Slight Violation of the Alday-Maldacena Duality for a Wavy Circle

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IMM2 arXiv : 0803.1547
see also IMM arXiv : 0712.0159
IM8 arXiv : 0712.2316

I) Introduction

- Gauge / gravity correspondence (& matrix models) continue to be the central themes of string theory

- ① BDS's (Bern, Dixon, Smirnov) conjectured exponentiation a la Sudarkov of the all order planar n- gluon amplitudes for perturbative N=4 SYM,

$$\Rightarrow \quad e^{-D_{\sqcap}}$$

which is now known to be slightly violated at n=6, L=2 loop level.

BDKKRSVV 08031465
DHKS 08031466

② A new version of gauge-string duality by Alday-Maldacena : computation at strong coupling by the minimal surface of an AdS_5 string

$$\Rightarrow \quad e^{-\kappa A_{\sqcap}}$$

- ① and ② $\Rightarrow$ A fruitful assessment of the issue $D_{\sqcap} \overset{?}{\approx} \kappa A_{\sqcap}$ and its reformulation are called for today.

Hints in the strong coupling limit would be supplied by IMM2

- Note that $D_{\sqcap} = \kappa A_{\sqcap}$, if it were true, would give a resolution to the AdS-minimal surface problem.

Contents

$$\triangle_{21} \equiv \triangle_0 - \mathcal{D}^2 + \mathcal{D}$$

II) symbolically

$$\text{“ } \mathcal{A}_n(\mathbf{p}_1,\ldots,\mathbf{p}_n|\lambda) \;=\; \mathcal{A}_{\text{tree}}\,\mathcal{A}_{\text{IR}}\,\mathcal{A}_{\text{finite}} \text{ ”}$$

factorizes & exponentiates

To be more precise,

$$\mathcal{A}_n = g^{n-2}\sum_{L=0}^{\infty} a^L \sum_{\rho} \mathrm{Tr}(T^{a_{\rho(1)}}\cdots T^{a_{\rho(n)}})A_n^{(L)}(\rho(1),\ldots,\rho(n))$$

$$\lambda = g^2 N$$

$$D = 4 - 2\epsilon$$

ρ : noncyclic perm

$$a = \frac{\lambda\mu^{2\epsilon}}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}$$

Define, with the help of MHV & N=4 SUSY,

$$M_n^{(L)}(\epsilon) \;\equiv\; \text{“ } A_n^{(L)}(\epsilon)/A_n^{(0)} \text{ ”} \; : \text{ scalar function}$$

$$\mathcal{A}_n = \mathcal{A}_n^{(0)} \sum_{L=0}^{\infty} a^L M_n^{(L)}(\epsilon)$$

$$\stackrel{?}{=} \mathcal{A}_n^{(0)} \exp\left[\sum_{\ell=1}^{\infty} a^\ell \left(f^{(\ell)}(\epsilon) M_n^{(1)}(\ell\epsilon) + c^{(\ell)} + o(\epsilon)\right)\right]$$

BDS conjecture

$$f^{(\ell)}(\epsilon) = f_0^{(\ell)} + \epsilon f_1^{(\ell)} + \epsilon^2 f_2^{(\ell)}$$

known

BES ↓

$$f(\lambda) \equiv 4 \sum_{\ell=1}^{\infty} a^\ell f_0^{(\ell)}$$ planar cusp anomalous dimension $\overset{\text{large}}{\sim} \sqrt{\lambda}$

$$g(\lambda) \equiv 2 \sum_{\ell=2}^{\infty} \frac{a^\ell}{\ell} f_1^{(\ell)}$$ planar collinear anomalous dimension

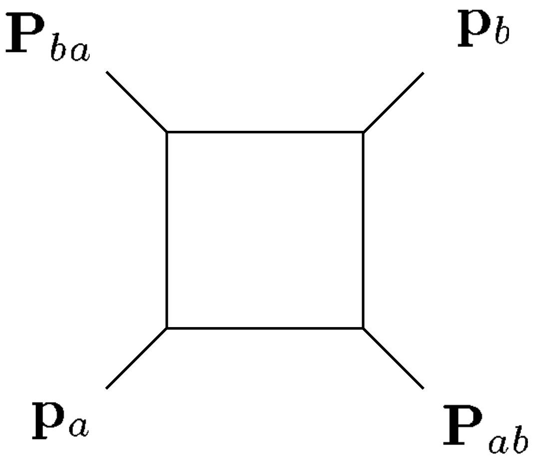
$$k(\lambda) \equiv -\frac{1}{2} \sum_{\ell=2}^{\infty} \frac{a^\ell}{\ell^2} f_2^{(\ell)}$$

III)

- $M_n^{(1)}$ is expressed as a sum of the so-called ``two mass easy box functions'' F^{2me} : (Bern, Dunbar, Dixon, Kosower)

$$M_n^{(1)} = \sum_{a<b} F^{2me}(\mathbf{p}_a, \mathbf{P}_{ab}, \mathbf{p}_b, \mathbf{P}_{ba}) \;, \quad \mathbf{P}_{ab} = \sum_{c=a+1}^{b-1} \mathbf{p}_c$$

massless massive



with

$$s = (p+P)^2 \;, \quad t = (p+Q)^2$$

$$a = \frac{s+t-P^2-Q^2}{st-P^2Q^2} \;, \quad \mathrm{Li}_p(z) = \sum \frac{z^p}{k^p}$$

- dilogarithmic rep. of F^{2me}

$$F^{2me}(p,P,q,Q) = -\frac{1}{\epsilon^2}\left[\left(\frac{-s}{\mu^2}\right)^{-\epsilon} + \left(\frac{-t}{\mu^2}\right)^{-\epsilon} - \left(\frac{-P^2}{\mu^2}\right)^{-\epsilon} - \left(\frac{-Q^2}{\mu^2}\right)^{-\epsilon}\right]$$
$$+\mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2) - \mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at)$$

- $D_\sqcap = \oint_\sqcap \oint_\sqcap \frac{dy^\mu dy'_\mu}{(y-y')^{2+\epsilon}}$ is known to reproduce, after computation, the rep. of $M_n^{(1)}$

  in momentum space

$\sqcap$: polygon made of light-like segment
$=$ external momenta $\mathbf{p}_a$

Drummond, Korchemsky, Sokatchev; Brandhuber, Heslop, Travaglini

$$= \sum (\text{pair of segments in } \sqcap)$$

$$= D_\sqcap^{(1)} + D_\sqcap^{(2)} + D_\sqcap^{(3)}$$

two identical adjacent nonadjacent

$= 0 \quad \sim \frac{1}{(-\epsilon)^2} \quad \sim$ dilog

- $D_{\sqcap}$ $\sim$ log of abelian Wilson loop

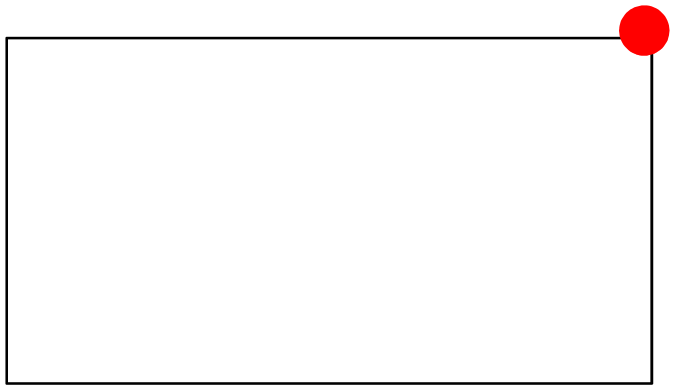
$$< \exp\left(i \oint_{\sqcap} dy^{\mu} A_{\mu}(y)\right) > \qquad \text{free field}$$

$$= \exp\left(-\frac{1}{2}\oint_{\sqcap} dy^{\mu} \oint_{\sqcap} dy'^{\nu} < A_{\mu}(y) A_{\nu}(y') >\right)$$

$$= \exp\left(\text{const } D_{\sqcap}\right)$$

$$M_n^{(1)} = (\text{const})\ D_{\sqcap}$$

- situation ; T-dualities operating



IV)

- AdS / CFT duality ; $\sqrt{\lambda} \equiv \sqrt{g^2 N} = \frac{R^2}{\alpha'}, \ \frac{1}{N} \sim g_s$ assumed

- compute the same gluon amplitude at strong coupling using tree level semiclassical string theory

- The original AdS_5 geometry

$$-(X^{-1})^2 - (X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = -R^2$$

embedding into $R^{2,4}$

$$X^{-1} + X^4 = \frac{R}{z}, \ X^{-1} - X^4 = R\,\frac{z^2 + x_\mu x^\mu}{z}, \ X^\mu = R\,\frac{x^\mu}{z}$$

$$ds^2 = dX^\mu dX_\mu = R^2\,\frac{dz^2 + dx_\mu dx^\mu}{z^2}$$

- place D3 brane at $z = z_{IR} \to \infty$ (IR regulator)

- take T-duality in $\mu = 0,1,2,3$ directions

$$\partial_a x^\mu = i \frac{R^2}{r^2} \epsilon_{ac} \partial_c y^\mu , \quad r = \frac{R^2}{z}$$

- T-dualized geometry, which is again AdS$_5$

$$ds^2 = R^2 \frac{dr^2 + dy_\mu dy^\mu}{r^2} , \quad r_{IR} = \frac{R^2}{z_{IR}} \rightarrow 0$$

- semiclassical string amplitudes

$$\sim \text{(prefactor)}\ e^{-S_E[y^\mu = y^\mu_{sa}, r = r_{cl}, k^I_\mu]}$$

- Equivalence of Neuman rep. and Dirichlet rep. (AdS$_5$ σ-model)

N : $I[z, x^\mu] = S_E[z, x^\mu] - i \int_{\mathcal{D}} d^2\xi \ J_\mu(\xi; k^\mu_I) x^\mu(\xi)$

$$J_\mu(\xi; k^\mu_I) = \sum_I k^\mu_I \delta^{(2)}(\xi - \xi_I)$$

$$\left.\frac{\delta I[z, x^\mu]}{\delta_1(z, x^\mu)}\right|_{z_{cl},\ x^\mu_{cl}} = 0$$

$$I[z_{cl}, x^\mu_{cl}] = \frac{\sqrt{\lambda}}{2\pi}\frac{1}{2}\int_{\mathcal{D}} d^2\xi \ \frac{\partial \ln z_{cl}}{\partial \xi^a}\frac{\partial \ln z_{cl}}{\partial \xi^a} + \frac{i}{2}\sum_I k^\mu_I x^{cl}_\mu(\xi_I)$$

D :

$$I[r_{cl}, y^{\mu}_{cl}] = \frac{\sqrt{\lambda}}{2\pi}\frac{1}{2}\int_{D} d^2\xi\, \frac{\partial \ln r_{cl}}{\partial \xi^a}\frac{\partial \ln r_{cl}}{\partial \xi^a} + \frac{\sqrt{\lambda}}{2\pi}(-\frac{1}{2})\epsilon^{ab}\int_{\partial D} d\xi^a y^{\mu}\frac{1}{r^2}\partial_b y_{\mu}$$

The 2nd term, after T-duality relation, and `` the D-instanton condition '' $y^{\mu}|_{\text{boundary}}(s) = \sum_I y^{\mu}_{0I}\Theta(s_I < s < s_{I+1})$

$$= \frac{i}{\sqrt{2}}\frac{\sqrt{\lambda}}{2\pi}\int_{D} ds \frac{dy^{\mu}}{ds} x_{\mu}(\xi)$$

$$\frac{2\pi}{\sqrt{\lambda}} k^{\mu}_{I} \equiv (y^{\mu}_{0I} - y^{\mu}_{0I-1})$$

$$= \frac{i}{2}\sum_I k^{\mu}_I x_{\mu}(\xi_I)$$

$$\therefore \; I[z_{cl}, x^{\mu}_{cl}] = S_E[r_{cl}, y^{\mu}_{cl}]$$

V)

- work on the Euclidean worldsheet
- choose $\xi^1 = y_1$, $\xi^2 = y_2$
- The 1st ansatz ; $y_3 = 0$ ⋯ ①

$$S_{E,NG} = \frac{\sqrt{\lambda}}{2\pi} \int dy_1 dy_2 \sqrt{\det H} \,, \quad H_{ij} = \frac{1}{r^2}(\delta_{ij} - \partial_i y_0 \partial_j y_0 + \partial_i r \partial_j r)$$

recognize this as $f(\lambda) \overset{\lambda \text{ large}}{\sim} \sqrt{\lambda}$

- The 2nd ansatz ; $1 = y_\mu y^\mu + r^2 \Leftrightarrow Y^4 = 0$ ⋯ ②

IMM1, IM8

- ① and ② form AdS_3 ansatz, which contains the Alday-Maldacena rhombus solution.

- Eq. of motion

$$\delta y_0 \;:\; \partial_1\left(\frac{H_{22}}{r^2\sqrt{\det H}}\partial_1 y_0\right)+\partial_2\left(\frac{H_{11}}{r^2\sqrt{\det H}}\partial_2 y_0\right) - \partial_1\left(\frac{H_{21}}{r^2\sqrt{\det H}}\partial_2 y_0\right)-\partial_2\left(\frac{H_{12}}{r^2\sqrt{\det H}}\partial_1 y_0\right)=0$$

$$\delta r \;:\; \partial_1\left(\frac{H_{22}}{r^2\sqrt{\det H}}\partial_1 r\right)+\partial_2\left(\frac{H_{11}}{r^2\sqrt{\det H}}\partial_2 r\right) - \partial_1\left(\frac{H_{21}}{r^2\sqrt{\det H}}\partial_2 r\right)-\partial_2\left(\frac{H_{12}}{r^2\sqrt{\det H}}\partial_1 r\right)+\frac{2\sqrt{\det H}}{r}=0$$

- linear approximation to NG

eliminate r^2 through $S_{E,NG}$ and ②, linearize w.r.t. y_0

$$\triangle_{21} y_0 = 0$$, where $\triangle_{21} = \triangle_0 - \mathcal{D}^2 + \mathcal{D}$

$\triangle_0 = 4\partial\bar{\partial},\ \mathcal{D} = z\partial + \bar{z}\bar{\partial}$

solution

$$y_0 = \sum_{k\geq 0} Re(\alpha_k z^k)\,{}_2F_1\left(\frac{k}{2},\frac{k-1}{2};k+1;z\bar{z}=x\right)$$

$$(1+k\sqrt{1-x})(1-\sqrt{1-x})^k \;/\; x^k$$

VI)

- how to deal with the issue $D_{\sqcap} \overset{?}{\approx} \kappa A_{\sqcap}$ fruitfully in the strong coupling side

- explicit examples containing ∞ ly many parameters needed

  $\Rightarrow$ an infinitesimal deformation of the unit circle into an arbitrary curve on the plane

- circle solution (formal $n = \infty$ limit of lightlike n-gon)

AdS$_3$ ansatz
$$\begin{cases} y_3 = 0 \\ 1 = r^2 + y_\mu y^\mu = r^2 - y_0^2 + y_i^2 \end{cases}$$

now $y_0 = 0$

- $L_{NG} = \frac{1}{r^2}\sqrt{1+(\partial_i r)^2}$

The only candidate to the solution which lie in these ansatze is

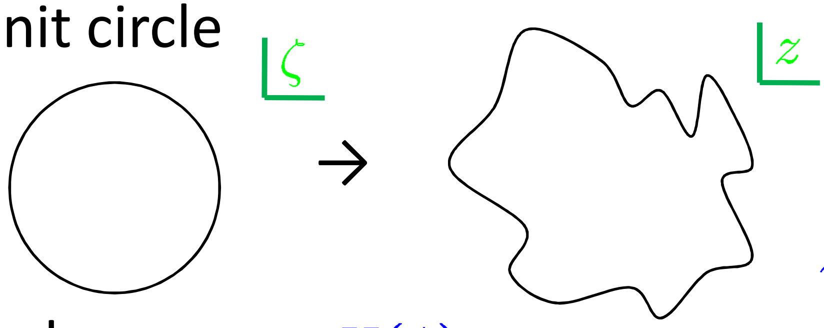
$r^2 = 1 - y_i^2$, which in fact solves

IMM1

$$r(\partial_i r)(\partial_j r)(\partial_i \partial_j r) = (2 + r\partial^2 r)(1 + (\partial_i r)^2)$$

- formulation

unit circle



bdd by Π

$z = y_1 + iy_2$

- consider the conformal map $z = H(\zeta)$
- find the shape of the minimal surface

$$r^2(z, \bar{z}) = 1 - \zeta\bar{\zeta} + a(\zeta, \bar{\zeta})$$

by solving the NG eq. for $a(\zeta, \bar{\zeta})$ subject to the b.c.

$$a|_{|\zeta|=1} = 0$$

action

- some simplification due to $\bar{\partial} z = 0$
- write $\partial z = 1 + \sum_{k=1}^{\infty} k h_k \zeta^{k-1} \equiv \partial H \equiv 1 + \partial h$

$$
\begin{aligned}
S_{NG}[a,h] &= \frac{\sqrt{\lambda}}{2\pi} \int d^2\zeta \frac{1}{r^2} \sqrt{|\partial H|^2 (|\partial H|^2 + 4 \partial r \bar{\partial} r)} \\
&= \frac{\sqrt{\lambda}}{2\pi} \int d^2\zeta \frac{|1+\partial h|^2 (1 - \zeta\bar{\zeta} + a + \frac{(\partial a - \bar{\zeta})(\bar{\partial} a - \zeta)}{|1+\partial h|^2})^{1/2}}{(1 - \zeta\bar{\zeta} + a)^{3/2}}
\end{aligned}
$$

- need to compute a=a(h) (at least) to the lowest order in h
- regularization needed

$$
0 = \partial \left(\frac{\partial \mathcal{L}}{\partial(\partial a)} \right) + \bar{\partial} \left(\frac{\partial \mathcal{L}}{\partial(\bar{\partial} a)} \right) - \frac{\partial \mathcal{L}}{\partial a} = \frac{1/4}{(1 - \zeta\bar{\zeta})^{3/2}} \triangle_{21} (a + \bar{\zeta} h + \zeta \bar{h}) + o(h^2)
$$

Eq. of motion

$$\triangle_{21}\psi = (\triangle_0 - \mathcal{D}^2 + \mathcal{D})\psi = 0,\ \triangle_0 = 4\partial\bar{\partial},\ \mathcal{D} = z\partial + \bar{z}\bar{\partial}$$

has appeared again in a-linearized problem

- The solution which satisfies the boundary condition is

$$a(\zeta,\bar{\zeta}) = 2\sum_{k=1}^{\infty} Re(h_k \zeta^{k-1}) A_k(\zeta\bar{\zeta})$$

$$A_k(x) = F_{k-1}(x) - x$$

$$F_k(x) = \frac{(1 + k\sqrt{1-x})(1-\sqrt{1-x})^k}{x^k}$$

- remaining procedure
  - substitute the solution into the regularized area
  - and evaluate it

- Evaluating the regularized area

$$\text{Area} = S_{NG}[a(h),h] = S_{circ} + S_0[h] + S_1[a,h] + S_2[a] + o(a^{3-j}h^j)$$

Ways

i) μ – reg. : replace r^2 in the denominate by $r^2+\mu^2$

ii) c – reg. : cut the integral over $x \equiv \zeta\bar{\zeta}$ at $x = 1-c^2$

iii) dimensional reg. : we did not accomplish

⋮

i) - direct evaluation without partial integration

$$(Area)_\mu = \frac{\pi^2}{\mu}\left(1 + \frac{1}{4}\sum_k^\infty k^2|h_k|^2\right) - \frac{\pi}{2}k(k-1)(k-2) - 2\pi + o(h^3)$$

- manage to find the local boundary counterterm

$$\frac{L}{2\pi} = \oint_\sqcap d\ell = 1 + \sum_k^\infty \frac{k^2|h_k|^2}{4} + o(h^3)$$

ii) ▪ partial integration to isolate the singularities

▪ hard to find local boundary counterterms

▪ shift the boundary condition to a(h) to the regularized boundary $\approx$ omitting the surface terms

$$\text{(Area)}_c^{bulk} = \sum_k^{\infty}\left(\frac{k^2}{2c} - \frac{k(k-1)(k-2)}{2}\right) + o(h^3)$$

- Double contour integral

- For example, in `` r'/r regularization " : two radii are s.t. $rr' = 1$

$$D_\Pi = \oint_{\Pi_r} \oint_{\Pi_{r'}} \frac{\frac{1}{2}(dzd\bar{z}' + d\bar{z}dz')}{(z - z')(\bar{z} - \bar{z}')} = D_\Pi^{(0)} + \underset{\substack{\| \\ 0}}{D_\Pi^{(1)}} + D_\Pi^{(2)} + o(h^3)$$

$$D_\Pi^{(0)} = \text{a Poisson integral} = 2(2\pi)^2 \left(\frac{1}{1 - a^2} - 1 \right) \ , \quad a \equiv \frac{r'}{r}$$

$$\begin{aligned} D_\Pi^{(2)} &= -2(2\pi)^2 \sum_k |h_k|^2 \left[\oint \frac{d\omega}{2\pi i} \frac{\omega^{-k} f_k(\omega)^2}{(1 - \omega)^2} \right] \\ &= -2(2\pi)^2 \sum_k |h_k|^2 \frac{k(k-1)(k-2)}{6} \end{aligned}$$

$$f_k(x) = \frac{1 - x^k}{1 - x} - k$$

- The alternative ; λ regularization

- <u>Results</u>

$$\frac{D_\sqcap}{2\pi} = \frac{L}{\lambda} - 2\pi - 4\pi\left[Q_\sqcap^{(2)} - Q_\sqcap^{(3,1)} - Q_\sqcap^{(3,2)}\right] + 4\pi Q_\sqcap^{(4)} + o(h^5)$$

$$\frac{A_\sqcap}{2\pi} = \frac{L}{4\mu} - 1 - \frac{3}{2}\left[Q_\sqcap^{(2)} - Q_\sqcap^{(3,1)} - 4Q_\sqcap^{(3,2)}\right] + o(h^4)$$

$$Q_\sqcap^{(2)} = \sum_{k=0}^{\infty} B_k |h_k|^2 \ , \quad B_k = \frac{k(k-1)(k-2)}{6}$$

$$Q_\sqcap^{(3)} = \underset{\substack{\uparrow \\ \text{diagonal}}}{Q_\sqcap^{(3,1)}} + \underset{\substack{\uparrow \\ \text{off-diagonal}}}{Q_\sqcap^{(3,2)}} = \frac{1}{2}\sum_{i,j=0}^{\infty} c_{ij}\left(h_i h_j \bar{h}_{i+j-1} + \bar{h}_i \bar{h}_j h_{i+j-1}\right)$$

$$c_{ij} = \frac{ij}{6}(i^2 + 3ij + j^2 - 6i - 6j + 7)$$

The red denote discrepancies

- comments on our result

- $\kappa_0 = \dfrac{8\pi}{3}$ in front of the bracket

  4 inside the bracket

  $\kappa_\Box = 8$ $\quad \dfrac{\kappa_0}{\kappa_\Box} = \dfrac{\pi}{3} \approx \dfrac{3.14}{3} \approx 1.05$ 5% discrepancy

- part in $D_\sqcap$ which is linear in $\bar{h}_\ell$ is consistent with the expression which is made of the Schwarzian derivative :

$$\oint_{\text{unit circle}} (\bar{z} - \bar{\zeta}) S_\zeta(z) \zeta^2 d\zeta \ , \quad S_\zeta(z) = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$$

- The planar nature of our wavy circle may have obscured some of the symmetry properties that this problem possesses.