Gluon Amplitudes and AdS_5 – Minimal Surfaces

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IMM1	arXiv:0712.0159	[hep-th]
IM8	arXiv:0712.2316	[hep-th]
IMM2	arXiv:0803.XXXX	[hep-th]
		this week

I). Introduction

- A new version of the string-gauge duality has appeared last year; Alday-Maldacena
- There is a chance that this version is formulated (pretty much) independently of properties of QFT, and instead as an algebraic identity, which is the case, for example, for 2d boson – fermion equivalence etc.
- A fruitful form of the assessment of this hypothesis has been called for.

Contents

II)

$\mathcal{N} = 4$ Planar SYM

- Conjecture by Bern, Dixon, Smirnov (and Kosower) on the full solution of perturbative n point planar amplitudes:

factorizes and exponentiates

$$\mathcal{A}(\mathbf{p_1}, \ldots \mathbf{p_n}|\lambda) = \mathcal{A}_{tree}\mathcal{A}_{IR}\mathcal{A}_{finite}$$

λ ; 't Hooft coupling

$\mathcal{A}_{tree}$; the tree amplitude (containing color factors)

IR divergent part

$$\mathcal{A}_{IR} \sim \exp\left(-\frac{1}{4}\sum_{l=1}^{\infty}\lambda^l\left[\gamma_{(l)} + 2l\epsilon g^{(l)}\right] I_n^{(1)}(l\epsilon)\right)$$

with 1-loop scalar box integral

$I_n^{(1)}(\epsilon) = \frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{s_{i,i+1}}\right)^{\epsilon}, \quad s_{i,i+1} = (p_i + p_j)^2$

$\gamma(\lambda) = \sum_l \gamma_{(l)}\lambda^l$; cusp anomalous dimension of the Wilson loop

ϵ, μ ; parameters in dim. reg.

IR finite part

$$\mathcal{A}_{finite} = \exp\left(\frac{1}{4}\gamma(\lambda)F_n^{(1)}(\mathbf{p_1}, \ldots \mathbf{p_n}) + \underset{\substack{\uparrow \\ \text{depends neither on } n \text{ nor kinematics}}}{c(\lambda)}\right)$$

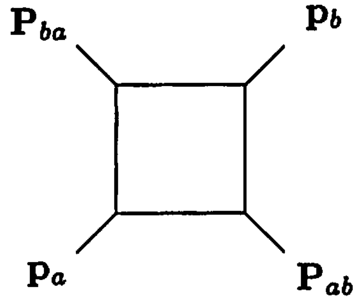
$\gamma(\lambda) \sim \sqrt{\lambda} + \text{const.} + O(1/\sqrt{\lambda})$ at λ large,

$\Longrightarrow$ research on integrability via Bethe ansatz, etc

$\cancel{F_n^{(1)}}$ $M_n^{(1)}$; one-loop ~~counterpart~~ in $\mathcal{N}=4$ SYM, is given by

$$M_n^{(1)} \cancel{F_n^{(1)}} = \sum_{a<b} F^{2me}(\mathbf{p}_a, \mathbf{P}_{ab}, \mathbf{p}_b, \mathbf{P}_{ba})$$

2-mass easy square diagrams



$$\mathbf{P}_{ab} = \sum_{c=a+1}^{b-1} p_c$$

The two on shell
The other two off shell

- dilogarithimic rep. of F^{2me} : OK cf. BDK

$$F^{2me}(\mathbf{p},\mathbf{P},\mathbf{q},\mathbf{Q}) = \frac{1}{\epsilon^2(-s)^\epsilon} + \frac{1}{\epsilon^2(-t)^\epsilon} + \frac{1}{\epsilon^2(-\mathbf{P}^2)^\epsilon} + \frac{1}{\epsilon^2(-\mathbf{Q}^2)^\epsilon}$$
$$+ Li_2(1-as) + Li_2(1-at) - Li_2(1-a\mathbf{P}^2) - Li_2(1-a\mathbf{Q}^2)$$

with $s = (\mathbf{p}+\mathbf{P})^2, \quad t = (\mathbf{p}+\mathbf{Q})^2$

$$a = \frac{s+t-\mathbf{P}^2-\mathbf{Q}^2}{st-\mathbf{P}^2\mathbf{Q}^2}, \quad Li_p(z) = \sum \frac{z^k}{k^p}$$

- Further conversion into a double contour integral done by Brandhuber, Heslop and Travaglini and advertised by Mironov, Morozov and Tomaras;

$$F_n^{(1)} = \oint_\Pi \oint_\Pi \frac{dy^\mu dy'_\mu}{(y-y')^{2+\epsilon}}$$

also denoted by $\equiv D_\Pi$ later in my talk

This is the expression seen in the linearized (or abelian) approximation to the Wilson loop overage, cf. Polyakov's book and is a purely geometric quantity.

Π ; polygon made out of extrernal momenta $\mathbf{p}_1, \cdots, \mathbf{p}_n$

III)

- Therefore, according to the AdS/CFT duality, $F_n^{(1)}$ should coincide with the area of a minimal surface in AdS_5 in string theory side

$$F_n^{(1)} = (\text{Minimal area})_\epsilon$$

In the n-point string amplitudes with T dualized string coordinates, the surface obeys Dirichlet b. c. and the boundary is the same polygon Π
; Alday-Maldacena

AdS_5 space

work on a metric

$$ds^2 = \frac{-dy_0^2 + dy_1^2 + dy_2^2 + dy_3^2 + dr^2}{r^2}$$

which is induced from the flat one in R_6^{--++++} on the hypersurface

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1$$

via

$$Y^\mu = \frac{y^\mu}{r}, \qquad \mu = 0, 1, 2, 3,$$
$$Y_{-1} + Y_4 = \frac{1}{r}$$
$$Y_{-1} - Y_4 = \frac{r^2 + y_\mu y^\mu}{r}$$

IV)

- Suggested approach IMM1

punch line: $\left\{ \begin{array}{l} \bullet \text{ solve the NG eq. for } r(y_1, y_2),\ y_0(y_1, y_2) \\ \bullet \text{ hunt for power series solution} \end{array} \right.$

(actually begin with assuming algebraic surfaces.)

- work with (over) simplified boundary conditions, at three levels:

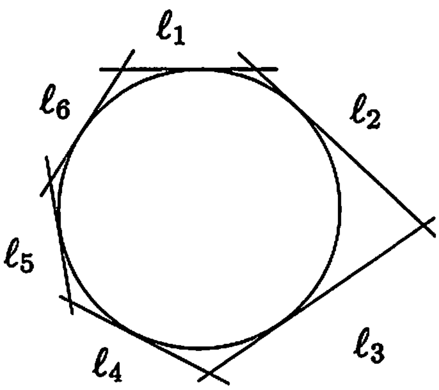
1). $y_3 = 0 \quad \leftrightarrow \quad Y_3 = 0$

also denote the projection of Π on y_1, y_2 plane by $\bar{\Pi}$

Π: 3d polygon　　$\bar{\Pi}$: 2d polygon

2). For $\bar{\Pi}$, $\exists$ an inscribing circle which touches all of the sides

I.C.



$n = 3$; triangle, always existing

$n = 4$; quadrilaterals,

$\exists$I.C. $\Leftrightarrow$ $\ell_1 + \ell_3 = \ell_2 + \ell_4$

$n > 4$; $\exists$I.C. $\overset{\Rightarrow}{\not\Leftarrow}$ $\sum_{i=1}^{n} (-)^i \ell_i = 0$

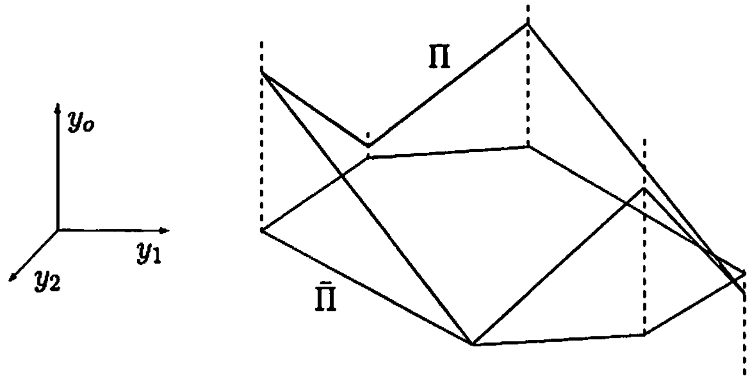
Π_n consists of light-like vectors

$$\therefore -y_0^2 + y_1^2 + y_2^2 = 1 \qquad \Pi_n \quad \leftrightarrow \quad Y_4 = 0$$

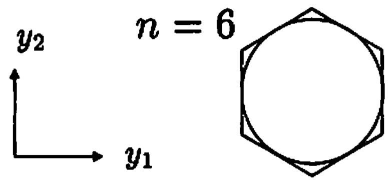
Then it is possible to work with the ansatz

$$y_0^2 = y_1^2 + y_2^2 + r^2 - 1$$

$\Rightarrow$ either $r(y_1, y_2)$ or $y_0(y_1, y_2)$ is an unknown function



3). Z_n symmetric $\bar{\Pi}_n$ and y_0 changes a direction at every vertex



• NG eq.

take (y_1, y_2) the worldsheet coordinates

$$\frac{\partial}{\partial y_1}\left(\frac{\partial y_0}{\partial y_1}\frac{H_{22}}{r^2 L_{NG}}\right) + \frac{\partial}{\partial y_2}\left(\frac{\partial y_0}{\partial y_2}\frac{H_{11}}{r^2 L_{NG}}\right) - \frac{\partial}{\partial y_1}\left(\frac{\partial y_0}{\partial y_2}\frac{H_{12}}{r^2 L_{NG}}\right) - \frac{\partial}{\partial y_2}\left(\frac{\partial y_0}{\partial y_1}\frac{H_{12}}{r^2 L_{NG}}\right) = 0,$$

$$H_{ij} = \frac{-\frac{\partial y_0}{\partial y_i}\frac{\partial y_0}{\partial y_j} + \frac{\partial r}{\partial y_i}\frac{\partial r}{\partial y_j} + \delta_{ij}}{r^2}$$

$$L_{NG} = \sqrt{\frac{(y_i \partial_i y_0 - y_0)^2 - (\partial_i y_0)^2 + 1}{(1 + y_0^2 - y_1^2 - y_2^2)^3}}$$

• boundary ring $\mathcal{R}_\Pi$

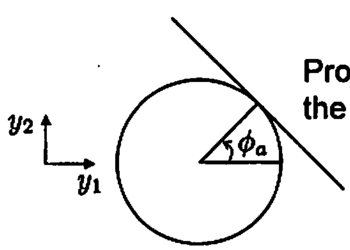
Def: a ring of polynomials in $y's$ which vanish at Π

- Due to our ansatz,

$$P_2 = y_0^2 + 1 - y_1^2 - y_2^2 \quad \text{is obviously} \quad \in \mathcal{R}_\Pi$$

- The solution to NG should be looked for inside the completion of $\mathcal{R}_\Pi$

• Construction of polygons of special type Z_n sym.



Projection of the a-th segment

$$\begin{cases} \cos\phi_a y_1 + \sin\phi_a y_2 = 1, \\ y_1 = (-)^a \sin\phi_a y_0 + \cos\phi_a, \\ y_2 = (-)^{a-1}\cos\phi_a y_0 + \sin\phi_a \end{cases}$$

or form complex variable

$$z = y_1 + iy_2$$

to write as

$$z = e^{i\phi_a}(1 + i(-)^{\overset{a\ ?}{\cancel{a-1}}} y_0)$$

Omitting some, we have found the following element of $\mathcal{R}_{\Pi}$

segment independent statement

$$\mathcal{P}_{n/2} = y_0 Q_{(n)}(y^2) - K_{n/2}(y_1, y_2) \qquad y^2 = y_1^2 + y_2^2$$

where
$$K_{n/2}(y_1, y_2) = \frac{1}{2^{n/2-1}} \mathrm{Im}(z^{n/2}),$$
$$Q_{(n)} = \frac{(1+\sqrt{1-y^2})^{n/2} - (1-\sqrt{1-y^2})^{n/2}}{2^{n/2}\sqrt{1-y^2}}$$

For $n = 4$

$\mathcal{P}_2 = y_0 - y_1 y_2$ so in this case

$\mathcal{P}_2 = 0$ leads to the A-M solution

- How to use the boundary ring to the approximate solution of NG. (sketchy)

$\exists$ two attitudes :

version I). traditional: impose b.c. after Z_n sym. dictated power series and truncate at level N:

$$y_0 = K_{n/2} \sum_{i,j\geq 0} c_{ij}^{(n)} K_{n/2}^{2i} y^{2j} = \sum_{i,j\geq 0} c_{ij}^{(n)} K_{n/2}^{2i+1} y^{2j} \qquad (*)$$

We found $c_{ij}^{(m)}$ relatively stable with the increase of N

In accuracies seem to increase in the vicinities of the polygon angles

—— bad for regularized area

version II). impose b. c. first, perturb, and develops a series solution from it

nontraditional

Ansatz

$y_0 \approx \frac{K}{Q}$.

and deform.

$$\mathcal{P}_{n/2} = y_0 P_2 B^{(n)}, \qquad B^{(n)} = \sum_{i,j\geq 0} b_{ij}^{(n)} y_0^{2i} y^{2j}$$

$$\therefore \quad y_0 = \frac{K}{Q - (1-y^2)B - y_0^2 B}$$

b. c. OK automatically, but due to uniform convergence prob.

$\exists$ still free parameters

$\therefore$ fix by minimizing the area

• Beyond Z_n sym. configurations

IM8

still $\exists$ inscribed circle

typically a rhombus.

- We developed a study of the boundary rings to these cases as well.
- ~~element of $\mathcal{R}_{\Pi}$ higher in y_0 seems to be important~~

• y_0-linear appx. to NG

from the experience of $(*)$, we thought of, and derived

$$\Delta y_0 = 0, \quad \text{where} \quad \Delta = \Delta_0 - \mathcal{D}^2 + \mathcal{D}$$

$$\Delta_0 = 4\partial\bar{\partial}, \quad \mathcal{D} = z\partial + \bar{z}\bar{\partial}$$

V)

$$(\star) \qquad \underline{D_\Pi \quad \overset{?}{=} \quad A_\Pi}$$

double contour integral regularized

regularized minimal area of AdS_5-NG strings

- reliable evidence come from the $n = 4$ (rhombus) amplitude (Alday-Maldacena)
- meanwhile, counter-arguments against this are mounting:
  - LHS contradicts?
    Some behavior of string scattering amplitudes for $n \geq 6$
  - D_Π, which is an abelian Wilson loop if you like, should not be confused with $\mathcal{N} = 4$ SYM nonabelian Wilson loop.

$$W_\Pi = \left\langle \mathrm{Tr}\, P\exp\left\{ i \oint_\Pi \left(A_\mu(\vec{y}) dy^\mu + \phi\, dl \right) \right\} \right\rangle_{\text{regularized}}$$

$\exists$ many people who accept~~s~~

$$A_\Pi = \log W_\Pi$$

For them, $D_\Pi \neq \log W_\Pi$ implies contradiction of $(\star)$

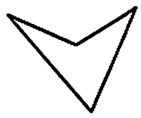
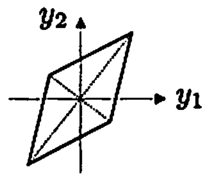
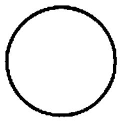
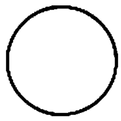
- discrepancy at $n = \infty$ rectangular Wilson loop etc. (Alday-Maldacena 3)

$\vdots$

Clearly, a more fruitful example, which may suggest a modification of $(\star)$, is called for

$\Rightarrow$ $^{\exists}$parameters

- (So far) Exactly solved cases

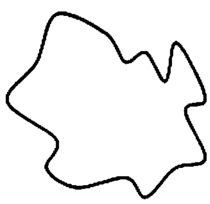
n	Π	$\bar{\Pi}$	
4			rhombus
∞			circle

At $n = \infty$, the width or height of a zig-zag path Π becomes zero.
Then, our ansatze are

$$\begin{cases} y_0 = y_3 = 0 \\ r^2 = 1 - y_1^2 - y_2^2 = 1 - z\bar{z} \end{cases}$$

and it is in fact an exact solution to NG eq. IMM1

- We consider deformulation of the circle
  into an arbitrary curve —— wavy circle on the plane.



containing infinitely many parameters

• **Formulation** :

• Consider the conformal map $\overset{\substack{y_1 + i y_2 \\ /\!/}}{z} = H(\zeta)$
of the interior of the unit circle in the complex ζ -plane
into the domain bdd by Π

$$H(\zeta) = \zeta + h(\zeta), \qquad h(\zeta) = \sum_{k=0}^{\infty} h_k \zeta^k$$

• find the shape of the minimal surface

$$r^2(z, \bar{z}) = 1 - \zeta\bar{\zeta} + a(\zeta, \bar{\zeta})$$

by solving the NG eq. for $a(\zeta, \bar{\zeta})$ subject to the b.c.

$$a|_{\mathrm{bndry}} = 0$$

• For nonvanishing h , $a = 0$ is not a solution and need to compute

a ; up to the ~~second~~ order in h

The relevant form of the NG eq. Is

$$\Delta_{NG}\Big(a + u(h)\Big) = O(a^2, ah, h^2) \qquad (\star\star)$$

where

$$\underset{\substack{\uparrow \\ \text{ordinary Laplace}}}{\Delta_{NG} = \Delta_0} - \mathcal{D}^2 + \underset{\substack{\uparrow \\ \text{dilatation}}}{\mathcal{D}}$$

as before

and

$$u(h) = 2\zeta\bar{\zeta} \sum_{k=1}^{\infty} \mathrm{Re}\Big(h_k \zeta^{k-1}\Big)$$

A generic solution to

$$\Delta_{NG}(a) = 0$$

$$a(\zeta, \bar{\zeta}) = 2\sum_{k=0}^{\infty} \mathrm{Re}\Big(a_k \zeta^k\Big) F_k(\zeta\bar{\zeta})$$

where

$$F_k(x) = \frac{(1 + k\sqrt{1-x})(1-\sqrt{1-x})^k}{x^k} \sim {}_2F_1\left(\frac{k}{2}, \frac{k-1}{2}; k+1; x\right)$$

The solution to $(\star\star)$ which obeys the b.c.

$$a(\zeta, \bar{\zeta}) = 2\sum_{k=1}^{\infty} \mathrm{Re}\Big(h_k \zeta^{k-1}\Big) A_k(\zeta\bar{\zeta}) + O(h^2),$$

$$A_k(x) = F_{k-1}(x) - x$$

- **Evaluating a regularized area:**

For example, in "c-regularization"

$$S_{NG}\{a, h\} = \int_{|\zeta|^2 \le 1-c^2} \frac{\sqrt{|\partial H|^2\Big(r^2|\partial H|^2 + |\partial r^2|^2\Big)}}{r^3} d^2\zeta$$

orders in a on-shell action

$$= S_0\{h\} + S_1\{a, h\} + S_2\{a\} + O(a^{3-j}h^j)$$

$$S_j\{a(h), h\} = \sum_{k=2}^{\infty} |h_k|^2 \sigma_k^{(j)} + O(h^3)$$

The bulk contributions are

$$\sigma_k^{(0)} = \frac{k^2}{2c} + I_1, \quad \sigma_k^{(1)} = I_2, \quad \sigma_k^{(2)} = 0, \qquad I_1 + I_2 = -\frac{k(k-1)(k-2)}{2}$$

The contributions from the bndry are nonuniversal

• Double contour integral

For example, quadratic contributions are

$$\sum_{k=0}^{\infty} |h_k|^2 \oint_{|z|=1} \frac{dz}{2\pi i} \frac{f_k(z)^2}{z^k (z-1)^2}, \qquad f_k(z) \equiv \frac{1-z^k}{1-z} - k$$
$$= + \frac{k(k-1)(k-2)}{6}$$

higher order contributions are in part controlled by the Schwarzian derivative

• Results

$D_\Pi \stackrel{?}{=} A_\Pi$ fails in an interesting way

The finite parts are

$$D_\Pi /_{2\pi} = -4\pi \left[Q_\Pi^{(2)} - Q_\Pi^{(3,1)} - Q_\Pi^{(3,2)} \right] + 4\pi Q_\Pi^{(4)} + O(h^5),$$
$$A_\Pi /_{2\pi} = - \frac{3}{2} \left[Q_\Pi^{(2)} \cancel{+} - \cancel{3} Q_\Pi^{(3,1)} - \frac{\cancel{12}}{4} Q_\Pi^{(3,2)} \right] + O(h^4)$$

$$Q_\Pi^{(2)} = \sum_{k=0}^{\infty} A_k |h_k|^2, \quad A_k = \frac{k(k-1)(k-2)}{6}$$

$$Q_\Pi^{(3)} = \underset{\text{diagonal}}{Q_\Pi^{(3,1)}} + \underset{\text{off diagonal}}{Q_\Pi^{(3,2)}} = \frac{1}{2} \sum_{i,j=0}^{\infty} c_{ij} (h_i h_j \bar{h}_{i+j-1} + \bar{h}_i \bar{h}_j h_{i+j-1})$$

$$c_{ij} = \frac{ij}{6} (i^2 + 3ij + j^2 - 6i - 6j + 7)$$

The discrepancies by factor ~~3, 12~~ 4 remain puzzling to us.