Nonperturbative studies of the fuzzy spheres in a matrix model with the Chern-Simons term

hep-th/0311***

Takehiro Azuma

Department of Physics, Kyoto University

Laboratory Camp 2003 at Kishuji Minabe

Nov. 22st 2003 17:00 ∼ 17:30

collaborated with S. Bal, K. Nagao and J. Nishimura

Contents

1 Introduction 2

2 Fuzzy sphere as classical solution 4

3 Nonperturbative stability of the fuzzy sphere 6

4 Connection to the Yang-Mills phase 12

5 Multi-fuzzy-sphere state 14

6 Conclusion 17

1 Introduction

Large-N reduced models are the most powerful candidates for the constructive definition of superstring theory.

The IIB matrix model

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

$$S = \frac{-1}{g^2} Tr_{N\times N}\left(\frac{1}{4}\sum_{\mu,\nu=0}^{9}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\psi}\sum_{\mu=0}^{9}\Gamma^\mu[A_\mu, \psi]\right).$$

The IIB matrix model has the following illuminating features:

- **We can describe the multi-body system of D-branes. The IIB matrix model is not the D-instanton action but the second quantization of superstring theory.**
- **Evidence of the gravitational interaction:**
  - ⋆ **When we regard the eigenvalues as the spacetime coordinates, this model incorporates the $\mathcal{N} = 2$ supersymmetry.** (hep-th/9612115)
  - ⋆ **Graviton-dilaton exchange:** (hep-th/9612115)
  - ⋆ **Diffeomorphism invariance:** (hep-th/9903217)
- **Derivation of 4-dimensional spacetime:** (hep-th/9802085,0204240,0211272)

Generalization of the IIB matrix model

Several alterations of the IIB matrix model have been proposed, to accommodate the curved-space background.

- **The matrix model with the Chern-Simons term:**
  (hep-th/0101102,0204256,0207115)

These matrix models accommodate the fuzzy sphere classical solutions:

The fuzzy sphere solutions are interesting in the following senses:

- **More manifest realization of the curved-space background:**
  Essential for an eligible framework for gravity.
- **The expansion of the reducible representation** $J_\mu^{(n)} \otimes 1_{k\times k}$ **leads to the** $U(k)$ **noncommutative gauge theory (**$J_\mu^{(n)} = n \times n$ **representation of** $su(2)$**).**
  We may get insight into the dynamical generation of the gauge group.

2 Fuzzy sphere as classical solution

Throughout this talk, we focus on the following bosonic action:

$$S = Tr_{N\times N}\left(-\frac{N}{4}[A_\mu, A_\nu]^2 + \frac{2i\alpha N}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).$$

- Defined in the three-dimensional Euclidean space $(\mu, \nu, \cdots = 1, 2, 3)$.
  $SO(3)$ rotational and $SU(N)$ gauge symmetry.
- Each A_μ is promoted to the $N \times N$ hermitian matrix.

Its classical equation of motion

$$[A_\mu, [A_\mu, A_\nu]] + i\alpha\epsilon_{\nu\rho\chi}[A_\rho, A_\chi] = 0.$$

accommodates the S^2 fuzzy sphere solution

$$A_\mu = \alpha J_\mu, \text{ where } [J_\mu, J_\nu] = i\epsilon_{\mu\nu\rho}J_\rho.$$

J_μ is an $N \times N$ irreducible representation of the $SU(2)$ Lie algebra.

The radius of the fuzzy sphere is given by the Casimir:

$$Q = A_1^2 + A_2^2 + A_3^2 = R^2 1_{N\times N}, \text{ where } R^2 = \alpha^2\frac{N^2-1}{4}.$$

Monte-Carlo simulation of the matrix model

We analyze this bosonic Chern-Simons matrix model through the heat bath algorithm of the Monte Carlo simulation.

In this sense, our approach is nonperturbative, unlike the foregoing perturbative approach:

- **Two-loop diagrammatic calculation:** (hep-th/0303120,0307007)
- **First order of the Gaussian expansion :** (hep-th/0303196)

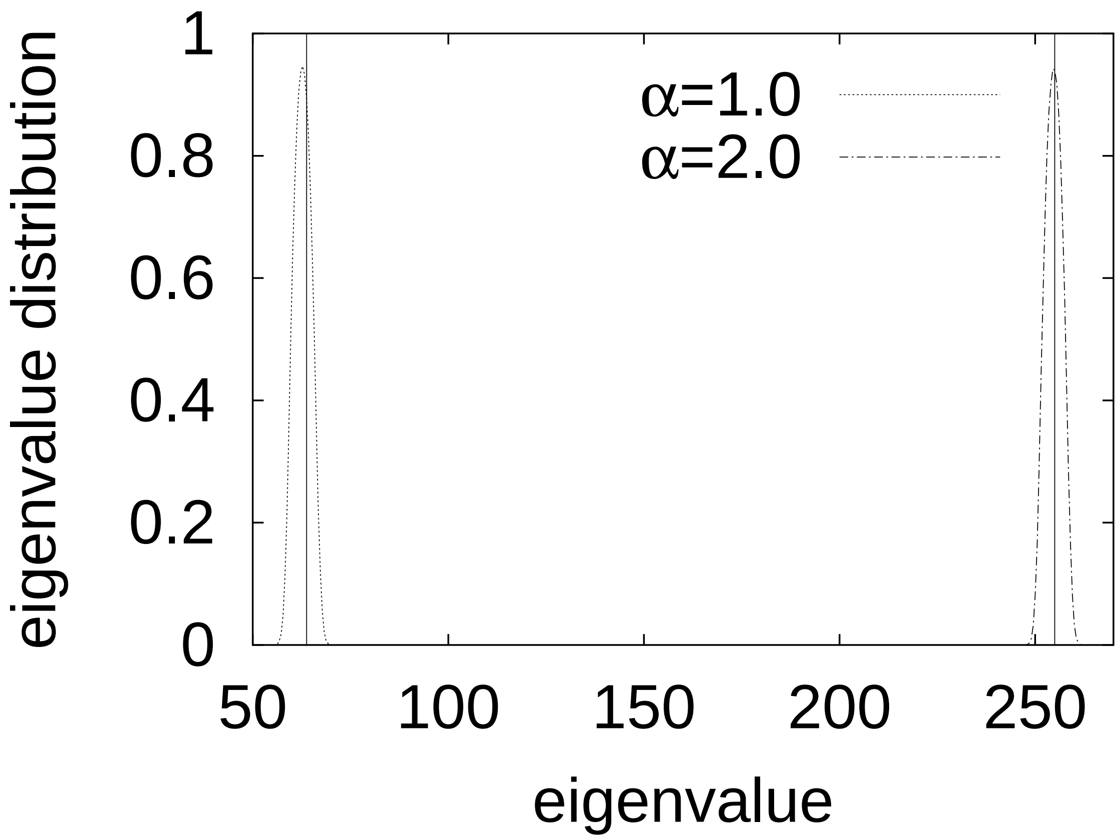
3 Nonperturbative stability of the fuzzy sphere

We start the Monte Carlo simulation from the initial condition

$$A_\mu^{(0)} = \alpha J_\mu,$$

for the $N = 16$, $\alpha = 1.0, 2.0$ case.

We plot the eigenvalue distribution of the Casimir $Q = A_1^2 + A_2^2 + A_3^2$.



The eigenvalues are peaked around

$$R^2 = \frac{1}{4}\alpha^2(N^2 - 1).$$

Nonperturbative stability of the fuzzy spheres!

The stability is ascribed to the small quantum effect at large α.

For the effective action $W = \int dA_\mu e^{-S}$

- Effect of the classical fuzzy sphere: $\mathcal{O}(\alpha^4 N^4)$.
- Effect of the path integral measure: $\mathcal{O}(N^2)$.

The quantum effect is small when $\alpha \gg \mathcal{O}(\frac{1}{\sqrt{N}})$.

We plot the miscellaneous quantities against $\tilde{\alpha} = \alpha\sqrt{N}$.

1. The action $\langle S \rangle$.
2. The spacetime extent $\langle \frac{1}{N} Tr A_\mu^2 \rangle$.
3. The bosonic Yang-Mills term $\langle \frac{1}{N} F_{\mu\nu}^2 \rangle$, where $F_{\mu\nu} = i[A_\mu, A_\nu]$.
4. The Chern-Simons term: $\langle M \rangle = \langle \frac{2i}{3N} Tr \epsilon_{\mu\nu\rho} A_\mu A_\nu A_\rho \rangle$.
5. Exact result derived from Schwinger-Dyson equation:

$$
\begin{aligned}
0 &= \int dA \frac{\partial}{\partial A_\mu^a} (Tr(t^a A_\mu) e^{-S}). \\
&\Downarrow \\
K &= \frac{1}{N} Tr F_{\mu\nu}^2 + 3\alpha M = 3(1 - \frac{1}{N^2}).
\end{aligned}
$$

First-order phase transition

We have a discontinuity at $\tilde{\alpha}_{cl}^{(l)} \sim 2.1$.

- The Yang-Mills phase:
  The quantum effect is large.
  The behavior resembles the bosonic IIB matrix model.
- The fuzzy sphere phase:
  The quantum effect is small.
  The model retains the classical fuzzy sphere.

One-loop exactness in the fuzzy sphere phase

In the fuzzy sphere phase, we have a one-loop exactness at the large N.

The one-loop calculation of the quantities:

$$\begin{aligned}
\frac{\langle S \rangle}{N^2} &= \underbrace{-\frac{1}{24}\tilde{\alpha}^4}_{\text{classical}} \underbrace{+1}_{\text{one-loop}}, \\
\frac{1}{N}\langle \frac{1}{N} Tr A_\mu^2 \rangle &= \underbrace{\frac{\tilde{\alpha}^2}{4}}_{\text{classical}} \underbrace{-\frac{1}{\tilde{\alpha}^2}}_{\text{one-loop}}, \\
\langle \frac{1}{N} Tr F_{\mu\nu}^2 \rangle &= \underbrace{\frac{\tilde{\alpha}^2}{2}}_{\text{classical}} \underbrace{+0}_{\text{one-loop}}, \\
\frac{1}{\sqrt{N}}\langle M \rangle &= \underbrace{-\frac{\tilde{\alpha}^3}{6}}_{\text{classical}} \underbrace{+\frac{1}{\tilde{\alpha}}}_{\text{one-loop}}.
\end{aligned}$$

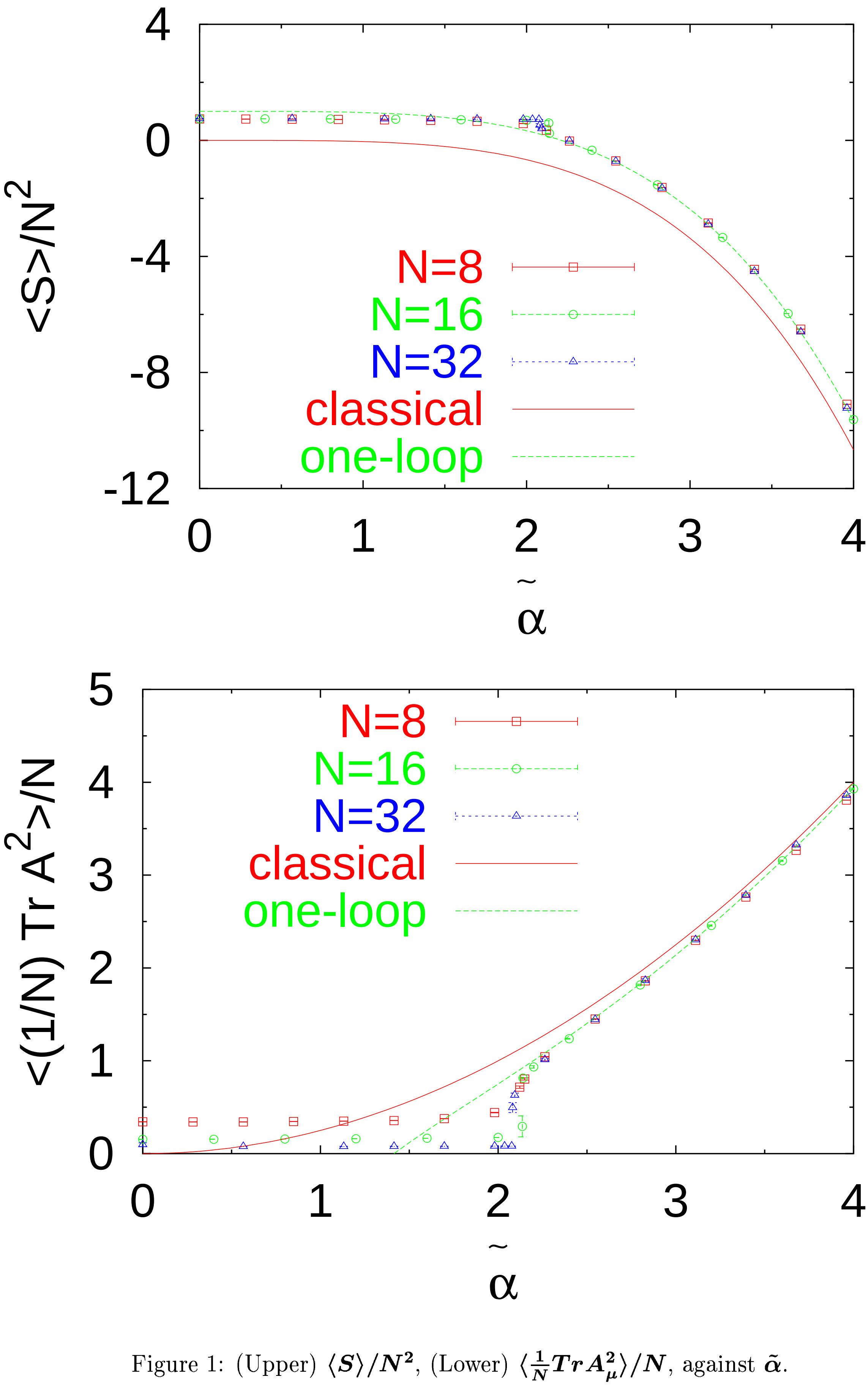


Figure 1: (Upper) $\langle S\rangle/N^2$, (Lower) $\langle \frac{1}{N} Tr A_\mu^2\rangle/N$, against $\tilde{\alpha}$.

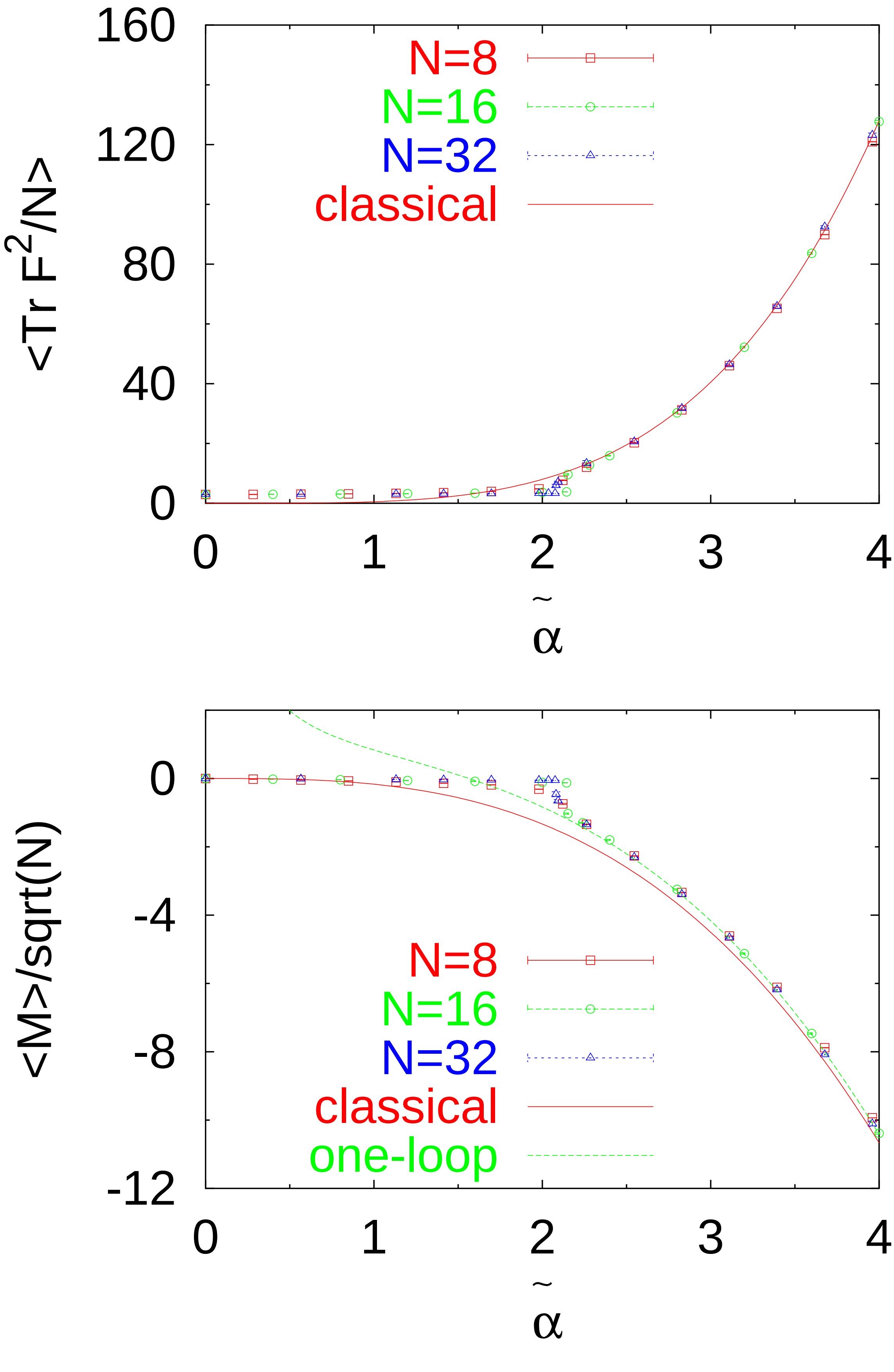


Figure 2: (Upper) $\langle \frac{1}{N} Tr F_{\mu\nu}^2 \rangle$, (Lower) $\langle M \rangle / \sqrt{N}$, against $\tilde{\alpha}$.

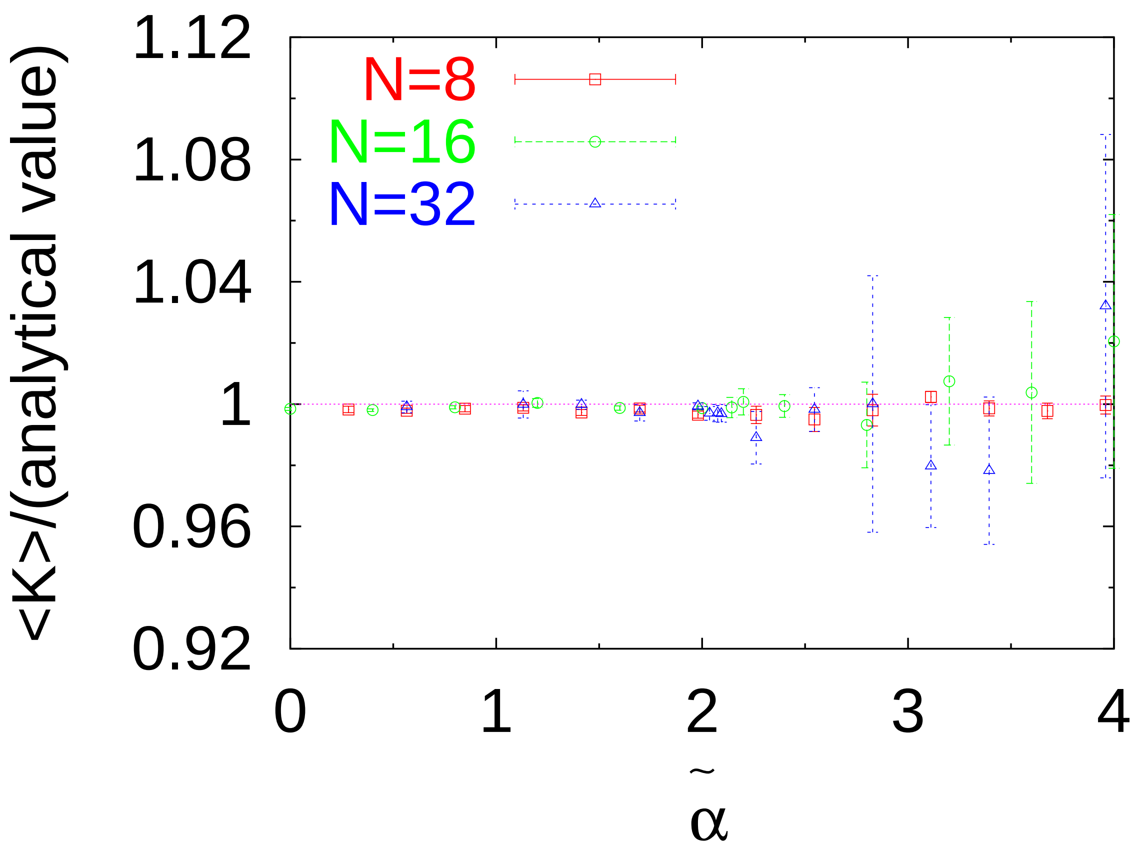


Figure 3: $\frac{\langle K\rangle}{\text{(analytical value)}} = \frac{\langle K\rangle}{3(1-1/N^2)}$

4 Connection to the Yang-Mills phase

We start from another initial configuration

$$A_\mu^{(0)} = 0.$$

The critical point is different from the fuzzy sphere initial condition!

$$\alpha_{cr}^{(u)} \sim 0.66.$$

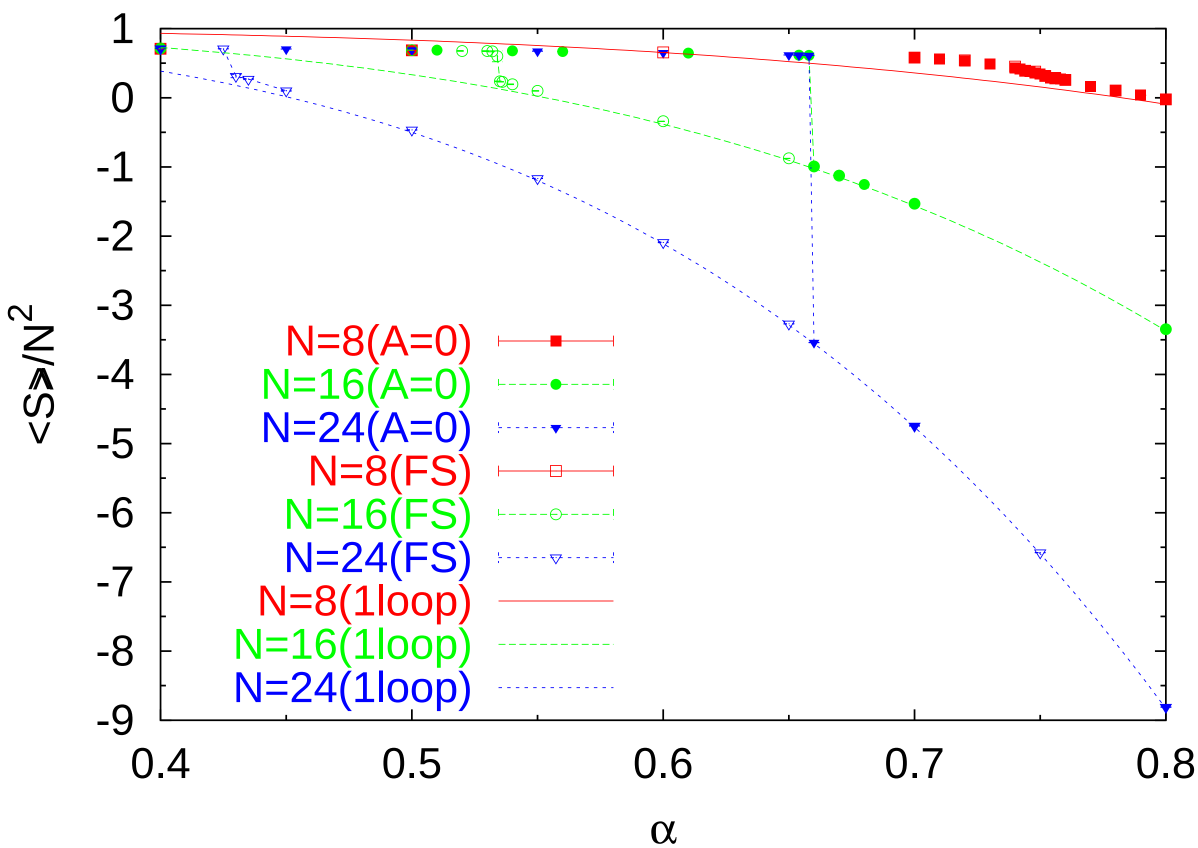
In the Yang-Mill phase,

$$\frac{\langle S \rangle}{N^2}, \quad \langle \frac{1}{N} Tr A_\mu^2 \rangle \sim \mathcal{O}(1).$$

Similar to the bosonic IIB matrix model ($\alpha = 0$).

T. Hotta, J. Nishimura and A. Tsuchiya hep-th/9811220.

We see a strong hysteresis at $N = 16$.



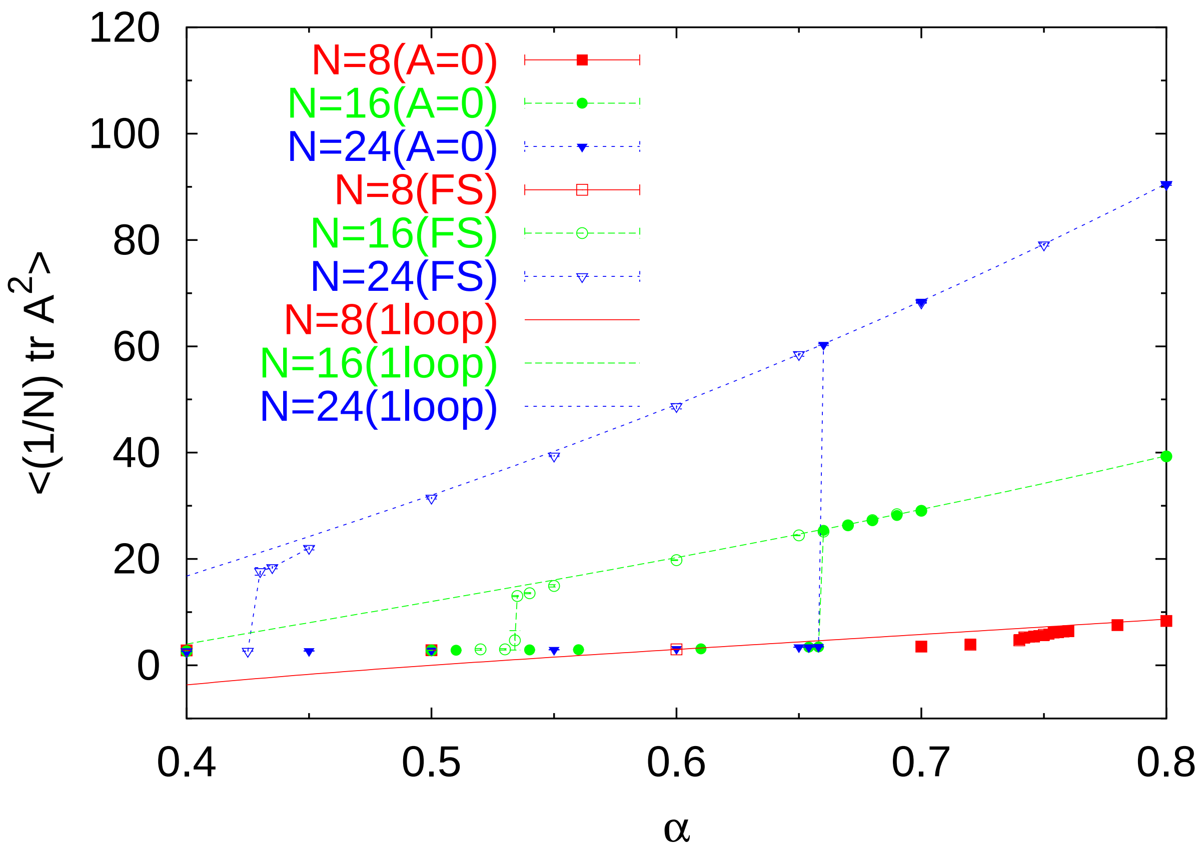


Figure 4: The hysteresis cycle of (Upper) $\langle \boldsymbol{S}\rangle/\boldsymbol{N^2}$ and (Lower) $\langle \frac{1}{N} \boldsymbol{Tr A_\mu^2}\rangle$.

5 Multi-fuzzy-sphere state

The matrix model accommodates the multi-fuzzy-sphere solution.

$$A_\mu = \alpha \begin{pmatrix} J_\mu^{(n_1)} & & & \\ & J_\mu^{(n_2)} & & \\ & & \ddots & \\ & & & J_\mu^{(n_k)} \end{pmatrix}.$$

- $J_\mu^{(n_a)}$: The n_a-dimensional irrep. of $SU(2)$. $n_1 + n_2 + \cdots + n_k = N$.
- The eigenvalues of Q are peaked at $r_a^2 = \frac{\alpha^2}{4}(n_a^2 - 1)$.
- The classical energy is $S = -\frac{\alpha^4 N}{24} \sum_{a=1}^k (n_a^3 - n_a)$. Higher than that of the one-fuzzy-sphere state $A_\mu = \alpha J_\mu$.

We initiate the simulation from $A_\mu^{(0)} = 0$ for

$$N = 16, \alpha = 2.0 \in \text{(fuzzy sphere phase)}.$$

The multi-fuzzy sphere is realized as a metastable state.

$$\underbrace{A_\mu^{(0)} = 0}_{\text{initial state}} \to \cdots$$

$$\to \underbrace{A_\mu = \alpha \begin{pmatrix} J_\mu^{(6\to5\to4\to3\to2\to1)} & 0 \\ 0 & J_\mu^{(10\to11\to12\to13\to14\to15)} \end{pmatrix}}_{\text{metastable vacuum}}$$

$$\to \underbrace{A_\mu = \alpha J_\mu}_{\text{stable vacuum}}.$$

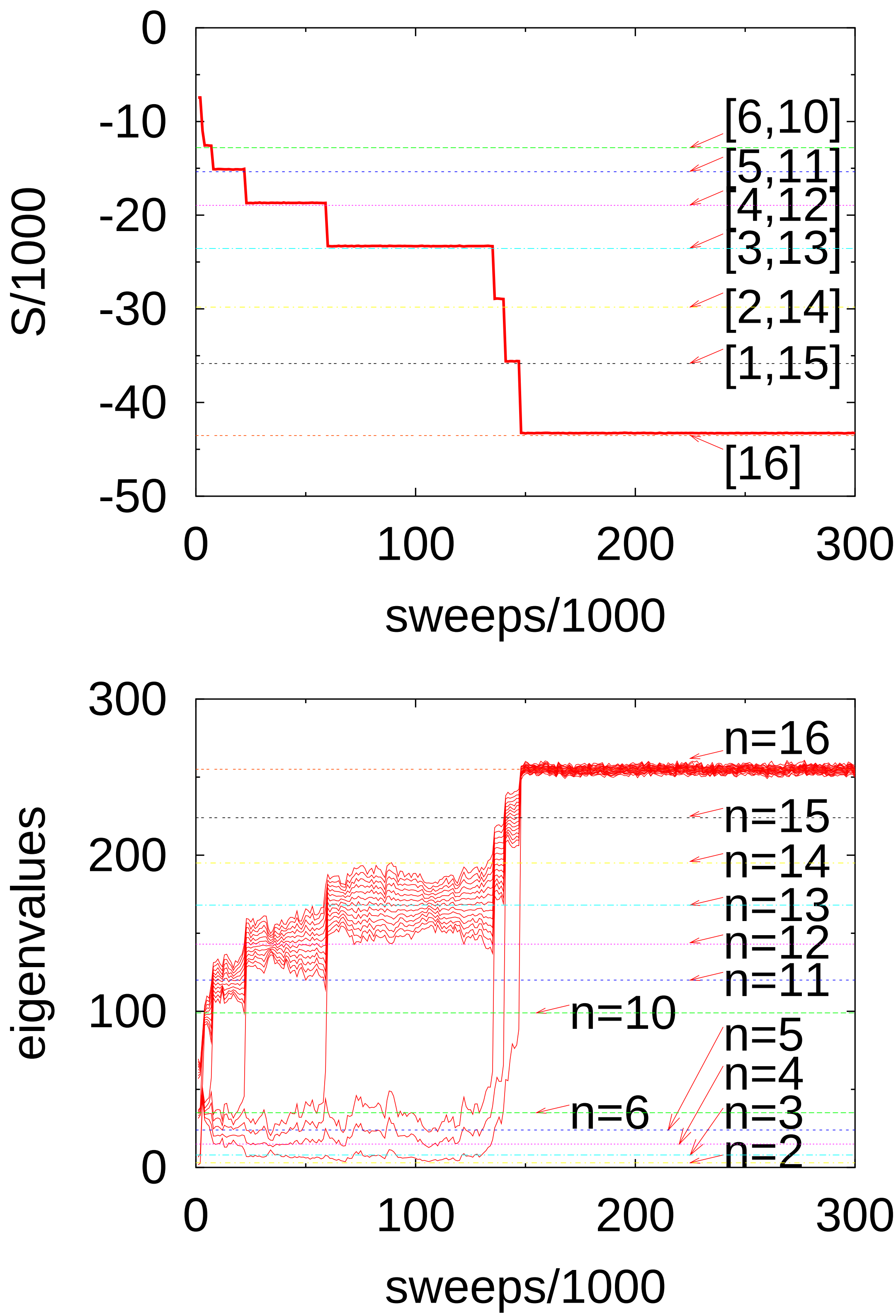


Figure 5: The history of the vacuum expectation value of the action $\langle S \rangle$ (left), and the eigenvalues of Q (right) against the sweeping time, for $N = 16$, $\alpha = 2.0$.

Metastability of the multi-fuzzy-sphere state

We compare the dependence of the multi-fuzzy-sphere state on k, α, N.

We initiate the simulation from

$$A_\mu^{(0)} = \alpha J_\mu^{(n)} \otimes 1_{k\times k},$$

namely when $n = n_1 = \cdots = n_k = \frac{N}{k}$.

- **k dependence: $N = 16$, $\alpha = 10.0$ fixed. $k = 2, 4, 8$. The sphere is more stable for smaller k (namely, larger $J_\mu^{(n)}$).**
- **α dependence: $N = 16$, $k = 8$ fixed. various α. The sphere is more stable for larger α.**
- **N dependence: $k = 2$, $\tilde{\alpha} = 40.0$ fixed. $N = 8, 16, 32$. The sphere is stable for larger N (commutative limit).**

6 Conclusion

In this work, we have investigated the stability of the fuzzy sphere in the matrix model with the Chern-Simons term.

- **The first-order phase transition between the Yang-Mills phase and the fuzzy sphere phase.**
- **One-loop exactness at the large N in the fuzzy-sphere phase.**

Future works:

- **Extension to the supersymmetric case.**
- **Extension to the higher-dimensional case.**
  fuzzy $2k$-sphere, $S^2 \times S^2, \cdots$.
- **Dynamical generation of the gauge group.**

Heat bath algorithm of the matrix model

(a) Warm-up: quadratic $U(N)$ one-matrix model

We start with the simplest case – quadratic $U(N)$ one-matrix model:

$$S = \frac{N}{2} Tr\phi^2.$$

We analyze this model via the heat bath algorithm. To this end, we rewrite the $U(N)$ matrix ϕ as

$$\phi_{ii} = \frac{a_i}{\sqrt{N}}, \quad \begin{cases} \phi_{ij} = \frac{x_{ij}+iy_{ij}}{\sqrt{2N}} \\ \phi_{ji} = \frac{x_{ij}-iy_{ij}}{\sqrt{2N}}, \end{cases} \quad (\text{for } i < j).$$

The N^2 real quantities a_i, x_{ij}, y_{ij} comply with the independent normal Gaussian distribution.

$$S = \frac{1}{2}\sum_{i=1}^{N} a_i^2 + \frac{1}{2}\sum_{i<j}((x_{ij})^2 + (y_{ij})^2).$$

$$Z = \int \prod_{i=1}^{N} da_i \prod_{1 \le i<j \le N} dx_{ij}dy_{ij} \exp\left(-\frac{1}{2}\sum_{i=1}^{N} a_i^2 - \frac{1}{2}\sum_{1\le i<j\le N}((x_{ij})^2 + (y_{ij})^2)\right).$$

a_i, x_{ij}, y_{ij} are updated by the Gaussian random number.

Generation of the uniform random number

We use the congruence method.

- We give the random seed z_1, such as $a_1 = \text{time}()$.
- We solve the recursion formula

$$z_{k+1} = az_k + c \quad (\text{mod } 2^{31} - 1).$$

  The choice $(a, c) = (5^{11}, 0)$ is known to give a good pseudo-random number.

- The sequence $\{\frac{z_k}{2^{31}-1}\}$ gives a uniform pseudo-random number [0:1].

Generation of the Gaussian random number

- We take two uniform random numbers $x, y \in [0:1]$.
- We introduce the quantity $r = \sqrt{-a^2 \log x^2}$. This complies with the probability distribution

$$P(r)dr = P(x)\frac{dx}{dr}dr = \frac{2r}{a^2}\exp\left(-\frac{r^2}{a^2}\right).$$

- We next introduce the quantities

$$X = r\cos(2\pi y), \quad Y = r\sin(2\pi y).$$

They comply with the probability distribution

$$P(r)drdy \propto \exp\left(-\frac{1}{a^2}(X^2+Y^2)\right).$$

(b) The bosonic IIB matrix model

T. Hotta, J. Nishimura and A. Tsuchiya hep-th/9811220.

We investigate the d-dimensional bosonic IIB matrix model via the the heat bath algorithm:

$$S = -\frac{N}{4}\sum_{\mu,\nu=1}^{d} Tr[A_\mu, A_\nu]^2 = -\frac{N}{2}\sum_{1\leq\mu<\nu\leq d} Tr\{A_\mu, A_\nu\}^2 + 2N\sum_{\mu<\nu} Tr(A_\mu^2 A_\nu^2).$$

This action is equivalent to $\tilde{S}$, after integrating out $Q_{\mu\nu}$ (where $G_{\mu\nu} = \{A_\mu, A_\nu\}$):

$$\begin{aligned}\tilde{S} &= \sum_{\mu<\nu}\left(\frac{N}{2}TrQ_{\mu\nu}^2 - NTr(Q_{\mu\nu}G_{\mu\nu}) + 2NTr(A_\mu^2 A_\nu^2)\right)\\ &= \frac{N}{2}\sum_{\mu<\nu} Tr(Q_{\mu\nu} - G_{\mu\nu})^2 + S.\end{aligned}$$

Then, $Q_{\mu\nu}$ is updated as

$$(Q_{\mu\nu})_{ii} = \frac{a_i}{\sqrt{N}} + (G_{\mu\nu})_{ii}, \ (Q_{\mu\nu})_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{2N}} + (G_{\mu\nu})_{ij},$$

We next update A_λ. We extract the dependence of A_λ.

$$\tilde{S} = -NTr(T_\lambda A_\lambda) + 2NTr(S_\lambda A_\lambda^2) + \cdots, \text{ where}$$
$$S_\lambda = \sum_{\mu\neq\lambda}(A_\mu^2),\ T_\lambda = \sum_{\mu\neq\lambda}(A_\mu Q_{\lambda\mu} + Q_{\lambda\mu}A_\mu).$$

- The diagonal part A_λ is updated by extracting the dependence of $(A_\lambda)_{ii}$:

$$\tilde{S} = 2N(S_\lambda)_{ii}(A_\lambda)_{ii}^2 - 4Nh_i(A_\lambda)_{ii}, \text{ where}$$
$$h_i = \frac{N}{4}[(T_\lambda)_{ii} - 2\sum_{j\neq i}((S_\lambda)_{ji}(A_\lambda)_{ij} + (S_\lambda)_{ij}(A_\lambda)_{ji})].$$

Then, $(A_\lambda)_{ii}$ is updated as

$$(A_\lambda)_{ii} = \frac{a_i}{\sqrt{4N(S_\lambda)_{ii}}} + \frac{h_i}{(S_\lambda)_{ii}}.$$

- The other components $(A_\lambda)_{ij}$ are updated likewise by extracting their dependence:

$$\tilde{S} = 2Nc_{ij}|(A_\lambda)_{ij}|^2 - 2Nh_{ji}(A_\lambda)_{ij}, \text{ where}$$
$$c_{ij} = (S_\lambda)_{ii} + (S_\lambda)_{jj},$$
$$h_{ij} = \frac{1}{2}(T_\lambda)_{ij} - \sum_{k\neq i}(S_\lambda)_{ik}(A_\lambda)_{kj} - \sum_{k\neq j}(S_\lambda)_{kj}(A_\lambda)_{ik}.$$

Then, $(A_\lambda)_{ij}$ are updated as

$$(A_\lambda)_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{4Nh_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$

(c) Extension to the bosonic IIB matrix model with the Chern-Simons term

The Chern-Simons term is *linear* with respect to *each* A_μ. We have only to replace T_λ as (for $d = 3$)

$$T_\lambda^{CS} = T_\lambda + 3g\epsilon_{\lambda\nu_1\nu_2}A_{\nu_1}A_{\nu_2}.$$