Destabilization of two fuzzy spheres at a distance

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1. Introduction

Large-N reduced models $\Rightarrow$ Powerful candidates for the constructive definition of superstring theory.

IIB (IKKT) matrix model:
N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115

$$S = \frac{1}{g^2}\mathrm{tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right).$$

- Defined in ten-dimensional Euclidean spacetime: $\mu, \nu = 1, 2, \cdots, 10$.
- A_μ, ψ: $N \times N$ hermitian matrices.
- Eigenvalues of $A_\mu \Rightarrow$ spacetime coordinates.
- $N \to \infty$: type IIB superstring theory.

Motivations of the fuzzy sphere studies

- Relation between the noncommutative field theory and the superstring study.
- Novel regularization scheme alternative to the lattice regularization.
- Prototype of the curved-space background in the large-N reduced model.

2. 3d bosonic Yang-Mills-Chern-Simons model

Toy model incorporating the fuzzy sphere background:
(defined in three-dimensional Euclidean spacetime: $\mu, \nu, \rho = 1, 2, 3$)

$$S = N\mathrm{tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).$$

Classical equation of motion:

$$[A_\nu, [A_\mu, A_\nu]] - i\alpha\epsilon_{\mu\nu\rho}[A_\nu, A_\rho] = 0.$$

Typical classical solution:

$$X = \begin{pmatrix} X_\mu^{(1)} & & & \\ & X_\mu^{(2)} & & \\ & & \ddots & \\ & & & X_\mu^{(l)} \end{pmatrix}, \text{ where}$$

$$[X_\mu^{(m)}, X_\nu^{(m)}] = i\epsilon_{\mu\nu\rho}(X_\rho^{(m)} - R_\rho^{(m)}),$$
$$[X_\mu, R_\nu] = [R_\mu, R_\nu] = 0.$$

$X_\mu^{(m)} = (n_m \times n_m$ irrep.), $\sum_{m=1}^{l} n_m = N$.

Two-fuzzy-sphere-solution

Solution representing two fuzzy spheres:

$$X_\mu = \begin{pmatrix} X_\mu^{(1)} & \\ & X_\mu^{(2)} \end{pmatrix}, \quad R_\mu = \begin{pmatrix} R_\mu^{(1)} & \\ & R_\mu^{(2)} \end{pmatrix},$$

$$R_\mu^{(I)} = r_\mu^{(I)} 1_{n_I \times n_I}, \quad X_\mu^{(I)} = \alpha J_\mu^{(n_i)} + R_\mu^{(I)},$$
$$[J_\mu^{(n_I)}, J_\nu^{(n_I)}] = i\epsilon_{\mu\nu\rho}J_\rho^{(n_I)}.$$

There two spheres are centered at $r_\mu^{(I)} = \frac{1}{n_I}\mathrm{tr}\, R_\mu^{(n_I)}$ $(I = 1, 2)$.

One-loop effective action for interaction of two spheres

S. Bal and H. Takata, hep-th/0108002

Perturbation around X_μ as $A_\mu = X_\mu + \tilde{A}_\mu$:

Gauge fixing term and ghost term:

$$S_{\mathrm{gf}} = -\frac{N}{2}\mathrm{tr}\,[X_\mu, A_\mu]^2, \quad S_{\mathrm{gh}} = -N\mathrm{tr}\,[X_\mu, \bar{C}][X_\mu, C].$$

We assume the following form of fluctuations:

$$\tilde{A}_\mu = \begin{pmatrix} a_\mu^{(1)} & b_\mu \\ b_\mu^\dagger & a_\mu^{(2)} \end{pmatrix}, \quad \tilde{\bar{C}} = \begin{pmatrix} \bar{c}_\mu^{(1)} & d_\mu \\ d_\mu^\dagger & \bar{c}_\mu^{(2)} \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} c_\mu^{(1)} & e_\mu \\ e_\mu^\dagger & c_\mu^{(2)} \end{pmatrix}.$$

$r_\mu = \alpha c_\mu =$ (distance of two spheres),

$$r_\mu^{(1)} = \frac{n_2}{n_1 + n_2}r_\mu, \quad r_\mu^{(2)} = -\frac{n_1}{n_1 + n_2}r_\mu.$$

Self-interaction of each sphere:

$$S_{2,\mathrm{B}}^{(\mathrm{self})} = N\sum_{I=1,2}\mathrm{tr}\left\{-\frac{1}{2}[a_\mu^{(I)}, X_\nu^{(I)}]^2 + 2i\epsilon_{\mu\nu\lambda}R_\lambda^{(I)}a_\mu^{(I)}a_\nu^{(I)}\right\},$$

$$S_{2,\mathrm{G}}^{(\mathrm{self})} = \alpha^2 N\left\{\sum_{I=1,2}\mathrm{tr}\,[L_\mu^{(I)}, \bar{c}^{(I)}][L_\nu^{(I)}, c^{(I)}]\right\}.$$

Interaction between two spheres $(k_I, l_I = 1, 2, \cdots, n_I)$:

$$S_{2,\mathrm{B}}^{(1)(2)} = \alpha^2 N(b_\mu^\dagger)_{k_2 k_1}\left[(H^2)\delta_{\mu\nu} - 2i\epsilon_{\mu\nu\lambda}c_\lambda \otimes 1\right]_{k_1 l_1 k_2 l_2}(b_\nu)_{l_1 l_2},$$

$$S_{2,\mathrm{G}}^{(1)(2)} = \alpha^2 N\{(d^\dagger)_{k_2 k_1}(H^2)_{k_1 l_1 k_2 l_2}(e)_{l_1 l_2} - (e^\dagger)_{k_2 k_1}(H^2)_{k_1 l_1 k_2 l_2}(d)_{l_1 l_2}\}.$$

where

$$(H_\mu)_{k_1 l_1 k_2 l_2} = (L_\mu^{(1)})_{k_1 l_1} \otimes 1_{k_2 l_2} - 1_{k_1 l_1} \otimes (L_\mu^{(2)})_{k_2 l_2}{}^{*}.$$

3. Stability of two fuzzy spheres

We set $c_\mu = (0, 0, c)$ without loss of generality.

Effective action for interaction of two spheres:

$$W_{\mathrm{eff}}^{(1)(2)} = \log\left[\det\left\{\alpha^2 N\frac{(H^2 + 2c)(H^2 - 2c)}{H^2}\right\}\right] = \log\left[\prod_{j=j_{\min}}^{j_{\max}} \alpha^2 N w_j\right],$$

where $W_{\mathrm{eff}}^{(1)(2)} = W_{\mathrm{B}}^{(1)(2)} + W_{\mathrm{G}}^{(1)(2)}$,

$$W_{\mathrm{B}}^{(1)(2)} = -\log\int db\, db^\dagger \exp\left(-b_\mu^\dagger\left[\alpha^2 N\{H^2\delta\mu\nu - 2i\epsilon_{\mu\nu\lambda}c_\lambda\}\right]b\nu\right) = -\log\left[\det\{\alpha^2 N\left(H^2 - 2i\epsilon\cdot c\right)\}\right]^{-1},$$

$$W_{\mathrm{G}}^{(1)(2)} = -\log\int dd\, dd^\dagger\, de\, de^\dagger \exp\left(-d^\dagger(\alpha^2 N H^2)e + e^\dagger(\alpha^2 N H^2)d\right) = -\log\left[\det \alpha^2 N H^2\right]^2,$$

$$j_{\min} = \frac{|n_1 - n_2|}{2}, \quad j_{\max} = \frac{n_1 + n_2}{2} - 1.$$

$w_j =$ (determinant of j-th block):

$$w_j = \frac{(c^2 - 2c(j+1) + j(j+1))}{(c^2 + 2cj + j(j+1))} \times \prod_{m=-j}^{j}(c^2 + 2c(-m+1) + j(j+1)).$$

The potential for cocentric two spheres ($c = 0$) reduces to

$$W_{\rm eff}^{(1)(2)} = \sum_{j=j_{\rm min}}^{j_{\rm max}} (2j+1) \log \left[\alpha^2 N j(j+1)\right].$$

Zero modes of effective action

We recall that the determinant is given by

$$\det(H^2 + 2ac)_j = \prod_{m=-j}^{j} [h(j,m)]. \text{ where}$$

$$h(j,m) = j(j+1) + c^2 + 2c(m+a).$$

For $a = -1$ ($\det(H^2 - 2c)$), $h(j,m)$ is negative for

$$(j+1) - \sqrt{j+1} < c < (j+1) + \sqrt{j+1} \text{ for } m = -j.$$

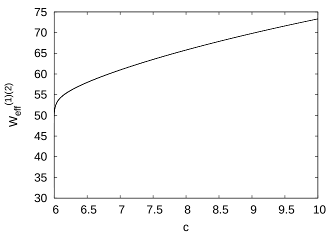
The fuzzy sphere is unstable if any one of $h(j,m)$ is negative.

Fuzzy spheres of the same size

Example: $n_1 = n_2 = 4$ case

regions	Number of negative w_j
$0.000 < c < 0.586$	1
$0.586 < c < 1.268$	2
$1.268 < c < 3.414$	3
$3.414 < c < 4.732$	2
$4.732 < c < 6.000$	1
$6.000 < c$	0

The fuzzy sphere is (meta)stable for $c > 6.0$.
Attractive force for this large c region:

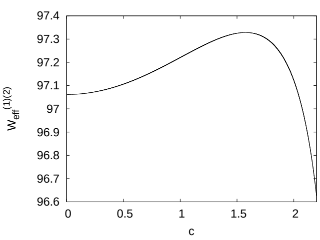


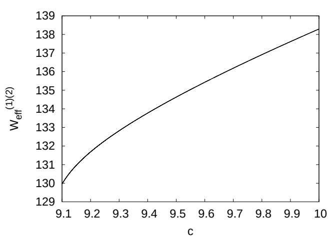
Fuzzy spheres of the different size

Example: $n_1 = 10, n_2 = 3$ case

regions	Number of negative w_j
$c < 2.379$	0
$2.379 < c < 3.155$	1
$3.155 < c < 3.950$	2
$3.950 < c < 6.621$	3
$6.621 < c < 7.845$	2
$7.845 < c < 9.050$	1
$9.050 < c$	0

The fuzzy sphere is (meta)stable for $c < 2.379, c > 9.050$.
Attractive force near $c = 0$ and at large c.





Relation to numerical studies

T. Azuma, S. Bal, K. Nagao and J. Nishimura, hep-th/0401038

Monte Carlo studies for the decay of two cocentric ($c = 0$) fuzzy spheres of different size:

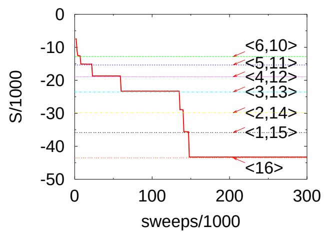
Simulation for $N = 16, \alpha = 2.0$ case, starting from $A_\mu = 0$.

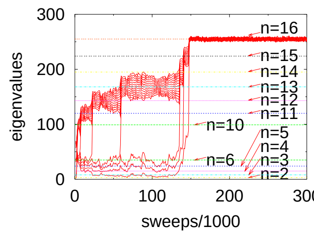
$$\underbrace{A_\mu^{(0)} = 0}_{\text{initial state}} \to \cdots$$

$$\to \underbrace{A_\mu = \alpha \begin{pmatrix} J_\mu^{(6\to5\to4\to3\to2\to1)} & 0 \\ 0 & J_\mu^{(10\to11\to12\to13\to14\to15)} \end{pmatrix}}_{\text{metastable vacuum}}$$

$$\to \underbrace{A_\mu = \alpha J_\mu}_{\text{stable vacuum}}.$$

The interaction potential near $c = 0$:



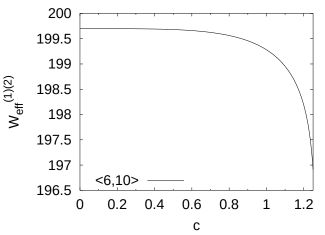


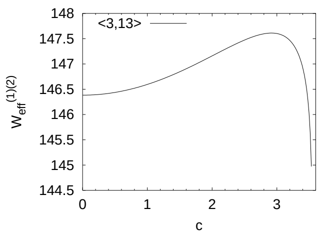
$$W_{\rm eff}^{(1)(2)} = \sum_{j=j_{\rm min}}^{j_{\rm max}} (2j+1)\log[\alpha^2 N j(j+1)] + a_{(n_1,n_2)} c^2 + O(c^4),$$

where

$$a_{(n_1,n_2)} = \frac{2}{3}\left[\frac{1}{n_1+n_2}\left(1+\frac{24}{n_1+n_2}\right) - \frac{1}{|n_1-n_2|}\left(1+\frac{24}{|n_1-n_2|}\right) + \sum_{j=j_{\rm min}}^{j_{\rm max}} \frac{1}{j}\right].$$

E.g. $a_{(6,10)} = -0.0005952\cdots$, $a_{(3,13)} = 0.2171\cdots$.
$\langle 3,13\rangle$ spheres are more stable than $\langle 6,10\rangle$ spheres, due to the coefficient $a_{(n_1,n_2)}$.





4. Supersymmetric case

Similar results holds for the three-dimensional supersymmetric case:

$$S = N{\rm tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho + \frac{1}{2}\bar\psi\sigma_\mu[A_\mu,\psi]\right).$$

Main results:

- Two fuzzy spheres separated sufficiently (or when one sphere is well inside the other) are metastable.
- There is an attractive force between these two spheres.