Quantum fluctuation of higher-dimensional fuzzy-sphere solution of a matrix model

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1. Constructive definition of superstring theory

A large N reduced model has been proposed as a constructive definition (nonperturbative formulation) of the superstring theory:
N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

$$S = -\frac{N}{4}\sum_{\mu,\nu=0}^{9} Tr_{N\times N}[A_\mu, A_\nu]^2 - \frac{N}{2}Tr\bar{\psi}\sum_{\mu=0}^{9}\Gamma^\mu[A_\mu,\psi].$$

- Dimensional reduction of $\mathcal{N}=1$ 10-dimensional SYM theory to 0 dimension.
  A_μ and ψ are $N\times N$ Hermitian matrices.
  - A_μ: 10-dimensional vectors
  - ψ: 10-dimensional Majorana-Weyl (i.e. 16-component) spinors
- Matrix regularization of the Schild action of the type IIB superstring theory.
- $SU(N)$ gauge symmetry and $SO(10)$ Lorentz symmetry $(SO(10)\times SU(N))$.
- $\mathcal{N}=2$ SUSY: This theory must contain spin-2 gravitons if it contains massless particles.

2. Fuzzy-sphere classical solution of the matrix model

The drawback of the IIB matrix model:
⇒ It has only a classical solution of the flat non-commutative space:

$$[A^\nu,[A_\mu,A_\nu]] = 0 \ \Rightarrow [A_\mu,A_\nu] = ic_{\mu\nu}\mathbf{1}.$$

In order to surmount this drawback, we consider the generalization of the IIB matrix model:
Y. Kimura hep-th/0204256, 0301055.

$$S = -\frac{N}{4}Tr[A_\mu,A_\nu]^2 - gN\epsilon^{\mu_1\cdots\mu_{2k+1}}TrA_{\mu_1}\cdots A_{\mu_{2k+1}}.$$

- This action is defined in the odd $(2k+1)$-dimensional Euclidean spacetime.
- $SO(2k+1)$ rotational symmetry and $SU(N)$ gauge symmetry.

The classical equation of motion

$$-[A_\nu,[A_\mu,A_\nu]] - g(2k+1)\epsilon_{\mu\nu_1\cdots\nu_{2k}}A_{\nu_1}\cdots A_{\nu_{2k}} = 0$$

incorporates the higher-dimensional fuzzy-sphere solution!

$$A_\mu = \alpha G_\mu \ (\text{with } g = \alpha^{3-2k}\frac{8k}{(2k+1)m_k}).$$

G_μ is given by the symmetric tensor product of the $(2k+1)$-dimensional gamma matrices:

$$G_\mu = (\Gamma_\mu^{(2k)}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1})\text{sym} + \cdots + (\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\Gamma_\mu^{(2k)})\text{sym}.$$

- $\Gamma_\mu^{(2k)}$ denotes the $2^k\times 2^k$ gamma matrices for the $(2k+1)$-dimensional Euclidean space.
- This symmetric tensor product is realized only for a limited size of the matrices. For the $(2k+1)$ dimensions, the size N_k is

$$N_1 = (n+1),$$
$$N_2 = \frac{(n+1)(n+2)(n+3)}{6},$$
$$N_3 = \frac{(n+1)(n+2)(n+3)^2(n+4)(n+5)}{360},$$
$$N_4 = \frac{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)}{302400}.$$

- G_μ gives the sphere's geometry in that

$$G_\mu G_\mu = n(n+2k)\mathbf{1}_{N_k\times N_k}.$$

- G_μ generally does not close with respect to the commutator. For $G_{\mu\nu}=[G_\mu,G_\nu]$, we obtain

$$G_{\mu\nu}G_{\mu\nu} = -8kn(n+2k)\mathbf{1}_{N_k\times N_k},$$
$$[G_{\mu\nu},G_\rho] = 4(-\delta_{\mu\rho}G_\nu+\delta_{\nu\rho}G_\mu),$$
$$[G_{\mu\nu},G_{\rho\sigma}] = 4(\delta_{\nu\rho}G_{\mu\sigma}+\delta_{\mu\sigma}G_{\nu\rho}-\delta_{\mu\rho}G_{\nu\sigma}-\delta_{\nu\sigma}G_{\mu\rho}).$$

- Self-dual condition:

$$\epsilon_{\mu\nu_1\cdots\nu_{2k}}G_{\nu_1}\cdots G_{\nu_{2k}} = m_kG_\mu.$$

The coefficient m_k satisfies the following recursive formula:

$$m_1 = 2i, \quad m_2 = 8(n+2), \quad m_3 = -48i(n+2)(n+4),$$
$$m_{k+1} = -2i(k+1)(n+2k)m_k.$$

However, the quantum stability of the fuzzy-sphere solution is still obscure.
⇒ We investigate the stability via the Monte-Carlo simulation.

3. Monte-Carlo simulation of matrix models

(a) Warm-up: quadratic $U(N)$ one-matrix model

We start with the simplest case − quadratic $U(N)$ one-matrix model:

$$S = \frac{N}{2}Tr\phi^2.$$

The Feynman diagram of this matrix model:

$$\langle\phi_{ij}\phi_{kl}\rangle = \frac{1}{N}\delta_{il}\delta_{jk}.$$

Then, the following quantities can be computed exactly:

$$\langle\frac{1}{N}Tr\phi^2\rangle = 1, \quad \langle\frac{1}{N}Tr\phi^4\rangle = 2+\frac{1}{N^2}, \quad \langle(\frac{1}{N}Tr\phi^2)^2\rangle = 1+\frac{2}{N^2}.$$

We analyze this model via the heat-bath algorithm. To this end, we rewrite the $U(N)$ matrix ϕ as

$$\phi_{ii} = \frac{a_i}{\sqrt{N}}, \quad \begin{cases}\phi_{ij} = \frac{x_{ij}+iy_{ij}}{\sqrt{2N}} \\ \phi_{ji} = \frac{x_{ij}-iy_{ij}}{\sqrt{2N}},\end{cases} \quad (\text{for } i<j).$$

The N^2 real quantities a_i, x_{ij}, y_{ij} comply with the independent normal Gaussian distribution.

$$S = \frac{1}{2}\sum_{i=1}^{N}a_i^2 + \frac{1}{2}\sum_{i<j}((x_{ij})^2+(y_{ij})^2).$$

(b) Quartic one-matrix model

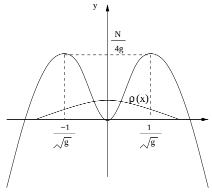
We analyze the one-matrix model via the heat-bath algorithm:

$$S = \frac{N}{2}Tr\phi^2 - \frac{gN}{4}Tr\phi^4.$$

This action is unbounded below. However, we can avoid the divergence in the large-N limit.

We introduce the auxiliary fields Q as ($\alpha=\sqrt{\frac{g}{2}}$) in order to render the action quadratic:

$$\tilde{S} = \frac{N}{2}Tr\phi^2 + \frac{N}{2}TrQ^2 - \alpha NTrQ\phi^2 = \frac{N}{2}Tr(Q-\alpha\phi^2)^2 + S.$$



We update Q as

$$Q_{ii} = \frac{a_i}{\sqrt{N}} + \alpha(\phi^2)_{ii}, \quad Q_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{2N}} + \alpha(\phi^2)_{ij},$$

where a_i, x_{ij}, y_{ij} comply with the normal Gaussian distribution.
In updating the diagonal part ϕ_{ii}, we extract the dependence of ϕ_{ii}:

$$\tilde{S} = \frac{N}{2}(\phi_{ii})^2 \underbrace{(1 - 2\alpha Q_{ii})}_{=c_i} - N\phi_{ii} \underbrace{(\alpha \sum_{j \neq i} (\phi_{ji} Q_{ij} + Q_{ji}\phi_{ij}))}_{=h_i}.$$

Then, ϕ_{ii} is updated as

$$\phi_{ii} = \frac{a_i}{\sqrt{Nc_i}} + \frac{h_i}{c_i}.$$

We likewise extract the ϕ_{ij} dependence:

$$\begin{aligned} \tilde{S} &= N \underbrace{(1 - \alpha(Q_{ii} + Q_{jj}))}_{=c_{ij}} |\phi_{ij}|^2 - N(\phi_{ij} h_{ji} + \phi_{ji} h_{ij}), \text{ where} \\ h_{ij} &= \alpha(\sum_{k \neq j} (\phi_{ik} Q_{kj} + \sum_{k \neq i} Q_{ik}\phi_{kj})). \end{aligned}$$

Then, ϕ_{ij} is updated as follows:

$$\phi_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{2Nc_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$

The legitimacy of the algorithm is ascertained by checking the following results (as $N \to \infty$):
E.Brezin, C.Itzykson, G.Parisi and J.Zuber, Comm. Math. Phys. **59**, 35 (1978).

$$\langle \frac{1}{N} Tr\phi^2 \rangle = \frac{1}{3} a^2 (4 - a^2), \text{ where } a^2 = \frac{2}{1 + \sqrt{1 - 12g}}.$$

The eigenvalue distribution is given by

$$\rho(x) = \frac{1}{2\pi}(-gx^2 - 2ga^2 + 1)\sqrt{4a^2 - x}.$$

(c) The bosonic IIB matrix model

T. Hotta, J. Nishimura and A. Tsuchiya hep-th/9811220.
We investigate the bosonic IIB matrix model via the the heat-bath algorithm:

$$S = -\frac{N}{4} Tr[A_\mu, A_\nu]^2 = -\frac{N}{2} \sum_{\mu < \nu} Tr\{A_\mu, A_\nu\}^2 + 2N \sum_{\mu < \nu} Tr(A_\mu^2 A_\nu^2).$$

This action is equivalent to $\tilde{S}$, after integrating out $Q_{\mu\nu}$ (where $G_{\mu\nu} = \{A_\mu, A_\nu\}$):

$$\begin{aligned} \tilde{S} &= \sum_{\mu < \nu} \left(\frac{N}{2} Tr Q_{\mu\nu}^2 - N Tr(Q_{\mu\nu} G_{\mu\nu}) + 2N Tr(A_\mu^2 A_\nu^2) \right) \\ &= \frac{N}{2} \sum_{\mu < \nu} Tr(Q_{\mu\nu} - G_{\mu\nu})^2 + S. \end{aligned}$$

Then, $Q_{\mu\nu}$ is updated as

$$(Q_{\mu\nu})_{ii} = \frac{a_i}{\sqrt{N}} + (G_{\mu\nu})_{ii}, \quad (Q_{\mu\nu})_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{2N}} + (G_{\mu\nu})_{ij},$$

We next update A_μ. We extract the dependence of A_λ.

$$\tilde{S} = -NTr(T_\lambda A_\lambda) + 2NTr(S_\lambda A_\lambda^2) + \cdots, \text{ where}$$
$$S_\lambda = \sum_{\mu \neq \lambda} (A_\mu^2), \quad T_\lambda = \sum_{\mu \neq \lambda} (A_\mu Q_{\lambda\mu} + Q_{\lambda\mu} A_\mu).$$

- The diagonal part A_μ is updated by extracting the dependence of $(A_\mu)_{ii}$:

$$\tilde{S} = 2N(S_\lambda)_{ii}(A_\mu)_{ii}^2 - 4Nh_i(A_\mu)_{ii}, \text{ where}$$
$$h_i = \frac{N}{4}[(T_\lambda)_{ii} - 2\sum_{j \neq i}((S_\lambda)_{ji}(A_\lambda)_{ij} + (S_\lambda)_{ij}(A_\lambda)_{ji})].$$

Then, $(A_\lambda)_{ii}$ is updated as

$$(A_\lambda)_{ii} = \frac{a_i}{\sqrt{4N(S_\lambda)_{ii}}} + \frac{h_i}{(S_\lambda)_{ii}}.$$

- The other components $(A_\mu)_{ij}$ are updated likewise by extracting their dependence:

$$\tilde{S} = 2Nc_{ij}|(A_\lambda)_{ij}|^2 - 2Nh_{ji}(A_\lambda)_{ij}, \text{ where}$$
$$c_{ij} = (S_\lambda)_{ii} + (S_\lambda)_{jj},$$
$$h_{ij} = \frac{1}{2}(T_\lambda)_{ij} - \sum_{k \neq i}(S_\lambda)_{ik}(A_\lambda)_{kj} - \sum_{k \neq j}(S_\lambda)_{kj}(A_\lambda)_{ik}.$$

Then, $(A_\mu)_{ij}$ are updated as

$$(A_\lambda)_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{4Nh_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$

The following Schwinger-Dyson equation serves as the consistency check of the algorithm.

$$-\langle \frac{1}{N} Tr[A_\mu, A_\nu]^2 \rangle = D(1 - \frac{1}{N^2}).$$

(d) Extension to the bosonic IIB matrix model with the Chern-Simons term

The Chern-Simons term is *linear* with respect to *each* A_μ.
We have only to replace T_λ as

$$T_\lambda^{CS} = T_\lambda + g(2k+1)\epsilon_{\lambda\nu_1\cdots\nu_{2k}} A_{\nu_1} \cdots A_{\nu_{2k}}.$$

The Schwinger-Dyson equation is replaced as

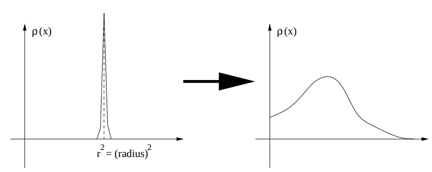
$$\begin{aligned} & -\langle \frac{1}{N} Tr[A_\mu, A_\nu]^2 \rangle - \langle \frac{g(2k+1)}{N} Tr\epsilon^{\mu_1 \cdots \mu_{2k+1}} A_{\mu_1} \cdots A_{\mu_{2k+1}} \rangle \\ = \; & D(1 - \frac{1}{N^2}). \end{aligned}$$

4. Stability of the fuzzy sphere

In order to see the stability of the fuzzy-sphere solution, we focus on the eigenvalues of the Casimir

$$C = X_1^2 + X_2^2 + \cdots + X_{2k+1}^2.$$

- We start by setting A_μ to be the fuzzy-sphere classical solution.
- We watch the behavior of the eigenvalue distribution, as we iterate the Monte-Carlo updating.



Our analysis is now under the way. We are faced with the setbacks in analyzing our case.

The difficulty comes from the unboundedness of the IIB matrix model with the Chern-Simons terms. This situation would be analogous to that for the one-matrix model.

In order to understand the behavior of the IIB matrix model with the Chern-Simons term, we should scrutinize the one-matrix model thoroughly.