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Gluon Amplitudes in $N=4$ SYM and AdS Minimal Surfaces

kandʒektʃər/

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I)
II) BDKKRSVV 0803.1465
III) BHT 0707.1153, MHT 0708.1625
IV) Al-Mal1 0705.0303, Alday review 0804.0951, Itoyama-Moxhay N.P.B293(1987) 685
V) IMM1 0712.0159, Al-Mal1
VI) IMM1, IM8 0712.2316
VII) IMM2 0803.1547
VIII) BKKKRSVV

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$$D_\pi \overset{?}{=} \kappa A_\pi$$

I) Introduction

- gauge/gravity correspondence (& matrix models) continue to be the central theme of string theory

② a new version of the string - gauge duality (strong - weak) appeared last year (Alday - Maldacena)

① BDS's conjectured exponentiation of the planar gluon amplitudes for $\mathcal{N}=4$ SYM

- ① + ② state $D_\pi \overset{?}{=} \kappa A_\pi$ giving a resolution to the AdS minimal surface problem.

There is (was) a chance that ~~[illegible]~~ this statement of duality boils down to some kind of identities which have nothing to do with QFT.

e.g. 2d boson - fermion equivalence

- The computat^n is a well-known exponentiation of directly related to the double logarithms of Sudakov form factor

II). BDS Conjecture, which now Calls for Modifications

• The previous term Basically goes in the same way as KIAS slides p3 ~ p5.

p8 ... Symbolically

$$\text{"} A_n(p_1 \cdots p_n | \lambda) = A_{\text{tree}}\, A_{IR}\, A_{\text{finite}} \text{"}$$

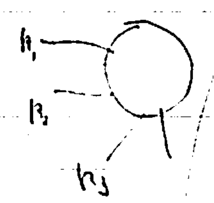
To be more precise

$\lambda = g^2 N$

$D = 4 - 2\epsilon$

ρ : noncyclic perm.

$$A_n = \sum_{L=0}^{\infty} A_n^{(L)}(\epsilon) \qquad a = \frac{\lambda^2 \mu^{2\epsilon}}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}$$



$$= g^{n-2}\, a^L \sum_{\rho} \mathrm{Tr}\left(T^{a_{\rho(1)}} \cdots T^{a_{\rho(n)}}\right) A_n^{(L)}(\rho(1), \cdots \rho(n))$$

Define $M_n^{(L)}(\epsilon) \equiv A_n^{(L)}(\epsilon) / A_n^{(0)}$ scalar factor

$$= A_n^{(0)} \underbrace{\sum_{L=0}^{\infty} a^L M_n^{(L)}(\epsilon)}_{\mathcal{M}_n(\epsilon)}$$

BDS conjecture

$$\mathcal{M}_n(\epsilon) = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + O(\epsilon)\right)\right]$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)} \qquad \text{3 term series}$$

the planar cusp anomalous dimension $= \frac{1}{4}\hat{\gamma}_K^{(l)}$

Define

$$f(\lambda) = 4\sum_{l=1}^{\infty} a^l f_0^{(l)}, \quad g(\lambda) = 2\sum_{l=2}^{\infty} \frac{a^l}{l} f_1^{(l)}, \quad k(\lambda) = -\frac{1}{2}\sum_{l=2}^{\infty} \frac{a^l}{l^2} f_2^{(l)}$$

collinear anomalous dim.

BDS conjecture may be written as

$$\ln \mathcal{M}_n(\epsilon) = \mathrm{Div}_n + \frac{f(\lambda)}{4} F^{(1)}(0) + n k(\lambda) + C(\lambda)$$

$$\mathrm{Div}_n = -\sum_{i=1}^{n}\left[\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda \mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^{\epsilon}}\right) + \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda \mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^{\epsilon}}\right)\right]$$

where $\left(\lambda \frac{d}{d\lambda}\right)^2 f^{(-2)}(\lambda) = f(\lambda) \quad \left(\lambda \frac{d}{d\lambda}\right) g^{-1}(\lambda) = g(\lambda)$

$\mu_{IR}^2 = 4\pi e^{-\gamma} \mu^2$

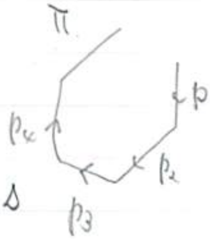
strong coupling behavior known $\sim \sqrt{\lambda}$

written 2008.6.7

III)

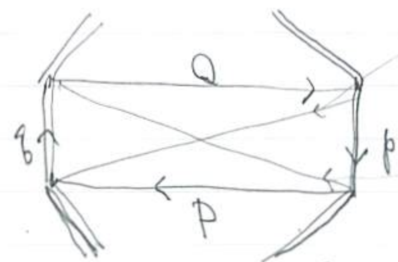
- Representation of $M_n^{(1)}$ is of interest — basically my KIAS slide p5.

$M_n^{(1)}$ may be expressed as a sum over 4 clusters in an auxiliary polygon Π



may be called a Wilson loop in the dual momentum space

consider a 4 cluster



$(p+P)^2 = s$

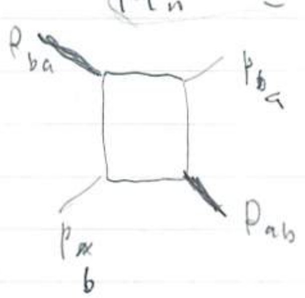
$(q+p)^2 = t$

contains two non intersecting sides in Π

4 vertices

4 diagonals of Π (2 actually are sides of the quadrilateral)

Furthermore



$$M_n^{(1)} = \sum_{a<b} F^{2me}(p_a, P_{ab}, p_b, P_{ba}) \qquad P_{ab} = \sum_{c=a+1}^{b-1} p_c$$

↑ namely the sum over the 4 clusters

The dilogarithmic repn of F^{2me} : DN formula ← two distinct ones

cf. BDK formula

needs refinements

· Relationship with the double contour integral

$$\boxed{D_\Pi = \oint_\Pi \oint_\Pi \frac{dy^\mu dy'_\mu}{(y-y')^{2+\epsilon}}}$$

a sum of the contributions from pairs of segments in Π

$$= D^{(1)}_\Pi + D^{(2)}_\Pi + D^{(3)}_\Pi$$

3 types

• $D^{(1)}_\Pi$ = ones from two identical segments p

$0 < \tau < 1$ parametrize the segment

$y^\mu(\tau) = p^\mu \tau$

$dy \cdot dy' = p^2 \, d\tau \, d\tau' = 0$

$= 0$

notation? • $D^{(2)}_\Pi$ = ones from two adjacent null segments p, q

$s = +(p+q)^2 = 2p\cdot q$

cf. $$\int_0^1 \int_0^1 \frac{(p\cdot q)\, d\tau_p d\tau_q}{(\tau_p p + \tau_q q)^{2(1+\epsilon)}} = \frac{1}{2}(2p\cdot q)\int_0^1 d\tau \int_0^1 d\tau' \frac{1}{(2\tau\tau' p\cdot q)^{1+\epsilon}}$$

$p = k_2 - k_1$, $q = k_3 - k_2$

$k_1(\tau_1) = k_1 + \tau_1 p$, $k_2(\tau_2) = k_2 + \tau_2 q$

$k_2(\tau_2) - k_1(\tau_1) = p(1-\tau_1) + q\tau_2$

$$= \frac{1}{2}(2p\cdot q)^{-\epsilon} \int_0^1 \frac{d\tau}{\tau^{1+\epsilon}} \int_0^1 \frac{d\tau'}{\tau'^{1+\epsilon}} = \frac{1}{2}(2p\cdot q)^{-\epsilon}\left(\left(-\frac{1}{\epsilon}\right)\tau^{-\epsilon}\Big|_0^1\right)^2$$

$$= \frac{1}{2}(2p\cdot q)^{-\epsilon}\frac{1}{(-\epsilon)^2}$$

$\epsilon < 0$ as an infrared regulator

• $D^{(3)}_\Pi$ = ones from nonadjacent null segments p, q

cf. $p = k_{p+1} - k_p$, $q = k_{q+1} - k_q$

$k_q \xrightarrow{\tau} k_{q+1}$, $k_p \xrightarrow{\tau'} k_{p+1}$

$k_p(\tau_p) = k_p + \tau_p p$, $k_q(\tau') = k_q + \tau_q q$

$$\left(k_q(\tau_q) - k_p(\tau_p)\right)^2 = \left(k_q - k_p + \tau_q q - \tau_p p\right)^2$$

$$= \left(+\sum_{i=p}^{q-1}(-k_i + k_{i+1}) + \tau_q q - \tau_p p\right)^2$$

$$= \Big(\underbrace{+\sum_{i=p+1}^{q-1}(k_i + k_{i+1})}_{P} + \tau_q q + (1-\tau_p)p\Big)^2 = P^2 + 2(pP)(1-\tau_p) + 2(qP)\tau_p + 2(pq)\tau_q(1-\tau_p)$$

written 2008.6.7

$$P + p + Q + q = 0$$

$$+s \equiv +(P+p)^2 = +P^2 + 2P\cdot p$$
$$+t \equiv +(P+q)^2 = +P^2 + 2P\cdot q$$

$$2pq = (p+q)^2 = (-P-Q)^2 = P^2+Q^2+2P\cdot Q$$
$$= P^2+Q^2-2P(P+p+q) = P^2+Q^2-s-t$$

$$\therefore \iint \frac{(p\cdot q)\, d\tau_p\, d\tau_q}{\left(k_q(\tau_q)-k_q(\tau_p)\right)^{2(1+\epsilon)}} = \frac{1}{2}\int_0^1\!\!\int_0^1 \frac{(P^2+Q^2-s-t)\, d\tau_p\, d\tau_q}{\left[P^2+(s-P^2)\tau_p+(t-P^2)\tau_q+(P^2+Q^2-s-t)\tau_p\tau_q\right]^{1+\epsilon}}$$

$$= \frac{1}{2}\int_0^1 d\tau_p \frac{(P^2+Q^2-s-t)}{(P^2+Q^2-s-t)\tau_p+t-P^2}\log\frac{t+(Q^2-t)\tau_p}{P^2+(s-P^2)\tau_p} + O(\epsilon)$$

$\leftarrow$ finite

$$= Li_2(1-as)+Li_2(1-at)-Li_2(1-aP^2)-Li_2(1-aQ^2)$$

$$a = \frac{P^2+Q^2-s-t}{P^2Q^2-st}$$

not yet seen.

- MHV amplitudes do not have multiparticle singularities

$F^{(2me)}$ do, but

$$M_n^{(1)}\Big|_{IR} = -\frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{-s_{i,i+1}}{\mu^2}\right)^{-\epsilon}$$

with $s_{i,i+1} = (p_i+p_{i+1})^2$

Key properties not ~~used~~ made explicit.

- maximal susy
- MHV

Conclusion

$$\boxed{M_n^{(1)} = D_\Pi}$$

including infrared divergent part? $\Rightarrow$ yes.

written 2008.6.17

D_π as Abelian Wilson Loop

It is easy to recognize D_π as log of the abelian Wilson loop

$$\left\langle \exp\left(i g \int_\pi dy^\mu A_\mu(y) \right) \right\rangle = 1 + \frac{i^2 g^2}{2!} \int_\pi dy^\mu \int_\pi dy'^\nu \langle A_\mu(y) A_\nu(y') \rangle$$

$$+ \frac{i^4 g^4}{4!} 3 \left(\int_\pi dy^\mu \int_\pi dy'^\nu \langle A_\mu(y) A_\nu(y') \rangle \right)^2 + \ldots$$

$$\frac{i^6\, 5\cdot 3}{6!} = \frac{i^6}{6\cdot 4\cdot 2} = \frac{i^6}{3!}\frac{1}{8}$$

$$= \exp\left(\frac{i^2}{2} g^2 \int_\pi dy^\mu \int_\pi dy'^\nu \langle A_\mu(y) A_\nu(y') \rangle \right)$$

$\parallel$ ← Feynman gauge

$$\text{const} \frac{\eta_{\mu\nu}}{(y-y')^2}$$

$$= \exp\left(\frac{1}{\kappa} \ldots D_\pi \right)$$

albeit the difference btw the position space and the momentum space

Q: why such lengthy explanation needed to reach the conclusion?

some kind of T-duality operating.

my impression

repn on D instanton

amplitudes

$\parallel$

repn on D3 branes

weak coupling

strong coupling

written 2008. 6. 9.

IV) AdS_5 Space and the T-dualized Four-gluon Gluon Amplitudes

We will now compute the same gluon amplitude at strong coupling; using semiclassical string theory. in String Theory

Ignore S^5 part.

The original AdS^5 geometry is, in embedding coordinates of $\mathbb{R}^{2,4}$ a hyperboloid

$$X^{-1} + X^4 = \frac{R}{z}, \quad X^{-1} - X^4 = R\,\frac{z^2 + x_\mu x^\mu}{z}, \quad X^\mu = R\,\frac{x^\mu}{z} \quad (-,+,+,+) \quad \mu = 0,1,2,3$$

$$-(X^{-1})^2 - X^{0\,2} + X^{1\,2} + X^{2\,2} + X^{3\,2} + X^{4\,2} = -R^2$$

$$ds^2 = dX^M dX_M = -d(X^{-1}+X^4)\,d(X^{-1}-X^4) + dX_\mu dX^\mu$$

$$= R^2\,\frac{dz^2 + dx_\mu dx^\mu}{z^2}$$ AdS_5 metric in the Poincare coordinates.

Gluon momenta k_I^μ $I = 1 \sim n$ are constrained in the 3+1 dim Minkowski surface. $\therefore$ D3 brane background must be present, which we place at $z = z_{IR}$ and identify as an IR regulator

We take the worldsheet to be Euclidean signature and go to the T-dualized geometry for $\mu = 0,1,2,3$, directions. For this,

$$w(z) = \frac{R}{z} = \frac{r}{R}$$

$$ds^2 = R^2 \frac{dz^2}{z^2} + w^2(z)\,\partial_a X_\mu\,\partial_b X^\mu\, d\sigma^a d\sigma^b$$ pull back to ws

Let

$$(*)\quad \partial_a X^\mu = iA\,\epsilon_{ac}\,\partial_c y^\mu \qquad , \text{and} \quad \frac{R^2}{z} = r$$

$$\delta_{ab}\delta_{cd} - \delta_{ad}\delta_{cb}$$

$$= R^2\frac{dr^2}{r^2} + i^2 A^2 w^2(z)\,\epsilon_{ac}\,\partial_c y_\mu\,\epsilon_{bd}\,\partial_d y^\mu\, d\sigma^a d\sigma^b$$

$$= R^2\frac{dr^2}{r^2} + A^2 w^2\left(\partial_a y_\mu \partial_b y^\mu\, d\sigma^a d\sigma^b - \partial_c y_\mu \partial^c y^\mu\, d\sigma_a d\sigma^a\right)$$

choose $A^2 w^2 = \frac{R^2}{r^2} = \frac{z^2}{R^2} R^2 = w^{-2}$ $\therefore A = w^{-2}$

$$ds^2 = R^2 \frac{dr^2 + dy^\mu dy_\mu - \partial_c y^\mu \partial^c y_\mu\, d\sigma d\sigma^a}{r^2}$$ — probably dilaton

with D insranton background present.

This is a string theory in a dual AdS5 geometry.

$\frac{R^2}{z_{IR}} = r_{IR} \to 0$ as $z_{IR} \to \infty$

The embedding coordinates of this dual AdS5 geometry are

$R^{2,4}$; $-Y^{-1\,2} - Y^{0\,2} + Y^{1\,2} + Y^{2\,2} + Y^{3\,2} + Y^{4\,2} = -1$

$Y^\mu = \frac{y^\mu}{r}$ $\quad Y^{-1} + Y^4 = \frac{1}{r}$, $Y^{-1} - Y^4 = \frac{r^2 + y_\mu y^\mu}{r}$

Once again
induced metric is not preserved

$\gamma^{\text{original}}_{ab}(z, X^\mu) = \frac{R^2}{z^2}\left(\partial_a z \partial_b z + \partial_a X^\mu \partial_b X_\mu\right)$

$\frac{R^2}{z} = r$ $\quad = \frac{R^2}{r^2}\left(\partial_a r \partial_b r + \partial_a y^\mu \partial_b y_\mu - \partial_c y^\mu \partial_c y_\mu \delta_{ab}\right)$

$\neq \gamma^{(\text{dual})}_{ab}(r, y^\mu)$

$-\frac{1}{2}\delta^{ij} y^2 \partial_i X^{I} \partial_j X_I$

$\approx -\frac{1}{2}\delta^{ij}\left(-\frac{1}{y^2} P_i^I P_{jI} + 2 P_j^I \partial_i X_I\right)$

integrate out X_I

$\Rightarrow$ constraints
$\delta^{ij} P_j^I = i\epsilon^{ij}\partial_j \tilde{X}^I$

$\approx -\frac{1}{2}\frac{1}{y^2}\partial_j \tilde{X}^I \partial_j \tilde{X}^I$

Does not preserve the induced metric

however, worldsheet of σ model action

$\sim \int d^2 z_E\, G_{IJ} \frac{dY^I}{dz_E^i}\frac{dY^J}{dz_E^j}\delta^{ij}$

'gauging an abelian isometry

or gaussian integration

written 2008.5.23

$$X^{-1}+X^4=\frac{R}{z}\ ,\quad X^{-1}-X^4=R\,\frac{z^2+x_\mu x^\mu}{z}\ ,\quad X^\mu=R\,\frac{x^\mu}{z}$$

Horizon ?

· $r_{IR}=0$, $z_{IR}=\infty$ what is meant by this?

In the original space in terms of the embedding coordinates

$$\sqrt{X^{0\,2}+X^{-1\,2}}=\sqrt{R^2+X^{1\,2}+X^{2\,2}+X^{3\,2}+X^{4\,2}}$$

$$\xrightarrow{\text{bndry}}\sqrt{X^{1\,2}+X^{2\,2}+X^{3\,2}+X^{4\,2}}$$

· The bndry is given by

$$\boxed{-X^{0\,2}-X^{-1\,2}+X^{1\,2}+X^{2\,2}+X^{3\,2}+X^{4\,2}=0}$$

Can take, without losing generality $X^0\neq 0$

$$\underbrace{1+\left(\frac{X^{-1}}{X^0}\right)^2}_{\geq 0}=\left(\frac{X^1}{X^0}\right)^2+\left(\frac{X^2}{X^0}\right)^2+\left(\frac{X^3}{X^0}\right)^2+\left(\frac{X^4}{X^0}\right)^2$$

The bndry of AdS^5 is $\overset{\text{line}}{R}\times S^3$, which is accomplished by $z=0$

Now $z=\infty$ implies $X^{-1}+X^4=0$

in the entire AdS^5

$$-X^{0\,2}+X^{1\,2}+X^{2\,2}+X^{3\,2}=-R^2$$

1 dim. line

4dim.

→ 3dim hyperboloid

can parametrize as

$$X^0+X^1=\frac{R}{w}\qquad X^0-X^1=R\,\frac{w^2+x_ix_i}{w}$$

$$X^{2,3}=R\,\frac{x^i}{w}$$

The bndry of this space is

$$※\begin{cases}X^{-1}=-X^4\\ -X^{0\,2}+X^{1\,2}+X^{2\,2}+X^{3\,2}=0\end{cases},$$

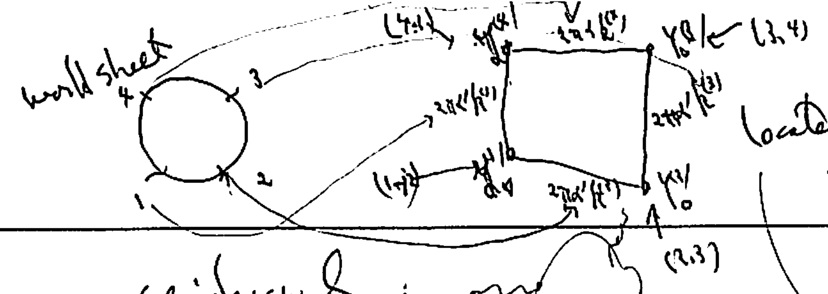
which is $S^2\times$ line

In fact, Alday-Maldacena solution reaches the bndry ※ as $u_i\to\pm\infty$

But $X^0, X^1, X^2\ (X^3)$ are all purely imaginary.

Namely they live in the complexified bndry.

map



located at $y_0^\mu(I)$

semiclassical

Gluon amplitudes in string theory

$$A(\zeta_I,k_I) = \left\langle\!\left\langle g \prod_I \int_{\partial M} dz_I \int d\theta_I\, \zeta_I\cdot D_I X\, e^{ik_I\cdot X(z_I,\bar z_I,\theta_I,\bar\theta_I)} \right\rangle\!\right\rangle_N$$

in flat 10d spacetime, NSR string.

$k_I^M = (k_I^\mu, 0)$.

$$S_E = \frac{1}{2\pi\alpha'}\frac{4}{2}\int_M \bar D X^M D X_M\, d^2z\, d\theta d\bar\theta$$

and Neumann B.C.

In our case

Minkowski contraction

$$e^{ik_I\cdot X} \equiv e^{ik_I^\mu X_\mu^\nu \eta_{\mu\nu}}$$

$$= e^{i k_I^{(pr)a} X^b g_{ab}}\Big|_{z=z_{IR}}$$

$X^b = X^\nu$

$$\therefore k_I^\mu = k_I^{(pr)\mu}\frac{R^2}{z_{IR}^2}$$

gluon momentum, kept fixed

$$\therefore k_I^{(pr)\mu} = k_I^\mu \frac{z_{IR}^2}{R^2} \xrightarrow{z_{IR}\to\infty} \text{large}$$

In fact $k_I^{(pr)a} k_J^{(pr)b} g_{ab} \sim k_I^\mu k_I^\nu \frac{z_{IR}^2}{R^2} \longrightarrow \infty$

So the action is going to be dominated by the saddle point. (Gross Mende)

Semiclassical (WKB) analysis good.

N→0 rule in flat space

amp on D3 brane with momentum k_I^M = amp on D instantons

with $y_I^M(\sigma=2\pi) - y_I^M(\sigma=0) = 2\pi\alpha' k_I^M$

is valid

$\to 2\pi\alpha' k_I^\mu$

Ignoring fermions,

$$\therefore A(k_I,\zeta_I) = \left\langle\!\left\langle \int_{\partial M} \epsilon^{ab} d\zeta_I^a\, \zeta_I \cdot \frac{d}{d\zeta_I^b} y(\zeta_I) \right\rangle\!\right\rangle_D$$

no $\cancel{e^{ikX}}$

averaging with $S_E[y^\mu, r; k_\mu^I] \dots$

Polygon boundary at $r=0$

semiclassical

$$\sim \text{prefactor}\; e^{-S_E[y^\mu=\dots,\, r=\dots;\, k_\mu^I]}$$

classical sol'n

minimal surface of NG string.

recognize as $\sim f_{strong}(\lambda) \sim \sqrt{\lambda}$ 5/5 -21

V) Nambu-Goto Eq., Alday-Maldacena Solution and Evaluation of the Area

- Start with the standard form of NG eq.

$$S_{NG} = \frac{\sqrt{\lambda}}{2\pi} \int d\tau_M d\sigma \sqrt{-\det \gamma_{..}} \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}$$

γ_{ij} is the induced metric of the dual AdS_5 geometry

$$ds^2 = R^2 \frac{dr^2 + dy^\mu dy_\mu}{r^2} \qquad \mu ; (-,+,+,+)$$

$$= \underbrace{G_{IJ} \frac{dy^I}{d\xi^a} \frac{dy^J}{d\xi^b}}_{\gamma_{ab}} d\xi^a d\xi^b \qquad y^I = (r, y^\mu)$$

anything

$i\tau_M = \tau$

- Choose the world sheet to be Euclidean (for a tunnelling process)

$$e^{iS_{NG}} = e^{-S_{ENG}} \qquad \text{sign?}$$

$$S_{ENG} = -i \frac{\sqrt{\lambda}}{2\pi} i \int d^2\xi_E \sqrt{\det \gamma} = \frac{\sqrt{\lambda}}{2\pi} \int dy_1 dy_2 \sqrt{\det H}$$

Reparametrization inv. $\Longrightarrow$ choose $\xi^1_E = y^1$, $\xi^2_E = y^2$. Let me denote the γ by H

and put the 1st ansatz ① $\boxed{y_3 = 0}$ $\quad ab \to ij$

$$H_{ij} = \frac{1}{r^2}\left(\delta_{ij} - \partial_i y_0 \partial_j y_0 + \partial_i r \partial_j r \right)$$

$$= \frac{1}{r^2} \begin{pmatrix} 1 - (\partial_1 y_0)^2 + (\partial_1 r)^2, & -\partial_1 y_0 \partial_2 y_0 + \partial_1 r \partial_2 r \\ -\partial_1 y_0 \partial_2 y_0 + \partial_1 r \partial_2 r, & 1 - (\partial_2 y_0)^2 + (\partial_2 r)^2 \end{pmatrix}$$

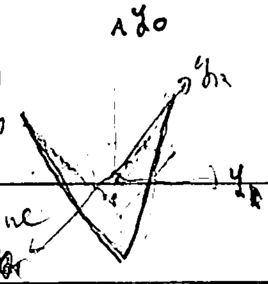
- Eq. of motion

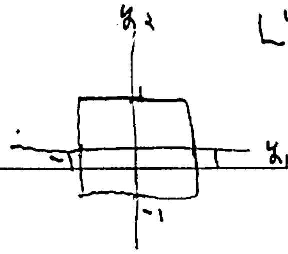
$$\delta y_0 ; \ \partial_1 \left(\frac{H_{22}}{r^2 \sqrt{\det H}} \partial_1 y_0 \right) + \partial_2 \left(\frac{H_{11}}{r^2 \sqrt{\det H}} \partial_2 y_0 \right) - \partial_1 \left(\frac{H_{21}}{r^2 \sqrt{\det H}} \partial_2 y_0 \right) - \partial_2 \left(\frac{H_{12}}{r^2 \sqrt{\det H}} \partial_1 y_0 \right) = 0$$

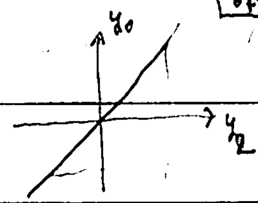
$$\delta r ; \ \partial_1 \left(\frac{H_{22}}{r^2 \sqrt{\det H}} \partial_1 r \right) + \partial_2 \left(\frac{H_{11}}{r^2 \sqrt{\det H}} \partial_2 r \right) - \partial_1 \left(\frac{H_{21}}{r^2 \sqrt{\det H}} \partial_2 r \right) - \partial_2 \left(\frac{H_{12}}{r^2 \sqrt{\det H}} \partial_1 r \right) + \frac{2\sqrt{\det H}}{r} = 0$$

written 2008. 5. 28

$\therefore$ denote the geometry of ~~the bndry polygon by~~ its projection onto $(y_1 y_2)$ plane







- Alday - Maldacena "square" solution $y_1^2+y_2^2$

Under ⊛
$$\begin{cases} 1+y_0^2 = y^2+r^2 & \text{—①} \\ y_0 = y_1 y_2 & \text{—③} \end{cases}$$

$$Y^{-1}+Y^4 = \frac{1}{r},\quad Y^{-1}-Y^4 = \frac{r^2+y_\mu y^\mu}{r} = \frac{1}{r}$$
$\therefore Y^4=0$

$\therefore$ ⊛ and ① form AdS_3 ansatz

At the boundary, it forms a lightlike polygon.

$$\text{or}\quad r^2 = (1-y_1^2)(1-y_2^2) \quad \text{—③}$$

Direct check :

$$r^4\det H = \det r^2 H = (1-(\partial_1 y_0)^2+(\partial_1 r)^2)(1-(\partial_2 y_0)^2+(\partial_2 r)^2) - (-\partial_1 y_0\partial_2 y_0+\partial_1 r\,\partial_2 r)^2$$
$$= 1-(\partial_i y_0)^2+(\partial_i r)^2-(\partial_1 r\partial_2 y_0-\partial_2 r\partial_1 y_0)^2$$
$$= 1-(\partial_i y_0)^2+(1-(\partial_2 y_0)^2)(\partial_1 r)^2 + (1-(\partial_1 y_0)^2)(\partial_2 r)^2 + 2\,\partial_1 y_0\partial_2 y_0\,\partial_1 r\,\partial_2 r$$

$$① \Rightarrow r\delta r = y_0\delta y_0 - y_i\delta y_i \quad \text{—①}'$$
$$③ \Rightarrow \delta y_0 = y_1\delta y_2 + y_2\delta y_1 \quad \text{—③}'$$

$$= 1-\underbrace{(\partial_i y_0)^2}_{\overset{③'}{=} y_i^2} + \frac{X}{r^2}$$

$$X \overset{①③'}{=} (1-(\partial_2 y_0)^2)(y_0 y_2-y_1)^2+(1-(\partial_1 y_0)^2)(y_0 y_1-y_2)^2 + 2\partial_1 y_0\partial_2 y_0\,(y_0 y_2-y_1)(y_0 y_1-y_2)$$
$$\overset{③}{=} (1-y_1^2)((y_2^2-1)y_1)^2+(1-y_2^2)((y_1^2-1)y_2)^2 + 2y_2 y_1\,((y_2^2-1)y_1)((y_1^2-1)y_2)$$
$$= (1-y_1^2)(1-y_2^2)\left(y_1^2(1-y_2^2)+y_2^2(1-y_1^2)+2y_1^2y_2^2\right)$$
$$= (1-y_1^2)(1-y_2^2)\,y_i^2 \overset{③}{=} r^2 y_i^2$$

$$\therefore r^2\sqrt{\det H} = 1$$

summary (2.5)

can check

$$H_{22}\partial_1 y_0 - H_{21}\partial_2 y_0 = \frac{y_2}{(1-y_2^2)^2} = \text{fn of } y_2 \text{ only}$$
$$H_{11}\partial_2 y_0 - H_{12}\partial_1 y_0 = \frac{y_1}{(1-y_1^2)^2} = \text{fn of } y_1 \text{ only}$$
$$H_{22}\partial_1 r - H_{21}\partial_2 r = -\frac{y_1}{r}\frac{1}{(1-y_2^2)}$$
$$H_{11}\partial_2 r - H_{12}\partial_1 r = -\frac{y_2}{r}\frac{1}{1-y_1^2}$$

$$\therefore \partial_1(H_{22}\partial_1 r - H_{21}\partial_2 r) = -\frac{1}{r}\frac{1}{1-y_2^2} - y_1\left(-\frac{1}{r^3}\right)\underbrace{(y_0y_2-y_1)}_{-y_1(1-y_2^2)}\frac{1}{1-y_2^2}$$
$$= -\frac{1}{r}\frac{1}{1-y_2^2} - \frac{y_1^2}{r^3} = -\frac{1-y_1^2+y_1^2}{r^3} = -\frac{1}{r^3}$$

Similarly

$$\partial_2(H_{11}\partial_2 r - H_{12}\partial_1 r) = -\frac{1}{r^3}$$

written 2008. 5. 30

Can use the AdS_2 ansatz to eliminate either y_0 or r

$$\sqrt{\det H} \overset{②}{=} \frac{1}{r^2}\sqrt{\frac{(y_i \partial_i r - r)^2 - (\partial_i r)^2 - 1}{y_1^2 + y_2^2 + r^2 - 1}}$$

or

$$\overset{①}{=} \sqrt{\frac{(y_i \partial_i y_0 - y_0)^2 - (\partial_i y_0)^2 + 1}{(1 + y_0^2 - y_1^2 - y_2^2)^3}}$$

IMM1 (1.15) (1.16)

① recall

$$r^4 (\det H) = 1 - (\partial_i y_0)^2 + (\partial_i r)^2 - \underbrace{(\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}_{= J}$$

$$= 1 - (\partial_i y_0)^2 + (y_1 \partial_1 y_0)^2 + (y_2 \partial_2 y_0)^2 + y_0^2 - 2 y_1 \partial_1 y_0 y_0 - 2 y_2 \partial_2 y_0 y_0 + 2 y_1 \partial_1 y_0 y_2 \partial_2 y_0$$

$$= 1 - (\partial_i y_0)^2 + (y_i \partial_i y_0 - y_0)^2$$

$$r \partial_i r = y_0 \partial_i y_0 - y_i$$

$$r^2 J = (y_0 \partial_i y_0 - y_i)^2 -$$

$$- \left((\cancel{y_0 \partial_1 y_0} - y_1) \partial_2 y_0 - (\cancel{y_0 \partial_2 y_0} - y_2) \partial_1 y_0 \right)^2$$

$$= y_0^2 (\partial_1 y_0)^2 - 2 y_0 \partial_1 y_0 y_1 + y_1^2$$
$$+ y_0^2 (\partial_2 y_0)^2 - 2 y_0 \partial_2 y_0 y_2 + y_2^2$$
$$- y_1^2 (\partial_2 y_0)^2 + 2 y_1 \partial_2 y_0 y_2 \partial_1 y_0$$
$$- y_2^2 (\partial_1 y_0)^2$$

$$\therefore r^6 \det H = (1 + y_0^2 - \cancel{y_i^2})(1 - (\partial_i y_0)^2)$$
$$+ y_0^2 (\partial_i y_0)^2 + y_i^2 - y_1^2 (\partial_2 y_0)^2 - y_2^2 (\partial_1 y_0)^2$$
$$- 2 (y_i \partial_i y_0) y_0 + 2 (y_1 \partial_1 y_0)(y_2 \partial_2 y_0)$$

$$= 1 + y_0^2 - \cancel{y_i^2}$$

$(\partial_i^2 y_i^2)((\partial_1 y_0)^2 + (\partial_2 y_0)^2)$

$$- (\partial_i y_0)^2 - y_0^2 (\partial_i y_0)^2 + y_j^2 (\partial_i y_0)^2$$
$$+ y_0^2 (\partial_i y_0)^2 + \cancel{y_i^2} - y_1^2 (\partial_2 y_0)^2 - y_2^2 (\partial_1 y_0)^2$$
$$- 2 y_i \partial_i y_0 y_0 + 2 y_1 \partial_1 y_0 y_2 \partial_2 y_0$$

written 2008.5.31

A-M square

rename

$y^1 = \xi_1 , \; y^2 = \xi_2$

$y^0 = y^1 y^2 = \xi_1 \xi_2$

$r = \sqrt{(1-\xi_1^2)(1-\xi_2^2)}$

· Generating the rhombus solution by boost

$\tilde{Y}^0 + \tilde{Y}^4 = e^{\eta} Y^0$ $\quad \eta$: rapidity

$\tilde{Y}^0 - \tilde{Y}^4 = e^{-\eta} Y^0$ $\quad Y^4 = 0 \quad Y^3 = 0$

$\tilde{Y}^{1,2,3} = Y^{1,2,3}$ $\quad Y^0 = \frac{y^0}{r}$

$\tilde{Y}^{-1} = -Y^{-1}$ $\quad Y^{1,2} = \frac{y^{1,2}}{r}$

$Y^{-1} = \frac{1}{r}$

also

$\tilde{Y}^{\mu} = \frac{\tilde{y}^{\mu}}{\tilde{r}} , \quad \tilde{Y}^{-1} + \tilde{Y}^4 = \frac{1}{\tilde{r}}$

$\tilde{Y}^{-1} - \tilde{Y}^4 = \frac{\tilde{r}^2 + \tilde{y}_{\mu} \tilde{y}^{\mu}}{\tilde{r}}$

$\therefore \frac{1}{\tilde{r}} = \frac{1}{r} + \sinh\eta \, \frac{y^0}{r}$ $\quad \sinh\eta = b$

$$\tilde{r} = \frac{\sqrt{(1-\xi_1^2)(1-\xi_2^2)}}{1 + b\xi_1\xi_2}$$

$$\tilde{y}^0 = \tilde{r} \cosh\eta \, \frac{y^0}{r} = \frac{\overset{B}{\sqrt{1+b^2}} \, \xi_1 \xi_2}{1+b\xi_1\xi_2}$$

$$\tilde{y}^1 = \tilde{r} \, \frac{y^1}{r} = \frac{\xi_1}{1+b\xi_1\xi_2}$$

$$\tilde{y}^2 = \tilde{r} \, \frac{y^2}{r} = \frac{\xi_2}{1+b\xi_1\xi_2}$$

This deviates from AdS_3 ansatz, of course.

Converting it into an AdS_3 form ZMM1. §2.6. summary (43)

$$\left(\tilde{r}^2 + \tilde{y}_\mu \tilde{y}^\mu - 1\right)(1+b\xi_1\xi_2)^2 = (1-\xi_1^2)(1-\xi_2^2) - \beta^2\xi_1^2\xi_2^2 + \xi_1^2 + \xi_2^2 - (1+b\xi_1\xi_2)^2$$

$$= \underbrace{(1-\beta^2-b^2)}_{1-1-b^2-b^2}\xi_1^2\xi_2^2 - 2b\xi_1\xi_2 = -2b\xi_1\xi_2(1+b\xi_1\xi_2)$$

$$\therefore \tilde{r}^2 + \tilde{y}_\mu\tilde{y}^\mu - 1 = \frac{-2b\xi_1\xi_2}{1+b\xi_1\xi_2} = -\frac{2b}{B}\tilde{y}^0$$

$$\therefore \tilde{r}^2 - \left(\tilde{y}^0 - \frac{b}{B}\right)^2 + \tilde{y}_i\tilde{y}^i + \underbrace{\left(\frac{b^2}{B^2} - 1\right)}_{-\frac{1}{B^2}} = 0$$

So define

$$y^0 = \left(\tilde{y}^0 - \frac{b}{B}\right)B = \frac{B^2\xi_1\xi_2 - b(1+b\xi_1\xi_2)}{1+b\xi_1\xi_2} = \frac{\xi_1\xi_2 - b}{1+b\xi_1\xi_2}$$

$$y^1 = \tilde{y}^1 B = \frac{B\xi_1}{1+b\xi_1\xi_2}$$

$$y^2 = \tilde{y}^2 B = \frac{B\xi_2}{1+b\xi_1\xi_2}$$

$$r = \tilde{r}B = B\frac{\sqrt{(1-\xi_1^2)(1-\xi_2^2)}}{1+b\xi_1\xi_2}$$

Then $r^2 + y_\mu y^\mu = 1$ or $y^{0\,2} = y_1^2 + y_2^2 + r^2 - 1$ → back to AdS_3.

also eliminate ξ_1, ξ_2

$$y^1y^2(1+b\xi_1\xi_2)^2 = B^2\xi_1\xi_2$$

$$(1+b\xi_1\xi_2)y^0 = \xi_1\xi_2 - b$$

$$\xi_1\xi_2 = \frac{y^0+b}{1-by^0}$$

$$y^1y^2\frac{(1-by^0+by^0+b^2)^2}{(1-by^0)^2} = B^2\frac{(y^0+b)}{(1-by^0)}$$

$$\therefore y^1y^2 B^{4\,2} = \cancel{B^2}(b+y^0)(1-by^0)$$

quadratic eq. in y^0

$$by^{0\,2} - (1-b^2)y^0 + B^2y^1y^2 - b = 0$$

$$y^0 = \frac{(1-b^2) - \sqrt{(1-b^2)^2 - 4b(B^2y^1y^2 - b)}}{2b}$$

$$= \frac{(1-b^2) - B^2\sqrt{1 - \frac{4by^1y^2}{B^2}}}{2b}$$

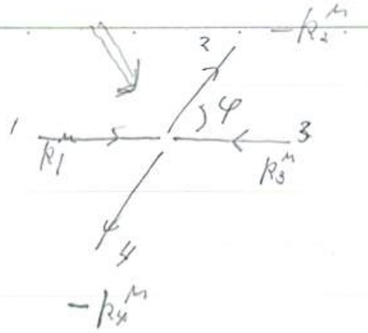
$$= -b + y_1y_2 + \frac{b}{B^2}(y_1y_2)^2 + \cdots$$

to be used later

written 2008.6.1

not yet.
I like to derive further.

· Kinematics of the scattering



Take center of mass frame

$k_1^\mu = (k, \vec{k})$ $k_3^\mu = (k, -\vec{k})$

$-k_2^\mu = (k, \vec{k}')$, $-k_4^\mu = (k, -\vec{k}')$

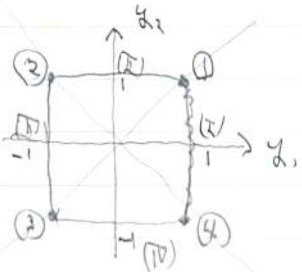
$k = |\vec{k}| = |\vec{k}'|$, $\vec{k}\cdot\vec{k}' = k^2 \cos\varphi$

$(-,+,+,+)$ metric

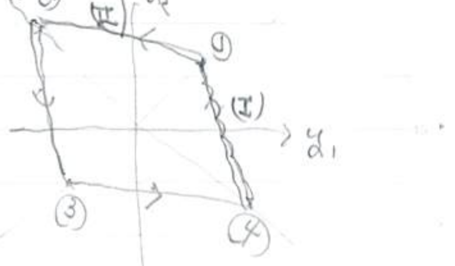
$s = -(k_1+k_2)^2 = -2k_1\cdot k_2 = -2(k^2 - k^2\cos\varphi) = -4k^2\sin^2\frac{\varphi}{2} \le 0$

$t = -(k_1+k_4)^2 = -2k_1\cdot k_4 = -2(k^2 + k^2\cos\varphi) = -4k^2\cos^2\frac{\varphi}{2} \le 0$

$u = -(k_1+k_3)^2 = 4k^2$, $s+t+u = 0$



$\Rightarrow$



Recall

$\Delta y^{\mu(I)}_{I} = y^{\mu(I)}_{(\sigma=\pi)} - y^{\mu(I)}_{(\sigma=0)}$

$= 2\pi k^\mu_{(I)}$

Vertices	y_1	y_2	y_0
(1)	$\frac{B}{1+b}$	$\frac{B}{1+b}$	$\frac{1+b}{1+b}$
(2)	$-\frac{B}{1-b}$	$\frac{B}{1-b}$	$-\frac{1+b}{1-b}$
(3)	$-\frac{B}{1+b}$	$-\frac{B}{1+b}$	$\frac{1-b}{1+b}$
(4)	$\frac{B}{1-b}$	$-\frac{B}{1-b}$	$-\frac{1+b}{1-b}$

(1)−(4) (3)−(1)

$$\therefore \frac{s}{(2\pi)^2} = -2\,\Delta y^{(I)}\cdot\Delta y^{(II)}$$

$$= -2\begin{pmatrix} \frac{1-b}{1+b} + \frac{1+b}{1-b} \\ \frac{B}{1+b} - \frac{B}{1-b} \\ \frac{B}{1+b} + \frac{B}{1-b} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1+b}{1-b} - \frac{1-b}{1+b} \\ -\frac{B}{1-b} - \frac{B}{1+b} \\ \frac{B}{1-b} - \frac{B}{1+b} \end{pmatrix}$$

not yet better to work out the angle

not complete

$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}$

$-t(2\pi)^2 = \frac{8a^2}{(1+b)^2}$

written 2008, 6.1

From NG to the Sigma model

Back to the NG action (5-1)
in the Minkowski signature.

To exploit the notation in the flat space

$$\gamma_{\tau_M \tau_M} = G_{IJ} \frac{dY^I}{d\tau_M}\frac{dY^J}{d\tau_M} = \dot{Y}\cdot\dot{Y}, \quad \gamma_{\tau_M\sigma} = \dot{Y}\cdot Y', \quad \gamma_{\sigma\sigma} = Y'\cdot Y'$$

$$S_{NG} = \frac{\sqrt{\lambda}}{2\pi}\iint d\tau_M d\sigma \sqrt{-(\dot{Y}\cdot\dot{Y})(Y'\cdot Y') + (\dot{Y}\cdot Y')^2}$$

$$\approx \frac{\sqrt{\lambda}}{2\pi}\frac{1}{2}\iint d\tau_M d\sigma \left(\dot{Y}\cdot\dot{Y} - Y'\cdot Y'\right)$$

under the Virasoro conditions

$$\begin{cases} \frac{1}{2}(\dot{Y}+Y')\cdot(\dot{Y}+Y') \approx 0 \\ \frac{1}{2}(\dot{Y}-Y')\cdot(\dot{Y}-Y') \approx 0 \end{cases}$$

or

$$\begin{cases} \dot{Y}\cdot\dot{Y} + Y'\cdot Y' \approx 0 \\ \dot{Y}\cdot Y' \approx 0 \end{cases}$$

In Euclidean signature, we get

$$S_{E\,NG} = -i S_{NG} = -i\frac{\sqrt{\lambda}}{2\pi}\frac{1}{2}\iint -i\,d\tau\,d\sigma\,(-)\left(\dot{Y}^e\cdot\dot{Y}^e + Y'\cdot Y'\right)$$

$$= \frac{\sqrt{\lambda}}{2\pi}\frac{1}{2}\iint d^2\xi_E \left(G_{IJ}\frac{dY^I}{d\xi_E^i}\frac{dY^J}{d\xi_E^j}\right)\delta^{ij}$$

$$\quad\quad = \eta_{MN}\frac{dY^M}{d\xi_E^i}\frac{dY^N}{d\xi_E^j}\delta^{ij}$$

provided

$$(A)\begin{cases} \dot{Y}^e\cdot\dot{Y}^e \approx Y'\cdot Y' \\ \dot{Y}\cdot Y' \approx 0 \end{cases}$$

Namely, the induced metric $ds^2 = H_{ij}\,d\xi_E^i\,d\xi_E^j$
is conformally flat

Now for A-M square solution

$$H_{ij} = \begin{pmatrix} \frac{1}{(1-y_1^2)^2} & 0 \\ 0 & \frac{1}{(1-y_2^2)^2} \end{pmatrix}$$

written 2008.6.2

$$ds^2 = \frac{dy_1^2}{(1-y_1^2)^2} + \frac{dy_2^2}{(1-y_2^2)^2} = du_1^2 + du_2^2 \quad \text{under } y_i = \tanh u_i$$

$\therefore (u_1, u_2) = (\zeta_{E1}, \zeta_{E2})$ obviously solves the Virasoro constraints

Conclusion: Once we choose u_1, u_2 as the worldsheet coordinates, A–M square solution becomes a solution to σ-model of, satisfying Vir. const. So is the AM rhombus solution as boost is a part of $SO(4,2)$, which is an isometry and is linearly realized in Y^M coordinates.

5-9

refer to the end on part

Evaluation of the Onshell Action

~~Rest~~ of Alday - Maldacena and I am not going to go through

Via D-brane version of the dimensional regularization

Just quote the answer A-Mal (3.35) ~ (3.38)

$$S_E = B_\epsilon \left(\frac{\pi \Gamma[-\epsilon/2]^2}{\Gamma[\frac{1-\epsilon}{2}]^2} \, {}_2F_1\left(\tfrac{1}{2}, -\tfrac{\epsilon}{2}, \tfrac{1-\epsilon}{2}; b^2\right) + 1 \right)$$

$$A = \text{scalar part} = \exp\left[- S_{E,div} + \frac{\sqrt{\lambda}}{8\pi} \left(\log \frac{s}{t}\right)^2 - \tilde{C} \right]$$

$$\tilde{C} = \frac{\sqrt{\lambda}}{4\pi} \left(1 - \frac{\pi^2}{3} - 2 \log 2 + (\log 2)^2 \right)$$

$$S_{E,div} = 2 S_{div,s} + 2 S_{div,t}$$

$$S_{E,div} = + \frac{1}{\epsilon^2} \frac{1}{2\pi} \sqrt{\frac{\lambda \mu^{2\epsilon}}{(-\Delta)^\epsilon}} + \frac{1}{\epsilon} \frac{1}{4\pi} (1 - \log 2) \sqrt{\frac{\lambda \mu^{2\epsilon}}{(-\Delta)^\epsilon}}$$

cf. natural cutoff r_c Alday review (3.23)

$$S_E = \frac{\sqrt{\lambda}}{2\pi} A \,, \quad A = \frac{1}{4} \left(\log \frac{r_c^2}{-8\pi^2 s} \right)^2 + \frac{1}{4} \left(\log \frac{r_c^2}{-8\pi^2 t} \right)^2 - \frac{1}{4} \log^2 \left(\frac{s}{t}\right)^2 + \text{const.}$$

$$M = e^{-S_E}$$ agrees with BDS with

$$f(\lambda)_{\text{strong}} = \frac{\sqrt{\lambda}}{\pi} \,, \quad g(\lambda)_{\text{strong}} = \frac{\sqrt{\lambda}}{2\pi} (1 - \ln 2)$$

written 2008. 6. 2.

VI) Use of the boundary ring and Approximate Solution to NG

- suggestions and simplifications

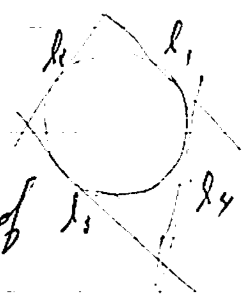
Fruitful to have more examples, in particular for higher point amplitudes

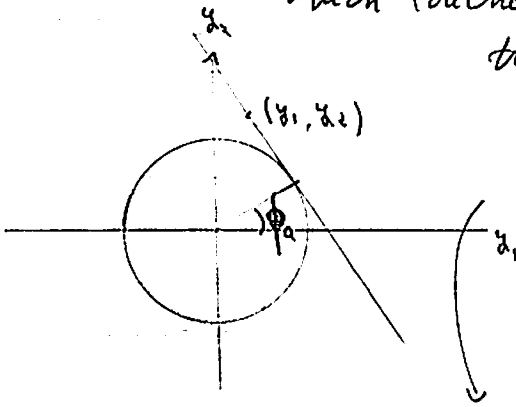
Suggestions
1) stay with NG, determine $r(y_1 y_2)$, $y_0(y_1 y_2)$, $y_3(y_1 y_2)$
2) assume, temporarily, that the minimal surface in question is algebraic
3) begin with the simplified B.C. ; the boundary light-like polygon is denoted by Π

Three levels of simplification

- $y_3 = 0$, $\Rightarrow$ convenient to consider the projection of Π onto $(y_1 y_2)$ plane, denoted by $\bar{\Pi}$
- assume that $\bar{\Pi}$ admits an inscribed circle which touches all of the sides

special ones $n=4$ $l_1 + l_3 = l_2 + l_4$

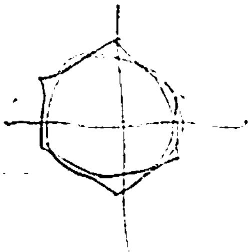




$$\begin{cases} 1 = c_a y_1 + s_a y_2 \\ y_0 = \sigma_a(-s_a y_1 + c_a y_2) \end{cases}$$

$\sigma_a = \pm 1$ and alternates with a

$\therefore 1 + y_0^2 = y_1^2 + y_2^2$



namely the boundary polygon Π lies in AdS^3.

Then it is possible to work with the ansatz

$y_0 = y_1^2 + y_2^2 + r^2 - 1$, which is $Y4 = 0$

$\Rightarrow$ either $r(y_1 y_2)$ or $y^0(y_1, y_2)$ unknown.

written, see 6.3

Z_N symmetric, y^0 changes direction at every vertex

~~KIAS slide~~ p13

- Boundary ring R_π

Definition

$\cdot$ $P_2 \in R_\pi$

we look for segment independent statement

- Elementary construction of Boundary Ring for Z_n symmetric Configuration

summary (18)

$$z = e^{i\phi_a}(1+i(-)^{a} y_0)$$

raise $\frac{n}{2}$

(*1) $\boxed{z^{\frac{n}{2}} = (-)^a (1+i(-)^{a} y_0)^{\frac{n}{2}}}$

∴ contained in the imaginary part of (*1).

⇓

Formula for yoline element of R_π

seen in p15 KIAS slide

n=4

$$LHS = y_1^2 - y_2^2 + 2i y_1 y_2$$

$$RSH = (-)^a(1-y_0^2) + 2i y_0$$

Imaginary part $y_1 y_2 = y_0$

AM square

n=6

$$LHS = y_1^3 - 3y_1 y_2^2 + i(3y_1^2 y_2 - y_2^3)$$

$$RHS = (-)^a(1+3i(-)^a y_0 - 3y_0^2 - i(-)^a y_0^3)$$

∴ Imaginary ~~part~~ $3y_0 - y_0^3 = 3y_1^2 y_2 - y_2^3$

subtract P_2 part

$$3y_0 - y_0(y_1^2+y_2^2-1) = 3y_1^2 y_2 - y_2^3$$

$$\therefore 4 g_3(y_0, y_1, y_2) = y_0(4 - y_1^2 - y_2^2) - y_2(3y_1^2 - y_2^2) = 0$$

not a solution to N.G.

written 2008.6.11

P16. P17

· Approximate Solutions to NG and the use of B.R.

KI45 slides P16 P17

$P_2 = P + y_0^2 - y_1^2 - y_2^2$ rhombus cut

- Beyond Z_N symmetric configuration, which are needed to do physics e.g. rhombus.

rhombus illustration

IMS (5.36), (6.41) ~ (6.45)

- The exact solution

$$0 = S_\Diamond = y_1 y_2 - \frac{1}{2}(1-y_0^2)\sin(2\phi) - y_0\cos 2\phi = -\cos 2\phi\left(y_0 - \frac{y_1 y_2}{\cos 2\phi} + \frac{1}{2}\tan 2\phi(1-y_0^2)\right)$$

$$\sin 2\phi = \frac{2b}{1+b^2} \qquad \cos 2\phi = \frac{1-b^2}{1+b^2}$$

- On the other hand, we developed a more general construction of y_0 linear B.R. elements

$$\frac{1}{\cos 2\phi} = \cosh\xi$$

$$\tan 2\phi = \sqrt{\cosh^2\xi - 1} = \sinh\xi$$

$$\downarrow$$

$$\mathcal{L}_\Diamond = y_0 - y_1 y_2 \cosh\xi + \left(1 - \frac{1}{2}(y_1^2 + y_2^2)\right)\sinh\xi \sim y_0 - \cosh\xi\, y_1 y_2 + \frac{1}{2}(1-y_0^2)\sinh\xi$$

$$\therefore S_\Diamond - \mathcal{L}_\Diamond = \left(\frac{1}{2}(1-y_0^2) - 1 + \frac{1}{2}(y_1^2+y_2^2)\right)\sinh\xi$$

$$= -\frac{1}{2}\sinh\xi\, P_2$$

$$\boxed{\therefore S_\Diamond = \mathcal{L}_\Diamond - \frac{1}{2}\sinh\xi\, P_2.}$$

$n=6$ case and higher IMS 6.63, R.

~~To~~ Obtain exact solution
~~Begin with~~
try to

can be complicated and not algebraic

$$S_\pi = \mathcal{L}_\pi + M_\pi(y_0, y_1, y_2)\, P_2$$

written 2008.6.11

· y_0 linear Approximation to NG eq.

Back to 5-3

$$\int L_{NG}\, dy_1 dy_2 = \int \sqrt{\frac{(y_i \partial_i y_0 - y_0)^2 - (\partial_i y_0)^2 + 1}{(1 + y_0^2 - y_i^2)^3}}\, dy_1 dy_2$$

Expand in y_0

$$\int \frac{dy_1 dy_2}{(1 - y_1^2 - y_2^2)^{3/2}} - \frac{1}{2}\int dy_1 dy_2 \left(\frac{(\partial_i y_0)^2 - (y_i \partial_i y_0 - y_0)^2}{(1 - y_i^2)^{3/2}} + \frac{3 y_0^2}{(1 - y_i^2)^{5/2}} \right)$$

E-L eq : $\partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i y_0)} = \frac{\partial \mathcal{L}}{\partial y_0}$

$$\partial_i \left(\frac{2\partial_i y_0 - 2(y_j \partial_j y_0 - y_0)\, y_i}{(1 - y_\ell^2)^{3/2}} \right) = \frac{6 y_0}{(1 - y_i^2)^{5/2}} + \frac{2(y_i \partial_i y_0 - y_0)}{(1 - y_i^2)^{3/2}}$$

$$\| $$

$$\frac{2\partial_i \partial_i y_0 - 4(y_j \partial_j y_0 - y_0) - 2\left(\partial_i (y_j \partial_j y_0) - \partial_i y_0\right) y_i}{(1 - y_\ell^2)^{3/2}} + \cancel{3}\,6\, \frac{y_i \left(\partial_i y_0 - (y_j \partial_j y_0 - y_0) y_i \right)}{(1 - y_\ell^2)^{5/2}}$$

$$\therefore\ 2\partial_i \partial_i y_0 - 4 y_j \partial_j y_0 + 4 y_0 - 2 y_j \partial_j y_0 - 2 \underbrace{y_i y_j \partial_i \partial_j y_0}_{\mathcal{D}^2 - \mathcal{D}} + 2 y_i (\partial_i y_0)$$

$$+ 6 y_i \partial_i y_0 = 2 y_i \partial_i y_0 - 2 y_0 + 6 y_0$$

$\Delta_0 = \partial_i \partial_i = 4\partial\bar{\partial}$, $\mathcal{D} = y^i \partial_i = z\partial + \bar{z}\bar{\partial}$

$\mathcal{D}^2 = y^j \partial_j y^i \partial_i = y^j y^i \partial_j \partial_i + y^i \partial_i$

$$2\Delta_0 y_0 - 4\mathcal{D} y_0 + \cancel{4 y_0} - \cancel{2\mathcal{D} y_0} - 2(\mathcal{D}^2 - \mathcal{D}) y_0 + \cancel{2\mathcal{D} y_0}$$

$$+ 6\mathcal{D} y_0 = \cancel{2\mathcal{D} y_0} - \cancel{2 y_0} + \cancel{6 y_0}$$

ZM8 (3.6)

$\Delta_{21} y_0 = 0$ where $\Delta_{21} = \Delta_0 - \mathcal{D}^2 + \mathcal{D}$

written 2008. 6. 5

$z = y_1 + i y_2, \quad \bar{z} = y_1 - i y_2$

The R^3 case $\quad y_0^{flat} = \sum_{k \geq 0} Re(\alpha_{k0} z^k)$

Our series solution in the AdS_3 case

$$y_0 = \sum_{k, d' \geq 0} (\alpha_{kd'} z^k + \bar{\alpha}_{kd'} \bar{z}^k)(z\bar{z})^{d'} = \sum_{k, d'} Re(\alpha_{kd'} z^k)(z\bar{z})^{d'}$$

In the linearized case, the recurrence relation

$$\alpha_{k,d'+1}^{linear} = \frac{(k+2d')(k+2d'-1)}{4(d'+1)(k+d'+1)} \alpha_{kd'}^{linear}$$

$$\alpha_{k,d'}^{lin} = \frac{k(k-1)}{4^{d'} d'!} \frac{(k+2d'-2)!}{(k+d')!} \alpha_{k0}^{lin}$$ seen in TMM 1. (4.2)

$$\therefore y_0^{lin} = \sum_{k \geq 0} Re(\alpha_{k0} z^k) \, {}_2F_1\left(\frac{k}{2}, \frac{k-1}{2}; k+1; z\bar{z}\right)$$

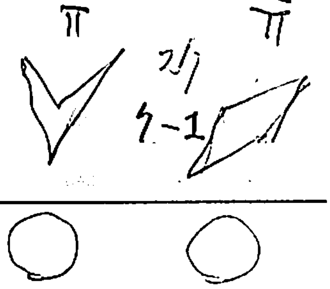
$${}_2F_1(a; b; c; x) = \sum_{d' \geq 0} \frac{\Gamma(a+d')\Gamma(b+d')}{d'! \, \Gamma(c+d')} x^{d'}$$

$${}_2F_1\left(\frac{k}{2}, \frac{k-1}{2}; k+1; x\right) \sim \frac{(1 + k\sqrt{1-x})(1-\sqrt{1-x})^k}{x^k}$$

$$= 2^k k(k-1) \, i \sqrt{\frac{2}{\pi}} \left(\frac{1-x}{x}\right)^{\frac{3}{4}} x^{\frac{1-k}{2}} Q_{k-1/2}^{-3/2}\left(\frac{1}{\sqrt{x}}\right)$$

$y^0_{a1} = (-)^a \tan\frac{\pi}{n} \xrightarrow{n\to\infty} 0$ $\qquad n=4$

$n=\infty$



VII) Computation of a Wavy Circle

We thought of one manageable case, other than Alday–Maldacena rhombus, a wavy circle on the plane. This is an infinitesimal deformation of the unit circle into an arbitrary curve on the plane, containing ∞ many parameters.

- $n = \infty$ circle solution $\qquad$ $y_3 = 0$ $\qquad$ AdS_3 : $1 = r^2 + y_\mu y^\mu$
$$= r^2 y_0^2 + y_i y_i$$

And now $y^0_a = 0$ namely the polygon flattens.

$$H_{ij} = \frac{1}{r^2}\begin{pmatrix} 1+(\partial_1 r)^2, & \partial_1 r\, \partial_2 r \\ \partial_1 r\, \partial_2 r, & 1+(\partial_2 r)^2 \end{pmatrix}$$

$$\det H = \frac{1}{r^4}\left(1+(\partial_1 r)^2 + (\partial_2 r)^2\right), \quad L_{NG} = \frac{1}{r^2}\sqrt{1+(\partial_i r)^2}$$

E-L eq.
$$\partial_i\left(\frac{1}{r^2}\frac{(\partial_i r)}{\sqrt{1+(\partial_j r)^2}}\right) = -\frac{2}{r^3}\sqrt{1+(\partial_i r)^2}$$

$$= -\frac{2(\partial_i r)^2}{r^3\sqrt{1+(\partial_j r)^2}} + \frac{\partial^2 r}{r^2\sqrt{1+(\partial_j r)^2}} - \frac{1}{2}\frac{2(\partial_j r)(\partial_i\partial_j r)(\partial_i r)}{r^2\left(\sqrt{1+(\partial_j r)^2}\right)^3}$$

$$\therefore -2 - \cancel{2(\partial_i r)^2} = -\cancel{2(\partial_i r)^2} + r\partial^2 r - \frac{r(\partial_i r)(\partial_j r)(\partial_i\partial_j r)}{1+(\partial_i r)^2}$$

$$\therefore \boxed{r(\partial_i r)(\partial_j r)(\partial_i\partial_j r) = (2 + r\partial^2 r)(1+(\partial_i r)^2)}$$

The only candidate which lies in the AdS_3 ansatz is

$$\boxed{r^2 = 1 - y_i^2}$$

$$r\,\partial_i r = -y_i$$
$$\partial_j r\, \partial_i r + r\,\partial_j\partial_i r = -\delta_{ij}$$
$$r\,\partial_i\partial_j r = -\delta_{ij} - \partial_i r\, \partial_j r$$
$$= -\delta_{ij} - \frac{y_i y_j}{r^2}$$

written 2008. 6. 5.

LHS $\cdot r^2 = y_i y_j (-\delta_{ij} - \partial_i r \partial_j r) = -y_i y_i - (y_i \partial_i r)^2 = -y^2 - \frac{(y^2)^2}{r^2}$

RHS $\cdot r^2 = -y_i y_i \left(1 + \frac{y^2}{r^2}\right)$ ok,

$\therefore$ $\boxed{r^2 = 1 - y^2 \text{ is a solution}}$

$y^0 = y^3 = 0$

- Formulation and the goal:

$z = y_1 + i y_2$

- Consider the conformal map $z = H(\zeta)$ of the interior of the unit circle in the complex ζ-plane into the domain bdd by π
- Find the shape of the minimal surface

$$r^2(z, \bar z) = 1 - \zeta \bar\zeta + a(\zeta, \bar\zeta)$$

by solving the NG eq. for $a(\zeta, \bar\zeta)$ subject to the b.c. $a\big|_{\substack{\text{bndry}\\ |\zeta|=1}} = 0$

$\zeta, \bar\zeta$: complex world sheet coordinates

- check $D_\pi \overset{?}{=} A_\pi$

- Action

$(1, i; 1, -i)$ $z, \bar z$

anything $d\zeta\, d\bar\zeta = -2i\, d^2\zeta$

$$S_E = \frac{\sqrt{\lambda}}{2\pi} \int_{D_E} \cancel{d^2 z} \sqrt{\det \gamma}\,..$$

The cleanest summary (26)

$$\gamma_{ab} = \frac{1}{r^2} \left(\partial_a y^\mu \partial_b y_\mu + \partial_a r \partial_b r \right)$$

and write

$$\partial z = 1 + \sum_{k=1}^{\infty} k h_k \zeta^{k-1} \equiv \partial H \equiv 1 + \partial h$$

$$\begin{array}{c|c} & \zeta \quad \bar\zeta \\ \hline \begin{array}{c}\zeta\\ \bar\zeta\end{array} & (\gamma) \end{array} = \frac{1}{r^2} \begin{pmatrix} \partial_\zeta z\, \partial_\zeta \bar z + \partial_\zeta r \partial_\zeta r, & \frac{1}{2}(\partial_\zeta z \partial_{\bar\zeta} \bar z + \partial_\zeta \bar z \partial_{\bar\zeta} z) + \partial_\zeta r \partial_{\bar\zeta} r \\ '' , & \partial_{\bar\zeta} z\, \partial_{\bar\zeta} \bar z + \partial_{\bar\zeta} r \partial_{\bar\zeta} r \end{pmatrix}$$

$$r^4 \det\gamma = -\frac{1}{4}\left\{|\partial z|^2 - |\bar\partial z|^2\right\}^2 - \left(|\partial z|^2 + |\bar\partial z|^2\right)|\partial r|^2 + \partial z\, \partial \bar z\, (\bar\partial r)^2 + \bar\partial z\, \bar\partial \bar z\, (\partial r)^2$$

In our case, some simplification
$\bar\partial z = 0$

$$S_\epsilon[a,h] = \frac{\sqrt{\lambda}}{2\pi}\int d^2\zeta \, \frac{1}{r^2}\sqrt{|\partial H|^2\left(|\partial H|^2 + 4\,\partial r \bar\partial r\right)}$$

reg. needed.

• Eq. of motion for $a(\zeta,\bar\zeta)$ — Basically goes as p24, p25 of KIAS talk slide

cf summary (40) (41)

up to the lowest nontrivial order in h

$$= \frac{\sqrt{\lambda}}{2\pi}\int d^2\zeta \, \frac{|\partial H|^2\sqrt{r^2 + \partial r^2 \bar\partial r^2/|\partial H|^2}}{r^3}$$

$$= \frac{\sqrt{\lambda}}{2\pi}\int d^2\zeta \, \frac{|1+2h|^2\left(1-\zeta\bar\zeta + a + \frac{(\partial a-\bar\zeta)(\bar\partial a - \zeta)}{|1+2h|^2}\right)^{1/2}}{(1-\zeta\bar\zeta+a)^{3/2}}$$

$$0 \underset{\text{eq. of motion}}{=} \partial_\zeta\left(\frac{\partial\mathcal{L}}{\partial(\partial_\zeta a)}\right) + \partial_{\bar\zeta}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\bar\zeta} a)}\right) - \frac{\partial\mathcal{L}}{\partial a} = \frac{\frac{1}{4}}{(1-\zeta\bar\zeta)^{3/2}}\left(\Delta_{21}^{NG}\, a - \partial\left((\partial h + \bar\partial \bar h)\zeta\bar\zeta\right) + 4(\partial h + \bar\partial\bar h)\right)$$

$$\Delta_{21}(\bar\zeta h + \zeta\bar h)$$

Generic solution in p25.

$$\therefore\ a(\zeta,\bar\zeta) - 2\zeta\bar\zeta\sum_{k=1}^{\infty}\mathrm{Re}(h_k\zeta^{k-1}) = \sum_{k=1}^{\infty}\mathrm{Re}(a_{k-1}\zeta^{k-1})\,F_{k-1}(\zeta\bar\zeta)$$

$$F_k(1) = 1$$

B.c. of the unregularized theory $\Rightarrow a_{k-1} = 2h_k$

$$\boxed{\begin{aligned} a(\zeta,\bar\zeta) &= 2\sum_{k=1}^{\infty}\mathrm{Re}(h_k\zeta^{k-1})\,A_k(\zeta,\bar\zeta) \\ A_k(x) &= F_{k-1}(x) - x \end{aligned}}$$

written 2018.6.5

· Evaluating a regularized area

· c-reg KIAS slide p26, the situation in c-reg is best explained in summary (43), which I partly recalled.

μ-reg replace $r^2 \to r^2+\mu^2$

$$S_E = \frac{\sqrt{\lambda}}{2\pi}\int_{|z|\le 1} d^2z \, \frac{\sqrt{|\partial H|^2(|\partial H|^2+4|\partial r|^2)}}{r^2+\mu^2}$$

· Double contour integral ZMM2 V2, Appendix II

· The result

KIAS slide p28, coefficients corrected in order to restore the normalization + ...
ZMM2 V2 (1.6)
(1.3)

two sides differ, only slightly

written 2008. 6. 12

· Comment on our result

- $K_0 = \frac{8\pi}{3}$: difference factor in front of the bracket

  4 inside [ ]

  $K_\square = 8$ $\therefore \frac{K_0}{K_\square} = \frac{\pi}{3} = \frac{3.14\cdots}{3} \simeq 1.05$ 5% discrepancy.

· part in D_Π which is linear in h_g is consistent with the expression which is made of the schwarzian derivative

$$\oint_{\text{unit circle}} (\bar{z}-\bar{\zeta})\, S_\zeta\{z\}\, \zeta' d\zeta \quad , \quad S_\zeta\{z\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$$

· perspectives

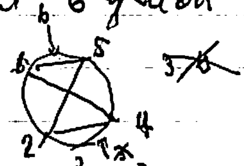
· issue of conformal symmetry

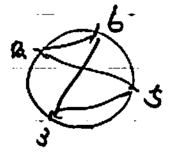
cf. Non MHV ⌊also begins with $n\geq 6$

VIII) Latest Status

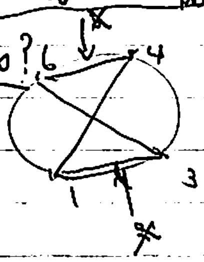
BDS now known to be violated at ⌊also 6 gluon 2-loop

$$\boxed{A_6 = A_{BDS} + f(u_1, u_2, u_3)}$$





Why $n\geq 6$?

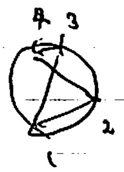


$$u_1 = \frac{x_{13}^2\, x_{46}^2}{x_{14}^2\, x_{36}^2} \qquad u_2 = \frac{x_{24}^2\, x_{15}^2}{x_{25}^2\, x_{14}^2} \qquad u_3 = \frac{x_{35}^2\, x_{26}^2}{x_{36}^2\, x_{25}^2}$$

invariant cross ratios

$$2\pi k_i = x_i - x_{i+1}$$

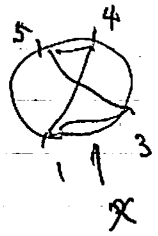
cf. $n=4$



$$\frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} \quad (x_{12}^2 = 0,\ x_{34}^2 = 0)$$

cannot form invariant cross ratios

$n=5$



$x_{45}^2 = 0, \ \ldots$

Why 2-loop: BDS conjecture is an exponentiation of 1-loop result. Nontrivial comparison must begin with 2-loop.

BDS conjecture

$$M_n^{(2)\,BDS}(\epsilon) = \frac{1}{2}\left(f^{(1)}(\epsilon) M_n^{(1)}(\epsilon) + C^{(1)}\right)^2 + f^{(2)}(\epsilon) M_n^{(1)}(2\epsilon) + C^{(2)}$$

$$\lim_{\epsilon\to 0}\left(M_n^{(2)}(\epsilon) - M_n^{(2)\,BDS}(\epsilon)\right) \equiv R_n^{(2)} \quad \text{; remainder fn.}$$

shown to be invariant numerically

current conjecture / very nontrivial & issue.

$$\lim_{\substack{\text{strong coupling} \\ \text{conformal const}}} W_\pi \overset{?}{=} e^{\kappa \sqrt{\lambda} A_\pi}$$

$$W_\pi = \left\langle \mathrm{Tr}\, P \exp\left\{ i \oint_\pi \left(A_\mu(\vec{x}) dy^\mu + \phi\, d\ell\right) + \cdots \right\}\right\rangle$$

considerably weaker than BDS,

written 2008.6.12