Gauge Theory on Fuzzy $S^2 \times S^2$ and Regularization on Noncommutative R^4

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Wolfgang Behr, Frank Meyer and Harold Steinacker

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Contents

[1] This slide is used for Takehiro Azuma's presentation at KEK. Therefore, it is not the authors *but the presenter Takehiro Azuma* that is responsible for any potential flaws in this slide.

1 Introduction

Motivations of fuzzy manifold studies:

- **Prototype of the curved-space background in the large-N reduced models.**
- **Dynamical generation of spacetime / gauge group.**

A plethora of works for the fuzzy sphere physics:

hep-th/0101102,0103192,0204256,0209057,0301055,0303120,0307007,0312241,0401038,0403242,0405096,0405277,0410263,

0412303,0412312,0504217 $\cdots$

Four-dimensional fuzzy manifolds:

$\mathbf{S}^2 \times \mathbf{S}^2$, $\mathbf{S}^4 = \mathbf{SO(5)/U(2)}$, $\mathbf{CP}^2 = \mathbf{SU(3)/U(2)}$, $\cdots$.

These fuzzy manifolds are compact, and thus realized by finite matrices.

We focus on the gauge theory on fuzzy $\mathbf{S}^2 \times \mathbf{S}^2$:

$\Rightarrow$ **We obtain $\mathbf{R}^4_\theta$ as a scalar field.**

2 Fuzzy spaces S^2_N and $S^2_{N_L} \times S^2_{N_R}$

Definitions of fuzzy S^2

$x_i = (x_1, x_2, x_3) =$ (Hermitian operators)

$$[x_i, x_j] = i\Lambda_N \epsilon_{ijk} x_k, \quad x_1^2 + x_2^2 + x_3^2 = R^2, \quad \Lambda_N = \frac{2R}{\sqrt{N^2-1}}.$$

Λ_N = (NC parameter) = O([length]^1).

They are obtained from the N-dimensional representation of su(2) λ_i:

$$x_i = \Lambda_N \lambda_i, \quad \Lambda_N = \frac{2R}{\sqrt{N^2-1}}, \text{ where } [\lambda_i, \lambda_j] = i\epsilon_{ijk}\lambda_k, \quad \lambda_i^2 = \frac{N^2-1}{4}.$$

(algebra of functions f on S^2_N) ⇔ (matrix algebra $\text{Mat}(N, C)$):

$$\int_{S^2_N} f = \frac{4\pi R^2}{N} \text{tr}\, f.$$

$N \to \infty$ (R fixed) limit : commutative S^2_N is recovered.

Definition of fuzzy $S^2_{N_L} \times S^2_{N_R}$

$\lambda^L_i, \lambda^R_j \Rightarrow N_L(N_R)$-dimensional representation of su(2) algebra:

$$[\lambda^L_i, \lambda^L_j] = i\epsilon_{ijk}\lambda^L_k, \ \ [\lambda^R_i, \lambda^R_j] = i\epsilon_{ijk}\lambda^L_k, \ \ [\lambda^L_i, \lambda^R_j] = 0, \ \ \text{where } (\lambda^{L,R}_i)^2 = \frac{N^2_{L,R} - 1}{4}.$$

This representation is realized by the tensor product:

$$\lambda^L_i = \lambda_i \otimes 1_{N_R}, \quad \lambda^R_i = 1_{N_L} \otimes \lambda_i.$$

(algebra of functions f on $S^2_{N_L} \times S^2_{N_R}$) = (matrix algebra $\mathrm{Mat}(\mathcal{N}, C)$),

where $\mathcal{N} = N_L N_R$.

Normalized coordinate function

$$x^{L,R}_i = \frac{2R}{\sqrt{(N_{L,R})^2 - 1}}\lambda^{L,R}_i, \quad (x^L_i)^2 = (x^R_i)^2 = R^2.$$

This space is a regularization of $S^2_{N_L} \times S^2_{N_R} \subset R^6$.

Normalized integral of a function $f \in S^2_{N_L} \times S^2_{N_R}$:

$$\int_{S^2_{N_L} \times S^2_{N_R}} f = \frac{V}{\mathcal{N}} \mathrm{tr}\, f, \ \ \text{where we define the volume } V = 16\pi^2 R^4.$$

Quantum plane limit R^4_θ

Tangential coordinate $x_{1,2}$ near the "north pole":

For $R^2 = N\theta/2$, we obtain

$$[x_1, x_2] = i\frac{2R}{N}x_3 = i\frac{2R}{N}\sqrt{R^2 - x_1^2 - x_2^2} = i\theta(1 + \mathrm{O}(1/N)).$$

Algebra of the quantum plane $[x_1, x_2] = i\theta$ ($R, N \to \infty$, θ fixed).

The fuzzy $\mathrm{S}^2_{N_L} \times \mathrm{S}^2_{N_R}$ case: ($R^2 = N_{L,R}\theta_{L,R}/2$)

$$[x_i^L, x_j^L] = i\epsilon_{ij}\theta^L, \quad [x_i^R, x_j^R] = i\epsilon_{ij}\theta^R, \quad [x_i^L, x_j^R] = 0.$$

Integral of a function $f(x) \in \mathrm{S}^2_{N_L} \times \mathrm{S}^2_{N_R}$:

$$\int_{\mathrm{S}^2_{N_L} \times \mathrm{S}^2_{N_R}} f(x) \to 4\pi^2\theta_L\theta_R \mathrm{tr}\, f(x) = \int_{\mathrm{R}^4_\theta} f(x).$$

3 Gauge theory on fuzzy $\mathrm{S}^2_{N_L} \times \mathrm{S}^2_{N_R}$

Construct $\mathrm{S}^2_{N_L} \times \mathrm{S}^2_{N_R}$ as "submanifold" of R^6.

Consider a multi-matrix model with 6 dynamical fields $B_i^{L,R}$:

$$S = \frac{1}{g^2}\int_{\mathrm{S}^2_{N_L}\times \mathrm{S}^2_{N_R}} \left(\frac{1}{2}F_{ia,jb}F_{ia,jb} + \varphi_L^2 + \varphi_R^2\right), \text{ where}$$

$$\varphi_{L(R)} = \frac{1}{R^2}\left(B_i^{L(R)}B_i^{L(R)} - \frac{N_{L(R)}^2 - 1}{4}\right),$$

$$F_{iL(R),jL(R)} = \frac{1}{R^2}\left(i[B_i^{L(R)}, B_j^{L(R)}] + \epsilon_{ijk}B_k^{L(R)}\right), \quad F_{iL,jR} = \frac{i}{R^2}[B_i^L, B_j^R].$$

Invariant under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and $\mathrm{U}(\mathcal{N})$.

Equation of motion

$$\{B_i^L, (B_j^L)^2 - \frac{N_L^2 - 1}{4}\} + (B_i^L + i\epsilon_{ijk}B_j^L B_k^L) + i\epsilon_{ijk}[B_j^L, (B_k^L + i\epsilon_{krs}B_r^L B_s^L)] + [B_j^R, [B_j^R, B_i^L]] = 0.$$

Classical solution $F = \varphi = 0 \Rightarrow B_i^{L(R)} = \lambda_i^{L(R)}$.

Expansion around the classical solution $B_i^a = \lambda_i^a + RA_i^a$.

Gauge transformation of the fluctuation : $A_i^{L(R)} \to A_i'^{L(R)} = UA_i^{L(R)}U^{-1} + U[\lambda_i^{L(R)}, U^{-1}]$.

The field strength takes the form

$$F_{iL(R),jL(R)} = \frac{i}{R}\left([\lambda_i^{L(R)}, A_j^{L(R)}] - [\lambda_j^{L(R)}, A_i^{L(R)}] + R[A_i^{L(R)}, A_j^{L(R)}]\right),$$

$$F_{iL,jR} = \frac{i}{R}\left([\lambda_i^{L}, A_j^{R}] - [\lambda_j^{R}, A_i^{R}] + R[A_i^{L}, A_j^{R}]\right).$$

Commutative limit ⇒ separate the radial/tangential degrees of freedom.

$\varphi_{L(R)}$ is bounded for configurations with finite action.

$$\varphi_{L(R)} = \frac{1}{R}(\lambda_i^{L(R)} A_i^{L(R)} + A_i^{L(R)} \lambda_i^{L(R)}) + A_i^{L(R)} A_i^{L(R)} \quad \Rightarrow x_i A_i^a + A_i^a x_i = \mathrm{O}(\frac{\varphi}{N}).$$

A_i^a is tangential in the commutative limit.

Standard electrodynamics on commutative $\mathrm{S}^2_{N_L} \times \mathrm{S}^2_{N_R}$:

$$S = \frac{1}{2g^2} \int_{\mathrm{S}^2_{N_L} \times \mathrm{S}^2_{N_R}} F^t_{ia,jb} F^t_{ia,jb}.$$

At the north pole, $\frac{i}{R}\mathrm{ad}\lambda_i^{L(R)} = -\epsilon_{ij} \frac{\partial}{\partial x_j^{L(R)}}$.

The field strength :$F^t_{iL,jR} = \partial_i^L A_j^{(\mathrm{cl})R} - \partial_j^R A_i^{(\mathrm{cl})L}$, $(A_i^{(\mathrm{cl})L(R)} = -\epsilon_{ij} A_j^{L(R)})$.

4 Formulation based on SO(6)

We cast the action on $S^2_{N_L} \times S^2_{N_R}$ into the formulation based on SO(6).

Embed $S^2_{N_L} \times S^2_{N_R} \subset R^6$ ($SO(3)_L \times SO(3)_R \subset SO(6)$):

$$B_\mu = (B_i^L, B_j^R), \quad \gamma_\mu = (\gamma_i^L, \gamma_i^R),$$
$$B^L = \frac{1}{2} + B_i^L \gamma_i^L, \quad B^R = \frac{i}{2} + B_i^R \gamma_i^R.$$

The gamma matrices' relations:

$$(\gamma_i^L)^\dagger = \gamma_i^L, \quad (\gamma_i^R)^\dagger = -\gamma_i^R,$$
$$\gamma_i^L \gamma_j^L = \delta_{ij} + i\epsilon_{ijk}\gamma_k^L, \quad \gamma_i^R \gamma_j^R = -\delta_{ij} - \epsilon_{ijk}\gamma_k^R, \quad [\gamma_i^L, \gamma_j^R] = 0.$$

(1) separate B into two $4\mathcal{N} \times 4\mathcal{N}$ matrices

We break the symmetry as SO(6) → SO(3) × SO(3).

$$\gamma_i^L = \sigma_i \otimes 1_{2\times 2}, \quad \gamma_i^R = 1_{2\times 2} \otimes i\sigma_i.$$

For the projector $P = \frac{1}{2}(1 + \sigma_i \otimes \sigma_i)$**, they satisfy** $\gamma_i^R = iP\gamma_i^L P$, $P^2 = 1$.

The degree of freedom reduces to two $2\mathcal{N} \times 2\mathcal{N}$ **matrices:**

$$B_L = \underbrace{\left(B_i^L \sigma_i + \frac{1}{2}\right)}_{=X_L} \otimes 1_{2\times 2}, \quad B_R = iP\left\{\underbrace{\left(B_i^R \sigma_i + \frac{1}{2}\right)}_{=X_R} \otimes 1_{2\times 2}\right\} P.$$

The action $S = S_6 - 2S_{\text{break}}$ **recovers the action of** $\mathrm{S}^2_{N_L} \times \mathrm{S}^2_{N_R}$**:**

$$\begin{aligned} S_6 &= 2\mathrm{Tr}\left(B_L^2 - B_R^2 - \frac{N^2}{2}\right)^2 + 2\mathrm{Tr}[B_L, B_R]^2, \\ S_{\text{break}} &= -2\mathrm{Tr}\left(B_L^2 - \frac{N^2}{4}\right)\left(-B_R^2 - \frac{N^2}{4}\right). \end{aligned}$$

(2) Embed B into one $8\mathcal{N} \times 8\mathcal{N}$ matrices

Construct the 8×8 gamma matrices:

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ (\gamma^\mu)^\dagger & 0 \end{pmatrix}, \text{ where } \{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}.$$

Consider a single Hermitean $8\mathcal{N} \times 8\mathcal{N}$ matrix:

$$C = \Gamma^\mu B_\mu + \underbrace{(C_0^L + C_0^R)}_{=C_0} = \underbrace{\begin{pmatrix} 0 & B^L \\ B^L & 0 \end{pmatrix}}_{=C^L} + \underbrace{\begin{pmatrix} 0 & B^R \\ -B^R & 0 \end{pmatrix}}_{=C^R}, \text{ where}$$

$$C_0^L = -\frac{i}{2}\Gamma_1^L\Gamma_2^L\Gamma_3^L = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_0^R = -\frac{i}{2}\Gamma_1^R\Gamma_2^R\Gamma_3^R = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The following action is close to that of $S^2_{N_L} \times S^2_{N_R}$:

$$S_6 = \mathrm{Tr}\left((C^2 - \frac{N^2}{2})^2\right) = 8\mathrm{tr}\,(B_\mu B_\mu - \frac{N^2-1}{2})^2 + 4\mathrm{tr}\,F_{\mu\nu}F_{\mu\nu}. \text{ where}$$

$$F_{ia,jb} = i[B_{ia}, B_{jb}] + \delta_{ab}\epsilon_{ijk}B_{ka}.$$

But we have $(B_\mu B_\mu - \frac{N^2-1}{2})^2$, instead of $(B_i^L B_i^L - \frac{N_L^2-1}{4})^2 + (B_i^R B_i^R - \frac{N_R^2-1}{4})^2$.

This is because its ground state should be some S^5.

We have to explicitly break the symmetry as $\mathbf{SO(6) \to SO(3) \times SO(3)}$.

We introduce the term

$$S_{\rm break} = {\rm Tr}(C_L^2 - \frac{N_L^2}{4})(C_R^2 - \frac{N_R^2}{4}).$$

The following action recovers that of $\mathbf{S^2_{N_L} \times S^2_{N_R}}$:

$$S = S_6 - 2S_{\rm break} = 8{\rm tr}\left((B_{iL}^2 - \frac{N_L^2 - 1}{4})^2 + (B_{iR}^2 - \frac{N_R^2 - 1}{4})^2 + \frac{1}{2} F_{\mu\nu} F_{\mu\nu} \right).$$

This formulation is useful in introducing fermions.

5 Quantization

The quantization is straightforward by a path integral over the Hermitian matrices:

No need to fix the gauge (since the gauge group U($\mathcal{N}$) is compact).

Finite path integral for any fixed $\mathcal{N}$.

(the square term of $B_i^{L(R)}$ further suppresses the path integral).

Perturbation around the fuzzy sphere → Gauge fixing by usual BRST prescription:

$$S_{\mathrm{BRST}} = S + \frac{1}{\mathcal{N}}\mathrm{tr}\left(\bar{c}[\lambda_\mu, [B_\mu, c]] - (\frac{\alpha}{2}b - [\lambda_\mu, B_\mu])b\right).$$

BRST transformation (such that $s^2 = 0$)

$$sB_\mu = [B_\mu, c], \;\; sc = cc, \;\; s\bar{c} = b, \;\; sb = 0.$$

6 Topologically nontrivial solutions on $S^2_{N_L} \times S^2_{N_R}$

Monopole solutions

$$B_i^L = \alpha^L \lambda_i^{N-m_L} \otimes 1_{N-m_R}, \quad B_i^R = \alpha^R 1_{N-m_L} \otimes \lambda_i^{N-m_R},$$
$$\text{where } \alpha^{L,R} = 1 + \frac{m_{L,R}}{N}, \ m_{L,R} \ll N.$$

Associated field strength and constraint term:

$$F_{iL(R),jL(R)} = -\frac{m^{L(R)}}{2R^3}\epsilon_{ijk}x_k^{L(R)}, \quad F_{iL,jR} = 0,$$
$$\lim_{N\to\infty}(B_i^{L(R)}B_i^{L(R)} - \frac{N^2-1}{4}) = 0.$$

Commutative limit: $F = -2\pi(m^L\omega^L + m^R\omega^R)$, where $\omega^{L(R)} = \frac{1}{4\pi R^3}\epsilon_{ijk}x_i^{L(R)}dx_j^{L(R)}dx_k^{L(R)}$.

This monopole solution is realized for matrix size $\mathcal{N} = (N-m_L)(N-m_R) \neq N^2$.

This mismatch can be reconciled by combining this with another type of solution.

Fluxon solutions

$$B_i^{L(R)} = \text{diag}(d_{i,1}^{L(R)}, \cdots, d_{i,n}^{L(R)}), \text{ where } \sum_i d_{i,k}^{L(R)} d_{i,k}^{L(R)} = \begin{cases} \frac{N^2-3}{4}, & \text{type A}, \\ 0, & \text{type B}. \end{cases}$$

Associated field strength and constraint term:

$$F_{iL(R),jL(R)} = \frac{\epsilon_{ijk}}{R^2} \text{diag}(d_{k,1}^L, \cdots, d_{k,n}^L), \quad F_{iL,jR} = 0,$$

$$\lim_{N \to \infty} (B_i^{L(R)} B_i^{L(R)} - \frac{N^2-1}{4}) = \begin{cases} -\frac{1}{2}, & \text{type A}, \\ -\frac{N^2-1}{4}, & \text{type B}. \end{cases}$$

Only the type A solution has a finite contribution:

$$S = \begin{cases} \frac{16\pi^2 R^4}{g^2 N^2} \left(\frac{n}{4R^4} + \frac{2n}{R^4} \frac{N^2-3}{4} \right) \to \frac{8\pi^2}{g^2} n, & \text{type A}, \\ \frac{16\pi^2 R^4}{g^2 N^2} \left(2 \times (\frac{N^2-1}{4})^2 \right) \to \infty, & \text{type B}. \end{cases}$$

We call these type A solutions "fluxons".

Combination of monopole and fluxon solutions

$$B_i^L = \begin{pmatrix} \alpha^L \lambda_i^{N-m_L} \otimes 1_{N-m_R} & 0 \\ 0 & \mathrm{diag}(d_{i,1}^L, \cdots, d_{i,n}^L) \end{pmatrix},$$

$$B_i^R = \begin{pmatrix} \alpha^R 1_{N-m_L} \otimes \lambda_i^{N-m_R} & 0 \\ 0 & \mathrm{diag}(d_{i,1}^R, \cdots, d_{i,n}^R) \end{pmatrix}, \text{ where}$$

$$m_L = -m_R = m, \quad n = m^2.$$

The total action of this solution:

$$S_{(m)} = \frac{4\pi^2}{g^2}(2m^2 + 2m^2), \text{ as } N \to \infty.$$

7 Gauge theory on R^4_θ from fuzzy $\mathrm{S}^2_{N_L} \times \mathrm{S}^2_{N_R}$

The most general noncommutative R^4_θ (for $\mu, \nu = 1, 2, 3, 4$)

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}, \text{ where } \theta_{\mu\nu} = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix}.$$

. We define the coordinates X_μ and ϕ as

$$X_{1,2} = \sqrt{\frac{2\theta_{12}}{N_L}} B^L_{1,2}, \quad X_{3,4} = \sqrt{\frac{2\theta_{34}}{N_R}} B^R_{1,2},$$

$$\phi^{L(R)} = B^{L(R)}_3 - \frac{N_{L(R)}}{2} + \frac{1}{N_{L(R)}} [(B^{L(R)}_1)^2 + (B^{L(R)}_2)^2].$$

Scaling limit $R^2 = \frac{1}{2} N_L \theta_{34} = \frac{1}{2} N_R \theta_{12} \to \infty$:

X are the covariant coordinates on the tangential R^4_θ as $N_{L,R} \to \infty$:

$$\begin{aligned}
\frac{1}{R^2}([B_1^L, B_1^R]) &= \frac{1}{\theta_{12}\theta_{34}}[X_1, X_3], \quad \text{etc.}, \\
\frac{1}{R^2}(B_1^L + i[B_2^L, B_3^L]) &= \sqrt{\frac{1}{\theta_{12}\theta_{34}R^2}}\,(X_1 + i[X_2, \phi^L] - \frac{i}{2\theta_{12}}[X_2, (X_1)^2]), \\
\frac{1}{R^2}(B_2^L + i[B_3^L, B_1^L]) &= \sqrt{\frac{1}{\theta_{12}\theta_{34}R^2}}\,(X_2 + i[X_1, \phi^L] - \frac{i}{2\theta_{12}}[X_1, (X_2)^2]), \\
\frac{1}{R^2}(B_3^L + i[B_1^L, B_2^L]) &= \frac{1}{\theta_{12}\theta_{34}}(\theta_{12} + i[X_1, X_2] + \frac{\theta_{12}\theta_{34}}{R^2}\phi_L - \frac{\theta_{12}\theta_{34}^2}{2R^4}((X_1)^2 + (X_2)^2)) \\
\frac{1}{R^2}(B_i^L B_i^L - \frac{N_L^2 - 1}{4}) &= \frac{1}{\theta_{34}}\phi^L + \frac{2}{R^2}((\phi^L)^2 + \frac{1}{4}) - \frac{1}{\theta_{12}R^2}\{\phi^L, (X_1)^2 + (X_2)^2\} \\
&\quad + \frac{1}{\theta_{12}^2 R^2}((X_1)^2 + (X_2)^2)^2.
\end{aligned}$$

The terms from the $\mathrm{S}^2_{N_L} \times \mathrm{S}^2_{N_R}$ action involving $\phi^{L,R}$:

$$\frac{1}{\theta_{34}^2}(\phi^L)^2 + \frac{1}{\theta_{12}^2}(\phi^R)^2 + \mathrm{O}(\frac{1}{R}).$$

At $R \to \infty$, we obtain the action

$$S = -\frac{1}{2g^2\theta_{12}^2\theta_{34}^2}\int([X_\mu, X_\nu] - i\theta_{\mu\nu})^2.$$

X_μ can be written as $X_\mu = x_\mu + i\theta_{\mu\nu}A_\nu$.

U(1) instantons on R^4_θ

We consider the self-dual case $\theta_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta_{\rho\sigma}$, namely $\theta_{12} = \theta_{34} (= \theta)$.

The action for U(1) gauge theory on R^4_θ:

$$S = \frac{(2\pi)^2}{2g^2\theta^2}\mathrm{tr}\, F_{\mu\nu}F_{\mu\nu}, \text{ where } F_{\mu\nu} = i([X_\mu, X_\nu] - i\theta_{\mu\nu}).$$

Complex coordinates: $x_{\pm L} = x_1 \pm i x_2, x_{\pm R} = x_3 \pm i x_4$.

Commutation relation: $[x_{+a}, x_{-b}] = 2\theta\delta_{ab}, [x_{+a}, x_{+b}] = [x_{-a}, x_{-b}] = 0$.

Basis of the Fock space $\mathcal{H}$ for these coordinates: $|n_1, n_2\rangle$, $n_1, n_2 \in \mathrm{N}$.

$$x_{-L}|n_1,n_2\rangle = \sqrt{2\theta}\sqrt{n_1+1}|n_1+1,n_2\rangle,\ x_{+L}|n_1,n_2\rangle = \sqrt{2\theta}\sqrt{n_1}|n_1-1,n_2\rangle,$$
$$x_{-R}|n_1,n_2\rangle = \sqrt{2\theta}\sqrt{n_2+1}|n_1,n_2+1\rangle,\ x_{+R}|n_1,n_2\rangle = \sqrt{2\theta}\sqrt{n_2}|n_1,n_2-1\rangle\,.$$

Complex covariant coordinates: $X_{\pm L} = X_1 \pm i X_2$, $X_{\pm R} = X_3 \pm i X_4$.

The action is rewritten as

$$S = \frac{\pi^2}{g^2\theta^2}\mathrm{tr}\left(\sum_a F_{+a,-a}F_{+a,-a} - \sum_{a,b} F_{+a,+b}F_{-a,-b}\right), \text{ where } F_{a\alpha,b\beta} = [X_{\alpha a}, X_{\beta b}] - 2\theta\epsilon_{\alpha\beta}\delta_{ab}.$$

Equation of motion: $\sum_{a,\alpha}[X_{\alpha,a}, (F_{\alpha a,\beta b})^\dagger] = 0$.

Finite-dimensional subvector space $V_n \subset \mathcal{H}$:

$$V_n = \{|i_k, j_k\rangle; k = 1, 2, \cdots, n\}.$$

Solutions of the equation of motion:

$$X^{(n)}_{+L(R)} = S x_{+L(R)} S^\dagger + \sum_{k=1}^{n} \gamma_k^{L(R)} |i_k, j_k\rangle\langle i_k, j_k|.$$

- $\gamma_k^{L,R} \in \mathrm{C}$: **position of fluxons**
- $S = \sum_{k=1}^{\infty} |i_{k+n}, j_{k+n}\rangle\langle i_k, j_k|$, **with** $S^\dagger S = 1,\ SS^\dagger = 1 - \underbrace{\sum_{k=1}^{n} |i_k, j_k\rangle\langle i_k, j_k|}_{=P_{V_n}}$.

  P_{V_n}: **projection operator onto** V_n.

$$F_{\mu\nu} = P_{V_n}\theta_{\mu\nu} \to S[X^{(n)}_{\pm,a}] = \frac{8\pi^2}{g^2}\mathrm{tr}\,(P_{V_n}) = \frac{8\pi^2 n}{g^2}.$$

This is the U(1)-instanton solutions on R^4_θ.

8 Fermions

Commutative Dirac operator on $\mathrm{S}^2 \times \mathrm{S}^2$

$$D_4 = \Gamma^\mu J_\mu + 2C_0.$$

The 8-component spinor Ψ_8 is split into two independent 4-component Dirac spinors.

Operators that anticommute with D_4:

- **6-dimensional chirality operator:** $\Gamma = i\Gamma_1^L\Gamma_2^L\Gamma_3^L\Gamma_1^R\Gamma_2^R\Gamma_3^R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

  This satisfies $\{D_4, \Gamma\} = 0$, $\Gamma^\dagger = \Gamma$ **and** $\Gamma^2 = 1$.

  The 8-component spinor is split as $\Psi_8 = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}_{\bar{\beta}} \end{pmatrix}$.

- **Operators** $\chi_{L(R)} = \Gamma^{iL(R)} x_{iL(R)}$, **which preserves** $\mathrm{SO}(3) \times \mathrm{SO}(3) \subset \mathrm{SO}(6)$.

  They satisfy $\{D_4, \chi_{L(R)}\} = \{\chi_L, \chi_R\} = 0$, $\chi_{L(R)}^2 = 1$.

We introduce a projector $P_{\pm} = \frac{1}{2}(1 \pm i\chi_L \chi_R)$.

$$P_{\pm}^2 = P_{\pm}, \;\; P_{+} + P_{-} = 1, \;\; P_{+}P_{-} = 0, \;\; [P_{\pm}, D_4] = [P_{\pm}, \Gamma] = 0.$$

The projected space is preserved by D_4 and Γ.

spinor Lagrangian : $S_{\rm f} = \Psi_8^{\dagger} D_4 \Psi_8 = \Psi_{+}^{\dagger} D_4 \Psi_{+} + \Psi_{-}^{\dagger} D_4 \Psi_{-}$, where $\Psi_{\pm} = P_{\pm}\Psi_8$.

Two ways to obtain a 4-component Dirac spinor:

- **Impose the constraint $P_{+}\Psi_8 = \Psi_8$.**
- **Add a mass term $M_{-}\Psi_8^{\dagger} P_{-} \Psi_8$, with $M_{-} \to \infty$.**

At the north pole $x_L = x_R =^t (1, 0, 0)$,

$$P_{+} = \frac{1}{2}\left\{ 1 + i \begin{pmatrix} -\gamma_1^L \gamma_1^R & 0 \\ 0 & \gamma_1^L \gamma_1^R \end{pmatrix} \right\} = \frac{1}{2}(1 + \sigma_3 \otimes \sigma_3 \otimes \sigma_3) = {\rm diag}(1, 0, 0, 1, 0, 1, 1, 0).$$

$P_{\pm}$ actually projects onto a 4-dimensional subspace exactly.

Gauged fuzzy Dirac and chirality operators

Fuzzy analogs of D_4 and χ.

Natural fuzzy spinor action : $S_F = \Psi^\dagger C \Psi$ ($\Psi = 8\mathcal{N} \times \mathcal{N}$ matrix).

C can be split into fuzzy Dirac operator $\hat{D}$ and operator $\hat{\chi}$:

$$\hat{\chi}\Psi = \sqrt{\frac{2}{N^2}}(\Gamma^\mu \Psi \lambda_\mu - C_0 \Psi), \text{ such that } \hat{\chi}^2 = 1.$$

Fuzzy Dirac operator:

$$\hat{D} = C - \sqrt{\frac{N^2}{2}}\hat{\chi} = \Gamma^\mu \hat{\mathcal{D}}_\mu + 2C_0, \text{ where } \hat{\mathcal{D}}_\mu \Psi = \underbrace{[\lambda_\mu, \Psi]}_{=\hat{J}_\mu \Psi} + A_\mu \Psi.$$

$\hat{\mathcal{D}}_\mu$ is a covariant derivative operator, i.e. $U\hat{\mathcal{D}}_\mu \psi = \hat{\mathcal{D}}'_\mu U \psi$

(where $D'_\mu \Psi = [\lambda_\mu, \Psi] + \underbrace{U A_\mu U^{-1}}_{=A'_\mu}$)

$\hat{D}$ satisfies $\{\hat{D}, \Gamma\} = 0$, however $\{\hat{D}, \chi\} \neq 0$:

$$\{\hat{D}, \hat{\chi}\} = O(\frac{1}{N}).$$

$$\hat{D}^2\psi = (\Sigma^{\mu\nu}F_{\mu\nu} + \underbrace{\hat{\mathcal{D}}_\mu\hat{\mathcal{D}}_\mu + \{\Gamma^\mu, C_0\}\hat{\mathcal{D}}_\mu}_{=\square} + 2)\psi.$$

This corresponds to the $\hat{D}^2$ **on the curved space.**

Projectors on the fuzzy spinors:

$$\hat{\chi}_{L(R)}\Psi = \frac{2}{N}(\Gamma^{iL(R)}\Psi\lambda_{iL(R)} + C_0^{L(R)}).$$

We define the projection operators $\hat{P}_\pm = \frac{1}{2}(1 \pm i\hat{\chi}_L\hat{\chi}_R)$.

$$\hat{\chi}^2_{L(R)} = 1, \ \{\hat{\chi}_L, \hat{\chi}_R\} = 0, \ (\hat{\chi}_L\hat{\chi}_R)^2 = 1,$$
$$[\hat{D}^2, \Gamma] = [\Sigma^{\mu\nu}F_{\mu\nu}, \Gamma] = [\hat{D}^2, \hat{P}_\pm] = [\Sigma^{\mu\nu}F_{\mu\nu}, \hat{P}_\pm] = 0.$$

The projector no longer commutes with the fuzzy Dirac operator: $[\hat{D}, \hat{\chi}_L\hat{\chi}_R] \neq 0$.

We have to add a mass term to reduce the degrees of freedom: $M_-\Psi_8^\dagger\hat{P}_-\Psi_8$.

The complete action for a Dirac fermion on fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$**:**

$$S_{\mathrm{Dirac}} = \int \left(\Psi_8^\dagger(\hat{D} + m)\Psi_8 + M_-\Psi_8^\dagger\hat{P}_-\Psi_8\right).$$

9 Summary

In this paper, we have studied the following:

- **Gauge theory on fuzzy $S^2 \times S^2$ as a multi-matrix model.**
- **Alternative formulations using "collective matrices" based on SO(6).**
- **Quantization by a finite path integral.**
- **The monopole and fluxon solution**
- **The quantum field theory in the flat noncommutative plane R^4_θ.**
- **Fermionic term and the chiral Dirac operator.**