Once-upon-a-time story of the matrix model as a constructive definition of string theory

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Contents

1 Introduction

The promising candidate of the constructive definition of superstring theory:

IIB matrix model

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

$$S = \frac{-1}{g^2} Tr_{N\times N}\left(\frac{1}{4}\sum_{\mu,\nu=0}^{9}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\psi}\sum_{\mu=0}^{9}\Gamma^\mu[A_\mu,\psi]\right).$$

IIB matrix model has the following illuminating features:

- We can describe the multi-body system of D-branes. IIB matrix model is not the D-instant action but the second quantization of superstring theory.
- Evidence of the gravitational interaction:
  - ⋆ When we regard the eigenvalues as the spacetime coordinates, this model incorporates $\mathcal{N} = 2$ SUSY. (hep-th/9612115)
  - ⋆ Graviton-dilaton exchange: (hep-th/9612115)
  - ⋆ Diffeomorphism invariance: (hep-th/9903217)
- Derivation of 4-dimensional spacetime: (hep-th/9802085,0204240,0211272)

How did the idea of "large-N reduced models" come about?

In order to see this, we explore the development of the old matrix model in the early 1990's.

2 Quantization of bosonic string for $D \leq 1$

J. Distler and H.Kawai, Nucl.Phys.B321:509,1989 .

Distler and Kawai succeeded in the quantization of bosonic-and-super string theory for $D \leq 1$.

$$Z = \int \frac{dgdX}{V_{diff \times Weyl}} e^{-S_M} = \int dX d\phi dbdce^{-(S_M + S_{bc})}.$$

- $S_M = \frac{1}{8\pi} \int_M d^2\xi \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu + \frac{1}{4\pi} \int_M d^2\xi \sqrt{g} R$.
  $g_{ab} =$ (metric of the worldsheet).
- $S_{bc} = \frac{1}{2\pi} \int_M d^2 z (b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}})$.
  Both S_M and S_{bc} are, *per se*, Weyl invariant.
- We integrate out the worldsheet metric $g_{ab} = e^\phi \hat{g}_{ab}$.
  $\phi =$ (parameter of Weyl transformation).
- In this section we set the Regge slope as $\alpha' = 2$.

Liouville mode for noncritical string

We have the Liouville mode in the Weyl transformation for the noncritical string $D \neq 26$.

$$[dX]_g = [dX]_{\hat{g}} \exp\left(\frac{D}{48\pi} S_L(\phi, \hat{g})\right),$$

$$[dbdc]_g = [dbdc]_{\hat{g}} \exp\left(-\frac{26}{48\pi} S_L(\phi, \hat{g})\right), \text{ where}$$

$$S_L = \int d^2\xi \sqrt{\hat{g}} \left(\frac{1}{2}\hat{g}^{ab} \partial_a \phi \partial_b \phi + \hat{R}\phi + \mu(e^\phi - 1)\right).$$

$\mu =$ (arbitrary integration constant).
For $D \neq 26$, we must integrate over the Liouville mode.

However, it is a conundrum to perform the path integral of the Liouville mode.

$$\| \delta\phi \|_g^2 = \int d^2\xi \sqrt{g}(\delta\phi)^2 = \int d^2\xi \sqrt{\hat{g}} e^{\phi} (\delta\phi)^2.$$

- The measure represents the distance in the functional space.
- The measure depends on ϕ itself!!

In order to evade this problem, we set an ansatz for the Jacobian of the Weyl transformation.

$$[dX d\phi db dc]_g = [dX d\phi db dc]_{\hat{g}} J, \text{ where}$$

$$J = \exp\left(-\frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}} (\hat{g}^{ab} \partial_a \phi \partial_b \phi - Q\hat{R}\phi + 4\mu_1 e^{\alpha\phi}) \right).$$

$$= \exp\left(-\frac{1}{2\pi} \int d^2 z (\partial\phi\bar{\partial}\phi - \frac{\sqrt{\hat{g}}}{4} Q\hat{R}\phi + \mu_1 \sqrt{\hat{g}} e^{\alpha\phi}) \right).$$

- The coefficient Q must be determined by the Weyl invariance of the partition function.

  The energy-momentum tensor of the Liouville mode:

$$T_L(z) = -\frac{1}{2} : \partial\phi\partial\phi(z) : -\frac{Q}{2}\partial^2\phi(z),$$

$$T_L(z) T_L(w) = \frac{1+3Q^2}{2(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{2}\partial T(w).$$

  The Weyl invariance implies

$$c_L + c_M + c_{gh} = 1 + 3Q^2 + D - 26 = 0, \ \Leftrightarrow Q = \sqrt{\frac{25-D}{3}}.$$

- α must be determined by the Weyl invariance of $g_{ab} = e^{\alpha\phi}\hat{g}_{ab}$:

$$(\text{conformal weight of } e^{\alpha\phi}) = -\frac{1}{2}\alpha(Q+\alpha) = 1.$$

Then, we have two choices:

$$\alpha_{\pm} = -\frac{1}{2\sqrt{3}}(\sqrt{25-D} \mp \sqrt{1-D}).$$

Since α_- does not coincide with the classical limit $D \to -\infty$, we obtain

$$\alpha = \alpha_+ = -\frac{1}{2\sqrt{3}}(\sqrt{25-D} - \sqrt{1-D}).$$

This quantization is well-defined only for $D \leq 1$.

- $1 < D < 25$: We have a tachyon vertex!

$$e^{\alpha\phi} = \exp\left(-\frac{\phi}{2\sqrt{3}}\sqrt{25-D} + \underbrace{\frac{i\phi}{2\sqrt{3}}\sqrt{D-1}}_{\text{tachyon vertex!}}\right).$$

Liouville mode is thus unstable.

- $D \geq 25$: Q, α are pure-imaginary.

In order to evade the unstable tachyon vertex, we should take $\phi \to -i\phi$.

However, ϕ is regarded as the ghost field.

Evaluation of critical exponent

We derive the string susceptibility for all genera.

$$Z(A) = K A^{\gamma-3}, \text{ where}$$
$$A = \int d^2\xi \sqrt{g} = (\text{area of world sheet}),$$
$$\gamma = (\text{string susceptibility}).$$

$$Z = \int [dX d\phi db dc]_{\hat{g}} e^{-(S_M + S_{bc} + J)} \delta\left(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\phi} - A\right).$$

The partition function is invariant under $\phi \to \phi + \frac{\rho}{\alpha}$:

$$J \to J - \frac{Q}{8\pi} \int d^2\xi \sqrt{\hat{g}} \hat{R} \frac{\rho}{\alpha} = J - \frac{(1-h)Q\rho}{\alpha},$$
$$\delta\left(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\phi} - A\right) \to e^{-\rho} \delta\left(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\phi} - e^{-\rho} A\right).$$

Then, we obtain

$$Z(A) = \exp\left(\rho\left(\frac{Q(1-h)}{\alpha} - 1\right)\right) Z(e^{-\rho} A), \Leftrightarrow Z = K A^{\frac{Q}{\alpha}(1-h)-1}.$$

$$\gamma = 2 + \frac{1-h}{12}\left(D - 25 - \sqrt{(25-D)(1-D)}\right).$$

This result coincides with the matrix-model analysis for all genera.

The coincidence of the string susceptibility gives an important touchstone for the legitimacy of the matrix model as a nonperturbative formulation of string theory.

Extension to 'super'string theory

The analysis of Distler and Kawai is extended to the 'super'string theory.

Again, the quantization is well-defined only for $D \leq 1$.

$$
\begin{aligned}
Q &= \sqrt{\frac{9-D}{2}}, \\
\alpha &= -\frac{1}{2\sqrt{2}}\left(\sqrt{9-D}-\sqrt{1-D}\right), \\
\gamma &= 2+\frac{1-h}{4}\left(D-9-\sqrt{(9-D)(1-D)}\right).
\end{aligned}
$$

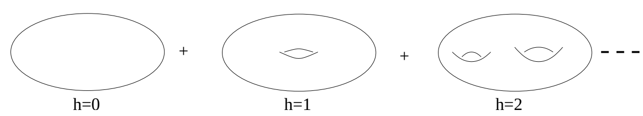
3 Random triangulation

Random triangulation is suggested as a constructive definition of the quantum gravity.

F. David, Nucl.Phys.B257:543,1985 .

The path integral of the string:

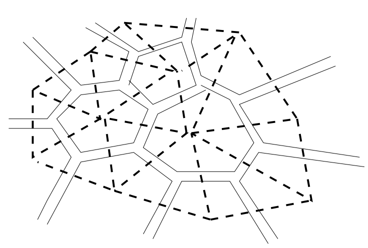
$$Z = \sum_{h=0}^{\infty} \int dg \exp(-\beta A + \gamma\chi).$$



- $A = \frac{1}{8\pi} \int d^2\xi \sqrt{g}$ = (area of world sheet)
- $S_M = \frac{1}{8\pi} \int d^2\xi \sqrt{g} g^{\alpha\beta} \partial_a X^\mu \partial_b X_\mu$ is equivalent to A for $D = 0$.
- $\chi = \frac{1}{4\pi} \int d^2\xi \sqrt{g} R = 2(1-h)$.

It is difficult to evaluate this path integral exactly. We resort to the discritization of the world sheet into many equilateral triangles.

$$\sum_{h=0}^{\infty} \int d^2\xi \Rightarrow \sum_{\text{random triangulation}} .$$

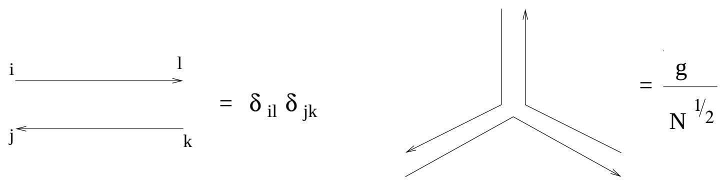


The triangulation is described by 0-dimensional ϕ^3 theory.

$$S = \frac{1}{2} TrM^2 - \frac{g}{\sqrt{N}} TrM^3.$$

- $M =$ ($N \times N$ hermitian matrix)
- Feynman rule:

$$\langle M_{ij} M_{kl} \rangle = \frac{\int d^{N^2} M M_{ij} M_{kl} e^{-\frac{1}{2} TrM^2}}{\int d^{N^2} M e^{-\frac{1}{2} TrM^2}} = \delta_{il} \delta_{jk}.$$



(Proof) We note that, due to the hermiticity of M, the trace is written as

$$\frac{1}{2} TrM^2 = \frac{1}{2} \sum_{i,j=1}^{N} M_{ij} M_{ji} = \sum_{1 \leq i < j \leq N} M_{ij} M_{ij}^{\star} + \frac{1}{2} \sum_{i=1}^{N} M_{ii} M_{ii}.$$

Especially, we separate M_{ij} into the real/imaginary part as

$$M_{ij} = \frac{X_{ij} + iY_{ij}}{\sqrt{2}} (= M_{ji}^{\star}).$$

Then, the quadratic term is

$$\frac{1}{2} TrM^2 = \frac{1}{2} \sum_{i=1}^{N} M_{ii} + \frac{1}{2} \sum_{1 \leq i < j \leq N} (X_{ij}^2 + Y_{ij}^2).$$

The derivation of the propagator reduces to the simple Gaussian integral:

$$\frac{1}{a} = \frac{\int_{-\infty}^{+\infty} dx x^2 \exp(-\frac{ax^2}{2})}{\int_{-\infty}^{+\infty} dx \exp(-\frac{ax^2}{2})}.$$

- ⋆ $\langle M_{ii} M_{ll} \rangle$ survives only for $i = l$.
- ⋆ For $\langle M_{ij} M_{kl} \rangle$ $(i \neq j)$, we note that
  - ∗ $\langle M_{ij} M_{ij} \rangle = \frac{1}{2} \langle (\underbrace{X_{ij} X_{ij} - Y_{ij} Y_{ij}}_{\text{cancelled}} + 2i \underbrace{X_{ij} Y_{ij}}_{(*)}) \rangle = \frac{1-1}{2} = 0.$

    (*) does not contribute ab initio, since this is a linear term of each X_{ij} and Y_{ij}.
  - ∗ $\langle M_{ij} M_{ji} \rangle = \frac{1}{2} \langle (X_{ij} X_{ij} + Y_{ij} Y_{ij}) \rangle = 1$ survives.

    (namely, $i = l, j = k$).

Then, the path integral is rewritten as

$$Z = e^{-F} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{g}{\sqrt{N}}\right)^n \int d^{N^2}M \exp\left(-\frac{1}{2}TrM^2\right)(TrM^3)^n.$$

$n =$ **(♯ of triangle)** $=$ **(area)** $= A$.

On the other hand, the power of N is $\mathcal{O}(N^{\chi})$.

(Proof) When we rescale the matrix as $M \to M\sqrt{N}$,

$$S = N\left(\frac{1}{2}TrM^2 - gTrM^3\right).$$

- Vertex: One vertex is clearly $\mathcal{O}(N)$.
- Propagator: Now, the propagator is $\langle M_{ij}M_{kl}\rangle = \frac{1}{N}\delta_{il}\delta_{jk}$.
  $\Rightarrow$ This has the power $\mathcal{O}(N^{-1})$.
- Loop: The contraction of the indices is $\sum_{i,j,k,l=1}^{N}\delta_{il}\delta_{jk} = N^2$.
  Together with the power of the propagator, one loop brings $\mathcal{O}(N^2N^{-1}) = \mathcal{O}(N)$.

For the diagram with V vertices, E edges and F triangles, the power is $\mathcal{O}(N^{V-E+F}) = \mathcal{O}(N^{\chi})$. (Q.E.D.)

We see the following correspondence:

$$g \Leftrightarrow e^{-\beta}, \quad N \Leftrightarrow e^{+\gamma}.$$

Extension to the square:
We consider 0-dimensional ϕ^4 theory.

$$S = \frac{1}{2}TrM^2 - \frac{g}{N}TrM^4.$$

4 Exact solution via orthogonal polynomial method

E. Brezin and V.A. Kazakov, Phys.Lett.B236:144-150,1990

Especially, we concentrate on the pure gravity:

- Pure gravity ($c = 0$):

  This is a system without matter field (since $c = 0$).

$$V(M) = \frac{1}{2g}\left(TrM^2 + \frac{1}{N}TrM^4\right).$$

We consider the following path integral:

$$\begin{aligned}
Z &= \int d^{N^2}M \exp(-V(M)) \\
&= \int dU_{ij} \prod_{i=1}^{N} d\lambda_i \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 e^{-V(\lambda)} \\
&= \int dU_{ij} \prod_{i=1}^{N} d\lambda_i (\det X)^2 e^{-V(\lambda)} \\
&= \int dU_{ij} \prod_{i=1}^{N} d\lambda_i (\det X')^2 e^{-V(\lambda)}.
\end{aligned}$$

$$X = \begin{pmatrix} 1 & 1 & \cdots & & 1 \\ \lambda_1 & \lambda_2 & \cdots & & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & \cdots & & \lambda_N^2 \\ \vdots & \vdots & & \ddots & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & & \lambda_N^{N-1} \end{pmatrix}, \quad X' = \begin{pmatrix} P_0(\lambda_1) & P_0(\lambda_2) & \cdots & & P_0(\lambda_N) \\ P_1(\lambda_1) & P_1(\lambda_2) & \cdots & & P_1(\lambda_N) \\ P_2(\lambda_1) & P_2(\lambda_2) & \cdots & & P_2(\lambda_N) \\ \vdots & \vdots & & \ddots & \vdots \\ P_{N-1}(\lambda_1) & P_{N-1}(\lambda_2) & \cdots & & P_{N-1}(\lambda_N) \end{pmatrix}.$$

Here, the orthogonal polynomials $P_n(x)$ of n-th degree are defined such that

- The coefficient of the highest power is 1; namely

  $P_0(x) = 1$, $P_n(x) = x^n + \sum_{j=0}^{n-1} a_{n,j}x^j$.

- $\int_{-\infty}^{+\infty} dx e^{-V(x)} P_n(x) P_m(x) = h_n \delta_{mn}$

(Proof of the measure): We verify the formula for the measure

$$d^{N^2}M = dU_{ij} \prod_{i=1}^{N} d\lambda_i \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2.$$

First, the matrix M is diagonalized by the unitary matrix U as

$$UMU^\dagger = D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N).$$

We separate the measure into the radius part and the angular part. [1]

$$d^{N^2}M = \underbrace{\prod_{i=1}^{N} d\lambda_i h(\lambda_1, \cdots, \lambda_N)}_{\text{radius part}} \underbrace{dU_{ij}}_{\text{angular part}} .$$

Our job reduces to determining the function h:

The infinitesimal form of the unitary matrix is given by

$$\delta U_{ij} = I_{ij} + i(E_{ij}\epsilon_{ij} + E_{ji}\epsilon_{ij}^\dagger).$$

For this δU_{ij}, M is obtained as

$$\begin{aligned} M &= (\delta U)^\dagger D(\delta U) = D - i[D, (E_{ij}\epsilon_{ij} + E_{ji}\epsilon_{ij}^\dagger)] \\ &= D - i(-\epsilon_{ij}E_{ij} + \epsilon_{ij}^\dagger E_{ji})(\lambda_i - \lambda_j). \end{aligned}$$

When, i, j sweeps over $1, \cdots, N$, we find that

$$h(\lambda_1, \cdots, \lambda_N) = \prod_{1\le i,j\le N, i\ne j} \{i(\lambda_i - \lambda_j)\} = \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2.$$

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[1] The analogy for the simpler case: For the spherical coordinate

$$(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta),$$

the measure is written as

$$dxdydz = \underbrace{r^2 dr}_{\text{radius part}} \underbrace{\sin\theta d\theta d\phi}_{\text{angular part}} .$$

Then, the partition function is given by

$$Z = N! \prod_{i=0}^{N-1} h_i = N! h_0^N \prod_{i=1}^{N-1} f_k^{N-k}, \text{ where } f_k = \frac{h_k}{h_{k-1}}.$$

Our job is to derive the recursive formulae for f_k. Next, we note that

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + f_n P_{n-1}(\lambda).$$

(Proof) We make an expansion

$$\lambda P_n(\lambda) = \sum_{i=0}^{n+1} c_{n,i} P_i(\lambda),$$

$$c_{n,i} = \begin{cases} 1 & (\text{for } i = n+1) \\ h_i^{-1} \int_{-\infty}^{+\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) P_i(\lambda) & (\text{for } i = 1, 2, \cdots, n) \end{cases}$$

- $c_{n,i} = 0$ for $i = 0, 1, \cdots n-2$. This is trivial since $\lambda P_i(\lambda)$ can be expressed by the linear combination of $P_0(\lambda), P_1(\lambda), \cdots, P_{n-1}(\lambda)$.
- $c_{n,n-1} = h_{n-1}^{-1} \int_{-\infty}^{+\infty} d\lambda e^{-V(\lambda)} P_n(\lambda)(P_n(\lambda) + \sum_{j=0}^{n-1} c_{n-1,j} P_j(\lambda)) = f_n$.
- $c_{n,n} = 0$ because the potential $V(\lambda)$ is an even function, whereas $\lambda(P_n(\lambda))^2$ is an odd function.

This completes the proof of this relation. (Q.E.D.)

We next derive the following recursive formula:

$$gn = f_n + \frac{2}{N} f_n (f_{n-1} + f_n + f_{n+1}).$$

(Proof)We evaluate the following integral in two ways:

$$\mathcal{I} = \int_{-\infty}^{+\infty} d\lambda e^{-V(\lambda)} \lambda P_n(\lambda) \frac{dP_n(\lambda)}{d\lambda}.$$

- Using $\lambda \frac{dP_n(\lambda)}{d\lambda} = \lambda(n\lambda^{n-1} + \sum_{j=1}^{n-1} a_{n,j} j \lambda^{j-1}) = n\lambda^n + \cdots$, we readily obtain $\mathcal{I} = nh_n$.
- The other way is to perform a partial integration. Here, we exploit an explicit form $V(\lambda) = \frac{1}{2g}(\lambda^2 + \frac{1}{N}\lambda^4)$. Then, the integral in question is

$$\mathcal{I} = \int_{-\infty}^{+\infty} d\lambda e^{-V(\lambda)} \frac{dP_n(\lambda)}{d\lambda} f_n P_{n-1} = f_n \int_{-\infty}^{+\infty} e^{-V(\lambda)} \frac{dV(\lambda)}{d\lambda} P_n(\lambda) P_{n-1}(\lambda).$$

Exploiting the fact that $\frac{dV(\lambda)}{d\lambda} = \frac{\lambda}{g} + \frac{2\lambda^3}{gN}$, this integral is finally evaluated as

$$\begin{aligned} \mathcal{I} &= \frac{1}{g} h_n f_n + \frac{2}{gN} f_n \int_{-\infty}^{+\infty} d\lambda P_n(\lambda) \lambda^2 (P_n(\lambda) + f_{n-1} P_{n-1}(\lambda)) = \cdots \\ &= \frac{1}{g} f_n h_n + \frac{2}{gN} f_n h_n (f_{n-1} + f_n + f_{n+1}). \end{aligned}$$

This completes the above relation. (Q.E.D.)

In order to solve this, we translate this recursive formula into the differential equation for a continuous function:

$$\frac{f_n}{N} = f(\xi), \quad \frac{f_{n\pm1}}{N} = f(\xi \pm \epsilon), \quad \text{where } \epsilon = \frac{1}{N}, \ \xi = \frac{n}{N}.$$

Planar limit

We take a limit $N \to \infty$ and discard the effect of ϵ.

$$g\xi = W(f(\xi)) = g_c + \frac{1}{2}W''(f_c)(f(\xi) - f_c)^2.$$

- $W(f(\xi)) = f(\xi) + 6f(\xi)^2$.
- The saddle point: $\frac{dW(f)}{df} = 0$ at $f(\xi) = f_c$, and $g_c = W(f_c)$.

Then, the string susceptibility is given by

$$f(\xi) - f_c \sim (g_c - g\xi)^{-\gamma}.$$

(Proof) First, the path integral is evaluated as

$$\begin{aligned} -\frac{1}{N^2}F &= \frac{1}{N^2}\log Z \sim \frac{1}{N}\sum_{k=0}^{N-1}(1 - \frac{k}{N})\log f_k \sim \int_0^1 d\xi(1-\xi)\log f(\xi) \\ &\sim \int_0^1 d\xi(1-\xi)\log(f_c + (g_c - g\xi)^{-\gamma}) \sim \int_0^1 d\xi(1-\xi)(g_c - g\xi)^{-\gamma} \\ &\sim (g_c - g)^{-\gamma+2} \sim \sum_{n=0}^{\infty} n^{\gamma-3}(\frac{g}{g_c})^n. \end{aligned}$$

From the correspondence between the random triangulation, the number of square n is identified with the area A. Therefore, γ is a string susceptibility!. (Q.E.D.)

Therefore, we read off the string susceptibility for the planar case

$$g\xi - g_c = \frac{1}{2}W''(f_c)(f(\xi) - f_c)^2 \Leftrightarrow \gamma = -\frac{1}{2}.$$

This agrees with the analysis of Distler and Kawai for $D = 0, h = 0$:

$$\gamma = \frac{1-0}{12}\left(0 - 25 - \sqrt{(25-0)(1-0)}\right) + 2 = \frac{-30}{12} + 2 = -\frac{1}{2}.$$

Nonplanar limit

Next, we include the $\epsilon = \frac{1}{N}$ effect.

$$\begin{aligned} g\xi &= g_c + \frac{1}{2}W''(r_c)(r(\xi) - r_c)^2 + 2r(\xi)(r(\xi+\epsilon) + r(\xi-\epsilon) - 2r(\xi)) \\ &= g_c + \frac{1}{2}W''(r_c)(r(\xi) - r_c)^2 + 2\epsilon^2 \frac{d^2 r}{d\xi^2}. \end{aligned}$$

We take the double-scaling limit $N \to \infty$, $g \to g_c$. Namely, the following quantity remains finite:

$$\kappa^{-1} = (g - g_c)^{\frac{5}{4}} N = (\text{const.}),$$

where $g - g_c = \kappa^{-\frac{4}{5}} a^2$, $\epsilon = \frac{1}{N} = a^{\frac{5}{2}}$ with $a \to 0$.

We introduce the variable z as $-g_c + g\xi = a^2 z$.
We set an ansatz $r(\xi) = r_c + a u(z)$.
This gives the Painlevè equation:

$$z = u^2(z) + \frac{d^2 u}{dz^2}.$$

Derivation of String susceptibility

We start with the asymptotic solution for $z \to \infty$

$$u(z) = \sqrt{z} \text{ for } z \to \infty.$$

This corresponds to the planar effect, in that

$$-g + g_c\xi = a^2 z = (\text{const.}) \Leftrightarrow a \to 0 \Leftrightarrow N = a^{-\frac{5}{2}} \to \infty.$$

Starting from this asymptotic solution, we read off the sub-leading effect:

$$u(z) = \sqrt{z} + az^b.$$

This coefficient turns out to be $(a, b) = (-\frac{1}{8}, -2)$.

We read off the string susceptibility as

$$u(z) = \frac{1}{a}(f(\xi) - f_c) \sim \sqrt{\frac{1}{a^2}(g_c - g\xi)} - \frac{1}{8}\left(\frac{1}{a^2}(g_c - g\xi)\right)^{-2} + \cdots.$$

The string susceptibility for genus $h = 1$ is $\gamma = 2$.

Likewise, the solution of the Painlevè equation is obtained as

$$u(z) = \sqrt{z}\left(1 + \sum_{h=1}^{\infty} u_h z^{-\frac{5h}{2}}\right).$$

Therefore, the string susceptibility for all genera is

$$\gamma_h = \frac{-1 + 5h}{2}.$$

This again agrees with the analysis of Distler and Kawai for $D = 0$:

$$\gamma_h = 2 + \frac{1 - h}{12}\left(0 - 25 - \sqrt{(25 - 0)(1 - 0)}\right) = \frac{-1 + 5h}{2}.$$

5 Conclusion

In this talk, we have reviewed the successful aspects of the one-matrix model as a constructive definition of the bosonic string.

- Distler and Kawai succeeded in the quantization of the string for $D \leq 1$, and derived the string susceptibility for all genera.
- David elucidated the correspondence between the one-matrix model and the bosonic string theory by the triangulation of the world sheet.
- Brezin and Kazakov solved the non-planar effect of the string theory by the orthogonal polynomial method.

This method per se is not useful for the constructive definition of 'super'string theory.

- The direct extension to the 'super'string faced with the similar setback as the treatment of the fermion in lattice gauge theory.
- The 'state-of-the-art' matrix models (such as IKKT) do not inherit the same techniques as the old matrix model.

Nevertheless, this story of the old matrix model legitimates the belief that

The constructive definition of the superstring theory is realized by the matrix model!!