Monte Carlo studies of the phase transition of finite-temperature large-N gauge theory

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1. Introduction

Lattice Monte-Carlo vs. **Holography** in 4dim Yang-Mills
Both methods have several merits and demerits.
→ We need to know their validity.

An analytic approach through a $1/D$ **expansion** is available in lower dimensional gauge theories.
→ It is valuable to compare these three methods in the lower dimensional gauge theories to understand the above problem.

Today we test these three methods in the following $SU(N)$ matrix quantum mechanics (MQM):

$$Z = \int dX_i dA e^{-S_{\rm YM}}, \qquad \text{where}$$

$$S_{\rm YM} = \frac{1}{g^2}\int_0^{\frac{1}{T}} dt \left\{ \frac{1}{2}\mathrm{tr}\sum_{i=1}^{D}(D_t X_i)^2 - \frac{1}{4}\mathrm{tr}\sum_{i,j=1}^{D}[X_i, X_j]^2 \right\}$$

Features and Demerits in the three approaches

Monte-Carlo O. Aharony et. al. hep-th/0406210,0508077 , N. Kawahara, J. Nishimura and S. Takeuchi arXiv:0706.3517, 0710.2188

- Feature : Non-perturbative. Any finite N, D OK.
- Demerit: $N \to \infty$ limit is difficult. Numerical errors.
  Cut off (lattice space) dependence.

Holography (N D1 branes on a Scherk-Schwarz circle)
O. Aharony et. al. hep-th/0406210,0508077

- Feature: Non-perturbative, $N \to \infty$, $D = 9$
- Demerit: Gravity describes a strong coupling 2dim SYM. We need to extraporate the information of the MQM from the SYM.
  Cut off (KK scale) dependence.

$1/D$ expansion G.Mandal, M.Mahato and T.Morita. arXiv:0910.4526

- Feature: Non-perturbative, $N \gg 1$, $D \gg 1$
- Demerit: The $1/D$ expansion is valid in $D \gg N \gg 1$ case.
  The validity in $N \gg D > 1$ case is subtle.

→ We will see that the Monte-Carlo and $1/D$ expansion are consistent even in small D. We also find some agreements including finite N effects. However, several results from the holography disagree.

Large N phase transition in the MQM

Phase transitions happen in the MQM in the large N limit.

- Analogues of the confinement/deconfinement transition.
- Correspond to a black string/black hole transition via holography.

→ We investigate how this transition is resolved through finite N effects, which correspond to quantum gravity effects in the holography.

2. Effective action via $1/D$ expansion

By taking a 't Hooft like limit $D \to \infty$, $g \to 0$ with a fixed coupling $\tilde{\lambda} = g^2 D N$, we can derive an effective action,

$$Z = \int dX_i dA e^{-S_{\rm YM}}$$
$$= \int dA d\Delta e^{-S_{\rm eff}(A,\Delta) + O(1/D)},$$

$$S_{\rm eff}/DN^2 = -\frac{\Delta^4}{8T\tilde{\lambda}^{\frac{1}{3}}} + \frac{\Delta}{2T} + \sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{1}{D} - \exp\left(-\frac{n\Delta}{T}\right)\right)|u_n|^2$$

where Δ is an auxiliary field and u_n are Wilson loops defined by:

$$u_n = \frac{1}{N}\mathrm{tr}\, U^n = \frac{1}{N}\sum_{a=1}^{N}\exp(in\alpha_a), \qquad \text{where}$$

$$U = \mathcal{P}\exp\left(i\int_0^{\frac{1}{T}} dt A(t)\right) = \mathrm{diag}(e^{i\alpha_1}, \cdots, e^{i\alpha_N}).$$

Especially if temperature is low and u_n are small, we can integrate out Δ and obtain a Landau-Ginzburg type effective action:

$$S_{\rm LG}/DN^2 = \frac{3\tilde{\lambda}^{\frac{1}{3}}}{8T} + b_1|u_1|^4 + \sum_{n=1}^{+\infty} a_n|u_n|^2,$$

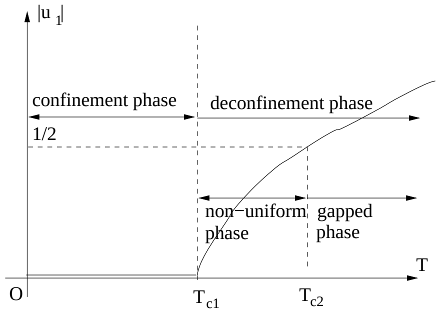
$$a_n = \frac{1}{n}\left(\frac{1}{D} - \exp\left(-\frac{n\tilde{\lambda}^{\frac{1}{3}}}{T}\right)\right), \qquad b_1 = \frac{\tilde{\lambda}^{\frac{1}{3}}}{3T}\exp\left(-\frac{2\tilde{\lambda}^{\frac{1}{3}}}{T}\right),$$

We will investigate the phase structure of this model in the next section.

3. Phase structure of the MQM from $1/D$ expansion

In large N, three phases and two critical temperatures T_{c1}, T_{c2} appear.
u_n are the order parameters of these phase transitions.

- Confinement phase ($T < T_{c1}$): $u_n = 0$ for all n.
- Deconfinement phase (non-uniform) ($T_{c1} < T < T_{c2}$):
  $u_1 = \sqrt{-a_1/2b_1} \leq 1/2$, $u_n = 0$ for $n \geq 2$.
- Deconfinement phase (gapped) ($T_{c2} < T$):
  $u_1 \geq 1/2$, $u_n \neq 0$ for $n \geq 2$.
- The transition at T_{c1} is second order and the transition at T_{c2} is Gross-Witten-Wadia type third order.



$$\frac{1}{T_{c1}} = \frac{\log D}{\tilde{\lambda}^{\frac{1}{3}}}\left(1 + \frac{0.523}{D}\right) + O(1/D^2)$$

$$\frac{1}{T_{c2}} = \frac{1}{T_{c1}} - \frac{1}{\tilde{\lambda}^{\frac{1}{3}}} \times \frac{\log D}{D}\left(\frac{1}{6} + \frac{0.137\log D + 0.293}{D}\right) + O(1/D^2)$$

(Here we have evaluated O(1/D) corrections in the effective action.)

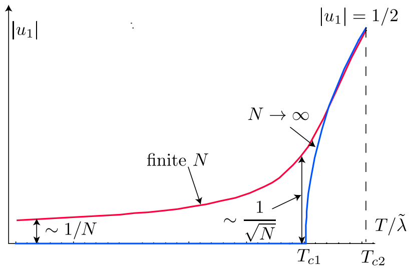
Resolution of the transitions through $1/N$ effects

We evaluate the leading finite N effects in the path-integral and observe that all u_n become non-zero :

$$\langle|u_1|\rangle \to \begin{cases} \frac{\sqrt{\pi}}{2N} & (T \to 0) \\ \frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt{N\pi}}\left(\frac{3D}{\log D}\right)^{\frac{1}{4}} & (T = T_{c1}) \end{cases}.$$

$$\langle|u_n|\rangle = \frac{1}{2N}\sqrt{\frac{\pi}{Da_n}}, \qquad (T \lesssim T_{c2}, n = 2, 3, 4, \cdots)$$

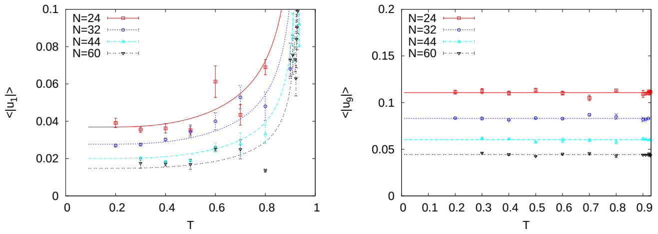
→ The order parameters are always non-zero. The transitions are resolved to crossovers.



4. $1/D$ expansion vs. Monte-Carlo

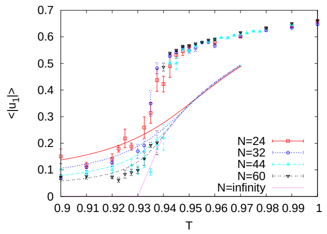
We evaluated the MQM through the Monte Carlo and obtained the following results:
(Curves in the plots are the results from the $1/D$ expansion up to T_{c2}.)

Behavior of u_n at low temperatures ($D = 6$)



The Monte-Carlo agrees with the $1/D$ expansion even in finite N.

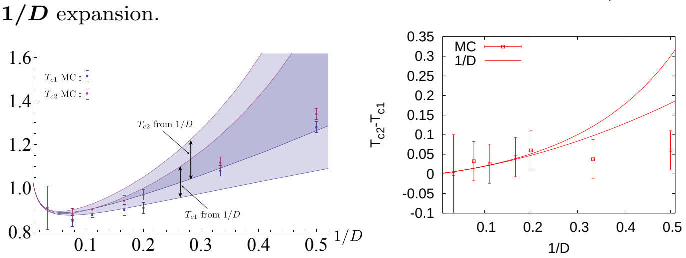
Behavior of u_1 around T_{c1} ($D = 6$)



- Numerical errors are large near T_{c1} but we can see some similarities.
- As is predicted from the $1/D$ expansion, there is no sharp phase transition at finite N.
  We need a special care to extrapolate the critical temperature at large N from the finite-N Monte Carlo data.

D dependence of the Critical Temperatures

Preliminary Monte Carlo results of critical temperature $T_{c1,c2}$ versus $1/D$ expansion.



- The nature of the phase transitions do not depend on D.
  (Always two phase transitions occur.)
- The critical temperatures are consistent. The differences between the Monte-Carlo and $1/D$ expansion are within $O(1/D^2)$ order.
  (the errorbar of the $1/D$ expansion's result is
  $T_{c1,c2}(1 \pm 1/D^2)$.)
- There is an ambiguity in the Monte Carlo results of $T_{c1,c2}$, which comes from the extrapolation from finite-N Monte Carlo results.
- $T_{c2} - T_{c1}$ for smaller D does not agree well. But the errors in the Monte-Carlo are also large and we need to investigate them further.

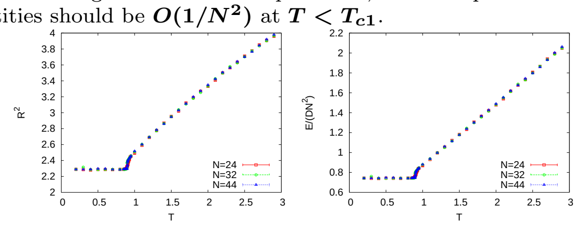
Physical quantities in the confinement phase ($T < T_{c1}$)

We evaluate the following two quantities:

$$R^2 = \frac{T}{g^2 N^2} \int_0^{\frac{1}{T}} \mathrm{tr}\, X_i^2(t) dt$$

$$\frac{E}{DN^2} = -\frac{3T}{4g^2 N^2 D} \int_0^{\frac{1}{T}} \mathrm{tr}\, [X_i(t), X_j(t)]^2 dt \quad \text{(Internal Energy)}$$

Due to the large N volume independence, the T dependence of these quantities should be $O(1/N^2)$ at $T < T_{c1}$.

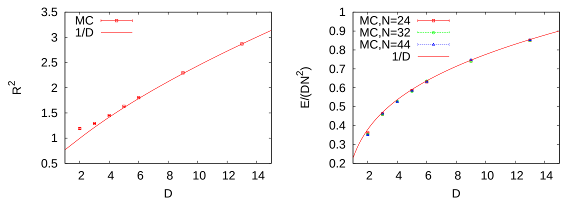


Results from the $1/D$ expansion at $T < T_{c1}$:

$$R^2 = \frac{\tilde{\lambda}^{\frac{1}{3}}}{2} \left(1 + \frac{0.2405}{D}\right) + O(1/N^2, 1/D^2)$$

$$\frac{E}{DN^2} = \tilde{\lambda}^{\frac{1}{3}} \left(\frac{3}{8} - \frac{0.1476}{D}\right) + O(1/N^2, 1/D^2)$$

These quantities also agree very well for various D ($T = 0.5, N = 44$):



5. Comments on holography

Witten proposed that we can extrapolate the p-dim non-supersymmetric gauge theory from N Dp branes wrapped on a Scherk-Schwarz circle.

E. Witten hep-th/9803131

According to his proposal, D1 brane geometries predict the MQM with $D = 9$ as follows:

- The confinement/deconfinement transition is first order.
  $\rightarrow$ Disagree with the 2nd+3rd order transitions in the Monte-Carlo and $1/D$ expansion.
- Internal energy in the confinement phase:

$$E/N^2 \propto -\lambda^{-1/2} L_{KK}^{-5/2} \quad \text{negative } (L_{KK}: \text{ cut off})$$

  $\rightarrow$ Disagree with the positive energy in the Monte-Carlo and $1/D$ expansion.

We need a special care for the application of holography to the non-supersymmetric gauge theories e.g. QCD, CMP.

6. Conclusion

- We calculated the finite N effects in the $1/D$ expansion and showed how the $1/N$ effects resolve the transitions.
- We compared the predictions from the $1/D$ expansion with Monte-Carlo simulation. We found several good agreements.
  $\rightarrow$ $1/D$ works even $D \geq 2$ and finite (but large) N.
- It seems that the $1/D$ expansion is available without the condition $D \gg N$.
- So far the Monte-Carlo does not work well near the critical points.
- Naive application of the holography for non-supersymmetric gauge theories is not always correct even qualitatively.

Further development

- Finite N effect vs. finite string coupling effect in holography.
- Improvement of the numerical calculation near the critical points.
- Numerical calculation of $S_{\mathrm{eff}}(A, \Delta)$
  $\rightarrow$ We can evaluate S_{eff} for any temperature. (partially done)
- Effects of matter fields on the confinement/deconfinement phase transition.

T. Azuma, T. Morita and S. Takeuchi, in progress