Nonperturbative studies of the fuzzy spheres in a matrix model with the Chern-Simons term
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Takehiro Azuma
Department of Physics, Kyoto University

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collaborated with S. Bal, K. Nagao and J. Nishimura

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1 Introduction

Large-$N$ reduced models are the most powerful candidates for the constructive definition of superstring theory.

**The IIB matrix model**

N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115.

\[
S = \frac{-1}{g^2} \text{Tr}_{N \times N} \left( \frac{1}{4} \sum_{\mu, \nu=0}^{9} [A_\mu, A_\nu]^2 + \frac{1}{2} \bar{\psi} \sum_{\mu=0}^{9} \Gamma^\mu [A_\mu, \psi] \right).
\]

The IIB matrix model has the following illuminating features:

- We can describe the multi-body system of D-branes. The IIB matrix model is not the D-instanton action but the second quantization of superstring theory.

- Evidence of the gravitational interaction:
  - When we regard the eigenvalues as the spacetime coordinates, this model incorporates the $\mathcal{N} = 2$ supersymmetry. (hep-th/9612115)
  - Graviton-dilaton exchange: (hep-th/9612115)
  - Diffeomorphism invariance: (hep-th/9903217)

- Derivation of 4-dimensional spacetime:
  (hep-th/9802085,0204240,0211272)
Generalization of the IIB matrix model

Several alterations of the IIB matrix model have been proposed, to accommodate the curved-space background.

- The matrix model with the Chern-Simons term:
  (hep-th/0101102,0204256,0207115)

These matrix models accommodate the fuzzy sphere classical solutions:

The fuzzy sphere solutions are interesting in the following senses:

- More manifest realization of the curved-space background:
  Essential for an eligible framework for gravity.

- The expansion of the reducible representation
  \( J^{(n)}_\mu \otimes 1_{k \times k} \) leads to the \( U(k) \) noncommutative gauge theory \( (J^{(n)}_\mu = n \times n \) representation of \( su(2) \)).
  We may get insight into the dynamical generation of the gauge group.
2 Fuzzy sphere as classical solution

Throughout this talk, we focus on the following bosonic action:

\[
S = Tr_{N \times N} \left( -\frac{N}{4} [A_\mu, A_\nu]^2 + \frac{2i\alpha N}{3} \epsilon_{\mu\nu\rho} A_\mu A_\nu A_\rho \right).
\]

- Defined in the three-dimensional Euclidean space \((\mu, \nu, \cdots = 1, 2, 3)\).
  - \(SO(3)\) rotational and \(SU(N)\) gauge symmetry.
- Each \(A_\mu\) is promoted to the \(N \times N\) hermitian matrix.

Its classical equation of motion

\[
[A_\mu, [A_\mu, A_\nu]] + i\alpha \epsilon_{\nu\rho\chi} [A_\rho, A_\chi] = 0.
\]

accommodates the \(S^2\) fuzzy sphere solution

\[
A_\mu = \alpha J_\mu, \text{ where } [J_\mu, J_\nu] = i\epsilon_{\mu\nu\rho} J_\rho.
\]

\(J_\mu\) is an \(N \times N\) irreducible representation of the \(SU(2)\) Lie algebra.

The radius of the fuzzy sphere is given by the Casimir:

\[
Q = A_1^2 + A_2^2 + A_3^2 = R^2 I_{N \times N}, \text{ where } R^2 = \alpha^2 \frac{N^2 - 1}{4}.
\]
Monte-Carlo simulation of the matrix model

We analyze this bosonic Chern-Simons matrix model through the heat bath algorithm of the Monte Carlo simulation.

In this sense, our approach is nonperturbative, unlike the foregoing perturbative approach:

- Two-loop diagrammatic calculation: (hep-th/0303120,0307007)
- First order of the Gaussian expansion: (hep-th/0303196)
We start the Monte Carlo simulation from the initial condition

\[ A_\mu^{(0)} = \alpha J_\mu, \]

for the \(N = 16, \alpha = 1.0, 2.0\) case.

We plot the eigenvalue distribution of the Casimir
\[ Q = A_1^2 + A_2^2 + A_3^2. \]

The eigenvalues are peaked around

\[ R^2 = \frac{1}{4} \alpha^2 (N^2 - 1). \]

Nonperturbative stability of the fuzzy spheres!
The stability is ascribed to the small quantum effect at large $\alpha$.
For the effective action $W = \int dA_\mu e^{-S}$

- Effect of the classical fuzzy sphere: $O(\alpha^4 N^4)$.
- Effect of the path integral measure: $O(N^2)$.

The quantum effect is small when $\alpha \gg O\left(\frac{1}{\sqrt{N}}\right)$.

We plot the miscellaneous quantities against $\tilde{\alpha} = \alpha \sqrt{N}$.

1. The action $\langle S \rangle$.
2. The spacetime extent $\langle \frac{1}{N} Tr A^2_\mu \rangle$.
3. The bosonic Yang-Mills term $\langle \frac{1}{N} F^2_{\mu\nu} \rangle$,
   where $F_{\mu\nu} = i [A_\mu, A_\nu]$.
4. The Chern-Simons term: $\langle M \rangle = \langle \frac{2i}{3N} Tr \epsilon_{\mu\nu\rho} A_\mu A_\nu A_\rho \rangle$.
5. Exact result derived from Schwinger-Dyson equation:

   \[
   0 = \int dA \frac{\partial}{\partial A_\mu^a} (Tr(t^a A_\mu) e^{-S}).
   \]

   \[\downarrow\]

   \[
   K = \frac{1}{N} Tr F^2_{\mu\nu} + 3\alpha M = 3\left(1 - \frac{1}{N^2}\right).
   \]
First-order phase transition

We have a discontinuity at $\tilde{\alpha}^{(l)}_{ci} \sim 2.1$.

- **The Yang-Mills phase:**
  The quantum effect is large.
  The behavior resembles the bosonic IIB matrix model.

- **The fuzzy sphere phase:**
  The quantum effect is small.
  The model retains the classical fuzzy sphere.

One-loop exactness in the fuzzy sphere phase

In the fuzzy sphere phase, we have a one-loop exactness at the large $N$.

The one-loop calculation of the quantities:

\[
\frac{\langle S \rangle}{N^2} = -\frac{1}{24} \tilde{\alpha}^4 + \frac{1}{\tilde{\alpha}}, \quad \text{classical one-loop}
\]

\[
\frac{1}{N} \langle \frac{1}{N} Tr A^2_{\mu} \rangle = \frac{\tilde{\alpha}^2}{4} - \frac{1}{\tilde{\alpha}^2}, \quad \text{classical one-loop}
\]

\[
\langle \frac{1}{N} Tr F^2_{\mu\nu} \rangle = \frac{\tilde{\alpha}^2}{2} + 0, \quad \text{classical one-loop}
\]

\[
\frac{1}{\sqrt{N}} \langle M \rangle = -\frac{\tilde{\alpha}^3}{6} + \frac{1}{\tilde{\alpha}}, \quad \text{classical one-loop}
\]
Figure 1: (Upper) $\langle S \rangle / N^2$, (Lower) $\langle \frac{1}{N} \text{Tr} A^2 \rangle / N$, against $\tilde{\alpha}$.
Figure 2: (Upper) $\langle \frac{1}{N}Tr F_{\mu\nu}^2 \rangle$, (Lower) $\langle M \rangle / \sqrt{N}$, against $\bar{\alpha}$. 
Figure 3: \[
\frac{\langle K \rangle}{\text{(analytical value)}} = \frac{\langle K \rangle}{3(1-1/N^2)}
\]
4 Connection to the Yang-Mills phase

We start from another initial configuration

\[ A_{\mu}^{(0)} = 0. \]

The critical point is different from the fuzzy sphere initial condition!

\[ \alpha_{cr}^{(u)} \sim 0.66. \]

In the Yang-Mill phase,

\[ \frac{\langle S \rangle}{N^2}, \quad \langle \frac{1}{N} Tr A_{\mu}^2 \rangle \sim O(1). \]

Similar to the bosonic IIB matrix model (\( \alpha = 0 \)).


We see a strong hysteresis at \( N = 16 \).
Figure 4: The hysteresis cycle of (Upper) $\langle S \rangle / N^2$ and (Lower) $\langle \frac{1}{N} Tr A^2 \rangle$. 
5 Multi-fuzzy-sphere state

The matrix model accommodates the multi-fuzzy-sphere solution.

\[ A_\mu = \alpha \begin{pmatrix} J^{(n_1)}_\mu & J^{(n_2)}_\mu & \cdots & J^{(n_k)}_\mu \end{pmatrix} . \]

- \( J^{(n_a)}_\mu \): The \( n_a \)-dimensional irrep. of \( SU(2) \).

\[ n_1 + n_2 + \cdots + n_k = N. \]

- The eigenvalues of \( Q \) are peaked at \( r^2_a = \frac{\alpha^2}{4} (n^2_a - 1) \).

- The classical energy is \( S = -\frac{\alpha^4 N}{24} \sum_{a=1}^{k} (n^3_a - n_a) \).

Higher than that of the one-fuzzy-sphere state \( A_\mu = \alpha J_\mu \).

We initiate the simulation from \( A^{(0)}_\mu = 0 \) for

\[ N = 16, \alpha = 2.0 \in \text{(fuzzy sphere phase)}. \]

The multi-fuzzy sphere is realized as a metastable state.

\[
\begin{align*}
A^{(0)}_\mu &= 0 \\
\text{initial state} \\
\rightarrow A_\mu &= \alpha \begin{pmatrix} J^{(6\rightarrow5\rightarrow4\rightarrow3\rightarrow2\rightarrow1)}_\mu & 0 \\
0 & J^{(10\rightarrow11\rightarrow12\rightarrow13\rightarrow14\rightarrow15)}_\mu \end{pmatrix} \\
\text{metastable vacuum} \\
\rightarrow A_\mu &= \alpha J_\mu . \\
\text{stable vacuum}
\end{align*}
\]
Figure 5: The history of the vacuum expectation value of the action $\langle S \rangle$ (left), and the eigenvalues of $Q$ (right) against the sweeping time, for $N = 16$, $\alpha = 2.0$. 
Metastability of the multi-fuzzy-sphere state

We compare the dependence of the multi-fuzzy-sphere state on $k, \alpha, N$.

We initiate the simulation from

$$A^{(0)}_\mu = \alpha J^{(n)}_\mu \otimes 1_{k \times k},$$

namely when $n = n_1 = \cdots = n_k = \frac{N}{k}$.

- $k$ dependence: $N = 16$, $\alpha = 10.0$ fixed. $k = 2, 4, 8$.
  The sphere is more stable for smaller $k$ (namely, larger $J^{(n)}_\mu$).

- $\alpha$ dependence: $N = 16$, $k = 8$ fixed. various $\alpha$.
  The sphere is more stable for larger $\alpha$.

- $N$ dependence: $k = 2$, $\tilde{\alpha} = 40.0$ fixed. $N = 8, 16, 32$.
  The sphere is stable for larger $N$ (commutative limit).
6 Conclusion

In this work, we have investigated the stability of the fuzzy sphere in the matrix model with the Chern-Simons term.

- The first-order phase transition between the Yang-Mills phase and the fuzzy sphere phase.
- One-loop exactness at the large $N$ in the fuzzy-sphere phase.

Future works:
- Extension to the supersymmetric case.
- Extension to the higher-dimensional case. fuzzy $2k$-sphere, $S^2 \times S^2$, $\cdots$.
- Dynamical generation of the gauge group.
Heat bath algorithm of the matrix model

(a) Warm-up: quadratic $U(N)$ one-matrix model

We start with the simplest case — quadratic $U(N)$ one-matrix model:

$$ S = \frac{N}{2} Tr \phi^2. $$

We analyze this model via the heat bath algorithm. To this end, we rewrite the $U(N)$ matrix $\phi$ as

$$ \phi_{ii} = \frac{a_i}{\sqrt{N}}, \quad \phi_{ij} = \frac{x_{ij} + i y_{ij}}{\sqrt{2N}}, \quad \phi_{ji} = \frac{x_{ij} - i y_{ij}}{\sqrt{2N}}, \quad \text{(for $i < j$)}. $$

The $N^2$ real quantities $a_i, x_{ij}, y_{ij}$ comply with the independent normal Gaussian distribution.

$$ S = \frac{1}{2} \sum_{i=1}^{N} a_i^2 + \frac{1}{2} \sum_{i<j} ((x_{ij})^2 + (y_{ij})^2). $$

$$ Z = \int \prod_{i=1}^{N} da_i \prod_{1 \leq i < j \leq N} dx_{ij} dy_{ij} \exp \left( -\frac{1}{2} \sum_{i=1}^{N} a_i^2 - \frac{1}{2} \sum_{1 \leq i < j \leq N} ((x_{ij})^2 + (y_{ij})^2) \right). $$

$a_i, x_{ij}, y_{ij}$ are updated by the Gaussian random number.

Generation of the uniform random number

We use the congruence method.

- We give the random seed $z_1$, such as $a_1 = \text{time}()$.
- We solve the recursion formula

$$ z_{k+1} = az_k + c \pmod{2^{31} - 1}. $$

The choice $(a, c) = (5^{11}, 0)$ is known to give a good pseudo-random number.
- The sequence $\left\{ \frac{z_k}{2^{31} - 1} \right\}$ gives a uniform pseudo-random number $[0:1]$. 

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Generation of the Gaussian random number

- We take two uniform random numbers \( x, y \in [0 : 1] \).
- We introduce the quantity \( r = \sqrt{-a^2 \log x^2} \). This complies with the probability distribution
  \[
P(r)dr = P(x)\frac{dx}{dr}dr = \frac{2r}{a^2} \exp \left(-\frac{r^2}{a^2}\right).
  \]
- We next introduce the quantities
  \[
  X = r \cos(2\pi y), \quad Y = r \sin(2\pi y).
  \]
  They comply with the probability distribution
  \[
P(r)dr dy \propto \exp \left(-\frac{1}{a^2}(X^2 + Y^2)\right).
  \]

(b) The bosonic IIB matrix model


We investigate the \( d \)-dimensional bosonic IIB matrix model via the heat bath algorithm:

\[
S = -\frac{N}{4} \sum_{\mu, \nu=1}^{d} Tr[A_\mu, A_\nu]^2 = -\frac{N}{2} \sum_{1 \leq \mu < \nu \leq d} Tr\{A_\mu, A_\nu\}^2 + 2N \sum_{\mu < \nu} Tr(A_\mu^2 A_\nu^2).
\]

This action is equivalent to \( \tilde{S} \), after integrating out \( Q_{\mu \nu} \) (where \( G_{\mu \nu} = \{A_\mu, A_\nu\} \)):

\[
\tilde{S} = \sum_{\mu < \nu} \left( \frac{N}{2} TrQ_{\mu \nu}^2 - NTr(Q_{\mu \nu} G_{\mu \nu}) + 2NTr(A_\mu^2 A_\nu^2) \right)
\]
\[
= \frac{N}{2} \sum_{\mu < \nu} Tr(Q_{\mu \nu} - G_{\mu \nu})^2 + S.
\]

Then, \( Q_{\mu \nu} \) is updated as

\[
(Q_{\mu \nu})_{ii} = \frac{a_i}{\sqrt{N}} + (G_{\mu \nu})_{ii}, \quad (Q_{\mu \nu})_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{2N}} + (G_{\mu \nu})_{ij},
\]
We next update $A_\lambda$. We extract the dependence of $A_\lambda$.

$$\tilde{S} = -N T r(T_\lambda A_\lambda) + 2 N T r(S_\lambda A_\lambda^2) + \cdots, \text{ where}$$

$$S_\lambda = \sum_{\mu \neq \lambda} (A_\mu^2), \quad T_\lambda = \sum_{\mu \neq \lambda} (A_\mu Q_{\lambda \mu} + Q_{\lambda \mu} A_\mu).$$

- The diagonal part $A_\lambda$ is updated by extracting the dependence of $(A_\lambda)_{ii}$:

$$\tilde{S} = 2N (S_\lambda)_{ii} (A_\lambda)_{ii}^2 - 4N h_i (A_\lambda)_{ii}, \text{ where}$$

$$h_i = \frac{N}{4} [(T_\lambda)_{ii} - 2 \sum_{j \neq i} ((S_\lambda)_{ji} (A_\lambda)_{ij} + (S_\lambda)_{ij} (A_\lambda)_{ji})].$$

Then, $(A_\lambda)_{ii}$ is updated as

$$(A_\lambda)_{ii} = \frac{a_i}{\sqrt{4N (S_\lambda)_{ii}}} + \frac{h_i}{(S_\lambda)_{ii}}.$$

- The other components $(A_\lambda)_{ij}$ are updated likewise by extracting their dependence:

$$\tilde{S} = 2N c_{ij} |(A_\lambda)_{ij}|^2 - 2N h_{ji} (A_\lambda)_{ij}, \text{ where}$$

$$c_{ij} = (S_\lambda)_{ii} + (S_\lambda)_{jj},$$

$$h_{ij} = \frac{1}{2} (T_\lambda)_{ij} - \sum_{k \neq i} (S_\lambda)_{ik} (A_\lambda)_{kj} - \sum_{k \neq j} (S_\lambda)_{kj} (A_\lambda)_{ik}.$$ 

Then, $(A_\lambda)_{ij}$ are updated as

$$(A_\lambda)_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{4N h_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$ 

\[\text{Extension to the bosonic IIB matrix model with the Chern-Simons term}\]

The Chern-Simons term is \textit{linear} with respect to each $A_\mu$.

We have only to replace $T_\lambda$ as (for $d = 3$)

$$T_\lambda^{CS} = T_\lambda + 3 g \epsilon_{\lambda \nu_1 \nu_2} A_{\nu_1} A_{\nu_2}.$$