Quantum fluctuation of higher-dimensional fuzzy-sphere solution of a matrix model

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1. Constructive definition of superstring theory

A large N reduced model has been proposed as a constructive definition (nonperturbative formualtion) of the superstring theory:

\[ S = -\frac{N}{4} \sum_{\mu, \nu=0}^{g} Tr \left[ A_{\mu} A_{\nu} \right] - \frac{N}{2} Tr \left[ \delta_{\mu} \left[ A_{\nu}, \psi \right] \right]. \]

- Dimensional reduction of \( N \) = 1 10-dimensional SYM theory to 0 dimension.
- \( A_{\mu} \) and \( \psi \) are \( N \times N \) Hermitian matrices.

- \( A_{\mu} \): 10-dimensional vectors
- \( \psi \): 10-dimensional Majorana-Weyl (i.e. 16-component) spinors

- Matrix regularization of the Schrödinger action of the type IIB superstring theory.

- \( SU(N) \) gauge symmetry and \( SO(10) \) Lorentz symmetry (\( SO(10) \times SU(N) \)).

\( N = 2 \) SUSY: This theory must contain spin-2 gravitons if it contains massless particles.

2. Fuzzy-sphere classical solution of the matrix model

The drawback of the IIB model model:
\[ [A^\mu, [A_\nu, A_\rho]] = 0 \Rightarrow [A^\mu, A_\nu] = i\epsilon^\mu_\nu. \]
In order to surmount this drawback, we consider the generalization of the IIB matrix model:

\[ S = -\frac{N}{4} Tr \left[ A_{\mu} A_{\nu} \right] - g N \epsilon^{\mu_1 \cdots \nu_{2k+1}} Tr A_{\nu_1} \cdots A_{\nu_{2k+1}}. \]

- This action is defined in the odd \( (2k+1) \)-dimensional Euclidean spacetime.

- \( SO(2k+1) \) rotational symmetry and \( SU(N) \) gauge symmetry.

The classical equation of motion:
\[ -[A_{\mu}, [A_\nu, A_\rho]] = g \epsilon^{\mu_1 \cdots \nu_{2k+1}} A_{\nu_1} \cdots A_{\nu_{2k+1}} = 0 \]
incorporates the higher-dimensional fuzzy-sphere solution!

\[ A_{\mu} = a G_{\mu} \text{ (with } g = -a^{2k-2} \frac{8k}{(2k+1)!}). \]

\( G_{\mu} \) is given by the symmetric tensor product of the \( (2k+1) \)-dimensional fuzzy-sphere solution:

\[ G_{\mu} = \left( \Gamma^{(2k)}_\mu \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \right)_{\text{sym}} + \cdots + \left( \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \Gamma^{(2k)}_\mu \right)_{\text{sym}}. \]

- \( \Gamma^{(2k)}_\mu \) denotes the \( 2^k \times 2^k \) gamma matrices for the \( (2k+1) \)-dimensional Euclidean space.

- This symmetric tensor product is realized only for a limited size of the matrices. For the \( (2k+1) \) dimensions, the size \( N_k \) is

\[
\begin{align*}
N_1 &= (n+1), \\
N_2 &= (n+1)(n+2)(n+3), \\
N_3 &= \frac{(n+1)(n+2)(n+3)^2(n+4)(n+5)}{6}, \\
N_4 &= \frac{360}{302400} (n+1)(n+2)(n+3)^2(n+4)(n+5)^2(n+6)(n+7).
\end{align*}
\]

- \( G_{\mu} \) gives the sphere’s geometry in that

\[ G_{\mu} G_{\nu} = n(n+2k)I_{N_k} \times I_{N_k}. \]

- \( G_{\mu} \) generally does not close with respect to the commutator. For \( G_{\mu \nu} = [G_{\mu}, G_{\nu}] \), we obtain

\[
\begin{align*}
G_{\mu \nu} G_{\mu \nu} &= -8kn(n+2k)I_{N_k} \times I_{N_k}, \\
G_{\mu \nu} G_{\nu \mu} &= 4(-\delta_{\mu,0} G_{\nu,0} + \delta_{\nu,0} G_{\mu,0}), \\
G_{\mu \nu} G_{\nu \mu} &= 4(-\delta_{\mu,0} G_{\nu,0} + \delta_{\nu,0} G_{\mu,0} - \delta_{\mu,0} G_{\nu,0}).
\end{align*}
\]

- Self-dual condition:

\[ \epsilon_{\mu_1 \cdots \nu_{2k}} G_{\mu_1} \cdots G_{\nu_{2k}} = m_k G_{\mu}. \]

The coefficient \( m_k \) satisfies the following recursive formula:

\[
\begin{align*}
m_1 &= 2, \\
m_2 &= 8(n+2), \\
m_3 &= -48(n+2)(n+4), \\
m_{k+1} &= -2(k+1)(n+2k)m_k.
\end{align*}
\]

However, the quantum stability of the fuzzy-sphere solution is still obscure.

* We investigate the stability via the Monte-Carlo simulation.

3. Monte-Carlo simulation of matrix models

(a) Warm-up: quadratic \( U(N) \) one-matrix model

We start with the simplest case - quadratic \( U(N) \) one-matrix model:

\[ S = -\frac{N}{2} Tr \phi^2. \]

The Feynman diagram of this model:

\[ \langle \phi_{ij} \phi_{kl} \rangle = \frac{1}{N} \delta_{ik} \delta_{jk}. \]

Then, the following quantities can be computed exactly:

\[ \langle \frac{1}{N} Tr \phi^2 \rangle = 1, \quad \langle \frac{1}{N} Tr \phi^4 \rangle = 2 + \frac{2}{N^2}, \quad \langle \left( \frac{1}{N} Tr \phi^2 \right)^2 \rangle = 1 + \frac{2}{N^2}. \]

We analyze this model via the heat-bath algorithm. To this end, we rewrite the \( U(N) \) matrix \( \phi \) as

\[ \phi_{ij} = \frac{1}{\sqrt{N}} \left\{ \begin{array}{l}
\phi_{ij} = \frac{a_{i+j}^{(x)}}{\sqrt{N}}, \\
\phi_{ij} = \frac{a_{i-j}^{(y)}}{\sqrt{N}} \end{array} \right. \text{ [for } i \neq j], \]

The \( N^2 \) real quantities \( a_i, \phi_{ij}, \phi_{ji} \) comply with the independent normal Gaussian distribution.

\[ S = \frac{1}{2} \sum_{i=1}^{N} a_i^2 + \frac{1}{2} \sum_{i < j} \left( (x_{ij})^2 + (y_{ij})^2 \right). \]

(b) Quartic one-matrix model

We analyze the one-matrix model via the heat-bath algorithm:

\[ S = -\frac{N}{2} Tr \phi^2 - \frac{N}{4} Tr \phi^4. \]

This action is unbounded below. However, we can avoid the divergence in the large-\( N \) limit.

We introduce the auxiliary fields \( Q \) as \( a = \sqrt{2} \) in order to render the action quadratic:

\[ \tilde{S} = \frac{N}{2} Tr \phi^2 + \frac{N}{2} Tr Q^2 - \alpha N Tr Q \phi^2 - \frac{N}{2} Tr (Q - \alpha \phi^2)^2 + S. \]
We update $Q$ as

$$Q_{ij} = \frac{a_i}{\sqrt{N}} + \alpha(\phi_i^2)_{ii}, \quad Q_{ij} = \frac{x_{ij} + y_{ij}}{\sqrt{2N}} + \alpha(\phi_i^2)_{ij},$$

where $a_i, x_{ij}, y_{ij}$ comply with the normal Gaussian distribution.

In updating the diagonal part $\phi_{ii}$, we extract the dependence of $\phi_{ii}:

$$\hat{S} = \frac{N}{2} \langle \phi_i^2 \rangle^2 \left[ 1 - 2a \phi_i^2 \right] - N \phi_i \left( \alpha \sum_{j \neq i} \langle \phi_j Q_{ij} + Q_{ij} \phi_j \rangle \right),$$

Then, $\phi_{ii}$ is updated as

$$\phi_{ii} = \frac{a_i}{\sqrt{N} c_i} + \frac{h_i}{c_i}.$$ We likewise extract the $\phi_{ij}$ dependence:

$$\hat{S} = \frac{N}{2} \left[ 1 - \alpha(\phi_i^2 + \phi_{ij}) \right] \phi_{ij} \left[ 1 - \alpha(\phi_j + \phi_{ij}) \right], \quad h_{ij} = \alpha \sum_{k \neq i} \langle \phi_k Q_{kj} + Q_{kj} \phi_k \rangle,$$

Then, $\phi_{ij}$ is updated as follows:

$$\phi_{ij} = \frac{a_{ij} + b_{ij}}{2N c_{ij}} + \frac{h_{ij}}{c_{ij}}.$$ The legitimacy of the algorithm is ascertained by checking the following results [as $N \to \infty$]:

$$\langle \frac{1}{N^2} Tr \phi^2 \rangle = \left\langle \frac{1}{2} a^2 (4 - a^2) \right\rangle, \quad \text{where} \quad a^2 = \frac{2}{1 + \sqrt{1 - 12g}}.$$ The eigenvalue distribution is given by

$$\rho(x) = \frac{1}{2} \left[ -2x^2 + 2x^2 + 1 \right] \sqrt{4x^2 - x}.$$ (c) The bosonic IIB matrix model

We investigate the bosonic IIB matrix model via the heat-bath algorithm:

$$S = \frac{N}{2} Tr[\alpha_{\mu} A_{\mu}]^2 - \frac{N}{2} \sum_{\mu < \nu} Tr[\alpha_{\mu} A_{\mu}]^2 + 2N \sum_{\mu < \nu} Tr(A_{\mu}^2 A_{\nu}^2).$$

This action is equivalent to $\hat{S}$, after integrating out $Q_{\mu \nu}$ (where $G_{\mu \nu} = \{A_{\mu}, A_{\nu}\}$):

$$\hat{S} = \sum_{\mu < \nu} \left( \frac{N}{2} Tr(Q_{\mu \nu} - N Tr(G_{\mu \nu} G_{\nu \mu}) + 2N Tr(A_{\mu}^2 A_{\nu}^2) \right)$$

$$= \sum_{\mu < \nu} Tr(Q_{\mu \nu} - G_{\mu \nu})^2 + S.$$ Then, $Q_{\mu \nu}$ is updated as

$$(Q_{\mu \nu})_{ii} = \frac{a_i}{\sqrt{N}} + (G_{\mu \nu})_{ii}, \quad (Q_{\mu \nu})_{ij} = \frac{x_{ij} + y_{ij}}{\sqrt{2N}} + (G_{\mu \nu})_{ij}.$$ We next update $A_{\nu}$. We extract the dependence of $A_{\nu}$:

$$\hat{S} = -N Tr(T_{\alpha} A_{\nu}) + 2N Tr(S_{\alpha} A_{\nu}^2) + \cdots, \quad S_{\alpha} = \sum_{\mu \neq \alpha} (A_{\mu}^2), \quad T_{\alpha} = \sum_{\mu \neq \alpha} (A_{\mu} Q_{\mu \nu} + Q_{\mu \nu} A_{\nu}).$$

$\bullet$ The diagonal part $A_{\nu}$ is updated by extracting the dependence of $(A_{\mu})_{ii}$:

$$\hat{S} = 2N (S_{\alpha})_{ii} (A_{\mu})_{ii}^2 - 4N h_{i} (A_{\nu})_{ii}, \quad h_{i} = \frac{N}{4} \left( T_{\alpha} \right)_{ii} - 2 \sum_{j \neq i} \langle (S_{\alpha})_{ij} (A_{\nu})_{ij} + (S_{\alpha})_{ij} (A_{\nu})_{ij} \rangle.$$ Then, $(A_{\nu})_{ii}$ is updated as

$$\langle A_{\nu} \rangle_{ii} = \frac{\alpha_{ii} + h_{i}}{\sqrt{4N (S_{\alpha})_{ii}^2}}.$$ $\bullet$ The other components $(A_{\mu})_{ij}$ are updated likewise by extracting their dependence:

$$\hat{S} = 2N c_{ij} [(A_{\mu})_{ij}]^2 - 2N h_{ij} (A_{\nu})_{ij}, \quad c_{ij} = \langle (S_{\alpha})_{ii} + (S_{\alpha})_{jj} \rangle,$$ $h_{ij} = \frac{1}{2} (T_{\alpha})_{ij} - \sum_{k \neq i} \langle (S_{\alpha})_{ik} (A_{\nu})_{ik} - \sum_{k \neq j} \langle (S_{\alpha})_{kj} (A_{\nu})_{kj} \rangle.$$ Then, $(A_{\nu})_{ij}$ are updated as

$$\langle A_{\nu} \rangle_{ij} = \frac{x_{ij} + y_{ij}}{\sqrt{2N c_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$ The following Schwinger-Dyson equation serves as the consistency check of the algorithm:

$$-\frac{1}{N} Tr [A_{\mu}, A_{\nu}]^2 = D(1 - \frac{1}{N^2}).$$

(d) Extension to the bosonic IIB matrix model with the Chern-Simons term

The Chern-Simons term is linear with respect to each $A_{\mu}$.

We have only to replace $T_{\alpha}$ as

$$T^G_{\alpha} = T_{\alpha} + g(2k + 1) \cdot A_{2k+1} \cdots A_{2k}.$$ The Schwinger-Dyson equation is replaced as

$$-\frac{1}{N} Tr [A_{\mu}, A_{\nu}]^2 = \left( \frac{g(2k + 1)}{N} \right) Tr^G_{\alpha} = \left( \frac{g(2k + 1)}{N} \right) Tr^G_{\alpha} (A_{2k+1} \cdots A_{2k}).$$

4. Stability of the fuzzy sphere

In order to see the stability of the fuzzy-sphere solution, we focus on the eigenvalues of the Casimir

$$C = x_1^2 + x_2^2 + \cdots + x_{2k+1}^2.$$ $\bullet$ We start by setting $A_{\mu}$ to be the fuzzy-sphere classical solution.

$\bullet$ We watch the behavior of the eigenvalue distribution, as we iterate the Monte-Carlo updating.

Our analysis is now under the way. We are faced with the setbacks in analyzing our case.

The difficulty comes from the unboundedness of the IIB matrix model with the Chern-Simons terms. This situation would be analogous to that for the one-matrix model.

In order to understand the behavior of the IIB matrix model with the Chern-Simons term, we should scrutinize the one-matrix model thoroughly.