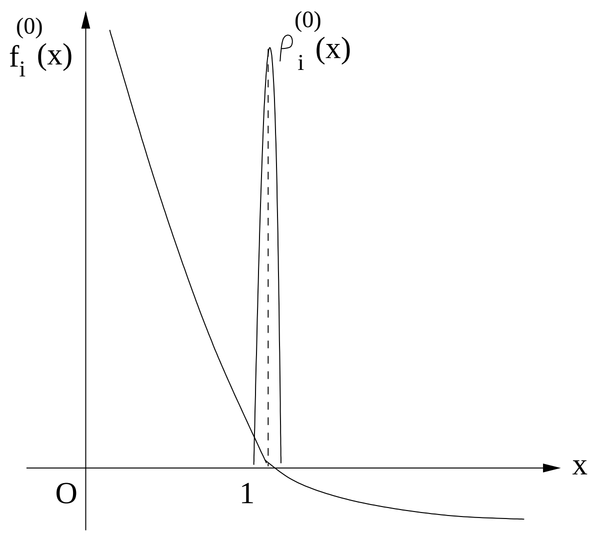
$w_n(x) > 0 \Rightarrow \langle\tilde{\lambda}_n\rangle$ **is the minimum of** $\mathcal{F}_n(x)$**:**

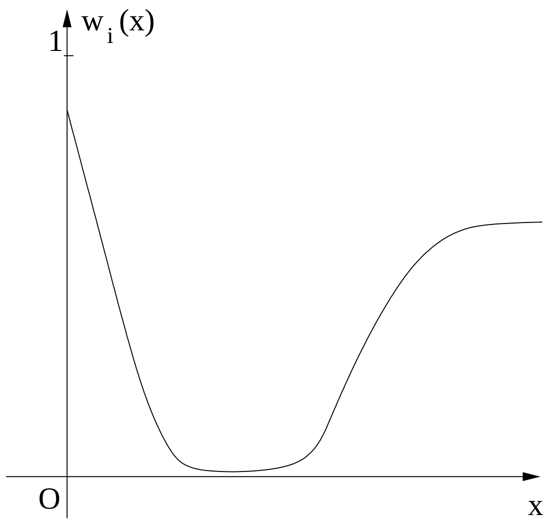
$$\mathcal{F}_n(x) = (\text{free energy density}) = -\frac{1}{N^2}\log\rho_n(x).$$

We solve $\mathcal{F}'_n(x) = 0$**, namely** $\frac{1}{N^2}f_n^{(0)}(x) = -\frac{d}{dx}\left\{\frac{1}{N^2}\log w_n(x)\right\}$.

Both $\frac{1}{N^2}\log w_n(x)$ **and** $\frac{1}{N^2}f_n^{(0)}(x)$ **scale at large** N **as**

$$\frac{1}{N^2}\log w_n(x) \to \Phi_n(x), \quad \frac{1}{N^2}f_n^{(0)}(x) \to F_n(x)$$





Behavior of $\Phi_n(x)$

Asymptotic behavior of $\Phi_n(x) = \frac{1}{N^2} \log w_n(x)$ at $x \ll 1$ and $x \gg 1$.
When we fix the n-th largest eigenvalue $\rightarrow$

- **$x \ll 1$ $(n = 2, 3, 4)$: $(5 - n)$ directions are shrunk**
  $\Rightarrow$ $(n - 1)$-dimensional configuration

- **$x \gg 1$ $(n = 1, 2, 3)$: $(4 - n)$ directions are shrunk**
  $\Rightarrow$ n-dimensional configuration

Fermion determinant $\det \mathcal{D}$ is complex conjugate under

$$A_n^P = A_n (n = 1, 2, 3), \quad A_4^P = -A_4$$

Ω_d =(d-dim. configuration such that $A_{d+1} = A_{d+2} = \cdots A_4 = 0$ after a certain SO(4) rotation)
3-dimensional configuration Ω_3 $\Rightarrow$ Fermion determinant is real.

J. Nishimura and G. Vernizzi, hep-th/0003223.

For d-dimensional configuration Ω_d,

$$\frac{\partial^n \Gamma}{\partial A_{\mu_1}^{a_1} \cdots \partial A_{\mu_n}^{a_n}} = 0 \text{ for } n = 1, \cdots, 3 - d$$

(Up to $(3-d)$-order perturbation $\Rightarrow$ configuration $\in \Omega_3$)

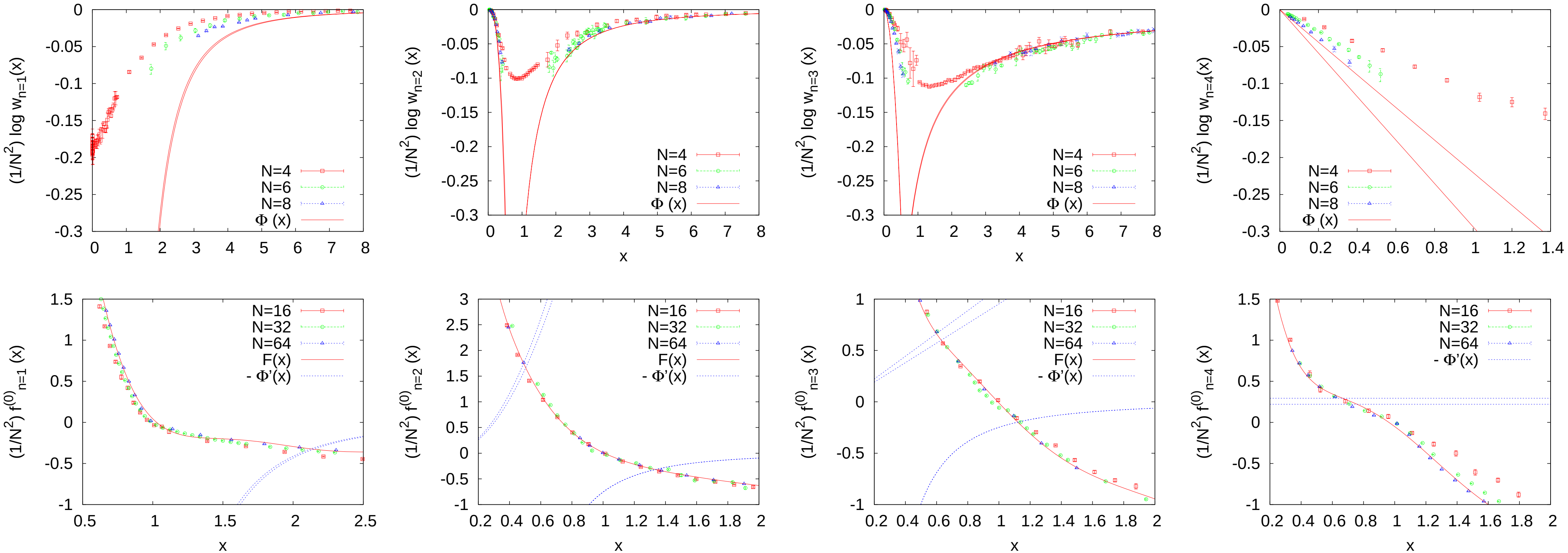
Expected power behaviors:

$$\Phi_n(x) \propto \begin{cases} c_{n,0} x^{5-n} + \cdots & (x \ll 1, n = 2, 3, 4) \\ \frac{d_{n,0}}{x^{4-n}} + \cdots & (x \gg 1, n = 1, 2, 3) \end{cases}$$

(*) x has the order of the eigenvalues of $T_{\mu\nu} = \frac{1}{N} \mathrm{tr}\,(A_\mu A_\nu)$.

Simulation for $r = 1$

$n = 1$ $\quad$ $n = 2$ $\quad$ $n = 3$ $\quad$ $n = 4$



Double-peak structure of $\rho_n(x)$ for $n = 2, 3$

Three solutions of $\frac{d}{dx} \log \rho_n(x) = 0$ $(\underbrace{x_s}_{\text{maximum}} < \underbrace{x_b}_{\text{minimum}} < \underbrace{x_l}_{\text{maximum}})$.

Which peak is higher?

$$\Delta_n = \frac{1}{N^2}(\log \rho_n(x_l) - \log \rho_n(x_s)) = (\Phi_n(x_l) - \Phi_n(x_s)) + \int_{x_s}^{x_l} dx \frac{1}{N^2} f_n^{(0)}(x)$$

Summary of the result for $r = 1$:

n	x_s	x_l	Δ_n	SO(2) (GEM)	SO(3) (GEM)
1	–	2.14(1)	–	1.4	1.17
2	0.49(1)	$\underbrace{1.317(1)}_{SO(2)}$	0.33(2)	1.4	1.17
3	$\underbrace{0.62(2)}_{SO(2)}$	$\underbrace{1.11(2)}_{SO(3)}$	0.11(4)	0.7	1.17
4	$\underbrace{0.71(5)}_{SO(3)}$	–	–	0.5	0.5

- $\langle\tilde{\lambda}_1\rangle > \langle\tilde{\lambda}_2\rangle > 1 > \langle\tilde{\lambda}_4\rangle \to$ SO(4) rotational symmetry breaking due to the effect of phase (it is subtle whether Δ_3 is positive or negative).

- $\langle\tilde{\lambda}_1\rangle \simeq 2.14(1)$ is far from $\langle\tilde{\lambda}_1\rangle_{\rm GEM} \simeq 1.4 \to$ still remaining overlap problem.
  $w_1(x)$ comes from the configuration $\langle\tilde{\lambda}_1\rangle > 1 > \langle\tilde{\lambda}_2\rangle > \langle\tilde{\lambda}_3\rangle > \langle\tilde{\lambda}_4\rangle \to$
  $w_1(x)$ is underestimated.

4 Multi-variable factorization method

Still remaining overlap problem (e.g. $\langle \tilde{\lambda}_{n=1} \rangle$).

We constrain the observables $\Sigma = \{\mathcal{O}_k | k = 1, 2, \cdots, n\}$.

Observables are normalized as $\tilde{\mathcal{O}}_k = \dfrac{\mathcal{O}_k}{\langle \mathcal{O}_k \rangle_0}$,

where $\langle \cdots \rangle_0$=(V.E.V. for the phase-quenched partition function Z_0).

Generalized distribution function $\rho(x_1, \cdots, x_n) = \left\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\rangle$ factorizes as

$$\begin{aligned}
\rho(x_1, \cdots, x_n) &\stackrel{\text{def}}{=} \langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \rangle \stackrel{\text{reweighting}}{=} \frac{\langle \prod_{k=1}^{n} \delta(x - \tilde{\mathcal{O}}_n) e^{i\Gamma} \rangle_0}{\langle e^{i\Gamma} \rangle_0} \\
&= \underbrace{\frac{1}{\langle e^{i\Gamma} \rangle_0}}_{=\frac{1}{C}} \times \underbrace{\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \rangle_0}_{=\rho^{(0)}(x_1, \cdots, x_n)} \times \underbrace{\frac{\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) e^{i\Gamma} \rangle_0}{\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \rangle_0}}_{=w(x_1, \cdots, x_n)} \\
&= \frac{1}{C} \rho^{(0)}(x_1, \cdots, x_n) w(x_1, \cdots, x_n), \text{ where} \\
\rho^{(0)}(x_1, \cdots, x_n) &= \left\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\rangle_0, \quad w(x_1, \cdots, x_n) = \langle e^{i\Gamma} \rangle_{x_1, \cdots, x_n}.
\end{aligned}$$

$\langle \cdots \rangle_{x_1, \cdots, x_n}$ = V.E.V. for partition function $Z_{x_1, \cdots, x_n} = \displaystyle\int dA e^{-S_0} \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k)$.

Evaluation of the observables $\langle \tilde{\mathcal{O}}_k \rangle$:

Peak of the distribution function $\rho(x_1, \cdots, x_n)$

⇒ solution of the saddle-point equation

$$\frac{d}{dx_k} \log \rho^{(0)}(x_1, \cdots, x_n) = -\frac{d}{dx_k} \log w(x_1, \cdots, x_n)$$

Set of observables in Gaussian toy model:

- **Single-variable factorization method: $\Sigma = \{\mathcal{O}_1 = \lambda_n\}$ for $n = 1, 2, 3, 4$ separately.**
- **Multi-variable factorization method: $\Sigma = \{\mathcal{O}_k = \lambda_k | k = 1, 2, 3, 4\}$ simultaneously.**

  Partition function to simulate:

$$Z_{x_1,x_2,x_3,x_4} = \int dA\, e^{-S_0} \prod_{k=1}^{4} \delta(x_k - \tilde{\lambda}_k),$$

Distribution function:

$$\rho(x_1, x_2, x_3, x_4) = \left\langle \prod_{k=1}^{4} \delta(x_k - \tilde{\lambda}_k) \right\rangle.$$

SO(3) vacuum

Solutions which satisfy $x_1 = x_2 = x_3 > 1 > x_4$.

Result of Gaussian Expansion Method: $\langle \tilde{\lambda}_1 \rangle = \langle \tilde{\lambda}_2 \rangle = \langle \tilde{\lambda}_3 \rangle = 1.17$, $\langle \tilde{\lambda}_4 \rangle = 0.5$ $(r = 1)$.

Minimum of the free energy density $\mathcal{F}(x)$

$$\frac{\partial}{\partial \zeta} \rho^{(0)}_{\mathrm{SO(3)}}(x, y) = -\frac{\partial}{\partial \zeta} w_{\mathrm{SO(3)}}(x, y) \quad (\zeta = x, y) \ , \ \text{where}$$

$$\rho^{(0)}_{\mathrm{SO(3)}}(x, y) = \rho^{(0)}(x, x, x, y) \ ,$$

$$w_{\mathrm{SO(3)}}(x, y) = w(x, x, x, y)$$

Calculation of $\langle \tilde{\lambda}_{n=3} \rangle$ at $r = 1$

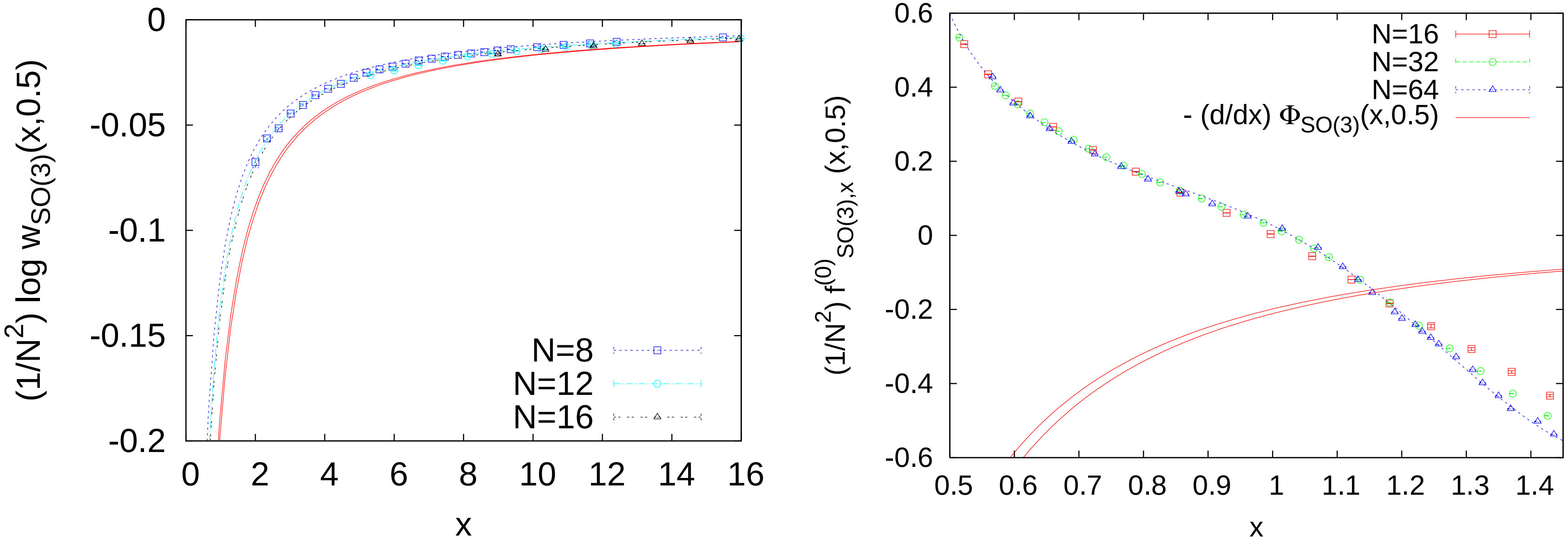
Calculation of $\langle \tilde{\lambda}_{n=3} \rangle$ for fixed $\langle \tilde{\lambda}_{n=4} \rangle = 0.5$.

$$\frac{1}{N^2} f^{(0)}_{\mathrm{SO}(3),x}(x, y) = -\frac{\partial}{\partial x} \Phi_{\mathrm{SO}(3)}(x, y),$$

where $f^{(0)}_{\mathrm{SO}(3),x}(x, y) = \frac{\partial}{\partial x} \log \rho^{(0)}_{\mathrm{SO}(3)}(x, y)$ **and** $\Phi_{\mathrm{SO}(3)}(x, y) = \frac{1}{N^2} \log w_{\mathrm{SO}(3)}(x, y)$.

Scaling behavior of the phase:

$$\Phi_{\mathrm{SO}(3)}(x, y = 0.5) \sim -\tilde{d}_1 x^{-1} + \tilde{d}_2 x^{-1.5}$$



Numerical Result: $\langle \tilde{\lambda}_{n=3} \rangle = 1.151(2)$, **(GEM result** $\langle \tilde{\lambda}_{n=3} \rangle_{\mathrm{GEM}} = 1.17$**).**

Calculation of $\langle \tilde{\lambda}_{n=4} \rangle$ at $r = 1$

Calculation of $\langle \tilde{\lambda}_{n=4} \rangle$ for fixed $\langle \tilde{\lambda}_{n=3} \rangle = 1.17$.

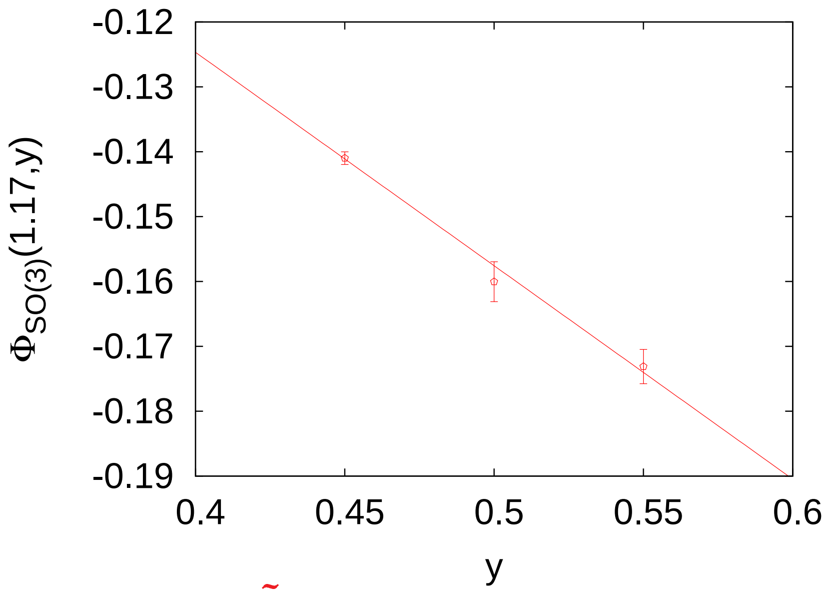
$$\frac{1}{N^2} f^{(0)}_{\mathrm{SO}(3),y}(x = 1.17, y) = -\frac{\partial}{\partial y} \Phi_{\mathrm{SO}(3)}(x = 1.17, y),$$

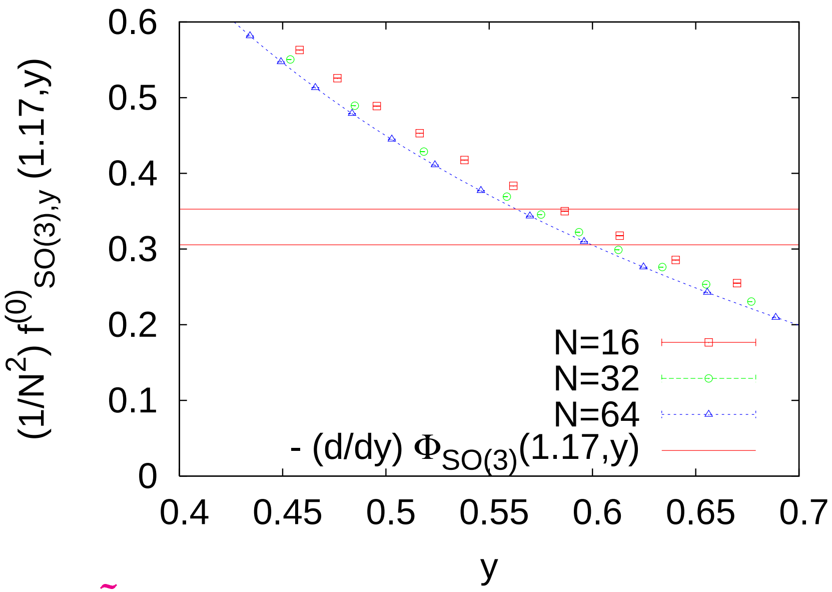
where $f^{(0)}_{\mathrm{SO}(3),y}(x, y) = \dfrac{\partial}{\partial y} \log \rho^{(0)}_{\mathrm{SO}(3)}(x, y)$.

$\Phi_{\mathrm{SO}(3)}(x = 1.17, y)$ **suffers finite-N effect at $y = 0.50 \Rightarrow$**

Calculate $\Phi_{\mathrm{SO}(3)}(x, y = 0.45)$, $\underbrace{\Phi_{\mathrm{SO}(3)}(x, y = 0.50)}_{\text{done in calculating } \langle \tilde{\lambda}_{n=3} \rangle}$, **and** $\Phi_{\mathrm{SO}(3)}(x, y = 0.55)$ **at** $x = 1.17$,

and obtain $-\dfrac{\partial}{\partial x} \Phi_{\mathrm{SO}(3)}(x = 1.17, y)$.





Numerical Result: $\langle \tilde{\lambda}_{n=4} \rangle = 0.59(2)$, **(GEM result** $\langle \tilde{\lambda}_{n=4} \rangle_{\mathrm{GEM}} = 0.50$**).**

SO(2) vacuum

Solutions which satisfy $x_1 = x_2 > 1 > x_3 > x_4$.

Result of Gaussian Expansion Method: $\langle \tilde{\lambda}_{1,2} \rangle = 1.4$, $\langle \tilde{\lambda}_3 \rangle = 0.7$, $\langle \tilde{\lambda}_4 \rangle = 0.5$ $(r = 1)$.

Minimum of the free energy density $\mathcal{F}(x)$

$$\frac{\partial}{\partial \zeta} \rho^{(0)}_{\mathrm{SO}(2)}(x, y, z) = -\frac{\partial}{\partial \zeta} w_{\mathrm{SO}(2)}(x, y, z) \quad (\zeta = x, y, z) \ , \text{ where}$$

$$\rho^{(0)}_{\mathrm{SO}(2)}(x, y, z) = \rho^{(0)}(x, x, y, z) \ ,$$

$$w_{\mathrm{SO}(2)}(x, y, z) = w(x, x, y, z)$$

Calculation of $\langle\tilde{\lambda}_{n=2}\rangle$ at $r = 1$

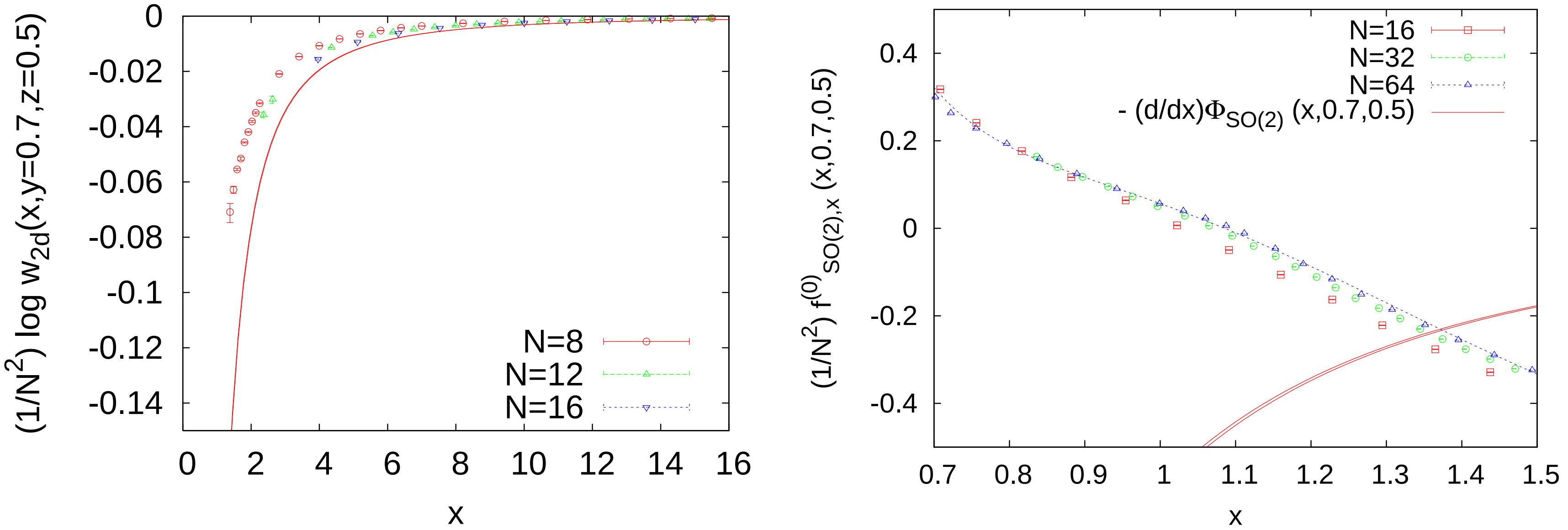
Calculation of $\langle\tilde{\lambda}_{n=2}\rangle$ for fixed $\langle\tilde{\lambda}_{n=3}\rangle = 0.7$ and $\langle\tilde{\lambda}_{n=4}\rangle = 0.5$.

$$\frac{1}{N^2} f^{(0)}_{SO(2),x}(x, y = 0.7, z = 0.5) = -\frac{\partial}{\partial x}\Phi_{SO(2)}(x, y = 0.7, z = 0.5),$$

where $f^{(0)}_{SO(2),x}(x,y,z) = \frac{\partial}{\partial x}\log \rho^{(0)}_{SO(2)}(x,y,z)$ **and** $\Phi_{SO(2)}(x,y,z) = \frac{1}{N^2}\log w_{SO(2)}(x,y,z)$.

Scaling behavior of the phase:

$$\Phi_{SO(2)}(x, y = 0.7, z = 0.5) \sim -\tilde{d}_1 x^{-2} + \tilde{d}_2 x^{-2.5}$$



Numerical Result: $\langle\tilde{\lambda}_{n=2}\rangle = 1.373(2)$, (GEM result $\langle\tilde{\lambda}_{n=2}\rangle_{GEM} = 1.4$).

Calculation of $\langle\tilde{\lambda}_{n=3}\rangle$ at $r = 1$

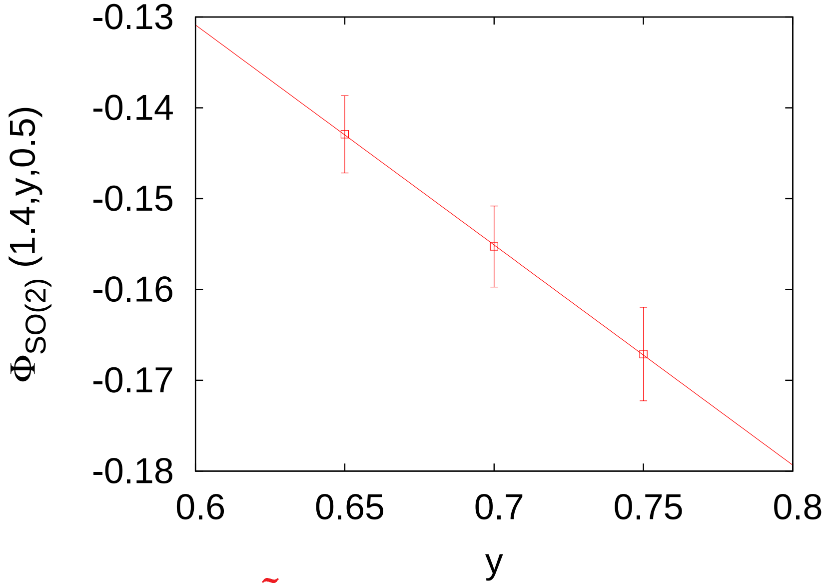
Calculation of $\langle\tilde{\lambda}_{n=3}\rangle$ for fixed $\langle\tilde{\lambda}_{n=2}\rangle = 1.4$ and $\langle\tilde{\lambda}_{n=4}\rangle = 0.5$.

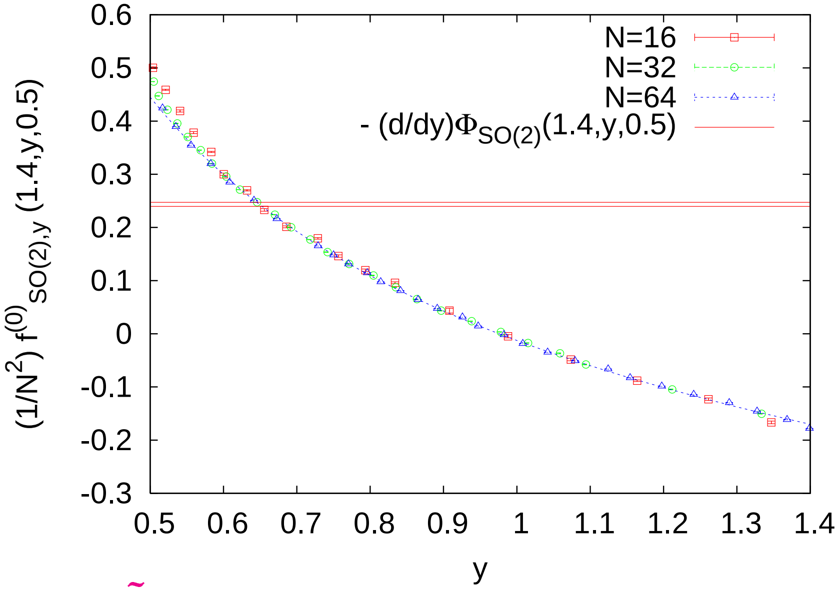
$$\frac{1}{N^2} f^{(0)}_{SO(2),y}(x = 1.4, y, z = 0.5) = -\frac{\partial}{\partial y}\Phi_{SO(2)}(x = 1.4, y, z = 0.5),$$

where $f^{(0)}_{SO(2),y}(x, y, z) = \dfrac{\partial}{\partial y}\log\rho^{(0)}_{SO(2)}(x, y, z)$.

Calculate $\Phi_{SO(2)}(x = 1.4, y = 0.65, z = 0.5)$, $\underbrace{\Phi_{SO(2)}(x = 1.4, y = 0.7, z = 0.5)}_{\text{done in calculating } \langle\tilde{\lambda}_{n=2}\rangle}$, **and**

$\Phi_{SO(2)}(x = 1.4, y = 0.75, z = 0.5)$, **and obtain** $-\dfrac{\partial}{\partial y}\Phi_{SO(2)}(x = 1.4, y, z = 0.5)$.





Numerical Result: $\langle\tilde{\lambda}_{n=3}\rangle = 0.649(4)$, **(GEM result** $\langle\tilde{\lambda}_{n=3}\rangle_{\rm GEM} = 0.7$**).**

Calculation of $\langle \tilde{\lambda}_{n=4} \rangle$ at $r = 1$

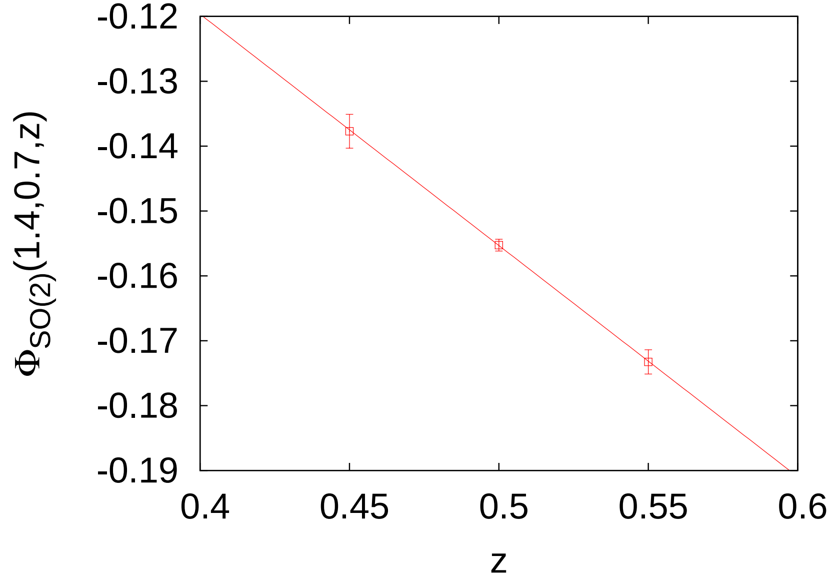
Calculation of $\langle \tilde{\lambda}_{n=4} \rangle$ for fixed $\langle \tilde{\lambda}_{n=2} \rangle = 1.4$ and $\langle \tilde{\lambda}_{n=3} \rangle = 0.7$.

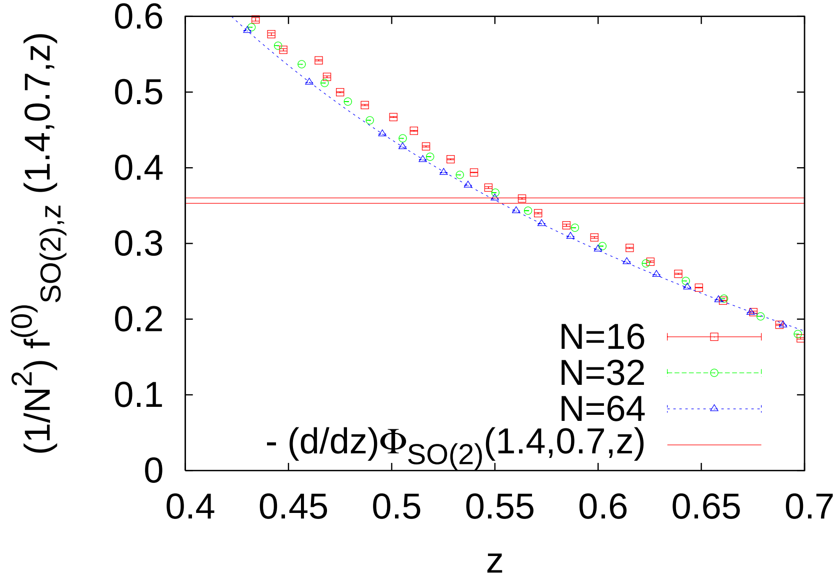
$$\frac{1}{N^2} f^{(0)}_{\mathrm{SO}(2),z}(x = 1.4, y = 0.7, z) = -\frac{\partial}{\partial z} \Phi_{\mathrm{SO}(2)}(x = 1.4, y = 0.7, z),$$

where $f^{(0)}_{\mathrm{SO}(2),z}(x, y, z) = \dfrac{\partial}{\partial z} \log \rho^{(0)}_{\mathrm{SO}(2)}(x, y, z)$.

Calculate $\Phi_{\mathrm{SO}(2)}(x = 1.4, y = 0.7, z = 0.45)$, $\underbrace{\Phi_{\mathrm{SO}(2)}(x = 1.4, y = 0.7, z = 0.50)}_{\text{done in calculating } \langle \tilde{\lambda}_{n=2} \rangle}$, **and**

$\Phi_{\mathrm{SO}(2)}(x = 1.4, y = 0.7, z = 0.55)$, **and obtain** $-\dfrac{\partial}{\partial z} \Phi_{\mathrm{SO}(2)}(x = 1.4, y = 0.7, z)$.





Numerical Result: $\langle \tilde{\lambda}_{n=4} \rangle = 0.551(2)$, **(GEM result** $\langle \tilde{\lambda}_{n=4} \rangle_{\mathrm{GEM}} = 0.50$**).**

Summary of the result for $r = 1$:

ansatz	SO(3)			SO(2)		
method	single-obs.	multi-obs.	GEM	single-obs.	multi-obs.	GEM
$\langle\tilde{\lambda}_1\rangle$	—	—	1.17	—	—	1.4
$\langle\tilde{\lambda}_2\rangle$	—	—	1.17	1.317(1)	1.373(2)	1.4
$\langle\tilde{\lambda}_3\rangle$	1.11(2)	1.151(2)	1.17	0.62(2)	0.649(4)	0.7
$\langle\tilde{\lambda}_4\rangle$	0.71(5)	0.59(2)	0.5	not available	0.551(2)	0.5

Comparison of the free energy

We evaluate $\Delta = -\mathcal{F}_{\mathrm{SO}(3)} + \mathcal{F}_{\mathrm{SO}(2)} = \frac{1}{N^2}\left\{\log \rho(\vec{x}_{\mathrm{SO}(3)}) - \rho(\vec{x}_{\mathrm{SO}(2)})\right\}$.

- $\Delta < 0 \Rightarrow$ SO(2) vacuum dominates. $\Delta > 0 \Rightarrow$ SO(3) vacuum dominates.
- $\vec{x}_{\mathrm{SO}(3)} = (X', X', X', Y'), \quad X' \simeq 1.17, \quad Y' \simeq 0.5.$
  $\vec{x}_{\mathrm{SO}(2)} = (X, X, Y, Z), \quad X \simeq 1.4, \quad Y \simeq 0.7, \quad Z = 0.5.$

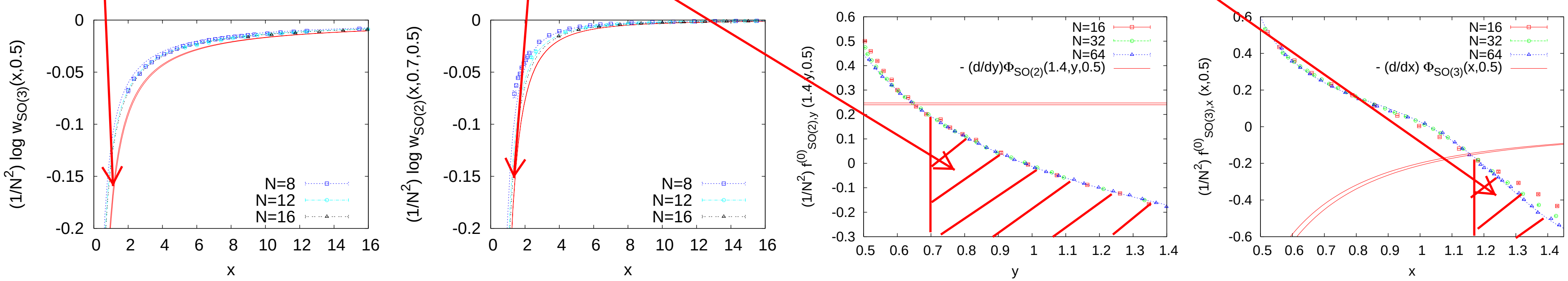
This is rewritten as

$$\Delta = \underbrace{\Phi_{\mathrm{SO}(3)}(X', Y')}_{\simeq -0.160(3)} - \underbrace{\Phi_{\mathrm{SO}(2)}(X, Y, Z)}_{\simeq -0.155(1)} + \underbrace{\int_{\vec{x}_{\mathrm{SO}(2)}}^{\vec{x}_{\mathrm{SO}(3)}} dx_j \frac{1}{N^2}\frac{\partial}{\partial x_j} \log \rho^{(0)}(x_1, x_2, x_3, x_4)}_{=\Xi\simeq 0.065} \simeq +0.060(4) > 0, \text{ where}$$

$$\Xi = \underbrace{\int_{0.7}^{1.4} \frac{1}{N^2} f^{(0)}_{\mathrm{SO}(2),y}(1.4, y, 0.5) dy}_{\simeq -0.014,\ (X,X,Y,Z)\to(X,X,X,Z)} - \underbrace{\int_{1.17}^{1.4} \frac{1}{N^2} f^{(0)}_{\mathrm{SO}(3),x}(x, 0.5) dx}_{\simeq -0.079, (X,X,X,Y'=Z)\to(X',X',X',Y')} .$$

Systematic errors of $\Phi_{\mathrm{SO}(3)}(X', Y')$ and $\Phi_{\mathrm{SO}(2)}(X, Y, Z) \Rightarrow$

It is difficult to determine Δ's sign.



Is there any more overlap problem?

Observables to constrain: $\Sigma = \{\mathcal{O}_k = \lambda_k | k = 1, 2, 3, 4\}$. **Is this enough?**

Partition function $Z_{\mathcal{O}} = \int dA e^{-S_0} \delta(x - \tilde{\mathcal{O}}) \prod_{n=1}^{4} \delta(x_n - \tilde{\lambda}_n)$

(here we constrain $\Sigma = \{\mathcal{O}, \lambda_1, \cdots, \lambda_4\}$**).**

Peak of the distribution function $\rho(x_1, x_2, x_3, x_4, x) = \langle \delta(x - \tilde{\mathcal{O}}) \prod_{k=1}^{4} \delta(x_k - \tilde{\lambda}_k) \rangle$.

$\rho_{\mathcal{O}}(x) = \rho(X = 1.4, X = 1.4, Y = 0.7, Z = 0.5, x)$ **(with** $x_1, \cdots, x_4$ **fixed at GEM results).**

Saddle-point equation: $\frac{d}{dx} \frac{1}{N^2} \log \rho^{(0)}_{\mathcal{O}}(x) = -\frac{d}{dx} \frac{1}{N^2} \log w_{\mathcal{O}}(x)$, **where**

$\rho^{(0)}_{\mathcal{O}}(x) = \langle \delta(x - \mathcal{O}) \rangle_{X,X,Y,Z}$

(VEV of partition function $Z_{X,X,Y,Z} = \int dA e^{-S_0} \delta(X - \tilde{\lambda}_1) \delta(X - \tilde{\lambda}_2) \delta(Y - \tilde{\lambda}_3) \delta(Z - \tilde{\lambda}_4)$.**)**

$w_{\mathcal{O}}(x) = \langle e^{i\Gamma} \rangle_{\mathcal{O}}$ **(VEV of partition function** $Z_{\mathcal{O}}$ **with** $x_1 = x_2 = X, x_3 = Y, x_4 = Z$**).**

Do the peaks of $\rho^{(0)}_{\mathcal{O}}(x)$ **and** $\rho_{\mathcal{O}}(x)$ **match?**

We consider $\mathcal{O} = -\frac{1}{N} \mathrm{tr}\, [A_\mu, A_\nu]^2$.

Simulation with $\tilde{\lambda}_n$ fixed at GEM result of SO(2) ansatz

$\langle \tilde{\lambda}_{1,2} \rangle = 1.4,\ \langle \tilde{\lambda}_3 \rangle = 0.7,\ \langle \tilde{\lambda}_4 \rangle = 0.5$ **($r = 1$).**

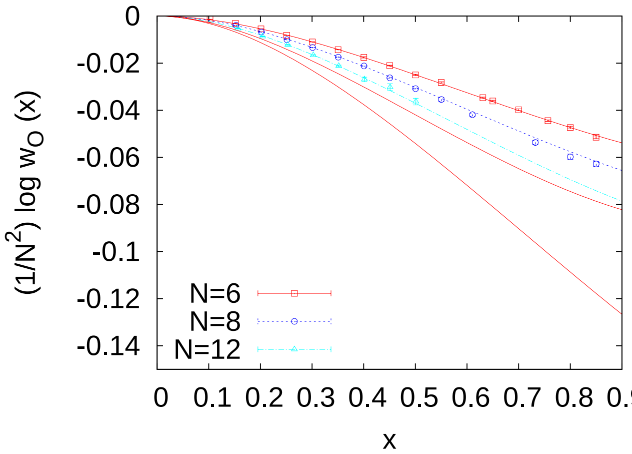
$w_{\mathcal{O}}(x) \to 1$ **as** $x \to 0$.

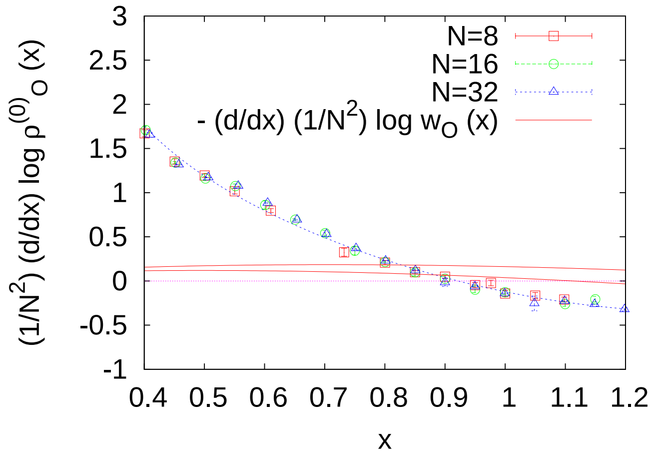
($[A_\mu, A_\nu] \to 0$ **as** $x \to 0 \Rightarrow$ **diagonal configurations become dominant.**

For $A_\mu = \mathrm{diag}(\alpha_\mu^{(1)}, \cdots, \alpha_\mu^{(N)})$, **we have** $\det \mathcal{D} = \prod_{i=1}^{N} \left(\sum_{\mu=1}^{4} (\alpha_\mu^{(i)})^2 \right) \geqq 0.$**)**

Asymptotic behavior: $\frac{1}{N^2} \log w_{\mathcal{O}}(x) = -c_{n,0} x^2 + c_{n,1} x^{2.5} + \cdots$.

- **Peak of** $\rho_{\mathcal{O}}^{(0)}(x)$**: Solution of** $\frac{1}{N^2} \frac{d}{dx} \log \rho_{\mathcal{O}}^{(0)}(x) = 0 \Rightarrow x \simeq 0.92$.
- **Peak of** $\rho_{\mathcal{O}}(x)$**: Solution of** $\frac{1}{N^2} \frac{d}{dx} \log \rho_{\mathcal{O}}^{(0)}(x) = -\frac{1}{N^2} \frac{d}{dx} \log w_{\mathcal{O}}(x) \Rightarrow x \simeq 0.85(3)$.
  Systematic error is around 10% ⇒ there is only a small overlap problem left.





5 Conclusion

Monte Carlo simulation of the toy model with similarity to the Euclidean IKKT model.

Factorization method to overcome ”overlap problem”.

- **Phase of the fermion determinant ⇒ crucial for rotational symmetry breaking.**
- **VEV's $\langle \tilde{\lambda}_n \rangle$ ⇒ consistent with the GEM.**

Future problems

Monte Carlo Simulation of the IKKT model Anagnostopoulos, T. A. and Nishimura, in progress

Effect of supersymmetry on dynamical generation of spacetime.

Application of factorization method to wider range of science.