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Complex Langevin analysis of the
spontaneous rotational symmetry
breaking in the Euclidean type IIB matrix
model

(arXiv:1712.07562)

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Mandelstam Institute, Apr 30th 2019, 14:00~



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Difficulties in simulating complex partition functions.

$$Z = \int dA \exp(-S_0 + i\Gamma), \quad Z_0 = \int dA e^{-S_0}$$

Sign problem:

The reweighting $\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\Gamma} \rangle_0}{\langle e^{i\Gamma} \rangle_0}$ requires configs. $\exp[O(N^2)]$

$\langle^* \rangle_0 =$ (V.E.V. for the phase-quenched partition function Z_0)

Various methods to address the sign problem:

(**Complex Langevin Method (CLM)**, factorization method, Lefschetz-thimble method...)

In the following, we discuss **CLM**.

2. Euclidean type IIB matrix model

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type IIB matrix model model (a.k.a. **IKKT model**)

⇒ Promising candidate for nonperturbative string theory

[N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115]

$$Z = \int dA d\psi e^{-(S_b + S_f)}$$
$$S_b = -\frac{N}{4} \text{tr}[A_\mu, A_\nu]^2, \quad S_f = N \text{tr} \bar{\psi}_\alpha (\Gamma^\mu)_{\alpha\beta} [A_\mu, \psi_\beta]$$

Euclidean case after Wick rotation $A_0 \rightarrow iA_D, \Gamma^0 \rightarrow -i\Gamma_D$.

⇒ Path integral is finite without cutoff.

[W. Krauth, H. Nicolai and M. Staudacher, hep-th/9803117, P. Austing and J.F. Wheeler, hep-th/0103059]

• $A_\mu, \Psi_\alpha \Rightarrow N \times N$ Hermitian traceless matrices.

$$\mu = 1, 2, \dots, D, \quad \alpha, \beta = \begin{cases} 1, 2, 3, 4 & (D = 6) \\ 1, 2, \dots, 16 & (D = 10) \end{cases}$$

• Originally defined in **D=10** (ψ : Majorana-Weyl)

We consider the **simplified D=6 case as well**

(ψ : Weyl, not Majorana $d\psi \rightarrow d\psi d\bar{\psi}$)

- Matrix regularization of the type IIB string action:

$$S_{\text{Sh}} = \int d^2\sigma \left\{ \sqrt{g} \alpha \left(\frac{1}{4} \{X_\mu, X_\nu\}^2 - \frac{i}{2} \bar{\psi} \Gamma^\mu \{X_\mu, \psi\} \right) + \beta \sqrt{g} \right\}.$$

$$-i[X, Y] \leftrightarrow \{X, Y\} = \frac{1}{\sqrt{g}} \varepsilon^{ab} \partial_a X \partial_b Y, \quad \text{tr} \leftrightarrow \int d^2\sigma \sqrt{g}.$$

- Eigenvalues of A_μ : spacetime coordinate $\Rightarrow \mathcal{N} = 2$ SUSY

$$\tilde{\delta}_\varepsilon^{(1)} = \delta_\varepsilon^{(1)} + \delta_\varepsilon^{(2)} \quad \tilde{\delta}_\varepsilon^{(2)} = i(\delta_\varepsilon^{(1)} - \delta_\varepsilon^{(2)}) \quad \text{where}$$

$$\delta_\varepsilon^{(1)} A_\mu = i\varepsilon(\mathcal{C}\Gamma_\mu)\psi, \quad \delta_\varepsilon^{(1)}\psi = \frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\varepsilon, \quad \delta_\varepsilon^{(2)} A_\mu = 0, \quad \delta_\varepsilon^{(2)}\psi = \varepsilon.$$

$$[\tilde{\delta}_\varepsilon^{(a)}, \tilde{\delta}_\xi^{(b)}] A_\mu = -2i\delta^{ab}\varepsilon(\mathcal{C}\Gamma_\mu)\xi, \quad [\tilde{\delta}_\varepsilon^{(a)}, \tilde{\delta}_\xi^{(b)}]\psi = 0, \quad (a, b = 1, 2).$$

2. Euclidean type IIB matrix model

Result of Gaussian Expansion Method (GEM)

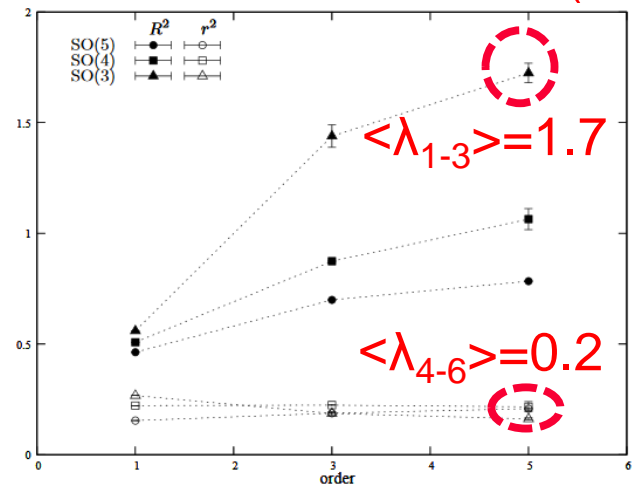
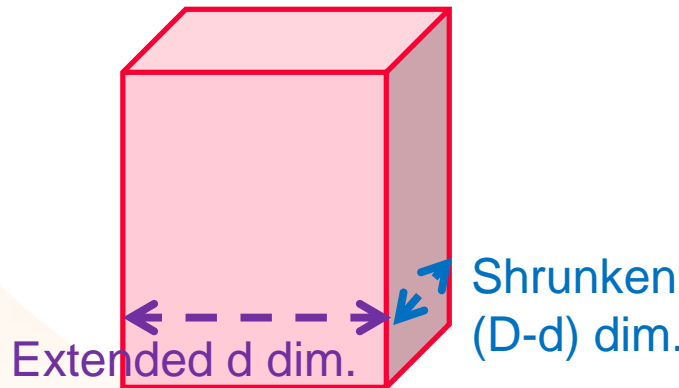
[T.Aoyama, J.Nishimura, and T.Okubo, arXiv:1007.0883, J.Nishimura, T.Okubo and F.Sugino, arXiv:1108.1293]

SSB **SO(6)** → **SO(3)** (In D=10, too, SO(10) → SO(3))
 Dynamical compactification to 3-dim spacetime.

$\lambda_n (\lambda_1 \geq \dots \geq \lambda_D)$: eigenvalues of $T_{\mu\nu} = \frac{1}{N} \text{tr}(A_\mu A_\nu)$

$$\rho_\mu = \frac{\langle \lambda_\mu \rangle}{\sum_{\nu=1}^D \langle \lambda_\nu \rangle} = \begin{cases} 0.30 & (\mu = 1, 2, 3) \\ 0.035 & (\mu = 4, 5, 6) \end{cases}$$

(D = 6) [arXiv:1007.0883](https://arxiv.org/abs/1007.0883) (D=6)



2. Euclidean type IIB matrix model

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$$Z = \int dA d e^{-S_b} \underbrace{\left(\int d\psi e^{-S_f} \right)}_{= \det/\text{Pf } \mathcal{M} = |\det/\text{Pf } \mathcal{M}| e^{i\Gamma}} = \int dA \underbrace{e^{-S}}_{e^{-\{S_b - \log(\det/\text{Pf } \mathcal{M})\}}}$$

- Integrating out ψ yields **det \mathcal{M}** in **D=6** (**Pf \mathcal{M}** in **D=10**)
- **det/Pf \mathcal{M} 's *complex phase*** contributes to the **Spontaneous Symmetry Breaking (SSB)** of **SO(D)**.

Under the parity transformation $A_D \Rightarrow -A_D$,

det/PfM is complex conjugate

\Rightarrow det/PfM is real for $A_D=0$ (hence (D-1)-dim config.).

For the d-dim config, $\frac{\partial^m \Gamma}{\partial A_{\mu_1} \cdots \partial A_{\mu_m}} = 0$ ($m=1, 2, \dots, (D-1)-d$)

The phase is more stationary for lower d.

2. Euclidean type IIB matrix model

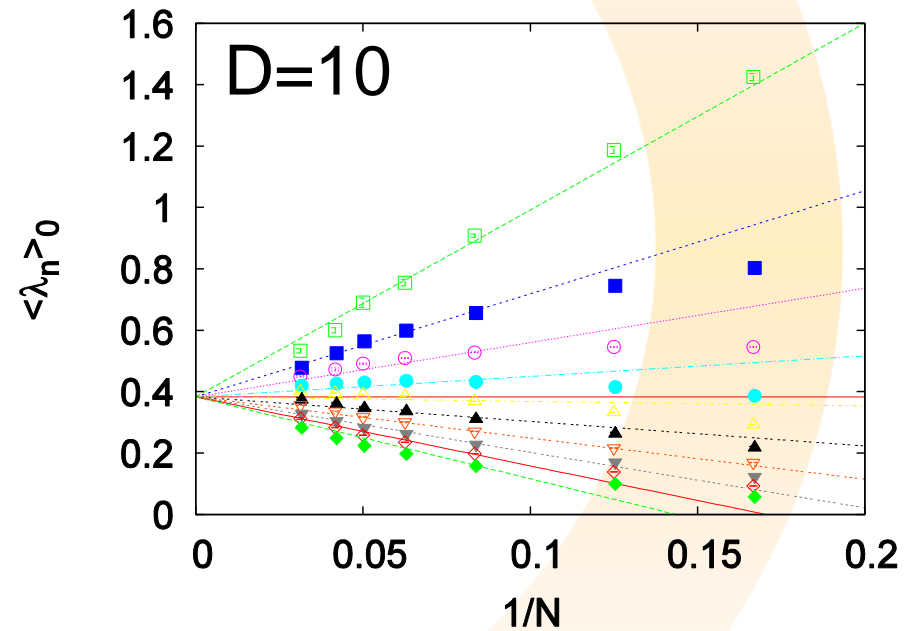
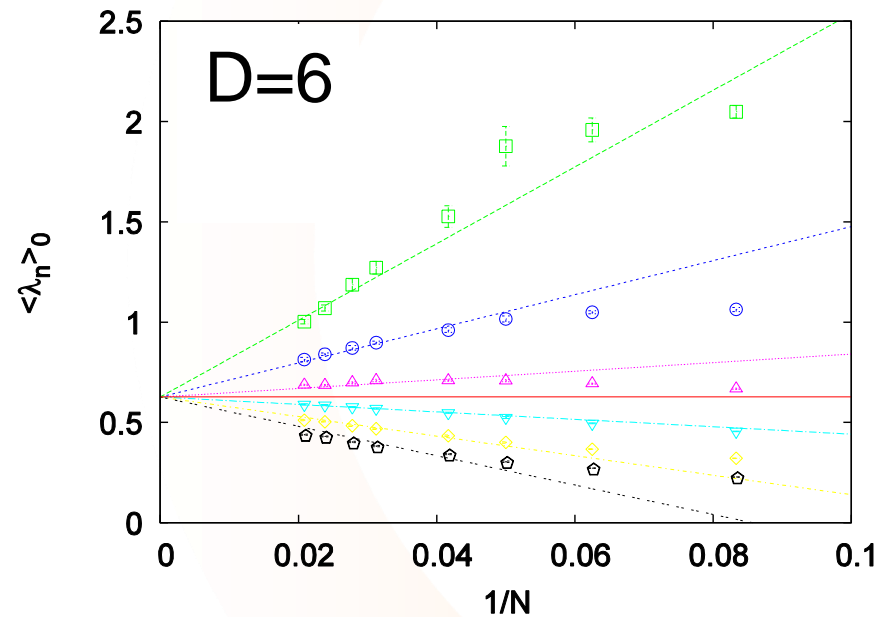
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No SSB with the phase-quenched partition function.

$$Z_0 = \int dA e^{-S_0} = \int dA e^{-S_b} |\det/\text{Pf } \mathcal{M}| \quad \langle * \rangle_0 = \text{V.E.V. for } Z_0$$

[J. Ambjorn, K.N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, hep-th/0003208,0005147, K.N. Anagnostopoulos, T. Azuma, J.Nishimura arXiv:1306.6135, 1509.05079]



Complex Langevin Method (CLM)

⇒ Solve the complex version of the Langevin equation.

[Parisi, Phys.Lett. 131B (1983) 393, Klauder, Phys.Rev. A29 (1984) 2036]

"Real" case

$x_k(t)$ and the action S are **real** (t : fictitious time)

$$\dot{x}_k^{(\eta)}(t) = - \underbrace{\frac{\partial S}{\partial x_k(t)}}_{\text{drift term}} + \eta_k(t)$$

• η_μ : White noise obeying the probability distribution

$$\exp\left(-\frac{1}{4} \int \eta_k^2(t) dt\right)$$

Probability distribution of $x^{(\eta)}_k(t)$

$$P(x;t) = \left\langle \prod_k \delta(x_k - x_k^{(\eta)}(t)) \right\rangle_{\eta} \quad \text{where}$$

$$\langle \dots \rangle_{\eta} = \frac{\int \mathcal{D}\eta \dots \exp\left(-\frac{1}{4} \int \eta_k^2(t) dt\right)}{\int \mathcal{D}\eta \exp\left(-\frac{1}{4} \int \eta_k^2(t) dt\right)} \quad \langle \eta_k(t_1) \eta_l(t_2) \rangle_{\eta} = 2\delta_{kl} \delta(t_1 - t_2)$$

This obeys the Fokker-Planck (FP) equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x_k} \left(\frac{\partial S}{\partial x_k} + \frac{\partial}{\partial x_k} \right) P$$

Time-independent solution $P_{\text{time-indep.}}(x) \propto e^{-S}$

Equivalent to the path integral.

3. Complex Langevin Method

Putting the **real Langevin equation** on a computer
 ⇒ discretized version

$$x_k^{(\eta)}(t + \Delta t) = x_k^{(\eta)}(t) - (\Delta t) \frac{\partial S}{\partial x_k} + \overbrace{(\Delta t) \eta_k(t)}^{= \sqrt{\Delta t} \tilde{\eta}_k(t)}$$

The white noise obeys $\exp\left(-\frac{1}{4} \int \eta_k^2(t) dt\right) \rightarrow \exp\left(-\frac{1}{4} \sum_t \overbrace{(\Delta t) \eta_k^2(t)}{= \tilde{\eta}_k^2(t)}\right)$

Derivation of the Fokker-Planck equation

$$\begin{aligned} & \langle f(x^{(\eta)}(t + \Delta t)) \rangle_\eta - \langle f(x^{(\eta)}(t)) \rangle_\eta = \int f(x) (P(x; t + \Delta t) - P(x; t)) dx \\ &= -(\Delta t) \left\langle \frac{\partial f}{\partial x_k} \frac{\partial S}{\partial x_k} \right\rangle_\eta + \frac{1}{2} \sqrt{(\Delta t)^2} \left\langle \frac{\partial^2 f}{\partial x_k \partial x_l} \tilde{\eta}_k(t) \tilde{\eta}_l(t) \right\rangle_\eta + O((\Delta t)^2) \\ &= (\Delta t) \left\{ - \left\langle \frac{\partial f}{\partial x_k} \frac{\partial S}{\partial x_k} \right\rangle_\eta + \frac{1}{2} \left\langle \underbrace{\frac{\partial^2 f}{\partial x_k \partial x_l}}_{\text{depends on } \eta(0), \dots, \eta(t-\Delta t)} \underbrace{\langle \tilde{\eta}_k(t) \tilde{\eta}_l(t) \rangle_\eta}_{= 2\delta_{kl}} \right\rangle_\eta \right\} + O((\Delta t)^2) \quad \text{Integration by part w.r.t. real x only} \\ &= (\Delta t) \left\{ \int \left(- \frac{\partial f}{\partial x_k} \frac{\partial S}{\partial x_k} + \frac{\partial^2 f}{\partial x_k^2} \right) P(x; t) dx \right\} + O((\Delta t)^2) = (\Delta t) \left\{ \int f(x) \left\{ \frac{\partial}{\partial x_k} \left(\frac{\partial S}{\partial x_k} + \frac{\partial}{\partial x_k} \right) P(x; t) \right\} dx \right\} + O((\Delta t)^2) \end{aligned}$$

3. Complex Langevin Method

Extension to complex actions

The action $S(x)$ is complex for real x .

$x(t)$ is complexified as $x \Rightarrow z = x + iy$

($S(z)$ is holomorphic by analytic continuation)

$$\dot{z}_k^{(\eta)}(t) = - \underbrace{\frac{\partial S}{\partial z_k(t)}}_{\text{drift term}} + \eta_k(t)$$

• η_μ : **real** white noise obeying $\exp\left(-\frac{1}{4} \int \eta_k^2(t) dt\right)$

Probability distribution $P(x, y; t) = \left\langle \prod_k \delta(x_k - x_k^{(\eta)}(t)) \delta(y_k - y_k^{(\eta)}(t)) \right\rangle_\eta$

$$\langle \dots \rangle_\eta = \frac{\int \mathcal{D}\eta \dots \exp\left(-\frac{1}{4} \int \eta_k^2(t) dt\right)}{\int \mathcal{D}\eta \exp\left(-\frac{1}{4} \int \eta_k^2(t) dt\right)} \quad \langle \eta_k(t_1) \eta_l(t_2) \rangle_\eta = 2\delta_{kl} \delta(t_1 - t_2)$$

3. Complex Langevin Method

$P(x,y;t)$ satisfies $\frac{\partial P}{\partial t} = L^\top P$

When the boundary term vanishes,

$$\int (Lf(x,y))g(x,y)dxdy = \int f(x,y)(L^\top g(x,y))dxdy$$

$$L^\top = \frac{\partial}{\partial x_k} \left\{ \text{Re} \left(\frac{\partial S}{\partial z_k} \right) + \frac{\partial}{\partial x_k} \right\} + \frac{\partial}{\partial y_k} \left\{ \text{Im} \left(\frac{\partial S}{\partial z_k} \right) \right\}$$

$$L = \left\{ -\text{Re} \left(\frac{\partial S}{\partial z_k} \right) + \frac{\partial}{\partial x_k} \right\} \frac{\partial}{\partial x_k} + \left\{ -\text{Im} \left(\frac{\partial S}{\partial z_k} \right) \right\} \frac{\partial}{\partial y_k}$$

To justify the CLM, does the following actually hold?

$$\int \underbrace{\mathcal{O}(x+iy)}_{\text{holomorphic}} P(x,y;t)dxdy \stackrel{?}{=} \int \mathcal{O}(x)\rho(x;t)dx$$

$=L_0^\top$

$$\frac{\partial \rho(x;t)}{\partial t} = \frac{\partial}{\partial x_k} \left(\frac{\partial S}{\partial x_k} + \frac{\partial}{\partial x_k} \right) \rho(x;t) \Rightarrow \rho_{\text{time-indep.}}(x) \propto e^{-S}$$

3. Complex Langevin Method

At $t=0$, we choose $P(x, y; t = 0) = \rho(x; t = 0) \delta(y)$

Time evolution at $t>0$: we define an observable $O(z;t)$

$$\frac{\partial}{\partial t} \mathcal{O}(z;t) = \underbrace{\left(\frac{\partial}{\partial z_k} - \frac{\partial S}{\partial z_k} \right)}_{=\tilde{L}} \frac{\partial}{\partial z_k} \mathcal{O}(z;t) \quad [\text{initial condition } \mathcal{O}(z; t = 0) = \mathcal{O}(z)]$$

Setting $y=0$,

$$\frac{\partial}{\partial t} \mathcal{O}(x;t) = \underbrace{\left(\frac{\partial}{\partial x_k} - \frac{\partial S}{\partial x_k} \right)}_{=L_0} \frac{\partial}{\partial x_k} \mathcal{O}(x;t) \quad [\mathcal{O}(x; t = 0) = \mathcal{O}(x)]$$

$$\int (L_0 f(x)) g(x) dx = \int f(x) (L_0^\top g(x)) dx$$

$S(z)$ is holomorphic $\Rightarrow O(z;t)$ remains holomorphic.

$$Lf(z) = \left\{ -\text{Re} \left(\frac{\partial S}{\partial z_k} \right) + \underbrace{\frac{\partial}{\partial x_k}}_{=\partial/\partial z_k} \right\} \underbrace{\frac{\partial f(z)}{\partial x_k}}_{=\partial f(z)/\partial z_k} + \left\{ -\text{Im} \left(\frac{\partial S}{\partial z_k} \right) \right\} \underbrace{\frac{\partial f(z)}{\partial y_k}}_{=i\partial f(z)/\partial z_k}$$

$f(z)$'s $\text{\textcircled{=}}$ holomorphy $\left\{ - \left(\frac{\partial S}{\partial z_k} \right) + \frac{\partial}{\partial z_k} \right\} \frac{\partial f(z)}{\partial z_k} = \tilde{L}f(z)$

3. Complex Langevin Method

Interpolating function $F(t, \tau) = \int dx dy \mathcal{O}(x + iy; \tau) P(x, y; t - \tau)$

$$\frac{\partial F(t, \tau)}{\partial \tau} = \int dx dy \left\{ \frac{\partial \mathcal{O}(x + iy; \tau)}{\partial \tau} P(x, y; t - \tau) + \mathcal{O}(x + iy; \tau) \frac{\partial P(x, y; t - \tau)}{\partial \tau} \right\}$$

$$= \int dx dy (\tilde{L} \mathcal{O}(x + iy; \tau)) P(x, y; t - \tau) - \int dx dy \mathcal{O}(x + iy; \tau) L^\top P(x, y; t - \tau)$$

integration

by part $\left(\nabla\right)$ $\int dx dy \overbrace{\{(\tilde{L} - L) \mathcal{O}(x + iy; \tau)\}}^{=0} P(x, y; t - \tau) = 0$

Similarly, $\frac{\partial}{\partial \tau} \underbrace{\int dx \mathcal{O}(x; \tau) \rho(x; t - \tau)}_{:=G(t, \tau)} \left(\nabla\right) 0$ Integration by part w.r.t. real x only.

$$\underbrace{\int \overbrace{\mathcal{O}(x + iy; t = 0)}^{=\mathcal{O}(z)} P(x, y; t) dx dy}_{=F(t, 0)} \left(\nabla\right) \underbrace{\int \overbrace{\mathcal{O}(x + iy; t)}^{=e^{i\tilde{L}} \mathcal{O}(x + iy)} \overbrace{P(x, y; t = 0)}^{=\rho(x; t=0) \delta(y)} dx dy}_{=F(t, t)} = \underbrace{\int \mathcal{O}(x; t) \rho(x; t = 0) dx}_{=G(t, t)} \left(\nabla\right) \underbrace{\int \overbrace{\mathcal{O}(x; t = 0)}^{=\mathcal{O}(x)} \rho(x; t) dx}_{=G(t, 0)}$$

1. Is the integration by part w.r.t. (x,y) justified?

2. Is $e^{t\tilde{L}} \mathcal{O}(z)$ well-defined at large t? $\frac{\partial \mathcal{O}(z; t)}{\partial t} = \tilde{L} \mathcal{O}(z; t) \Rightarrow \mathcal{O}(z; t) = e^{t\tilde{L}} \mathcal{O}(z)$

1. Integration by part is justified when $P(x,y;t)$ damps rapidly
 - in the imaginary direction
 - around the singularity of the drift term

[G. Aarts, F.A. James, E. Seiler and O. Stamatescu, arXiv:1101.3270,
K. Nagata, J. Nishimura and S. Shimasaki, arXiv:1508.02377]

2.
$$\int dx dy \{ e^{\tau \tilde{L}} \mathcal{O}(z) \} P(x, y; t) = \sum_{n=0}^{+\infty} \frac{\tau^n}{n!} \int dx dy \{ \tilde{L}^n \mathcal{O}(z) \} P(x, y; t)$$

This series should have a finite convergence radius
 \Rightarrow Probability of the **drift term** should **fall exponentially**.

[K. Nagata, J. Nishimura and S. Shimasaki, arXiv:1606.07627]

Look at the **drift term** \Rightarrow **Get the drift of CLM!!**

Complex Langevin equation for the IKKT model:

$$\frac{d(A_\mu)_{ij}}{dt} = -\frac{\partial S}{\partial (A_\mu)_{ji}} + \eta_{\mu,ij}(t)$$
$$\frac{\partial S}{\partial (A_\mu)_{ji}} = \frac{\partial S_b}{\partial (A_\mu)_{ji}} - c_d \text{Tr} \left(\frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1} \right) \quad c_d = \begin{cases} 1 & (D = 6 \rightarrow \det \mathcal{M}) \\ \frac{1}{2} & (D = 10 \rightarrow \text{Pf} \mathcal{M}) \end{cases}$$

- A_μ : **Hermitian** \rightarrow **general complex** traceless matrices.
- η_μ : Hermitian white noise obeying $\exp \left(-\frac{1}{4} \int \text{tr} \eta^2(t) dt \right)$

CLM does not work when it encounters these problems:

(1) Excursion problem: A_μ is too far from Hermitian
 \Rightarrow **Gauge Cooling** minimizes the **Hermitian norm**

$$\mathcal{N} = \frac{-1}{DN} \sum_{\mu=1}^D \text{tr}[(A_\mu - (A_\mu)^\dagger)^2] \quad [\text{K. Nagata, J. Nishimura and S. Shimasaki, arXiv:1604.07717}]$$

A_μ : **Hermitian** \rightarrow **general complex** traceless matrices.
 \Rightarrow We make use of this **extra symmetry**:

After each step of discretized Langevin equation,

$$A_\mu \rightarrow g A_\mu g^{-1}, \quad g = e^{\alpha H}, \quad H = \frac{-1}{N} \sum_{\mu=1}^D [A_\mu, A_\mu^\dagger]$$

α : real parameter, such that \mathcal{N} is minimized.

(2) Singular drift problem:

The drift term $dS/d(A_\mu)_{ji}$ diverges due to \mathcal{M} 's **near-zero** eigenvalues.

We trust CLM when the distribution $p(u)$ of the **drift norm**

$$u = \sqrt{\frac{1}{DN^3} \sum_{\mu=1}^D \sum_{i,j=1}^N \left| \frac{\partial S}{\partial (A_\mu)_{ji}} \right|^2} \quad \text{falls exponentially as } p(u) \propto e^{-au}.$$

[K. Nagata, J. Nishimura and S. Shimasaki, arXiv:1606.07627]

Mass deformation [Y. Ito and J. Nishimura, arXiv:1609.04501]

• SO(D) symmetry breaking term $\Delta S_b = \frac{1}{2} N \varepsilon \sum_{\mu=1}^D m_{\mu} \text{tr}(A_{\mu})^2$

Order parameters for SSB of SO(D): $\lambda_{\mu} = \text{Re} \left\{ \frac{1}{N} \text{tr}(A_{\mu})^2 \right\}$

• Fermionic mass term:

$$\Delta S_f = N m_f \text{tr} (\bar{\Psi}_{\alpha} \gamma_{\alpha\beta} \Psi_{\beta}), \quad \gamma = \begin{cases} \Gamma_6 & (D = 6) \\ i\Gamma_8 \Gamma_9^{\dagger} \Gamma_{10} & (D = 10) \end{cases}$$

Avoids the singular eigenvalue distribution of \mathcal{M} .

This breaks SO(6) \rightarrow SO(5) (SO(10) \rightarrow SO(7))

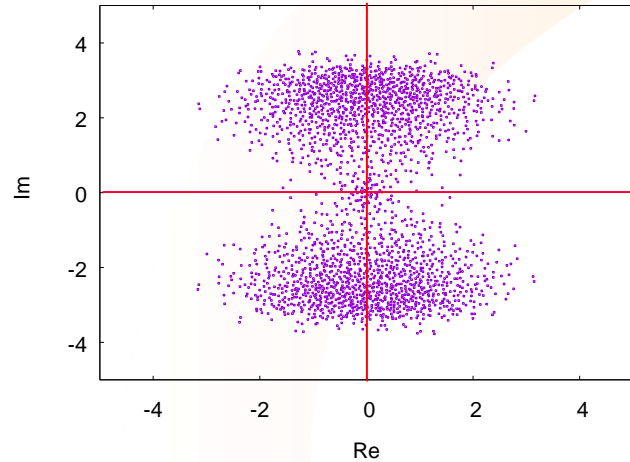
We study the SSB of the remaining symmetry.

Extrapolation (i) $N \rightarrow \infty \Rightarrow$ (ii) $\varepsilon \rightarrow 0 \Rightarrow$ (iii) $m_f \rightarrow 0$.

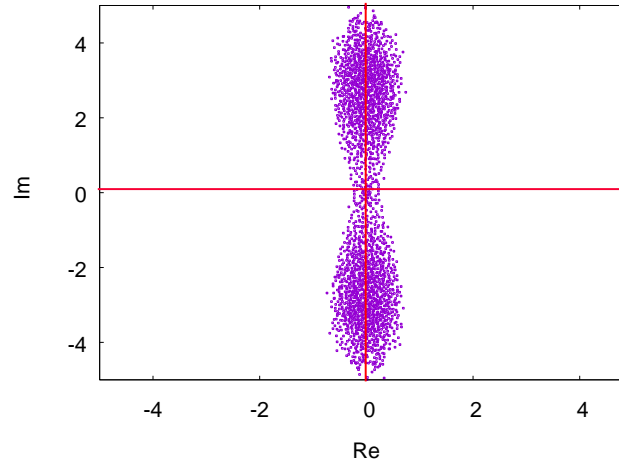
4. Result for D=6

The effect of adding these mass terms

$(\epsilon, m_f) = (0.00, 0.00)$



$(\epsilon, m_f) = (0.25, 0.00)$

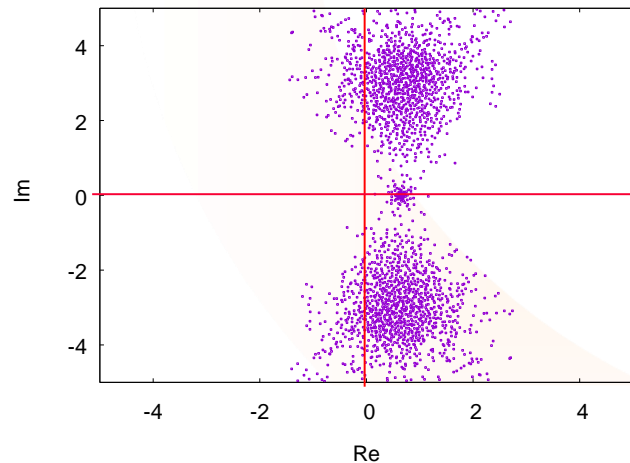


Scattering plots of the eigenvalues of the $4(N^2-1) \times 4(N^2-1)$ matrix \mathcal{M} for $D=6$, $N=24$.

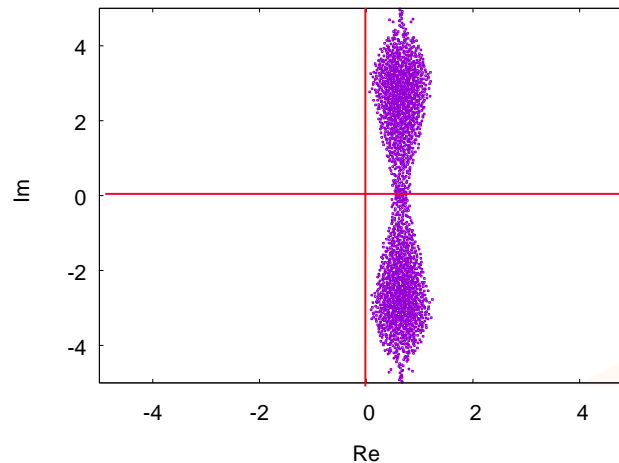
ΔS_b narrows the eigenvalue distribution.

ΔS_f shifts the eigenvalues, to evade the origin.

$(\epsilon, m_f) = (0.00, 0.65)$



$(\epsilon, m_f) = (0.25, 0.65)$



4. Result for D=6

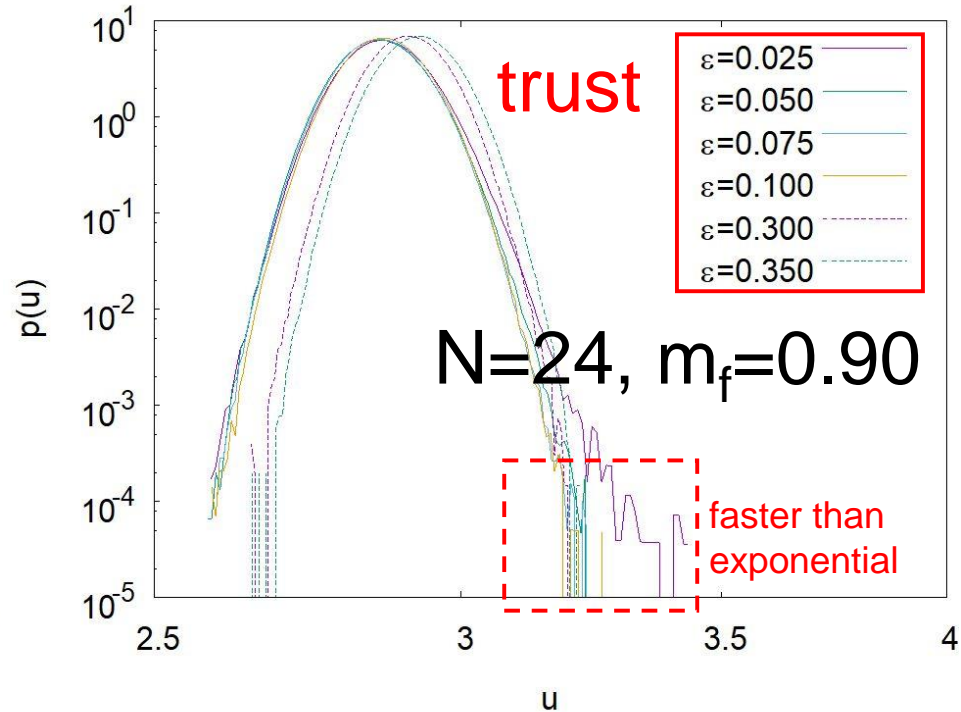
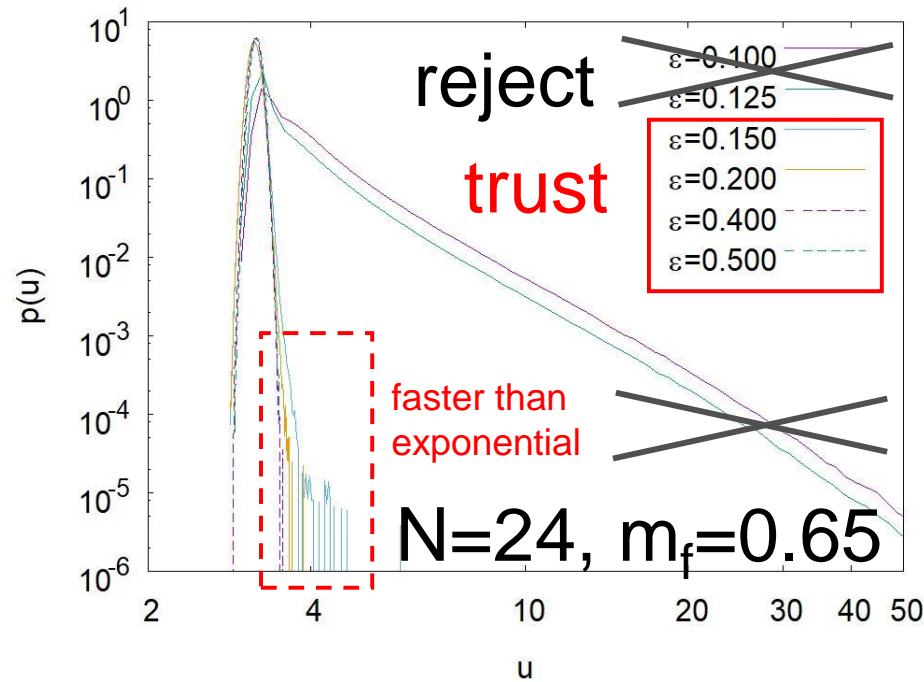
$$\Delta S_b = \frac{1}{2} N \varepsilon \sum_{\mu=1}^D m_{\mu} \text{tr}(A_{\mu})^2$$

$$\Delta S_f = N m_f \text{tr}(\bar{\Psi}_{\alpha} (\Gamma_D)_{\alpha\beta} \Psi_{\beta}) \quad (D = 6)$$

$$m_{\mu} = (0.5, 0.5, 1, 2, 4, 8)$$

$$u = \sqrt{\frac{1}{DN^3} \sum_{\mu=1}^D \sum_{i,j=1}^N \left| \frac{\partial S}{\partial (A_{\mu})_{ji}} \right|^2}$$

's distribution $p(u)$ (log-log)



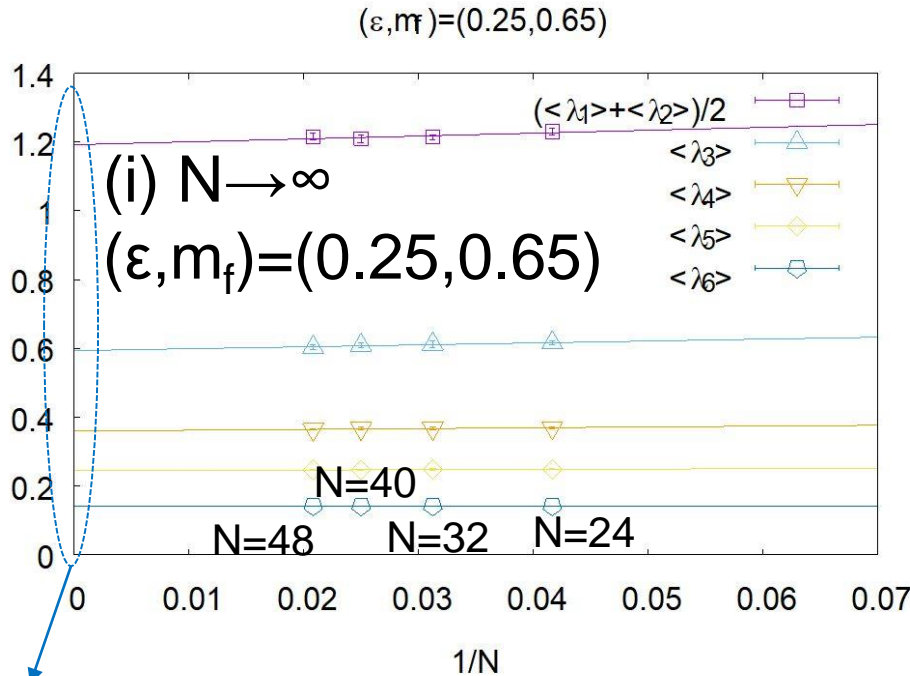
4. Result for D=6

$$\Delta S_b = \frac{1}{2} N \epsilon \sum_{\mu=1}^D m_{\mu} \text{tr}(A_{\mu})^2$$

$$\Delta S_f = N m_f \text{tr}(\bar{\psi}_{\alpha} (\Gamma_D)_{\alpha\beta} \psi_{\beta}) \quad (D = 6)$$

$$m_{\mu} = (0.5, 0.5, 1, 2, 4, 8)$$

(i) $N \rightarrow \infty$ limit for **fixed** (ϵ, m_f)



$(\epsilon, m_f) \rightarrow (0, 0)$ extrapolation
for **finite N**
 \Rightarrow We cannot observe
SSB of $SO(D)$.

$\langle \lambda_{\mu} \rangle_{\epsilon, m_f}$ at large N

4. Result for D=6

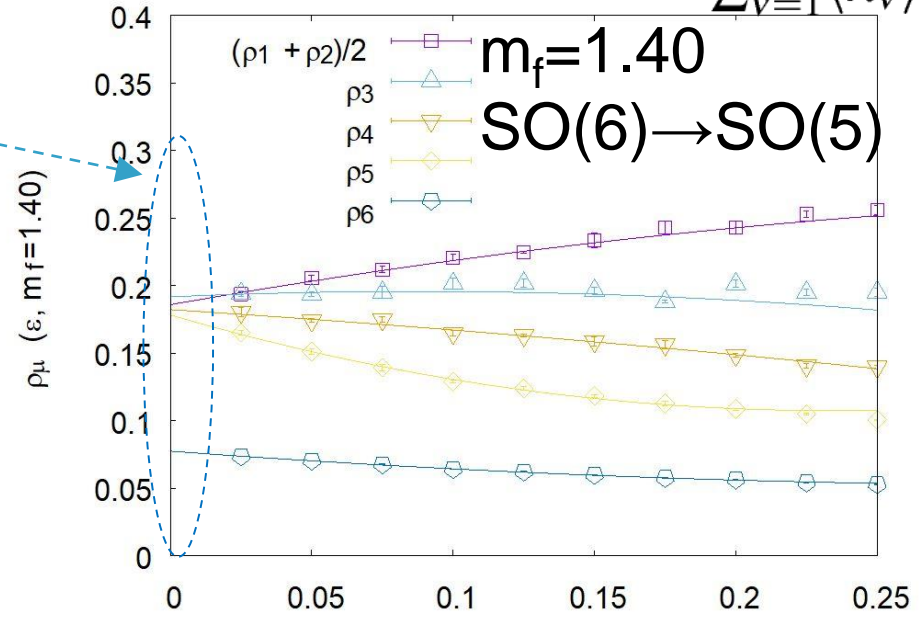
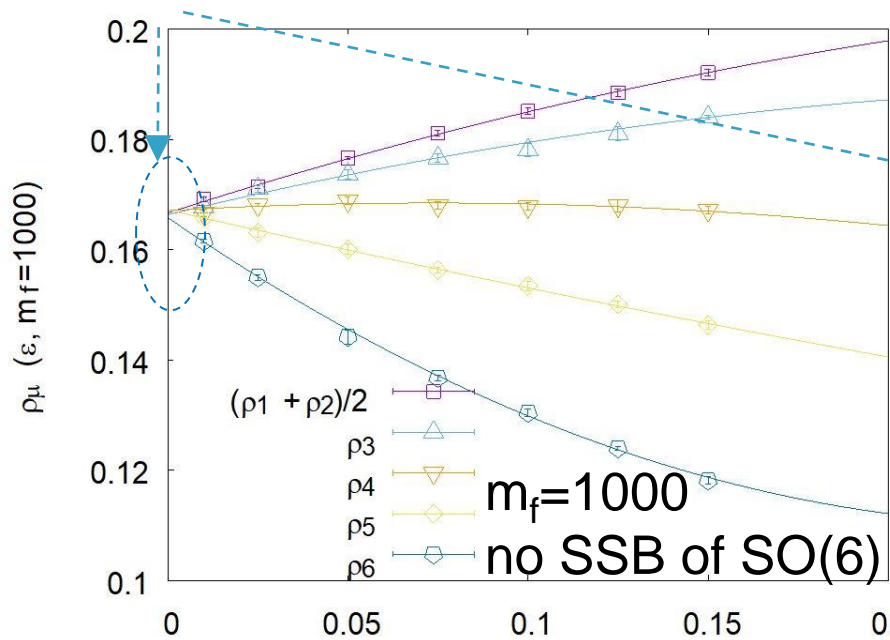
$$\Delta S_b = \frac{1}{2} N \varepsilon \sum_{\mu=1}^D m_\mu \text{tr}(A_\mu)^2$$

$$\Delta S_f = N m_f \text{tr}(\bar{\Psi}_\alpha (\Gamma_D)_{\alpha\beta} \Psi_\beta) \quad (D = 6)$$

$$m_\mu = (0.5, 0.5, 1, 2, 4, 8)$$

$$\rho_\mu(\varepsilon, m_f) = \frac{\langle \lambda_\mu \rangle_{\varepsilon, m_f}}{\sum_{\nu=1}^D \langle \lambda_\nu \rangle_{\varepsilon, m_f}}$$

(ii) $\varepsilon \rightarrow 0$ after $N \rightarrow \infty$



- $m_f \rightarrow \infty$: Ψ decouples from A_μ and reduces to the bosonic IKKT.
- The bosonic IKKT S_b does not break $SO(D)$.

[T. Hotta, J. Nishimura and A. Tsuchiya, hep-th/9811220]

- The SSB of $SO(D)$ is not an artifact of $\varepsilon \rightarrow 0$ but **a physical effect**.

4. Result for D=6

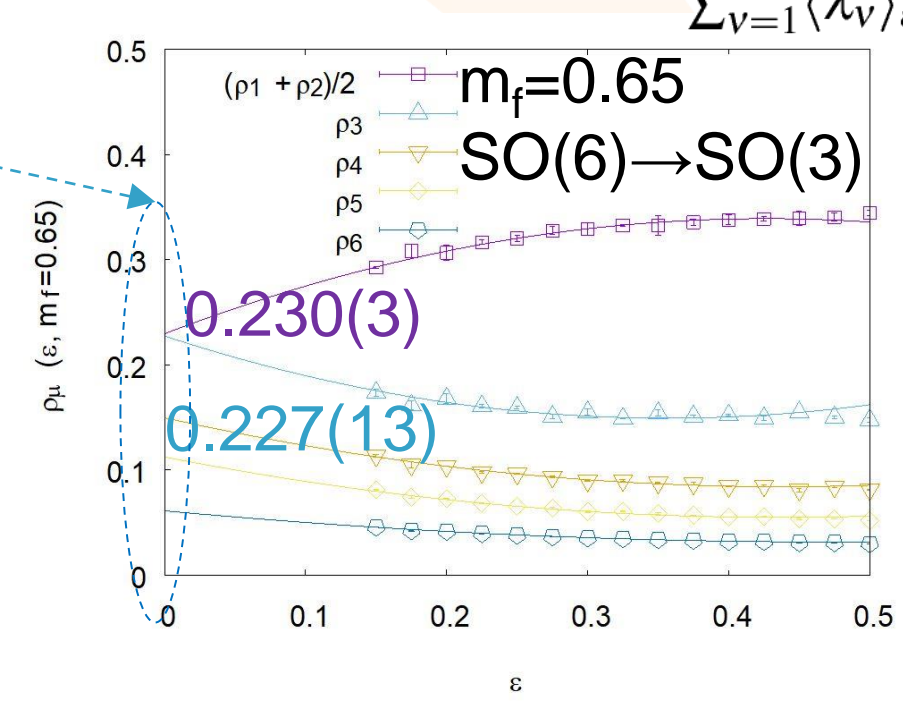
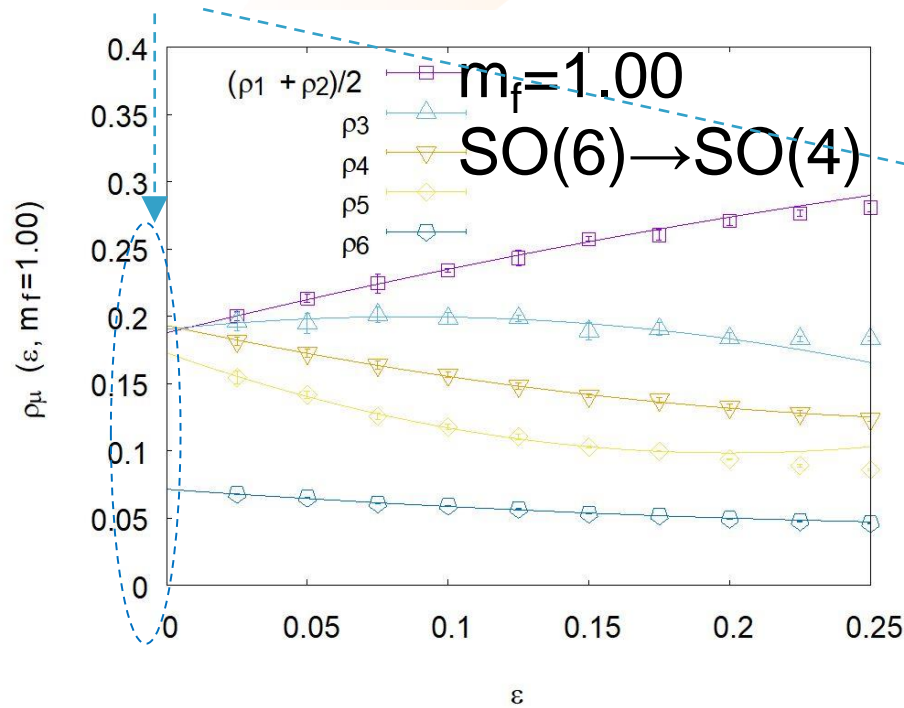
$$\Delta S_b = \frac{1}{2} N \varepsilon \sum_{\mu=1}^D m_{\mu} \text{tr}(A_{\mu})^2$$

$$\Delta S_f = N m_f \text{tr}(\bar{\psi}_{\alpha} (\Gamma_D)_{\alpha\beta} \psi_{\beta}) \quad (D = 6)$$

$$m_{\mu} = (0.5, 0.5, 1, 2, 4, 8)$$

$$\rho_{\mu}(\varepsilon, m_f) = \frac{\langle \lambda_{\mu} \rangle_{\varepsilon, m_f}}{\sum_{\nu=1}^D \langle \lambda_{\nu} \rangle_{\varepsilon, m_f}}$$

(ii) $\varepsilon \rightarrow 0$ after $N \rightarrow \infty$



4. Result for D=6

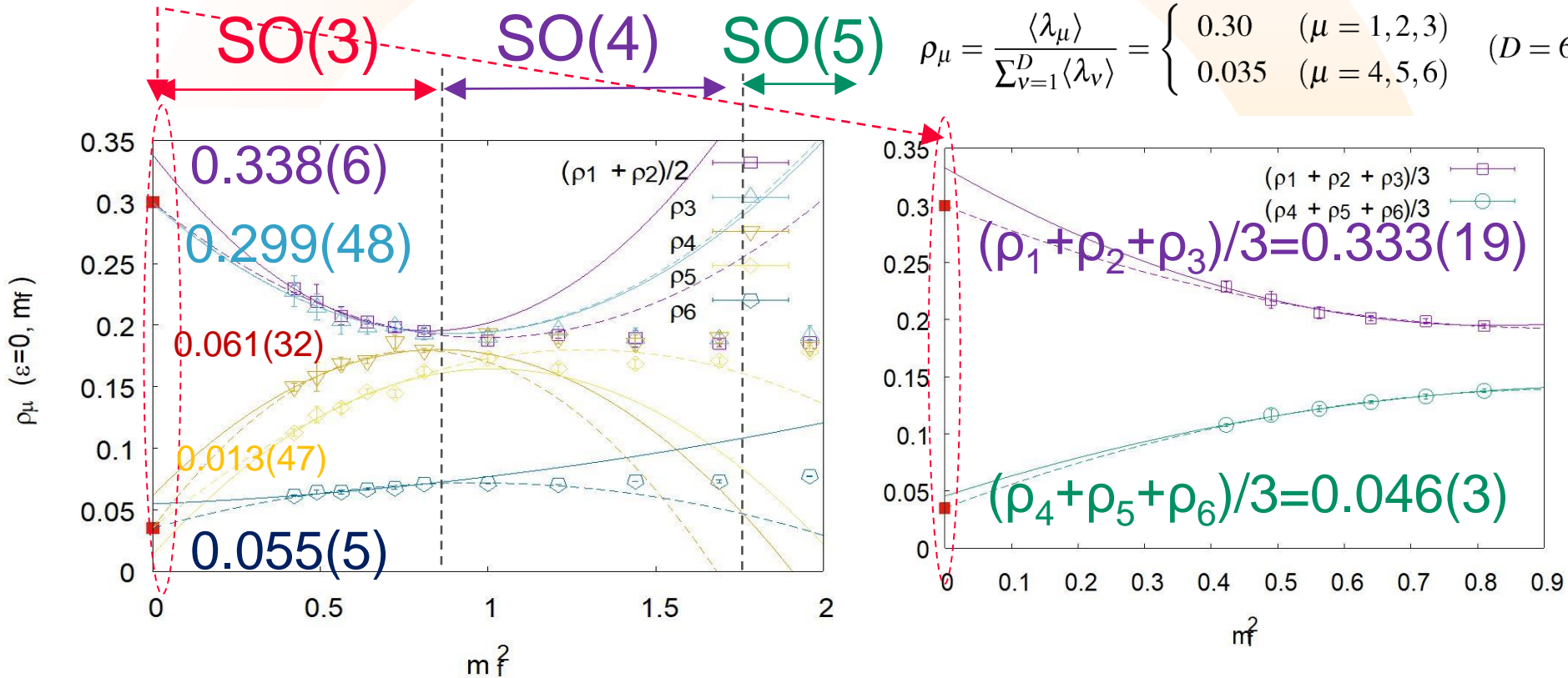
$$\Delta S_b = \frac{1}{2} N \varepsilon \sum_{\mu=1}^D m_\mu \text{tr}(A_\mu)^2$$

$$\Delta S_f = N m_f \text{tr}(\bar{\Psi}_\alpha (\Gamma_D)_{\alpha\beta} \Psi_\beta) \quad (D = 6)$$

$$m_\mu = (0.5, 0.5, 1, 2, 4, 8)$$

(iii) $m_f \rightarrow 0$ after $\varepsilon \rightarrow 0$

$$\rho_\mu = \frac{\langle \lambda_\mu \rangle}{\sum_{\nu=1}^D \langle \lambda_\nu \rangle} = \begin{cases} 0.30 & (\mu = 1, 2, 3) \\ 0.035 & (\mu = 4, 5, 6) \end{cases} \quad (D = 6)$$



(dotted line: $m_f \rightarrow 0$ limit fixed to GEM results)

SSB $SO(6) \rightarrow$ at most $SO(3)$

Consistent with GEM.

5. Result for D=10

$$m_\mu = (0.5, 0.5, 1, 2, 4, 8, 8, 8, 8, 8)$$

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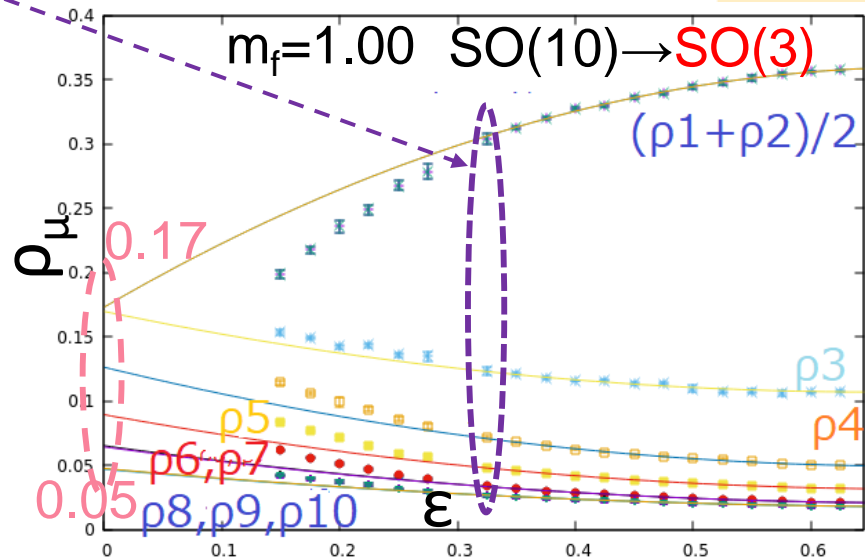
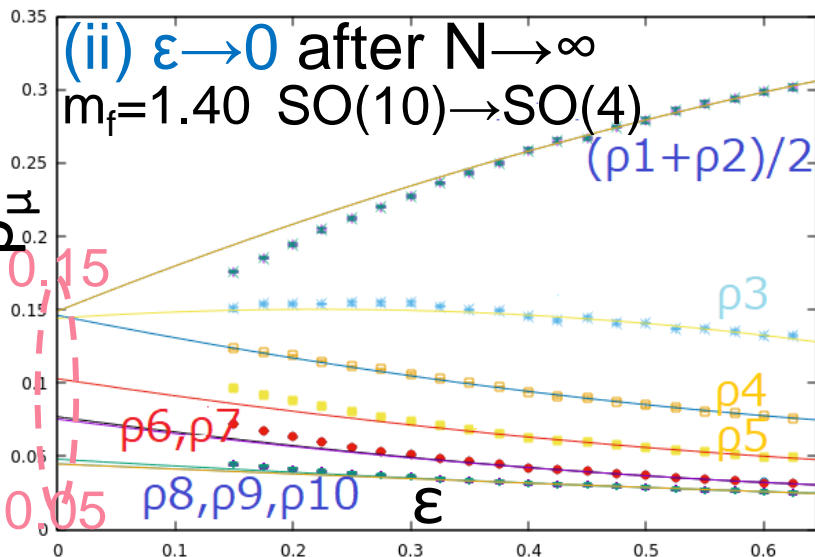
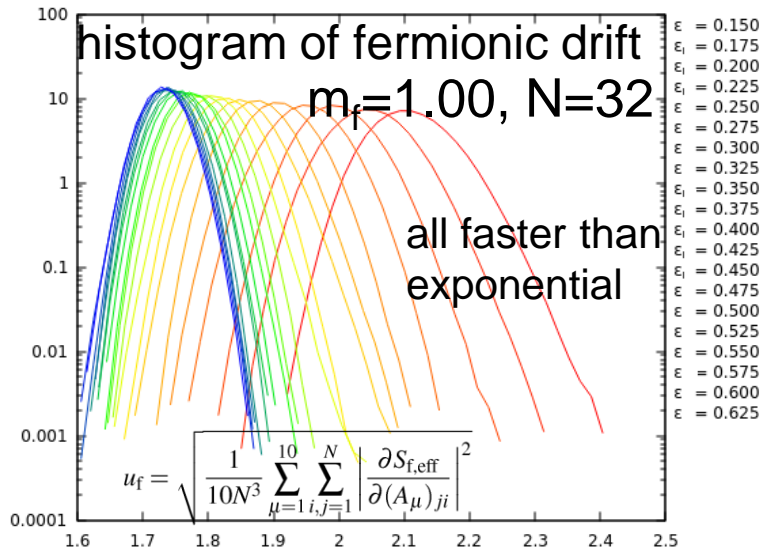
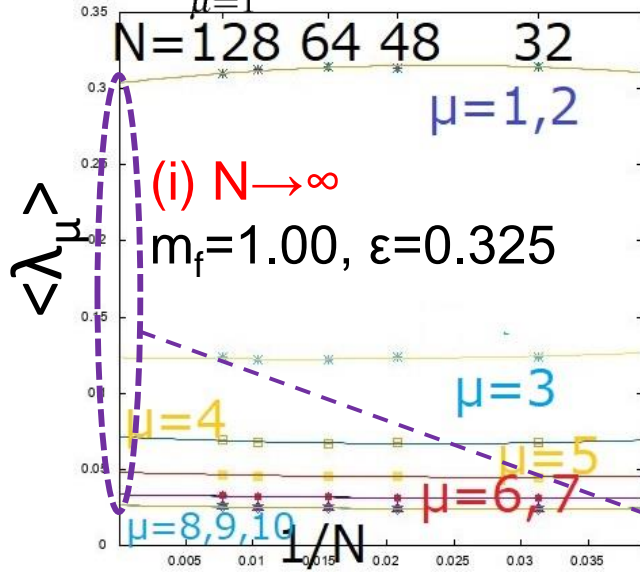


SETSUDAI

$$\Delta S_b = N \frac{\epsilon}{2} \sum_{\mu=1}^{10} m_\mu \text{tr}(A_\mu)^2$$

$$\Delta S_f = N m_f \text{tr} \left(\bar{\psi}_\alpha (i \Gamma_8 \Gamma_9^\dagger \Gamma_{10}) \alpha \beta \psi_\beta \right)$$

$$\rho_\mu(\epsilon, m_f) = \frac{\langle \lambda_\mu \rangle_{\epsilon, m_f}}{\sum_{\nu=1}^{10} \langle \lambda_\nu \rangle_{\epsilon, m_f}}$$



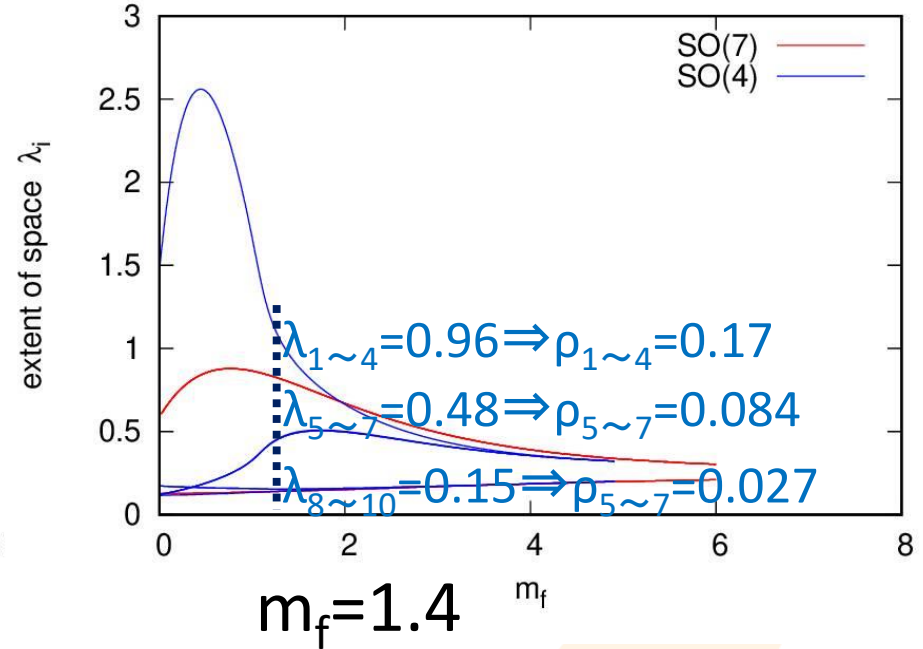
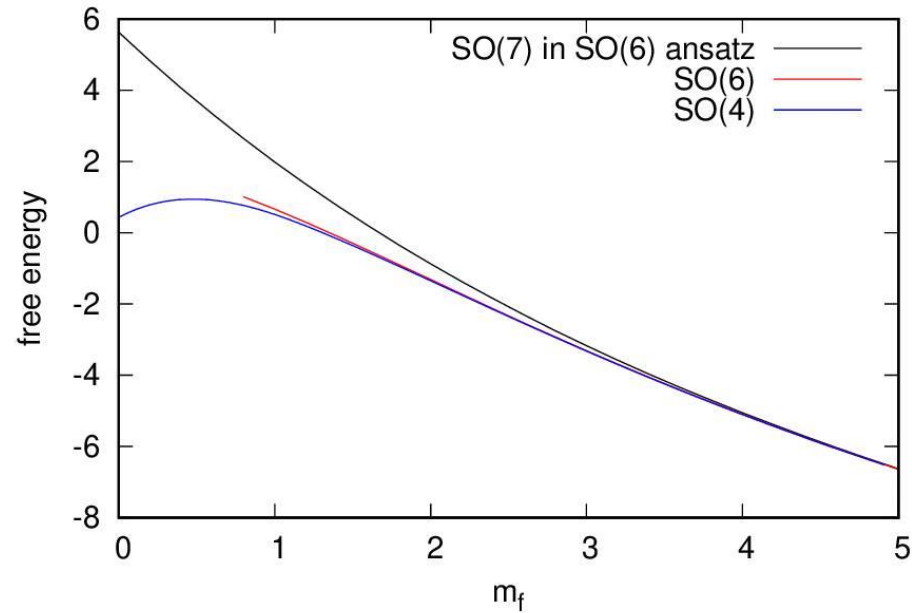
4. Result

$$m_\mu = (0.5, 0.5, 1, 2, 4, 8, 8, 8, 8, 8)$$

$$\Delta S_f = Nm_f \text{tr} \left(\bar{\psi}_\alpha (i\Gamma_8 \Gamma_9^\dagger \Gamma_{10})_{\alpha\beta} \psi_\beta \right) \quad (\text{no } \Delta S_b = N \frac{\epsilon}{2} \sum_{\mu=1}^{10} m_\mu \text{tr}(A_\mu)^2 \text{ term})$$

GEM result for $m_f > 0$ at 3 loop solutions of SO(4) and SO(6) ansatz

$$\rho_\mu(\epsilon, m_f) = \frac{\langle \lambda_\mu \rangle_{\epsilon, m_f}}{\sum_{\nu=1}^{10} \langle \lambda_\nu \rangle_{\epsilon, m_f}}$$



Dynamical compactification of the spacetime in the Euclidean type IIB matrix model.

"Complex Langevin Method" \Rightarrow trend of $SO(D) \rightarrow SO(3)$.

Future works

Application of CLM to other cases

Lorentzian version of the type IIB matrix model

generalization to Gross-Witten-Wadia model

$$S_g = N(\text{atr}U + \text{btr}U^\dagger) \quad [\text{P. Basu, K. Jaswin and A. Joseph arXiv:1802.10381}]$$

BFSS model $S = S_b + S_f$ ($D=5,9 \Rightarrow \det/\text{Pf } M$ is complex)

Example: Gaussian action $S(x) = \frac{\beta}{2}(x - i)^2 = \underbrace{\frac{\beta}{2}(x^2 - 1)}_{=\text{Re}S(x)} + i \underbrace{(-\beta x)}_{=\text{Im}S(x)}$ [Suri Kagaku2023/1 p14]

large $\beta \Rightarrow$ mimics large DOF ($\beta \sim V$)

$$\langle x^2 \rangle = \frac{\langle e^{-i\text{Im}S(x)} x^2 \rangle_{\text{reweighting}}}{\langle e^{-i\text{Im}S(x)} \rangle_{\text{reweighting}}} \stackrel{\text{V.E.V. w.r.t. ReS}}{=} \frac{\int e^{-\text{Re}S(x)} e^{-i\text{Im}S(x)} x^2 dx}{\int e^{-\text{Re}S(x)} e^{-i\text{Im}S(x)} dx} \div \frac{\int e^{-\text{Re}S(x)} dx}{\int e^{-\text{Re}S(x)} dx}$$

$$= \left(\underbrace{\int_{-\infty}^{+\infty} x^2 e^{i\beta x} e^{-\frac{\beta}{2}(x^2-1)} dx}_{=(\beta^{-1}-1)\sqrt{\frac{2\pi}{\beta}}} \div \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{\beta}{2}(x^2-1)} dx}_{=e^{\beta/2}\sqrt{\frac{2\pi}{\beta}}} \right) \div \left(\underbrace{\int_{-\infty}^{+\infty} e^{i\beta x} e^{-\frac{\beta}{2}(x^2-1)} dx}_{=\sqrt{\frac{2\pi}{\beta}}} \div \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{\beta}{2}(x^2-1)} dx}_{=e^{\beta/2}\sqrt{\frac{2\pi}{\beta}}} \right)$$

highly oscillatory at large β

$$= \frac{(\beta^{-1} - 1)e^{-\beta/2}}{e^{-\beta/2}} \stackrel{\text{numeric}}{\approx} \frac{(\beta^{-1} - 1)e^{-\beta/2} \pm O(1/\sqrt{N_{\text{config.}}})}{e^{-\beta/2} \pm O(1/\sqrt{N_{\text{config.}}})}$$

(Standard deviation of $\bar{X} = \frac{1}{N_{\text{config.}}} \sum_{k=1}^{N_{\text{config.}}} X_k$) $\propto O\left(\frac{1}{\sqrt{N_{\text{config.}}}}\right)$

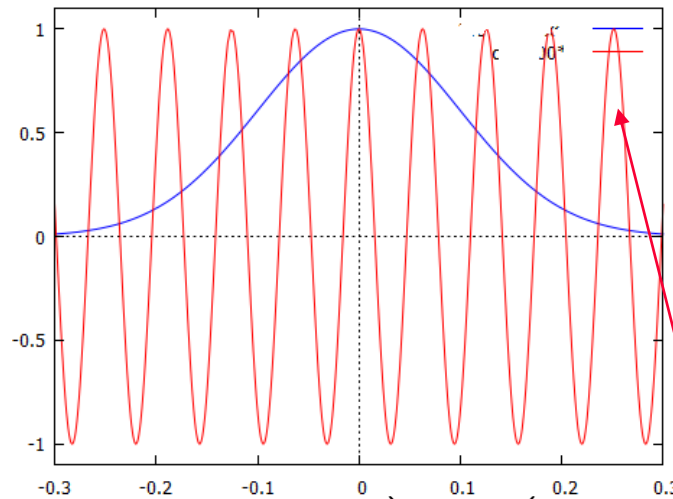
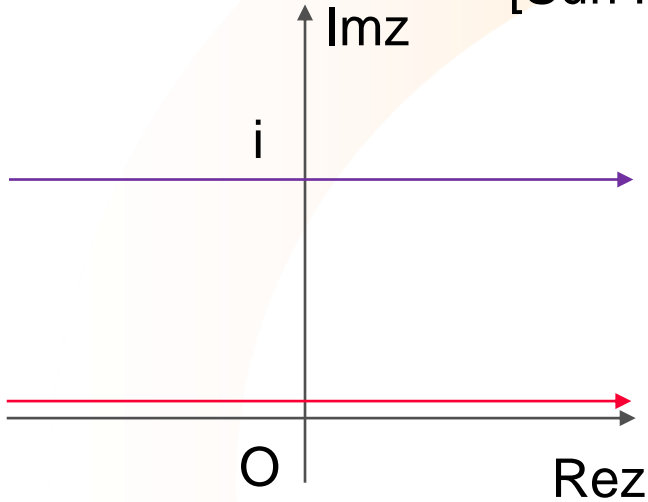
Necessary config.: $N_{\text{config.}} \geq e^{O(\beta)}$

Example:

Gaussian action

$$S(x) = \frac{\beta}{2}(x - i)^2 = \frac{\beta}{2}(x^2 - 1) + i \underbrace{(-\beta x)}_{= \text{Im}S(x)}$$

[Suri Kagaku2023/1 p14]



highly oscillatory

Contour \rightarrow : $\langle x^2 \rangle = \left(\int_{-\infty}^{+\infty} x^2 e^{i\beta x} e^{-\frac{\beta}{2}(x^2-1)} dx \right) \div \left(\int_{-\infty}^{+\infty} e^{i\beta x} e^{-\frac{\beta}{2}(x^2-1)} dx \right)$

Contour \rightarrow : $x \rightarrow z = u + i$ ($-\infty < u < +\infty$) No oscillation

$$\langle x^2 \rangle = \left(\int_{-\infty}^{+\infty} (u + i)^2 e^{-\frac{\beta}{2}u^2} du \right) \div \left(\int_{-\infty}^{+\infty} e^{-\frac{\beta}{2}u^2} du \right)$$

Cauchy's theorem \Rightarrow both are equivalent

Example [G. Aarts, arXiv:1512.05145]

$S(x) = \frac{1}{2} \underbrace{(a+ib)}_{=\sigma} x^2$, ($a, b \in \mathbf{R}$, $a > 0$) $S(x)$ is complex for **real** x .
Complexify to **$z=x+iy$** .

$$S(z) = \frac{1}{2} \sigma z^2 = \frac{1}{2} (a+ib) \overbrace{(x+iy)^2}^{=z^2} = \frac{a(x^2 - y^2)}{2} + ibxy, \quad \frac{\partial S}{\partial z} = \sigma z = (a+ib)(x+iy)$$

Complex Langevin equation for this action

$$\dot{x}(t) = -\text{Re} \left(\frac{\partial S}{\partial z} \right) + \eta(t) = (-ax + by) + \eta(t) \quad \dot{y}(t) = -\text{Im} \left(\frac{\partial S}{\partial z} \right) = (-ay - bx)$$

The **real** white noise satisfies

$$\langle \eta(t_1) \eta(t_2) \rangle = 2\delta(t_1 - t_2) \quad \langle \dots \rangle = \frac{\int \mathcal{D}\eta \dots \exp \left(-\frac{1}{4} \int \eta^2(t) dt \right)}{\int \mathcal{D}\eta \exp \left(-\frac{1}{4} \int \eta^2(t) dt \right)}$$

Solution of the Langevin equation

$$x(t) = e^{-at} \underbrace{[x(0) \cos bt + y(0) \sin bt]}_{=A(t)} + \int_0^t \eta(s) e^{-a(t-s)} \cos[b(t-s)] ds$$

$$y(t) = e^{-at} [y(0) \cos bt - x(0) \sin bt] - \int_0^t \eta(s) e^{-a(t-s)} \sin[b(t-s)] ds$$

$$\begin{aligned} \langle x^2 \rangle &= \lim_{t \rightarrow +\infty} \langle x^2(t) \rangle = \lim_{t \rightarrow +\infty} \left\{ \underbrace{e^{-2at} A(t)^2}_{\rightarrow 0} + 2e^{-at} A(t) \int_0^t \underbrace{\langle \eta(s) \rangle}_{=0} e^{-a(t-s)} \cos[b(t-s)] ds \right. \\ &\quad \left. + \int_0^t \int_0^t \underbrace{\langle \eta(s) \eta(s') \rangle}_{=2\delta(s-s')} e^{-a(2t-s-s')} \cos[b(t-s)] \cos[b(t-s')] ds ds' \right\} \\ &= \lim_{t \rightarrow +\infty} \left\{ 2 \int_0^t e^{-2a(t-s)} \cos^2[b(t-s)] \right\} ds = \frac{2a^2 + b^2}{2a(a^2 + b^2)} \end{aligned}$$

Similarly, $\langle y^2 \rangle = \frac{b^2}{2a(a^2 + b^2)}$, $\langle xy \rangle = \frac{-b}{2(a^2 + b^2)}$

This replicates $\langle z^2 \rangle = \langle x^2 \rangle - \langle y^2 \rangle + 2i\langle xy \rangle = \frac{a - ib}{a^2 + b^2} = \frac{1}{\sigma}$

Fokker-Planck equation

$$\frac{\partial P}{\partial t} = L^\top P \quad \text{where} \quad L^\top = \frac{\partial}{\partial x} \left\{ \underbrace{\operatorname{Re} \left(\frac{\partial S}{\partial z} \right)}_{=ax-by} + \frac{\partial}{\partial x} \right\} + \frac{\partial}{\partial y} \left\{ \underbrace{\operatorname{Im} \left(\frac{\partial S}{\partial z} \right)}_{=ay+bx} \right\}$$

Ansatz for its static solution:

$$P(x, y) = N \exp(-\alpha x^2 - \beta y^2 - 2\gamma xy) = N \exp \left(-\beta \left(y + \frac{\gamma x}{\beta} \right)^2 - \underbrace{\left(\alpha - \frac{\gamma^2}{\beta} \right)}_{=a(a^2+b^2)/(2a^2+b^2)} x^2 \right)$$

$$0 = \partial_t P = L^\top P = \left[\underbrace{(2a - 2\alpha)}_{=0 \rightarrow a=\alpha} + x^2 \underbrace{(4\alpha^2 - 2a\alpha - 2b\gamma)}_{=0 \rightarrow \gamma=a^2/b} + y^2 \underbrace{(4\gamma^2 + 2b\gamma - 2a\beta)}_{=0 \rightarrow \beta=a(1+2a^2/b^2)} + xy \underbrace{(4(2\alpha - a)\gamma + 2b(\alpha - \beta))}_{=0} \right] P$$

Using $\frac{\int_{-\infty}^{+\infty} t^2 e^{-At^2} dt}{\int_{-\infty}^{+\infty} e^{-At^2} dt} = \frac{1}{2A}$ ($A > 0$) we have

$$\langle x^2 \rangle = \frac{\iint x^2 P(x, y) dx dy}{\iint P(x, y) dx dy} = \frac{1}{2} \cdot \frac{a(a^2 + b^2)}{2a^2 + b^2} = \frac{2a^2 + b^2}{2a(a^2 + b^2)}$$

Noisy estimator: method to calculate $\text{Tr } A$
using Gaussian random numbers (A : $n \times n$ matrix)

X_k, Y_k independently obey the standard normal
distribution $N(0,1)$.

$$\chi_k = \frac{X_k + iY_k}{\sqrt{2}} \Rightarrow \langle \chi_j^* \chi_k \rangle = \delta_{jk} \quad (j, k = 1, 2, \dots, n)$$

$$\sum_{j,k=1}^n \langle \chi_j^* A_{jk} \chi_k \rangle = \sum_{j,k=1}^n A_{jk} \langle \chi_j^* \chi_k \rangle = \sum_{j,k=1}^n A_{jk} \delta_{jk} = \text{Tr} A$$

backup: noisy estimator

Integrate out $\Psi \Rightarrow \int d\bar{\Psi} d\Psi e^{-(S_f + \Delta S_f)} = \det \mathcal{M} \quad (D=6 \Rightarrow p=4) \quad \gamma = \begin{cases} \Gamma_6 & (D=6) \\ i\Gamma_8 \Gamma_9^\dagger \Gamma_{10} & (D=10) \end{cases}$

$\int d\Psi e^{-(S_f + \Delta S_f)} = \text{Pf} \mathcal{M} \quad (D=10 \Rightarrow p=16)$

$$S_f + \Delta S_f = N \left\{ \text{tr} \left(\bar{\Psi}_\alpha (\Gamma_\mu)_{\alpha\beta} [A_\mu, \Psi_\beta] \right) + m_f \text{tr} \left(\bar{\Psi}_\alpha (\gamma_{\alpha\beta}) \Psi_\beta \right) \right\}$$

Tracelessness of $\Psi \Rightarrow \mathcal{M}$ is a $p(N^2-1) \times p(N^2-1)$ matrix

$$\begin{aligned} \mathcal{M}_{a_1 a_2 \alpha, b_1 b_2 \beta} &= [(a_1 + (a_2 - 1)N + (\alpha - 1)(N^2 - 1), b_1 + (b_2 - 1)N + (\beta - 1)(N^2 - 1)) \text{ element}] \\ &= \mathcal{M}'_{a_1 a_2 \alpha, b_1 b_2 \beta} - \mathcal{M}'_{NN\alpha, b_1 b_2 \beta} \delta_{a_1 a_2} - \mathcal{M}'_{a_1 a_2 \alpha, NN\beta} \delta_{b_1 b_2} + \mathcal{M}'_{NN\alpha, NN\beta} \delta_{a_1 a_2} \delta_{b_1 b_2} \end{aligned}$$

$a_1, a_2, b_1, b_2 = 1, 2, \dots, N$, except for $(a_1, a_2) = (N, N), (b_1, b_2) = (N, N) \quad \alpha, \beta = 1, 2, \dots, p$

$$\mathcal{M}'_{a_1 a_2 \alpha, b_1 b_2 \beta} = (\Gamma_\mu)_{\alpha\beta} \left\{ (A_\mu)_{a_2 b_1} \delta_{a_1 b_2} - (A_\mu)_{b_2 a_1} \delta_{a_2 b_1} \right\} + m_f \gamma_{\alpha\beta} \delta_{a_1 b_2} \delta_{a_2 b_1}$$

$a_1, a_2, b_1, b_2 = 1, 2, \dots, N$, including $(a_1, a_2) = (N, N), (b_1, b_2) = (N, N) \quad \alpha, \beta = 1, 2, \dots, p$

\mathcal{M} in the scattering plots
(without altering $\det/\text{Pf} \mathcal{M}$ up to a constant)

$$\mathcal{M}'_{a_1 a_2 \alpha, b_1 b_2 \beta} = (\Gamma_\mu \gamma^{-1})_{\alpha\beta} \left\{ (A_\mu)_{a_2 b_2} \delta_{a_1 b_1} - (A_\mu)_{b_1 a_1} \delta_{a_2 b_2} \right\} + m_f \delta_{\alpha\beta} \delta_{a_1 b_1} \delta_{a_2 b_2}$$

unit matrix

\mathcal{M} is a $p(N^2-1) \times p(N^2-1)$ matrix ($p=4$ for $D=6$ and $p=16$ for $D=10$)

⇒ Naively calculating $\text{Tr} \left(\frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1} \right)$ takes CPU cost $O(N^6)$.

Instead, we use the **noisy estimator**

χ =(random number vector)

$$\text{Tr} \left(\frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1} \right) = \left\langle \chi^* \frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \underbrace{\mathcal{M}^{-1} \chi}_{=\zeta} \right\rangle$$

$\mathcal{M}^\dagger \mathcal{M} \zeta = \mathcal{M}^\dagger \chi$ by **conjugate gradient (CG)** method.

- $\mathcal{M}^\dagger \mathcal{M}$ is symmetric and positive definite.
- \mathcal{M} is sparse ⇒ CPU cost $O(N^3)$ per CG iteration.
- In solving Langevin eq., we use one noisy estimator $\chi^* \frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1} \chi$ instead of the average $\left\langle \chi^* \frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1} \chi \right\rangle$

Conjugate Gradient (CG) method:

Iterative algorithm to solve the linear equation $Ax=b$

(A : symmetric, positive-definite $n \times n$ matrix)

Initial config. $\mathbf{x}_0 = \mathbf{0}$ $\mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0$ $\mathbf{p}_0 = \mathbf{r}_0$

(for brevity, no preconditioning on \mathbf{x}_0 here)

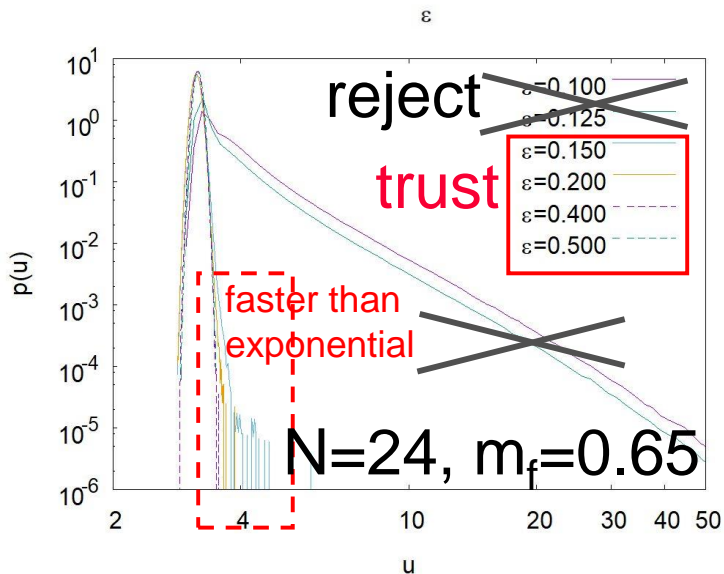
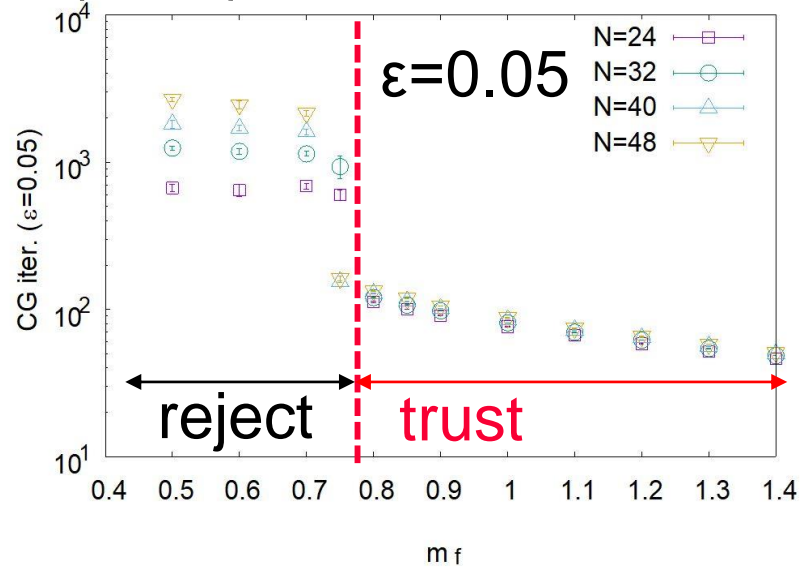
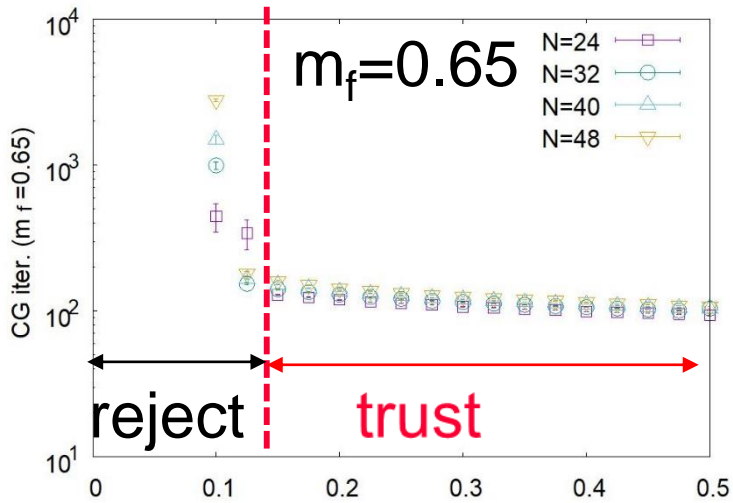
$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k \quad \mathbf{r}_{k+1} = \mathbf{r}_k - \alpha_k A \mathbf{p}_k \quad \alpha_k = \frac{(\mathbf{r}_k, \mathbf{r}_k)}{(\mathbf{p}_k, A \mathbf{p}_k)}$$

$$\mathbf{p}_{k+1} = \mathbf{r}_{k+1} + \frac{(\mathbf{r}_{k+1}, \mathbf{r}_{k+1})}{(\mathbf{r}_k, \mathbf{r}_k)} \mathbf{p}_k$$

Iterate this until $\sqrt{\frac{(\mathbf{r}_{k+1}, \mathbf{r}_{k+1})}{(\mathbf{r}_0, \mathbf{r}_0)}} < (\text{tolerance}) \simeq 10^{-4}$

The approximate answer of $Ax=b$ is $\mathbf{x} = \mathbf{x}_{k+1}$.

Required CG iteration time (D=6)



When we can trust CLM, there is small dependence of CG iter. on N .

In total, the CPU cost for

$$\text{Tr} \left(\frac{\partial \mathcal{M}}{\partial (A_\mu)_{ji}} \mathcal{M}^{-1} \right) \text{ is } O(N^3).$$