Generalized/Doubled/Nongeometric String Backgrounds

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Lower dimensional supergravity related to this issue

J. Maharana, J.H. Schwarz hep-th/9207016

L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré, T. Magri hep-th/9605032

N. Kaloper, R.C. Myers hep-th/9901045

T.W. Grimm hep-th/0507153

Generalized geometry

N.J. Hitchin math/0209099

M. Gualtieri math/0401221

M. Graña, R. Minasian, M. Petrini, A. Tomasiello hep-th/0406137 hep-th/0409176 hep-th/0505212 hep-th/0609124

M. Graña, J. Louis, D. Waldram hep-th/0505264 hep-th/0612237

M. Graña hep-th/0509003

I. Benmachiche, T.W. Grimm hep-th/0602241

Workshop at DESY, Feb. 2007

Doubled formalism

C.M. Hull hep-th/0406102 hep-th/0605149 hep-th/0701203

C.M. Hull, R.A. Reid-Edwards hep-th/0503114 arXiv:0711.4818

A. Dabholkar, C.M. Hull hep-th/0512005

A. Lawrence, M.B. Schulz, B. Wecht hep-th/0602025

G. Dall'Agata, M. Prezas, H. Samtleben, M. Trigiante arXiv:0712.1026

and more...

Introduction

- $\mathcal{N} = 2$ supergravity

  highly symmetric (controllable), dynamical (non-trivial), connectable to Seiberg-Witten, etc..

  dictated by holomorphic functionals (prepotentials)

- $\mathcal{N} = 1$ supergravity

  highly dynamical, less symmetric, connectable to (SUSY) GUTs, etc..

  dictated by Kähler potential and superpotential

many ways to derive them from type II and heterotic string theories

Moduli in $\mathcal{N} = 2$ supergravity: Appendix

	vector multiplet	hypermultiplet
generic	coord. of Hodge-Kähler	coord. of quaternionic
IIA on Calabi-Yau	Kähler moduli	complex moduli $+$ RR
IIB on Calabi-Yau	complex moduli	Kähler moduli $+$ RR

Duality relations in $\mathcal{N} = 2$ theories:

type IIA	$\longleftrightarrow$	type IIB	T-duality, mirror symmetry
type $\mathrm{II}/\mathrm{CY}_3$	$\longleftrightarrow$	heterotic$/[K3 \times T^2]$	S-duality

Reduction to $\mathcal{N} = 1$ supergravity is given in terms of orientifold planes

$$K^{\mathsf{KS}} = -\log\left(\tfrac{4}{3}\int_{\mathrm{CY}_3} J \wedge J \wedge J\right)$$

$$K^{\mathsf{CS}} = -\log\left(i\int_{\mathrm{CY}_3} \Omega \wedge \overline{\Omega}\right)$$

$$W_{\mathrm{IIA},\mathsf{RR}} = i\,\mathrm{e}^{\phi}\int_{\mathrm{CY}_3} G_{\mathrm{A}} \wedge \mathrm{e}^{-B-iJ}$$

$$W_{\mathrm{IIB},\mathsf{RR}} = i\,\mathrm{e}^{\phi}\int_{\mathrm{CY}_3} G_{\mathrm{B}} \wedge \Omega$$

$$W_{H\text{-}\mathsf{flux}} = \int_{\mathrm{CY}_3} H_3 \wedge \Omega$$

$$F_n \;=\; \mathrm{d}C_{n-1} - H_3 \wedge C_{n-3} \;\equiv\; \mathrm{e}^{B} G$$

$$G_{\mathrm{A}} \;=\; G_0 + G_2 + G_4 + G_6 \qquad\qquad G_{\mathrm{B}} \;=\; G_3$$

$$J \wedge J \wedge J \;=\; \frac{3i}{4}\Omega \wedge \overline{\Omega} \qquad\qquad J \wedge \Omega \;=\; 0 \;=\; B \wedge \Omega$$

Question 1: Generic supersymmetric effective theory beyond Calabi-Yau geometry?

☹ condition on geometry from supersymmetry? $\dashrightarrow$ $SU(3)$-structure manifold Appendix

☹ identify "light" modes?

☹ generic form of Kähler potentials and superpotentials?

$$\begin{aligned} \mathrm{d}s^2_{1,9} &= \mathrm{e}^{2A(y)} g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + g_{ij}\, \mathrm{d}y^i \otimes \mathrm{d}y^j \\ \delta\psi_i &= \Big(\partial_i + \frac{1}{4} \omega_{iab}\, \gamma^{ab} \Big) \eta - \frac{1}{4} H_{ijk} \gamma^{jk} \eta + \ldots \ \equiv \ 0 \\ \delta\lambda &= -\frac{1}{4} \Big(\gamma^i \partial_i \phi - \frac{1}{6} H_{ijk} \gamma^{ijk} \Big) \eta + \ldots \ \equiv \ 0 \end{aligned}$$

$$(\mathrm{d} - H_3 \wedge)(\mathrm{e}^{4A} *_6 F) = 0 \qquad\qquad (\mathrm{d} - H_3 \wedge) F = \delta(\text{source})$$

$$\mathrm{d}(\mathrm{e}^{4A-2\phi} *_6 H_3) = \mp \mathrm{e}^{4A} F_n \wedge *_6 F_{n+2}$$

$$\mathrm{d}H_3 = 0$$

Question 2: Modification of dualities among string theories by fluxes?

- ☹ T-duality (mirror symmetry) from (non-)Calabi-Yau to what?
- ☹ S-duality and U-duality symmetries?
- ☹ Find more non-trivial relations?

$$\begin{array}{ccc} \wedge^{\text{even}} T^* \mathcal{M}_6 & & \wedge^{\text{odd}} T^* \mathcal{M}_6 \\ \mathrm{e}^{-B - iJ} & \longleftrightarrow & \Omega \\ G_{\mathrm{A}} = G_0 + G_2 + G_4 + G_6 & & G_{\mathrm{B}} = G_3 \end{array}$$

Generically, a Calabi-Yau with non-trivial fluxes does not yield a supersymmetric solution...

How should we derive modified Kähler/superpotentials?

Extend geometrical information of compactified space

Decomposition of vector bundle on ten-dimensional spacetime:

$$T\mathcal{M}_{1,9} = T_{1,3} \oplus F$$

$$\begin{cases} T_{1,3}: & \text{a real } SO(1,3) \text{ vector bundle} \\ F: & \text{an } SO(6) \text{ vector bundle which admits a pair of } SU(3) \text{ structures} \end{cases}$$

10-dimensional spacetime itself is not decomposed yet, i.e., do not yet consider truncation of modes.

Decomposition of Lorentz symmetry:

$$Spin(1,9) \to Spin(1,3) \times Spin(6) = SL(2,\mathbb{C}) \times SU(4)$$

$$\mathbf{16}_1 = (\mathbf{2},\mathbf{4})_1 \oplus (\overline{\mathbf{2}},\overline{\mathbf{4}})_1 \qquad \mathbf{16}_2 = (\mathbf{2},\overline{\mathbf{4}})_2 \oplus (\overline{\mathbf{2}},\mathbf{4})_2$$

Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\begin{cases} \epsilon^1_{\mathrm{IIA}} = \xi^1_+ \otimes (a\eta^1_+) + \xi^1_- \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\mathrm{IIA}} = \xi^2_+ \otimes (\overline{b}\eta^2_-) + \xi^2_- \otimes (b\eta^2_+) \end{cases} \qquad \begin{cases} \epsilon^1_{\mathrm{IIB}} = \xi^1_+ \otimes (a\eta^1_+) + \xi^1_- \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\mathrm{IIB}} = \xi^2_+ \otimes (b\eta^2_+) + \xi^2_- \otimes (\overline{b}\eta^2_-) \end{cases}$$

Set $SU(3)$ invariant spinor η^A_+ s.t. $D^{(T)}\eta^A_+ = 0$ $(A = 1,2)$: Appendix

$$\text{a pair of } SU(3) \text{ on } F \ (\eta^1_+, \eta^2_+) \quad \longleftrightarrow \quad \text{a single } SU(3) \text{ on } F \ (\eta^1_+ = \eta^2_+ = \eta_+)$$

Requirement that we have a pair of $SU(3)$ structures means there is a sub-supermanifold

$$\mathcal{N}^{1,9|4+4} \subset \mathcal{M}^{1,9|16+16}$$

$$\left(\begin{array}{ll} \text{(1,9):} & \text{bosonic degrees} \\ \text{4+4:} & \text{eight Grassmann variables as spinors of } Spin(1,3) \text{ and singlet of } SU(3)\text{s} \end{array} \right)$$

Equivalence such as

$$\boxed{\begin{array}{c} \text{eight SUSY theory reformulation of type II supergravity} \\ \Updownarrow \\ \text{a pair of } SU(3) \text{ structures on vector bundle } F \\ \Updownarrow \\ \text{an } SU(3) \times SU(3) \text{ structure on extended } F \oplus F^* \end{array}}$$

$$\uparrow$$

Entrance Gate to generalized geometry

Information from Killing spinor eqs. with torsion $D^{(T)}\eta_{\pm} \; = \; 0$ ($^{\exists}$complex Weyl $\eta_{\pm}$)

- Invariant p-forms on $SU(3)$-structure manifold:

a real two-form $J_{ij} = \mp 2i\, \eta_{\pm}^{\dagger}\, \gamma_{ij}\, \eta_{\pm}$

a holomorphic three-form $\Omega_{ijk} = -2i\, \eta_{-}^{\dagger}\, \gamma_{ijk}\, \eta_{+}$

$$\mathrm{d}J \; = \; \frac{3}{2}\mathrm{Im}(\overline{\mathcal{W}}_1 \Omega) + \mathcal{W}_4 \wedge J + \mathcal{W}_3 \qquad \mathrm{d}\Omega \; = \; \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}}_5 \wedge \Omega$$

- Five classes of (intrinsic) torsion are given as

component	interpretation	$SU(3)$-representation
$\mathcal{W}_1$	$J \wedge \mathrm{d}\Omega$ or $\Omega \wedge \mathrm{d}J$	$\mathbf{1} \oplus \mathbf{1}$
$\mathcal{W}_2$	$(\mathrm{d}\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
$\mathcal{W}_3$	$(\mathrm{d}J)_0^{2,1} + (\mathrm{d}J)_0^{1,2}$	$\mathbf{6} \oplus \overline{\mathbf{6}}$
$\mathcal{W}_4$	$J \wedge \mathrm{d}J$	$\mathbf{3} \oplus \overline{\mathbf{3}}$
$\mathcal{W}_5$	$(\mathrm{d}\Omega)^{3,1}$	$\mathbf{3} \oplus \overline{\mathbf{3}}$

In case of heterotic string, see, for instance, K. Becker, M. Becker, J.-X. Fu, L.-S. Tseng, S.-T. Yau hep-th/0604137

T. Kimura, P. Yi hep-th/0605247 etc.

- Vanishing torsion classes in special $SU(3)$-structure manifolds:

complex	hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = 0$
	balanced	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = 0$
	special hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	Kähler	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	conformally Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 3\mathcal{W}_4 + 2\mathcal{W}_5 = 0$
almost complex	symplectic	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	nearly Kähler	$\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	almost Kähler	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	quasi Kähler	$\mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	semi Kähler	$\mathcal{W}_4 = \mathcal{W}_5 = 0$
	half-flat	$\mathrm{Im}\mathcal{W}_1 = \mathrm{Im}\mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$

A configuration of six-torus T^6 in the presence of H-flux in $\boxed{\text{five-brane}}$ solution:

$$\to \begin{cases} \mathrm{d}s^2 = \boxed{\mathrm{d}s^2_{\mathbb{R}^{1,2}} + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}y^3)^2} + V\Big\{\mathrm{d}\xi^2 + (\mathrm{d}x^3)^2 + (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2\Big\} \\ H_3 = *_4\mathrm{d}V = \lambda\,\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}x^3 \\ \mathrm{e}^{2\phi} = V = \lambda\xi \end{cases}$$

Perform T-duality along all x^i-directions with respect to the Buscher's rule:

$$\boxed{\begin{gathered} \mathrm{d}s^2 = \mathrm{d}s^2_{\mathbb{R}^{1,2}} + (\mathrm{d}\widetilde{x}^1)^2 + (\mathrm{d}\widetilde{x}^2)^2 + (\mathrm{d}y^3)^2 + V^{-1}(\mathrm{d}\widetilde{x}^3 - \lambda y^1 \mathrm{d}y^2)^2 + V\Big\{\mathrm{d}\xi^2 + (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2\Big\} \\ \widetilde{H}_3 = 0 \qquad \mathrm{e}^{2\widetilde{\phi}} = 1 \end{gathered}}$$

Choose $\quad e^1 = \mathrm{d}\widetilde{x}^1 + i\sqrt{V}\mathrm{d}y^1 \qquad e^2 = \mathrm{d}\widetilde{x}^2 + i\sqrt{V}\mathrm{d}y^2 \qquad e^3 = \dfrac{1}{\sqrt{V}}(\mathrm{d}\widetilde{x}^3 - \lambda y^1 \mathrm{d}y^2) + i\mathrm{d}y^3$

Two- and three-forms: $J = -i\delta_{m\overline{n}}\, e^m \wedge \overline{e^n}$ and $\Omega \equiv e^1 \wedge e^2 \wedge e^3$ with

$$\mathrm{d}J = -\frac{2\lambda}{\sqrt{V}}\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 \neq 0 \quad \text{and} \quad J \wedge \mathrm{d}J = 0$$

$$\mathrm{d}\Omega = -\frac{\lambda}{\sqrt{V}}\mathrm{d}\widetilde{x}^1 \wedge \mathrm{d}\widetilde{x}^2 \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2 \qquad \text{i.e.,} \quad \mathrm{Re}\,\mathrm{d}\Omega \neq 0 \quad \text{and} \quad \mathrm{Im}\,\mathrm{d}\Omega = 0$$

This is a (torsionful) half-flat manifold $\quad\to\quad$ Entrance Gate to doubled formalism

N.J. Hitchin

Generalized geometry

J on $T\mathcal{M}_d$, ω on $T^*\mathcal{M}_d$ $\dashrightarrow$ $\mathcal{J}_\pm$ on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$

"Cliff(6) pure spinor $\eta_\pm$" on $T\mathcal{M}_6$

$\dashrightarrow$ "Cliff(6,6) pure spinor $\Phi_\pm$" on $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$

Evaluate spaces of $\Phi_\pm$ to provide Kähler/superpotentials in supergravity

C.M. Hull

Doubled formalism

T^d with B-field $\dashrightarrow$ $T^d \times T^d$ (with B-field)

Regard (T)-duality transformation as a part of transition function

Go beyond (non)-abelian gauged supergravity with B-field

and its duality transformation

Generalized geometry

Introduce a generalized almost complex structure $\mathcal{J}$ on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$ s.t.

$$\mathcal{J} : \ T\mathcal{M}_d \oplus T^*\mathcal{M}_d \ \longrightarrow \ T\mathcal{M}_d \oplus T^*\mathcal{M}_d$$

$$\mathcal{J}^2 \ = \ -\mathbb{1}_{2d}$$

$$^{\exists}\ O(d,d) \text{ invariant metric } L, \text{ s.t. } \ \mathcal{J}^T L \mathcal{J} \ = \ L$$

Structure group on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$:

$^{\exists}L$	$GL(2d)$	$\dashrightarrow$	$O(d,d)$
$\mathcal{J}^2 = -\mathbb{1}_{2d}$	$O(d,d)$	$\dashrightarrow$	$U(d/2,d/2)$
$\mathcal{J}_1, \mathcal{J}_2$	$U_1(d/2,d/2) \cap U_2(d/2,d/2)$	$\dashrightarrow$	$U(d/2) \times U(d/2)$
integrable $\mathcal{J}_{1,2}$	$U(d/2) \times U(d/2)$	$\dashrightarrow$	$SU(d/2) \times SU(d/2)$

- Integrability is discussed by "(0,1)" part of the complexified $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$:

$$\Pi \equiv \frac{1}{2}(\mathbb{1}_{2d} - i\mathcal{J})$$

$$\Pi A = A \quad \text{where } A = v + \zeta \text{ is a section of } T\mathcal{M}_d \oplus T^*\mathcal{M}_d$$

We call this A i-eigenbundle $L_{\mathcal{J}}$, whose dimension is $\dim L_{\mathcal{J}} = d$.

Integrability condition of $\mathcal{J}$ is

$$\overline{\Pi}\big[\Pi(v+\zeta), \Pi(w+\eta)\big]_{\mathrm{C}} = 0 \qquad v, w \in T\mathcal{M}_d \qquad \zeta, \eta \in T^*\mathcal{M}_d$$

$$[v+\zeta, w+\eta]_{\mathrm{C}} = [v,w] + \mathcal{L}_v \eta - \mathcal{L}_w \zeta - \frac{1}{2}\mathrm{d}(\iota_v \eta - \iota_w \zeta) : \text{ Courant bracket}$$

- Two typical examples of generalized almost complex structures:

$$\mathcal{J}_1 = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I^T \end{pmatrix} \qquad \text{w/ } I^2 = -\mathbb{1}_d\text{: almost complex structure}$$

$$\mathcal{J}_2 = \begin{pmatrix} \mathbf{0} & -J^{-1} \\ J & \mathbf{0} \end{pmatrix} \qquad \text{w/ } J\text{: almost symplectic form}$$

$$\begin{aligned} \text{integrable } \mathcal{J}_1 \quad &\leftrightarrow \quad \text{integrable of } I \\ \text{integrable } \mathcal{J}_2 \quad &\leftrightarrow \quad \text{integrable of } J \end{aligned}$$

On a usual geometry, J_{ij} can be given by an $SU(3)$ invariant (pure) spinor η_+ as

$$J_{ij} = -2i\eta_+^\dagger \gamma_{ij} \eta_+ \qquad \gamma^m \eta_+ = 0 \qquad \gamma^{\overline{n}} \eta_+ \neq 0$$

In a similar analogy, we want to find Cliff$(6,6)$ pure spinor(s) Φ.

$\because$) Compared to almost complex structures, (pure) spinors can be easily utilized in supergravity framework.

On $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$, we can define Cliff$(6,6)$ algebra and $Spin(6,6)$ spinor Φ:

$$\{\Gamma^i, \Gamma^j\} = 0 \qquad \{\Gamma^i, \Gamma_j\} = \delta^i_j \qquad \{\Gamma_i, \Gamma_j\} = 0$$

Irreducible repr. of $Spin(6,6)$ spinor is a Majorana-Weyl

$\longrightarrow$ a generic $Spin(6,6)$ spinor bundle S splits to $S^{\pm}$ (Weyl)

Weyl spinor bundles $S^{\pm}$ are isomorphic to bundles of forms on $T^*\mathcal{M}_6$:

$$S^+ \text{ on } T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \quad \sim \quad \wedge^{\text{even}} T^*\mathcal{M}_6$$
$$S^- \text{ on } T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \quad \sim \quad \wedge^{\text{odd}} T^*\mathcal{M}_6$$

Thus we often regard a Cliff$(6,6)$ spinor as a form on $\wedge^{\text{even/odd}} T^*\mathcal{M}_6$

A form-valued representation of the algebra

$$\Gamma^i = \mathrm{d}x^i \wedge \qquad \Gamma_j = \iota_j$$

IF Φ is annihilated by half numbers of the Cliff$(6,6)$ generators:

$\longrightarrow$ Φ is called a pure spinor

cf.) $SU(3)$ invariant spinor η_+ is a Cliff(6) pure spinor: $\gamma^m \eta_+ = 0$

An equivalent definition of a Cliff$(6,6)$ pure spinor is given by "Clifford action":

$$(v+\zeta)\cdot\Phi \;=\; v^i \iota_{\partial_i}\Phi + \zeta_i \, \mathrm{d}x^i \wedge \Phi \qquad \text{w/ } v\text{: vector} \qquad \zeta\text{: one-form}$$

Define the annihilator of a spinor as

$$L_\Phi \;\equiv\; \Big\{ v+\zeta \in T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \,\Big|\, (v+\zeta)\cdot\Phi = 0 \Big\}$$

$$\dim L_\Phi \;\leq\; d$$

If $\dim L_\Phi = 6$ (maximally isotropic) $\rightarrow$ Φ is a pure spinor

Correspondence between pure spinors and generalized almost complex structures:

$$\mathcal{J} \leftrightarrow \Phi \qquad \text{if} \;\; L_{\mathcal{J}} = L_{\Phi} \quad \text{with} \;\; \dim L_{\Phi} = 6$$

More precisely: $\mathcal{J} \leftrightarrow$ a line bundle of pure spinor Φ

$\because$) rescaling Φ does not change its annihilator L_{Φ}

Then, we can rewrite the generalized almost complex structure as

$$\boxed{\mathcal{J}_{\pm\Pi\Sigma} \;=\; \big\langle \mathrm{Re}\Phi_{\pm}, \Gamma_{\Pi\Sigma}\, \mathrm{Re}\Phi_{\pm} \big\rangle}$$

w/ Mukai pairing:

even forms: $\big\langle \Psi_+, \Phi_+ \big\rangle = \Psi_6 \wedge \Phi_0 - \Psi_4 \wedge \Phi_2 + \Psi_2 \wedge \Phi_4 - \Psi_0 \wedge \Phi_6$

odd forms: $\big\langle \Psi_-, \Phi_- \big\rangle = \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Phi_3 + \Psi_1 \wedge \Phi_5$

$\mathcal{J}$ is integrable $\longleftrightarrow$ $^{\exists}$ vector v and one-form ζ s.t. $\mathrm{d}\Phi = (v \llcorner + \zeta \wedge)\Phi$

generalized CY $\longleftrightarrow$ $^{\exists}\Phi$ is pure s.t. $\mathrm{d}\Phi = 0$

"twisted" GCY $\longleftrightarrow$ $^{\exists}\Phi$ is pure, and H is closed s.t. $(\mathrm{d} - H\wedge)\Phi = 0$

A Cliff$(6,6)$ spinor can also be mapped to a bispinor:

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \cdots i_k} \gamma_{\alpha\beta}^{i_1 \cdots i_k}$$

On a geometry of a single $SU(3)$-structure, the following two $SU(3,3)$ spinors:

$$\Phi_{0+} = \eta_+ \otimes \eta_+^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_+^\dagger \gamma_{i_1 \cdots i_k} \eta_+ \, \gamma^{i_1 \cdots i_k} = \frac{1}{8} \mathrm{e}^{-iJ}$$

$$\Phi_{0-} = \eta_+ \otimes \eta_-^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_-^\dagger \gamma_{i_1 \cdots i_k} \eta_+ \, \gamma^{i_1 \cdots i_k} = -\frac{i}{8} \Omega$$

Check purity: $(\delta + iI)_i{}^j \, \gamma_j \, \eta_+ \otimes \eta_\pm^\dagger = 0 = \eta_+ \otimes \eta_\pm^\dagger \, \gamma_j (\delta \mp iI)^j{}_i$

One-to-one correspondence: $\Phi_{0-} \leftrightarrow \mathcal{J}_1, \quad \Phi_{0+} \leftrightarrow \mathcal{J}_2$

On a generic geometry of a pair of $SU(3)$-structure defined by (η_+^1, η_+^2): Appendix

$$\Phi_{0+} = \eta_+^1 \otimes \eta_+^{2\dagger} = \frac{1}{8} \big(\overline{c}_{\parallel} \mathrm{e}^{-ij} - i \overline{c}_{\perp} w \big) \wedge \mathrm{e}^{-iv \wedge v'}$$

$$\Phi_{0-} = \eta_+^1 \otimes \eta_-^{2\dagger} = -\frac{1}{8} \big(c_{\perp} \mathrm{e}^{-ij} + i c_{\parallel} w \big) \wedge (v + iv')$$

$$|c_{\parallel}|^2 + |c_{\perp}|^2 = 1$$

$$\Phi_{\pm} = \mathrm{e}^{-B} \Phi_{0\pm}$$

Each $\Phi_\pm$ defines an $SU(3,3)$ structure on E. Common structure is $SU(3) \times SU(3)$.

(F is extended to E by including $\mathrm{e}^{-\mathcal{B}}$)

Compatibility requires

$$\big\langle \Phi_+, V \cdot \Phi_- \big\rangle \;=\; \big\langle \overline{\Phi}_+, V \cdot \Phi_- \big\rangle \;=\; 0 \qquad \text{for } \forall V = x + \xi$$

$$\big\langle \Phi_+, \overline{\Phi}_+ \big\rangle \;=\; \big\langle \Phi_-, \overline{\Phi}_- \big\rangle$$

Start with a real form $\chi_f \in \wedge^{\text{even/odd}} F^*$ (associated with a real $Spin(6,6)$ spinor χ_s)

Regard χ_f as a stable form satisfying

$$\begin{aligned} q(\chi_f) &= -\frac{1}{4}\langle \chi_f, \Gamma_{\Pi\Sigma}\chi_f\rangle\langle \chi_f, \Gamma^{\Pi\Sigma}\chi_f\rangle \ \in \ \wedge^6 F^* \otimes \wedge^6 F^* \\ U &= \{\chi_f \in \wedge^{\text{even/odd}} F^* \ : \ q(\chi_f) < 0\} \end{aligned}$$

Define a Hitchin function

$$H(\chi_f) \ \equiv \ \sqrt{-\frac{1}{3} q(\chi_f)} \ \in \ \wedge^6 F^*$$

which gives an integrable complex structure on U

Then we can get another real form $\hat{\chi}_f$ and a complex form Φ_f by Mukai pairing

$$\langle \hat{\chi}_f, \chi_f \rangle \ = \ -\mathrm{d}H(\chi_f) \qquad \text{i.e.,} \quad \hat{\chi}_f \ = \ -\frac{\partial H(\chi_f)}{\partial \chi_f}$$

$$\dashrightarrow \qquad \Phi_f \ \equiv \ \frac{1}{2}(\chi_f + i\hat{\chi}_f) \qquad H(\Phi_f) \ = \ i\langle \Phi_f, \overline{\Phi}_f \rangle$$

Hitchin showed: Φ_f is a (form corresponding to) pure spinor!

N.J. Hitchin math/0010054 math/0107101 math/0209099

Consider the space of pure spinors Φ ...

$$\boxed{\begin{array}{ccc} \text{Mukai pairing } \langle *, * \rangle & \longrightarrow & \text{symplectic structure} \\ \text{Hitchin function } H(*) & \longrightarrow & \text{complex structure} \end{array}}$$

$$\Downarrow$$

The space of pure spinor is Kähler (or, rather rigid special Kähler)!

Quotienting this space by the $\mathbb{C}^*$ action $\Phi \to \lambda\Phi$ for $\lambda\mathbb{C}^*$

$\dashrightarrow$ The space becomes a local special Kähler geometry with Kähler potential K:

$$\mathrm{e}^{-K} \;=\; H(\Phi) \;=\; i\big\langle \Phi, \overline{\Phi} \big\rangle \;=\; i\big(\overline{\mathcal{Z}}^I \mathcal{F}_I - \mathcal{Z}^I \overline{\mathcal{F}}_I\big) \;\in\; \wedge^6 F^*$$

$\mathcal{Z}^I$: holomorphic homogeneous coordinates

$\mathcal{F}_I$: derivative of prepotential $\mathcal{F}$, i.e., $\mathcal{F}_I = \partial \mathcal{F} / \partial \mathcal{Z}^I$

These are nothing but objects which we want to introduce in $\mathcal{N} = 2$ supergravity!

Space of pure spinors $\Phi_{\pm}$ on $F \oplus F^*$ with $SU(3) \times SU(3)$ structure

$$\|$$

special Kähler geometry of local type = Hodge-Kähler geometry

$$\mathrm{e}^{-K_{\pm}} \;=\; H(\Phi_{\pm}) \;=\; i\langle \Phi_{\pm}, \overline{\Phi}_{\pm} \rangle \;=\; i\big(\overline{\mathcal{Z}}{}^I_{\pm} \mathcal{F}_{\pm I} - \mathcal{Z}^I_{\pm} \overline{\mathcal{F}}_{\pm I}\big) \in \; \wedge^6 F^*$$

For a single $SU(3)$-structure case:

$$\Phi_+ \;=\; -\frac{1}{8}\mathrm{e}^{-\mathcal{B}-iJ} \qquad\qquad K_+ \;=\; -\log\left(\frac{1}{48} J \wedge J \wedge J\right)$$

$$\Phi_- \;=\; -\frac{i}{8}\mathrm{e}^{-\mathcal{B}}\Omega \qquad\qquad K_- \;=\; -\log\left(\frac{i}{64} \Omega \wedge \overline{\Omega}\right)$$

Structure of forms is exactly same as the one in the case of Calabi-Yau compactification!

We should truncate Kaluza-Klein massive modes from these forms to obtain 4-dimensional supergravity.

As introduced, we want to obtain four-dimensional $\mathcal{N} = 1, 2$ supergravity theories

Type $\mathrm{IIA}/\mathrm{IIB}$ supergravity theories have 32 supercharges w/ field multiplets

	1	gravity multiplet	
$\rightarrow$	6	gravitino multiplets	$\leftarrow$ *should be truncated*
	15	vector multiplets	
	9	hypermultiplets	
	1	tensor multiplet	

in the language of "$\mathcal{N} = 2$" multiplets

Consider truncation of 6 gravitino multiplets in terms of group theoretical descriptions

Let us discuss group-theoretical properties of massless fields

on a generalized tangent bundle $T_{3,1} \oplus F \oplus F^*$ with $SU(3) \times SU(3)$ structure

First, consider decomposition of $\mathbf{8_S}$, $\mathbf{8_C}$, $\mathbf{8_V}$ of $SO(8)$ (i.e., light-cone gauge)

$SO(8)$	$\to$	$SO(2) \times SO(6)$	$\to$	$SO(2) \times SU(3)$
$\mathbf{8_S}$	$\to$	$\mathbf{4}_{\frac{1}{2}} \oplus \overline{\mathbf{4}}_{-\frac{1}{2}}$	$\to$	$\mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{\frac{1}{2}} \oplus \overline{\mathbf{3}}_{-\frac{1}{2}}$
$\mathbf{8_C}$	$\to$	$\mathbf{4}_{-\frac{1}{2}} \oplus \overline{\mathbf{4}}_{\frac{1}{2}}$	$\to$	$\mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{-\frac{1}{2}} \oplus \overline{\mathbf{3}}_{\frac{1}{2}}$
$\mathbf{8_V}$	$\to$	$\mathbf{1}_{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{6}_{0}$	$\to$	$\mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{0} \oplus \overline{\mathbf{3}}_{0}$

Using this, consider the decompositions of (NS,R), (R,NS), (NS,NS) and (R,R) sectors...

$\mathbf{a_b}$ denotes a field in the $SU(3)$ repr. $\mathbf{a}$ and 4-dimensional helicity $\mathbf{b}$. $\mathbf{T}$ denotes an antisymmetric tensor.

- Fermions: (R,NS) and (NS,R) sectors:

	$SO(8)_{\rm L} \times SO(8)_{\rm R}$	$\rightarrow$	$SO(2) \times SU(3)_{\rm L} \times SU(3)_{\rm R}$
IIA/IIB	$(\mathbf{8}_{\rm S}, \mathbf{8}_{\rm V})$	$\rightarrow$	$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\frac{3}{2},-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-\frac{3}{2},\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\pm\frac{1}{2}}$ $\oplus (\mathbf{3},\mathbf{3})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-\frac{1}{2}}$
IIB	$(\mathbf{8}_{\rm V}, \mathbf{8}_{\rm S})$	$\rightarrow$	$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{\frac{3}{2},-\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{-\frac{3}{2},\frac{1}{2}}$ $\oplus (\mathbf{3},\mathbf{3})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-\frac{1}{2}}$
IIA	$(\mathbf{8}_{\rm V}, \mathbf{8}_{\rm C})$	$\rightarrow$	$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{-\frac{3}{2},\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\frac{3}{2},-\frac{1}{2}}$ $\oplus (\mathbf{3},\mathbf{3})_{-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{\frac{1}{2}}$

$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2}}$: 2 gravitinos in gravity multiplet

$(\mathbf{3},\mathbf{1})_{\pm\frac{3}{2}}$ etc.: 6 gravitinos in gravitino multiplets

$(\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}}$ etc.: fermions in gravitino multiplets

(should not be included in $\mathcal{N} = 2$ theory)

- Bosons: (NS,NS) sector:

$$\mathbf{8_V} \times \mathbf{8_V} \;=\; \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \;=\; (\phi, \mathcal{B}_{MN}, \mathcal{G}_{MN})$$

$SO(8)_{\rm L} \times SO(8)_{\rm R}$	$\longrightarrow$	$SO(2) \times SU(3)_{\rm L} \times SU(3)_{\rm L}$	
		$\mathcal{E}_{\mu\nu}$	$(\mathbf{1},\mathbf{1})_{\pm\mathbf{2}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{T}}$
		$\mathcal{E}_{\mu i}$	$(\mathbf{1},\mathbf{3})_{\pm\mathbf{1}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\pm\mathbf{1}}$
$\mathcal{E}_{MN} = \mathcal{G}_{MN} + \mathcal{B}_{MN}$	$\longrightarrow$	$\mathcal{E}_{i\nu}$	$(\mathbf{3},\mathbf{1})_{\pm\mathbf{1}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm\mathbf{1}}$
		$\mathcal{E}_{ij}$	$(\mathbf{3},\mathbf{3})_{\mathbf{0}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\mathbf{0}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{\mathbf{0}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{\mathbf{0}}$

- Bosons: (R,R) sector:

	$SO(8)_{\rm L} \times SO(8)_{\rm R}$	$\rightarrow$	$SO(2) \times SU(3)_{\rm L} \times SU(3)_{\rm R}$
IIA	$(\mathbf{8}_{\rm S}, \mathbf{8}_{\rm C})$	$\rightarrow$	$(\mathbf{1},\mathbf{1})_{\pm 1,0} \oplus (\mathbf{3},\mathbf{3})_{0} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{0} \oplus (\mathbf{3},\overline{\mathbf{3}})_{1} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-1}$
IIB	$(\mathbf{8}_{\rm S}, \mathbf{8}_{\rm S})$	$\rightarrow$	$(\mathbf{1},\mathbf{1})_{\pm 1,0} \oplus (\mathbf{3},\mathbf{3})_{1} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-1} \oplus (\mathbf{3},\overline{\mathbf{3}})_{0} \oplus (\overline{\mathbf{3}},\mathbf{3})_{0}$

Field expressions:

$$
\begin{array}{c|rcll}
\mathrm{IIA} & \mathcal{A}_0^- &=& \mathcal{A}_{(0,1)} + \mathcal{A}_{(0,3)} + \mathcal{A}_{(0,5)} & \simeq\ (\mathbf{1},\mathbf{1})_{0} \oplus (\mathbf{3},\mathbf{3})_{0} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{0} \\
 & \mathcal{A}_1^+ &=& \mathcal{A}_{(1,0)} + \mathcal{A}_{(1,2)} + \mathcal{A}_{(1,4)} + \mathcal{A}_{(1,6)} & \simeq\ (\mathbf{1},\mathbf{1})_{\pm 1} \oplus (\mathbf{3},\overline{\mathbf{3}})_{1} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-1} \\
\hline
\mathrm{IIB} & \mathcal{A}_0^+ &=& \mathcal{A}_{(0,0)} + \mathcal{A}_{(0,2)} + \mathcal{A}_{(0,4)} + \mathcal{A}_{(0,6)} & \simeq\ (\mathbf{1},\mathbf{1})_{0} \oplus (\mathbf{3},\overline{\mathbf{3}})_{0} \oplus (\overline{\mathbf{3}},\mathbf{3})_{0} \\
 & \mathcal{A}_1^- &=& \mathcal{A}_{(1,1)} + \mathcal{A}_{(1,3)} + \mathcal{A}_{(1,5)} & \simeq\ (\mathbf{1},\mathbf{1})_{1} \oplus (\mathbf{3},\mathbf{3})_{1} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-1}
\end{array}
$$

where $\mathcal{A}_{(p,q)}$ is a "4-dimensional" p-form and a "6-dimensional" q-form

RR field strength is $\mathcal{G}^{\pm} = {\rm d}\mathcal{A}_0^{\mp}$, whose gauge potential is $\mathcal{C} = {\rm e}^{\mathcal{B}}\mathcal{A}$ w/ $\mathcal{F} = {\rm d}\mathcal{C} - \mathcal{H}_3 \wedge \mathcal{C} = {\rm e}^{\mathcal{B}}\mathcal{G}$

- Reduction: effective theory with two gravitinos

$\rightarrow$ all repr. of the form $(\mathbf{3},\mathbf{1}),(\overline{\mathbf{3}},\mathbf{1}),(\mathbf{1},\mathbf{3}),(\mathbf{1},\overline{\mathbf{3}})$ (6 gravitino multiplets) are projected out!

type IIA multiplet	$SU(3)\times SU(3)$ repr.	bosonic field content
gravity multiplet	$(\mathbf{1},\mathbf{1})$	$g_{\mu\nu}$ $\mathcal{A}_1^+$
tensor multiplet	$(\mathbf{1},\mathbf{1})$	$\mathcal{B}_{\mu\nu}$ ϕ $\mathcal{A}_0^-$
vector multiplet	$(\mathbf{3},\overline{\mathbf{3}})$	$\mathcal{A}_1^+$ $\delta\Phi^+$
hypermultiplet	$(\mathbf{3},\mathbf{3})$	$\delta\Phi^-$ $\mathcal{A}_0^-$

type IIB multiplet	$SU(3)\times SU(3)$ repr.	bosonic field content
gravity multiplet	$(\mathbf{1},\mathbf{1})$	$g_{\mu\nu}$ $\mathcal{A}_1^-$
tensor multiplet	$(\mathbf{1},\mathbf{1})$	$\mathcal{B}_{\mu\nu}$ ϕ $\mathcal{A}_0^+$
vector multiplet	$(\mathbf{3},\mathbf{3})$	$\mathcal{A}_1^-$ $\delta\Phi^-$
hypermultiplet	$(\mathbf{3},\overline{\mathbf{3}})$	$\delta\Phi^+$ $\mathcal{A}_0^+$

Notice that all fields are still living on 10-dimensional space, i.e., all KK modes are included.

In case of a tangent bundle $T_{3,1} \oplus F \oplus F^*$ w/ a single $SU(3)$-structure (i.e., $\eta^1_+ = \eta^2_+$):

Ten-dimensional fields are decomposed as

$\mathcal{G}_{MN}$	$g_{\mu\nu}$	$\mathbf{1}_{\pm 2}$
	$\mathcal{G}_{\mu i}$	$(\mathbf{3} + \overline{\mathbf{3}})_{\pm 1}$
	$\mathcal{G}_{ij}$	$\mathbf{1}_{0} + (\mathbf{6} + \overline{\mathbf{6}})_{0} + \mathbf{8}_{0}$
$\mathcal{B}_{MN}$	$\mathcal{B}_{\mu\nu}$	$\mathbf{1}_{\mathbf{T}}$
	$\mathcal{B}_{\mu i}$	$(\mathbf{3} + \overline{\mathbf{3}})_{\pm 1}$
	$\mathcal{B}_{ij}$	$\mathbf{1}_{0} + (\mathbf{3} + \overline{\mathbf{3}})_{0} + \mathbf{8}_{0}$
ϕ	ϕ	$\mathbf{1}_{0}$

Ψ_M	Ψ_μ	$\mathbf{1}_{\pm\frac{3}{2}} + \mathbf{3}_{\pm\frac{3}{2}}$
	Ψ_i	$\mathbf{1}_{\pm\frac{1}{2}} + \mathbf{3}_{\pm\frac{1}{2}} + 2 \times \overline{\mathbf{3}}_{\pm\frac{1}{2}} + \mathbf{6}_{\pm\frac{1}{2}} + \mathbf{8}_{\pm\frac{1}{2}}$
λ	λ	$\mathbf{1}_{\pm\frac{1}{2}} + \mathbf{3}_{\pm\frac{1}{2}}$

$\mathcal{C}_M$	$\mathcal{C}_\mu$	$\mathbf{1}_{\pm 1}$
	$\mathcal{C}_i$	$(\mathbf{3} + \overline{\mathbf{3}})_{0}$
$\mathcal{C}_{MNP}$	$\mathcal{C}_{\mu\nu k}$	$(\mathbf{3} + \overline{\mathbf{3}})_{\mathbf{T}}$
	$\mathcal{C}_{\mu jk}$	$\mathbf{1}_{\mathbf{T}} + (\mathbf{3} + \overline{\mathbf{3}})_{\pm 1} + \mathbf{8}_{\pm 1}$
	$\mathcal{C}_{ijk}$	$(\mathbf{1} + \mathbf{1})_{0} + (\mathbf{3} + \overline{\mathbf{3}})_{\pm 1} + (\mathbf{6} + \overline{\mathbf{6}})_{0}$

$\mathcal{C}_0$	$\mathcal{C}_0$	$\mathbf{1}_{0}$
$\mathcal{C}_{MN}$	$\mathcal{C}_{\mu\nu}$	$\mathbf{1}_{\mathbf{T}}$
	$\mathcal{C}_{\mu i}$	$(\mathbf{3} + \overline{\mathbf{3}})_{\pm 1}$
	$\mathcal{C}_{ij}$	$\mathbf{1}_{0} + (\mathbf{3} + \overline{\mathbf{3}})_{0} + \mathbf{8}_{0}$
$\mathcal{C}_{MNPQ}$	$\mathcal{C}_{\mu jkl}$	$\frac{1}{2}[(\mathbf{1} + \mathbf{1})_{\pm 1} + (\mathbf{3} + \overline{\mathbf{3}})_{\pm 1} + (\mathbf{6} + \overline{\mathbf{6}})_{\pm 1}]$
	$\mathcal{C}_{ijkl}/\mathcal{C}_{\mu\nu kl}$	$\mathbf{1}_{0} + (\mathbf{3} + \overline{\mathbf{3}})_{0} + \mathbf{8}_{0}$

Standard four-dimensional $\mathcal{N} = 2$ supergravity $=$ "absence of 6 gravitino multiplets"

IIA multiplets	$SU(3)$ repr.	field contents
gravity multiplet	$\mathbf{1}$	$g_{\mu\nu} \quad \mathcal{C}_{\mu} \quad \Psi_{\mu}$
tensor multiplet	$\mathbf{1}$	$\mathcal{B}_{\mu\nu} \quad \phi \quad \mathcal{C}_{ijk} \quad \lambda$
vector multiplet	$\mathbf{8}+\mathbf{1}$	$\mathcal{C}_{\mu jk} \quad \mathcal{G}_{ij} \quad \mathcal{B}_{ij} \quad \Psi_{i}$
hypermultiplet	$\mathbf{6}$	$\mathcal{G}_{ij} \quad \mathcal{C}_{ijk} \quad \Psi_{i}$

IIB multiplets	$SU(3)$ repr.	field contents
gravity multiplet	$\mathbf{1}$	$g_{\mu\nu} \quad \mathcal{C}_{\mu jkl} \quad \Psi_{\mu}$
tensor multiplet	$\mathbf{1}$	$\mathcal{B}_{\mu\nu} \quad \mathcal{C}_{\mu\nu} \quad \phi \quad \mathcal{C}_{0} \quad \lambda$
vector multiplet	$\mathbf{6}$	$\mathcal{C}_{\mu jkl} \quad \mathcal{G}_{ij} \quad \Psi_{i}$
hypermultiplet	$\mathbf{8}+\mathbf{1}$	$\mathcal{G}_{ij} \quad \mathcal{B}_{ij} \quad \mathcal{C}_{ij} \quad \mathcal{C}_{ijkl} \quad \Psi_{i}$

Notice that all fields are still living on 10-dimensional space, i.e., all KK modes are included.

Analyze potential (interaction) terms:

given in the supersymmetry transformation of 4-dimensional $\mathcal{N}=2$ gravitinos ψ_μ^A

$$\hat{\Psi}_\mu^A \equiv \Psi_\mu^A + \frac{1}{2}\gamma_\mu{}^i\Psi_i^A = \psi_{A\mu+}\otimes\eta_\pm^A + \psi_{A\mu-}\otimes\eta_\mp^A + \ldots$$

$$\delta\psi_{A\mu} = D_\mu\xi_A + i\gamma_\mu\, S_{AB}\,\xi^B \qquad A=1,2$$

$$S_{AB} = \frac{i}{2}\mathrm{e}^{\frac{1}{2}K_V}\,\sigma_{AB}^x\,\mathcal{P}^x \qquad \sigma_{AB}^x = \begin{pmatrix} \delta^{x1} - i\delta^{x2} & -\delta^{x3} \\ -\delta^{x3} & -\delta^{x1} - i\delta^{x2} \end{pmatrix} \qquad x=1,2,3$$

$\mathcal{P}^x$: $\mathcal{N}=2$ Killing prepotentials, which yield $\mathcal{N}=1$ superpotentials

M. Graña, J. Louis, D. Waldram hep-th/0612237

To get S_{AB}, project the SUSY transformation $\delta\hat{\Psi}_\mu$ onto $SU(3)$-singlet parts from

$$
\begin{aligned}
\delta\Psi_M \;&=\; D_M\epsilon - \frac{1}{96}\mathrm{e}^{-\phi}\Big(\gamma_M{}^{PQR}\mathcal{H}_{PQR} - 9\gamma^{PQ}\mathcal{H}_{MPQ}\Big)\mathcal{P}\,\epsilon \\
&\quad - \sum_n \frac{1}{64n!}\mathrm{e}^{\frac{5-n}{4}\phi}\Big[(n-1)\gamma_M{}^{N_1\cdots N_n} - n(9-n)\delta_M{}^{N_1}\gamma^{N_2\cdots N_n}\Big]\mathcal{F}_{N_1\cdots N_n}\mathcal{P}_n\,\epsilon
\end{aligned}
$$

In type IIB case $\big($w/ $\mathcal{F}^- = \mathcal{F}_1 + \mathcal{F}_3 + \mathcal{F}_5$, $\sigma(\mathcal{F}^-) = -\mathcal{F}_1 + \mathcal{F}_3 - \mathcal{F}_5\big)$:

$$
\begin{aligned}
\begin{pmatrix} \delta\psi^1_{\mu+} \\ \delta\psi^2_{\mu+} \end{pmatrix} \;&=\; \begin{pmatrix} D_\mu\xi^1_+ \\ D_\mu\xi^2_+ \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \gamma_\mu\xi^1_-\ \overline{\eta}{}^1_-\gamma^i D_i\eta^1_+ \\ \gamma_\mu\xi^2_-\ \overline{\eta}{}^2_-\gamma^i D_i\eta^2_+ \end{pmatrix} + \frac{1}{48}\begin{pmatrix} \gamma_\mu\xi^1_-\ \mathcal{H}_{ijk}\,\overline{\eta}{}^1_-\gamma^{ijk}\eta^1_+ \\ -\gamma_\mu\xi^2_-\ \mathcal{H}_{ijk}\,\overline{\eta}{}^2_-\gamma^{ijk}\eta^2_+ \end{pmatrix} \\
&\quad - \frac{1}{8}\begin{pmatrix} -\gamma_\mu\xi^2_-\ \mathrm{e}^\phi\frac{1}{n!}\mathcal{F}^-_{i_1\cdots i_n}\,\overline{\eta}{}^1_-\gamma^{i_1\cdots i_n}\eta^2_+ \\ \gamma_\mu\xi^1_-\ \mathrm{e}^\phi\frac{1}{n!}\sigma(\mathcal{F}^-)_{i_1\cdots i_n}\,\overline{\eta}{}^2_-\gamma^{i_1\cdots i_n}\eta^1_+ \end{pmatrix}
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
S_{11} &= \frac{i}{2}\,\overline{\eta}{}^1_- \gamma^i D_i \eta^1_+ - \frac{i}{48}\mathcal{H}_{ijk}\,\overline{\eta}{}^1_- \gamma^{ijk}\eta^1_+ \;=\; -\frac{1}{8}\big\langle \Phi_-, \mathrm{d}\Phi_+ \big\rangle \\
S_{22} &= \frac{i}{2}\,\overline{\eta}{}^2_- \gamma^i D_i \eta^2_+ + \frac{i}{48}\mathcal{H}_{ijk}\,\overline{\eta}{}^2_- \gamma^{ijk}\eta^2_+ \;=\; \frac{1}{8}\big\langle \Phi_-, \mathrm{d}\overline{\Phi}_+ \big\rangle \\
S_{12} &= \frac{i}{8n!}\mathrm{e}^{\phi}\mathcal{F}^-_{i_1\cdots i_n}\,\overline{\eta}{}^1_- \gamma^{i_1\cdots i_n}\eta^2_+ \;=\; \frac{1}{8}\big\langle \Phi_-, \mathcal{G}^- \big\rangle \\
S_{21} &= \frac{i}{8n!}\mathrm{e}^{\phi}\sigma(\mathcal{F})^-_{i_1\cdots i_n}\overline{\eta}{}^2_- \gamma^{i_1\cdots i_n}\eta^1_+ \;=\; \frac{1}{8}\big\langle \Phi_-, \mathcal{G}^- \big\rangle
\end{aligned}
$$

$$
\mathcal{F} \;=\; \mathrm{d}\mathcal{C} - \mathcal{H}_3 \wedge \mathcal{C} \;=\; \mathrm{e}^{\mathcal{B}}\mathcal{G} \qquad \mathcal{C} \;=\; \mathrm{e}^{\mathcal{B}}\mathcal{A} \qquad \mathcal{G}^{\pm} \;=\; \mathrm{d}\mathcal{A}^{\mp}_0
$$

Summarizing information, we obtain (also for type IIA)

$$
S^{(4)}_{AB}(\mathrm{IIB}) \;=\; \frac{1}{8}\mathrm{e}^{\frac{1}{2}K_-}\begin{pmatrix} -\mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}}\big\langle \Phi_-, \mathrm{d}\Phi_+ \big\rangle & -\mathrm{e}^{2\phi^{(4)}}\big\langle \Phi_-, \mathcal{G}^- \big\rangle \\ -\mathrm{e}^{2\phi^{(4)}}\big\langle \Phi_-, \mathcal{G}^- \big\rangle & \mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}}\big\langle \Phi_-, \mathrm{d}\overline{\Phi}_+ \big\rangle \end{pmatrix}
$$

$$
S^{(4)}_{AB}(\mathrm{IIA}) \;=\; \frac{1}{8}\mathrm{e}^{\frac{1}{2}K_+}\begin{pmatrix} \mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}}\big\langle \Phi_+, \mathrm{d}\Phi_- \big\rangle & \mathrm{e}^{2\phi^{(4)}}\big\langle \Phi_+, \mathcal{G}^+ \big\rangle \\ \mathrm{e}^{2\phi^{(4)}}\big\langle \Phi_+, \mathcal{G}^+ \big\rangle & -\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}}\big\langle \Phi_+, \mathrm{d}\overline{\Phi}_- \big\rangle \end{pmatrix}
$$

$$
g^{(4)}_{\mu\nu} \;=\; \mathrm{e}^{-2\phi^{(4)}} g_{\mu\nu} \qquad \phi^{(4)} \;=\; \phi - \frac{1}{4}\log\det\mathcal{G}_{ij}
$$

$\mathcal{N}=1$ superpotentials and Kähler potentials can be read as

$$\delta\psi_\mu \;=\; D_\mu\xi + i\mathrm{e}^{K/2}\, W\,\gamma_\mu \xi^c \qquad\qquad K \;=\; K_+ + K_- + 2\phi^{(4)}$$

Most generic form of $\mathcal{N}=1$ superpotentials on $SU(3)\times SU(3)$ structure:

$$W_{\mathrm{IIA}} \;=\; \cos^2\alpha\, \mathrm{e}^{i\beta}\langle\Phi_+, \mathrm{d}\Phi_-\rangle - \sin^2\alpha\, \mathrm{e}^{-i\beta}\langle\Phi_+, \mathrm{d}\overline{\Phi}_-\rangle + \sin 2\alpha\, \mathrm{e}^{\phi}\langle\Phi_+, \mathcal{G}^+\rangle$$

$$W_{\mathrm{IIB}} \;=\; -\cos^2\alpha\, \mathrm{e}^{i\beta}\langle\Phi_-, \mathrm{d}\Phi_+\rangle + \sin^2\alpha\, \mathrm{e}^{-i\beta}\langle\Phi_-, \mathrm{d}\overline{\Phi}_+\rangle - \sin 2\alpha\, \mathrm{e}^{\phi}\langle\Phi_-, \mathcal{G}^-\rangle$$

$$\mathcal{G}^+ \;=\; \mathcal{G}_0 + \mathcal{G}_2 + \mathcal{G}_4 + \mathcal{G}_6 \qquad\qquad \mathcal{G}^- \;=\; \mathcal{G}_1 + \mathcal{G}_3 + \mathcal{G}_5$$

$$\mathcal{G}^\pm \;=\; \mathrm{d}\mathcal{A}_0^\mp \qquad\qquad \mathcal{C} = \mathrm{e}^{\mathcal{B}}\mathcal{A} \qquad\qquad \mathcal{F} = \mathrm{d}\mathcal{C} - \mathcal{H}_3\wedge\mathcal{C} \;=\; \mathrm{e}^{\mathcal{B}}\mathcal{G}$$

Reducing to single $SU(3)$-structure by $\eta_+^1 = \eta_+^2 \equiv \eta_+$, we obtain well-known forms:

$2\alpha = -\beta = \dfrac{\pi}{2}$ in W_{IIB}	$W_{\mathsf{GVW}} \;=\; -i\mathrm{e}^{\phi}\langle\mathcal{F}_3 - \tau\mathcal{H}_3, \Omega\rangle$
$\alpha = \dfrac{\pi}{4}$, $\mathrm{d}\Phi_- = 0$ in W_{IIA}	$W_{\mathrm{IIA,RR}} \;=\; \mathrm{e}^{\phi}\langle\mathrm{e}^{-\mathcal{B}-iJ}, \mathcal{G}^+\rangle$
$\beta = \dfrac{\pi}{2}$, $\mathcal{G}^+ = 0$ in W_{IIA}	$W_{\mathsf{half\text{-}flat}} \;=\; i\langle\mathrm{e}^{-\mathcal{B}-iJ}, \mathrm{d}(\mathrm{Re}\Omega)\rangle$

$a = \cos\alpha\mathrm{e}^{-i\beta/2}$, $b = \sin\alpha\,\mathrm{e}^{i\beta/2}$, $\tau = \mathcal{C}_0 + i\mathrm{e}^{-\phi}$

We have obtained Kähler potentials and superpotentials

which should appear in four-dimensional $\mathcal{N} = 1, 2$ supergravity theories

in the language of ten-dimensional fields:

$$\mathrm{e}^{-K_\pm} = i\big\langle \Phi_\pm, \overline{\Phi}_\pm \big\rangle = i\big(\overline{\mathcal{Z}}{}^I_\pm \mathcal{F}_{\pm I} - \mathcal{Z}^I_\pm \overline{\mathcal{F}}_{\pm I} \big)$$

$$W_{\mathrm{IIA/IIB}} = \pm \cos^2\alpha\, \mathrm{e}^{i\beta} \big\langle \Phi_\pm, \mathrm{d}\Phi_\mp \big\rangle \mp \sin^2\alpha\, \mathrm{e}^{-i\beta} \big\langle \Phi_\pm, \mathrm{d}\overline{\Phi}_\mp \big\rangle \pm \sin 2\alpha\, \mathrm{e}^{\phi} \big\langle \Phi_\pm, \mathcal{G}^\pm \big\rangle$$

Next task is to find a suitable truncation of massive modes

by decomposition $\mathcal{M}_{1,9} = \mathcal{M}_{1,3} \times_{\mathrm{W}} \mathcal{M}_6$ with $T_{1,3} \equiv T\mathcal{M}_{1,3}$ and $F \equiv T\mathcal{M}_6$

We truncate ten-dimensional theory

with keeping only a finite number of light modes in the spectrum.

Generically, however, the distinction between heavy and light modes

in a Kaluza-Klein expansion on $\mathcal{M}_{1,9} = \mathcal{M}_{1,3} \times_{\mathrm{W}} \mathcal{M}_6$ is not straightforward!

✓ If $\mathcal{M}_6$ is a Calabi-Yau

All the field deformations give massless modes in four-dimensional viewpoint

$$\|$$

Corresponding fields on $\mathcal{M}_6$ are harmonic and are finite in number

✓ If $\mathcal{M}_6$ is a generic geometry (w/ torsion)

Existence of finite number of harmonic forms are not guaranteed..

Instead, we assume existence of a certain finite-dimensional subspace of $\wedge^* T^* \mathcal{M}_6$

If there exists harmonic forms on $\mathcal{M}_6$, we can evaluate the dimensions of the forms via Index theorem: T. Kimura arXiv:0704.2111

Assumption the existence of finite-dimensional subset of p-forms:

$$\wedge^p_{\text{finite}} \subset \wedge^p T^* \mathcal{M}_6 \qquad\qquad U^{\text{finite}} = U \cap \wedge^*_{\text{finite}}$$

Note: the truncation should not break supersymmetry

$\dashrightarrow$ special Kähler geometry on U should give special Kähler geometry on U^{finite}

i.e., we require $\begin{cases} \text{Mukai pairing } \langle *, * \rangle \text{ is non-degenerate on } \wedge^p_{\text{finite}} \\ \text{if } \chi \in U^{\text{finite}}, \text{ then } \hat{\chi} \in U^{\text{finite}} \end{cases}$

First we introduce a set of basis forms (w/ Mukai pairing as symplectic structure):

$$\text{even forms}: \quad \Sigma_+ \ = \ \big\{\omega_A, \widetilde{\omega}^B\big\}, \quad \int_{\mathcal{M}_6} \big\langle \omega_A, \widetilde{\omega}^B \big\rangle \ = \ \delta_A{}^B, \quad A, B \ = \ 0, \ldots, b^+$$

$$\text{odd forms}: \quad \Sigma_- \ = \ \big\{\alpha_K, \beta^L\big\}, \quad \int_{\mathcal{M}_6} \big\langle \alpha_K, \beta^L \big\rangle \ = \ \delta_K{}^L, \quad K, L \ = \ 0, \ldots, b^-$$

Using this, the pure spinors $\Phi_\pm$ are expanded

$$\Phi_+ \ = \ \mathrm{e}^{-\mathcal{B}} \Phi_{0+} \ = \ \mathcal{X}^A \omega_A - \mathcal{G}_A \tilde{\omega}^A$$

$$\Phi_- \ = \ \mathrm{e}^{-\mathcal{B}} \Phi_{0-} \ = \ \mathcal{Z}^K \alpha_K - \mathcal{F}_K \beta^K$$

The compatibility is read as (w/ using $\forall V \ = \ x + \xi \in E$)

$$\big\langle \omega_A, V \cdot \alpha_K \big\rangle \ = \ \big\langle \omega_A, V \cdot \beta^K \big\rangle \ = \ \big\langle \widetilde{\omega}^A, V \cdot \alpha_K \big\rangle \ = \ \big\langle \widetilde{\omega}^A, V \cdot \beta^K \big\rangle \ = \ 0$$

The truncated Kähler potentials by $\int_{\mathcal{M}_6} \langle \omega_A, \widetilde{\omega}^B \rangle = \delta_A{}^B$ and $\int_{\mathcal{M}_6} \langle \alpha_K, \beta^L \rangle = \delta_K{}^L$ are

$$\mathrm{e}^{-K_+} = i \int_{\mathcal{M}_6} \big\langle \Phi_+, \overline{\Phi}_+ \big\rangle = i\Big(\overline{\mathcal{X}}^A \mathcal{G}_A - \mathcal{X}^A \overline{\mathcal{G}}_A \Big)$$

$$\mathrm{e}^{-K_-} = i \int_{\mathcal{M}_6} \big\langle \Phi_-, \overline{\Phi}_- \big\rangle = i\Big(\overline{\mathcal{Z}}^K \mathcal{F}_K - \mathcal{Z}^K \overline{\mathcal{F}}_K \Big)$$

RR fields are also expanded as

type IIA: $\begin{cases} \mathcal{A}_0^- = \xi^K \alpha_K + \widetilde{\xi}_L \beta^L \\ \mathcal{A}_1^+ = A_1^A \omega_A + \widetilde{A}_{1B} \widetilde{\omega}^B \end{cases}$ w/ $\begin{array}{rl} \xi^K, \widetilde{\xi}_L : & \text{scalars} \\ A_1^A, \widetilde{A}_{1B} : & \text{vectors} \end{array}$

type IIB: $\begin{cases} \mathcal{A}_0^+ = \xi^A \omega_A + \widetilde{\xi}_B \widetilde{\omega}^B \\ \mathcal{A}_1^- = A_1^K \alpha_K + \widetilde{A}_{1L} \beta^L \end{cases}$ w/ $\begin{array}{rl} \xi^A, \widetilde{\xi}_B : & \text{scalars} \\ A_1^K, \widetilde{A}_{1L} : & \text{vectors} \end{array}$

Convenient to define dual antisymmetric tensor fields of $\mathcal{A}_0^-$ and $\mathcal{A}_0^+$:

$$\mathcal{A}_0^- \leftrightarrow \mathcal{A}_2^- \equiv \widetilde{C}_2^K \alpha_K + C_{2L} \beta^L \qquad \mathcal{A}_0^+ \leftrightarrow \mathcal{A}_2^+ \equiv \widetilde{C}_2^A \omega_A + C_{2B} \widetilde{\omega}^B$$

$$\xi^K \leftrightarrow C_{2K} \qquad \widetilde{\xi}_K \leftrightarrow \widetilde{C}_2^K \qquad \xi^A \leftrightarrow C_{2A} \qquad \widetilde{\xi}_A \leftrightarrow \widetilde{C}_2^A$$

The most general differential conditions which can be imposed on basis forms are

$$\mathrm{d}\alpha_K \sim p_K{}^A \omega_A + e_{KA} \widetilde{\omega}^A \qquad \mathrm{d}\beta^K \sim q^{KA} \omega_A + m^K{}_A \widetilde{\omega}^A$$

$$\mathrm{d}\omega_A \sim m^K{}_A \alpha_K - e_{KA} \beta^K \qquad \mathrm{d}\widetilde{\omega}^A \sim -q^{KA} \alpha_K + p_K{}^A \beta^K$$

$p_K{}^A$, q^{KA}, e_{KA} and $m^K{}_A$ are $(b^+ + 1) \times (b^- + 1)$-dimensional constant matrices

Not necessary to be closed as in Calabi-Yau

Introduce a notation $\Sigma_+ = \begin{pmatrix} \omega_A \\ \widetilde{\omega}^B \end{pmatrix}$, $\Sigma_- = \begin{pmatrix} \alpha_K \\ \beta^L \end{pmatrix}$ and $\mathcal{Q} = \begin{pmatrix} p_K{}^A & e_{KB} \\ q^{LA} & m^L{}_B \end{pmatrix}$.

In terms of them the above differential condition is

$$\mathrm{d}\Sigma_- \sim \mathcal{Q}\Sigma_+ \qquad \mathrm{d}\Sigma_+ \sim \mathcal{S}_+ \mathcal{Q}^T (\mathcal{S}_-)^{-1} \Sigma_-$$

$\mathcal{S}_\pm$: the symplectic structures on $U^\pm$

If we impose $\mathrm{d}^2 = 0$ on the charged matrix as $\mathcal{Q}\mathcal{S}_+\mathcal{Q}^T = 0 = \mathcal{Q}^T(\mathcal{S}_-)^{-1}\mathcal{Q}$, we obtain

$$q^{KA} m_A{}^L - m^K{}_A q^{AL} = 0 \qquad p_K{}^A e_{AL} - e_{KA} p^A{}_L = 0 \qquad p_K{}^A m_A{}^L - e_{KA} q^{AL} = 0$$

$$q^{AK} p_K{}^B - p^A{}_K q^{KB} = 0 \qquad m_A{}^K e_{KB} - e_{AK} m^K{}_B = 0 \qquad m_A{}^K p_K{}^B - e_{AK} q^{KB} = 0$$

Kinetic terms $|\mathcal{G}_n|^2$ generate mass terms via truncation of fields:

- Type IIA:

$$\begin{aligned}
\mathcal{G}_{2p} &= \mathrm{d}\mathcal{A}_{2p-1} \sim \mathrm{d}_6\mathcal{A}_2^- + \mathrm{d}_4\mathcal{A}_1^+ \equiv D_2^A\,\omega_A + \widetilde{D}_{2A}\,\widetilde{\omega}^A \\
D_2^A &= \mathrm{d}_4 A_1^A + \widetilde{C}_2^K\, p_K{}^A + C_{2K}\, q^{AK} \\
\widetilde{D}_{2A} &= \mathrm{d}_4 \widetilde{A}_1^A + \widetilde{C}_2^K\, e_{AK} + C_{2K}\, m^K{}_A
\end{aligned}$$

- Type IIB:

$$\begin{aligned}
\mathcal{G}_{2p+1} &= \mathrm{d}\mathcal{A}_{2p} \sim \mathrm{d}_6\mathcal{A}_2^+ + \mathrm{d}_4\mathcal{A}_1^- \equiv D_2^K\,\alpha_K + \widetilde{D}_{2K}\,\beta^K \\
D_2^K &= \mathrm{d}_4 A_1^K - \widetilde{C}_2^A\, m^K{}_A + C_{2A}\, q^{AK} \\
\widetilde{D}_{2K} &= \mathrm{d}_4 \widetilde{A}_1^K + \widetilde{C}_2^A\, e_{AK} - C_{2A}\, p_K{}^A
\end{aligned}$$

Then charge matrices give massive modes of RR fields:

	e_{AK}	$m^K{}_A$	$p_K{}^A$	q^{KA}
IIA	massive A_μ^A	massive A_μ^A	massive $\widetilde{C}_2^K$	massive C_{2K}
IIB	massive A_μ^K	massive $\widetilde{C}_2^A$	massive A_μ^K	massive C_{2A}

- Type IIA Killing prepotentials $\mathcal{P}^x$ in S_{AB} w/ $\mathcal{G}^+ = \mathrm{d}\mathcal{A}_0^- + G^A_{\text{(RR)}}\omega_A + \widetilde{G}_{\text{(RR)}A}\widetilde{\omega}^A$:

$$\begin{aligned}
\mathcal{P}^1 + i\mathcal{P}^2 &= -2\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_+, \mathrm{d}\Phi_- \rangle \\
&= 2\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \Big(- \mathcal{X}^A e_{AK} \mathcal{Z}^K + \mathcal{X}^A m_A{}^K \mathcal{F}_K - \mathcal{G}_A p^A{}_K \mathcal{Z}^K + \mathcal{G}_A q^{AK} \mathcal{F}_K \Big) \\
\mathcal{P}^3 &= \mathrm{e}^{2\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_+, \mathcal{G}^+ \rangle \\
&= \mathrm{e}^{2\phi^{(4)}} \Big[\mathcal{X}^A \big(\widetilde{G}_{\text{(RR)}A} + e_{AK} \xi^K + m_A{}^K \widetilde{\xi}_K \big) + \mathcal{G}_A \big(G^A_{\text{(RR)}} + p^A{}_K \xi^K + q^{AK} \widetilde{\xi}_K \big) \Big]
\end{aligned}$$

$\mathcal{N} = 1$ superpotential W_{IIA} is given by

$$W_{\text{IIA}} = \cos^2\alpha\, \mathrm{e}^{i\beta} \int_{\mathcal{M}_6} \langle \Phi_+, \mathrm{d}\Phi_- \rangle - \sin^2\alpha\, \mathrm{e}^{-i\beta} \int_{\mathcal{M}_6} \langle \Phi_+, \mathrm{d}\overline{\Phi}_- \rangle + \sin 2\alpha\, \mathrm{e}^{\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_+, \mathcal{G}^+ \rangle$$

- Type IIB Killing prepotentials $\mathcal{P}^x$ in S_{AB} w/ $\mathcal{G}^- = \mathrm{d}\mathcal{A}_0^+ + G_{\mathsf{(RR)}}^K \alpha_K + \widetilde{G}_{\mathsf{(RR)}L} \beta^L$:

$$\begin{aligned}
\mathcal{P}^1 - i\mathcal{P}^2 &= -2\mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_-, \mathrm{d}\Phi_+ \rangle \\
&= 2\mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}} \Big(- \mathcal{Z}^K e_{KA} \mathcal{X}^A - \mathcal{Z}^K p_K{}^A \mathcal{G}_A + \mathcal{F}_K m^K{}_A \mathcal{X}^A + \mathcal{F}_K q^{KA} \mathcal{G}_A \Big) \\
\mathcal{P}^3 &= -\mathrm{e}^{2\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_-, \mathcal{G}^- \rangle \\
&= -\mathrm{e}^{2\phi^{(4)}} \Big[\mathcal{Z}^K \big(\widetilde{G}_{\mathsf{(RR)}K} - e_{KA} \xi^A + p_K{}^A \widetilde{\xi}_A \big) + \mathcal{F}_K \big(G_{\mathsf{(RR)}}^K + m^K{}_A \xi^A - q^{KA} \widetilde{\xi}_A \big) \Big]
\end{aligned}$$

$\mathcal{N} = 1$ superpotential W_{IIB} is given by

$$W_{\mathrm{IIB}} = -\cos^2\alpha\, \mathrm{e}^{i\beta} \int_{\mathcal{M}_6} \langle \Phi_-, \mathrm{d}\Phi_+ \rangle + \sin^2\alpha\, \mathrm{e}^{-i\beta} \int_{\mathcal{M}_6} \langle \Phi_-, \mathrm{d}\overline{\Phi}_+ \rangle - \sin 2\alpha\, \mathrm{e}^{\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_-, \mathcal{G}^- \rangle$$

Generically, scalar potential V in four-dimensional theory is

$$V = \mathrm{e}^K \Big(g^{a\bar{b}} D_a W \, \overline{D_b W} - 3|W|^2 \Big)$$

$$g_{a\bar{b}} = \partial_a \overline{\partial}_{\bar{b}} \big(K_+ + K_- + 2\phi^{(4)} \big) \qquad D_a W = \big(\partial_a + \partial_a K \big) W$$

Expanded the scalar potential V by "scalar fields" $\{\mathcal{X}^A, \xi^A, \widetilde{\xi}_A, \mathcal{Z}^K, \xi^K, \widetilde{\xi}_K\}$,

we would obtain non-trivial mass terms in $\mathcal{N} = 1$ theory

$\dashrightarrow$ so-called moduli stabilization

- Introduce a pair of $SU(3)$ structures on F $\sim$ $SU(3) \times SU(3)$ structure on $F \oplus F^*$
- Define generalized complex structures $\mathcal{J}_i$
- Construct $Spin(6,6)$ pure spinors $\Phi_\pm$
- Evaluate the space of pure spinors, and define Hitchin functional $H(\Phi_\pm)$
- Derive Kähler potentials $K_\pm$ and superpotentials $W_{\mathrm{IIA/IIB}}$
- Truncation of ten-dimensional fields

Remaining problem of flux compactification in type $\mathrm{IIA/IIB}$ is...

to find concrete dimensions $b^\pm$ of (non-)harmonic forms on compactified geometry $\mathcal{M}_6$!

$\longrightarrow$ a (mathematical) future problem

Doubled geometry

Start from low energy effective field theory for ten-dimensional string theory including

$$S = \int \mathrm{d}^{10}x\sqrt{-\mathcal{G}}\,\mathrm{e}^{-\phi}\Big\{\mathcal{R} + (\nabla\phi)^2 - \frac{1}{12}\mathcal{H}_{MNP}\mathcal{H}^{MNP}\Big\}$$

$$\mathcal{H} = \mathrm{d}\mathcal{B}$$

Consider the field theory compactified on (twisted) torus in the presence of B-field

VEV of three-form $\mathcal{H}$ gives rise to the structure constant of a certain Lie algebra

Decomposition of fields by Kaluza-Klein Compactification on a flat d-torus

$$
\begin{aligned}
\mathrm{d}s^2 &= g_{\mu\nu}(x,y)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \mathcal{G}_{ij}(x,y)\big(\mathrm{d}y^i + \mathcal{V}^i{}_\mu(x,y)\mathrm{d}x^\mu\big) \otimes \big(\mathrm{d}y^j + \mathcal{V}^j{}_\nu(x,y)\mathrm{d}x^\nu\big) \\
\mathcal{B} &= \frac{1}{2}\mathcal{B}_{\mu\nu}(x,y)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \mathcal{B}_{\mu i}(x,y)\mathrm{d}x^\mu \wedge \mathrm{d}y^i + \frac{1}{2}\mathcal{B}_{ij}(x,y)\mathrm{d}y^i \wedge \mathrm{d}y^j
\end{aligned}
$$

with Ansatz (truncation of massive KK modes)

$$
\begin{aligned}
g_{\mu\nu}(x,y) &= g_{\mu\nu}(x) & \mathcal{G}_{ij}(x,y) &= g_{ij}(x) & \mathcal{V}^i{}_\mu(x,y) &= V^i{}_\mu(x) \\
\mathcal{B}_{\mu\nu}(x,y) &= \mathcal{B}_{\mu\nu}(x) & \mathcal{B}_{\mu i}(x,y) &= \mathcal{B}_{\mu i}(x) & \mathcal{B}_{ij}(x,y) &= B_{ij}(x)
\end{aligned}
$$

$$
\phi(x,y) = \varphi(x) + \frac{1}{2}\log\big|\det g_{ij}(x)\big|
$$

Reduced degrees of freedom to demonstrate manifest gauge invariance:

$$
\begin{aligned}
B_{\mu i} &= \mathcal{B}_{\mu i} + B_{ij}V^j{}_\mu \\
B_{\mu\nu} &= \mathcal{B}_{\mu\nu} + V^i{}_{[\mu}B_{\nu]i} - B_{ij}V^i{}_\mu V^j{}_\nu
\end{aligned}
$$

The reduced action in D-dimensions (w/ setting $D = 10 - d$):

$$S = \int \mathrm{d}^D x \sqrt{-g}\, \mathrm{e}^{-\varphi} \Big\{ R + (\nabla\varphi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{8} L_{IJ} \nabla_\mu \mathcal{M}^{JK} L_{KL} \nabla^\mu \mathcal{M}^{LI} - \frac{1}{4} F^I_{\mu\nu} L_{IJ} \mathcal{M}^{JK} L_{KL} F^{L\mu\nu} \Big\}$$

This theory has $U(1)^{2d}$ gauge symmetry and a manifest global $O(d,d)$ symmetry with

$$\mathcal{M}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik}\, g^{kl}\, B_{lj} & B_{ik}\, g^{kj} \\ -g^{ik}\, B_{kj} & g^{ij} \end{pmatrix} \quad : \quad \text{scalar moduli matrix}$$

$$F^I = \mathrm{d}A^I \qquad A^I_\mu = \begin{pmatrix} V^i{}_\mu \\ B_{\mu i} \end{pmatrix} \qquad H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2} A^I_\mu L_{IJ} F^J_{\nu\rho} + \text{(cyclic permutations)}$$

$$L^{IJ} \equiv \begin{pmatrix} \mathbf{0}_d & \mathbb{1}_d \\ \mathbb{1}_d & \mathbf{0}_d \end{pmatrix} \quad : \quad O(d,d) \text{ invariant metric}, \quad \forall M^I{}_J \in O(d,d),\ M^I{}_K L^{KL} M_L{}^J = L^{IJ}$$

Non-abelian gauge symmetry by considering a $2d$-dimensional subgroup G of $O(d,d)$:

The fundamental repr. of $O(d,d)$ becomes the adjoint repr. of G under the embedding

$$[T_I, T_J] \;=\; f_{IJ}{}^K T_K \qquad T_I \;=\; \frac{1}{2}\Theta_I{}^{JK} t_{JK} \qquad \left\{ \begin{array}{rl} T_I : & \text{generators of } G \text{ with structure constant } f_{IJ}{}^K \\ t_{JK} : & \text{generators of } O(d,d) \\ \Theta_I{}^{JK} : & \text{embedding tensor} \end{array} \right.$$

Then, D-dimensional theory with gauge symmetry G is

$$\begin{aligned} S \;=\; \int \mathrm{d}^D x \sqrt{-g}\, \mathrm{e}^{-\varphi} \Big\{ & R + (\nabla\varphi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \\ & + \frac{1}{8} L_{IJ} \mathcal{D}_\mu \mathcal{M}^{JK} L_{KL} \mathcal{D}^\mu \mathcal{M}^{LI} - \frac{1}{4} F^I_{\mu\nu} \mathcal{M}^{JK} L_{KL} F^{L\mu\nu} - g^2 W(\mathcal{M}) \Big\} \end{aligned}$$

with covariantized form (w/ $f_{IJK} = f_{IJ}{}^L L_{KL}$)

$$\begin{aligned} \mathcal{D}_\mu \mathcal{M}^{IJ} \;&=\; \partial_\mu \mathcal{M}^{IJ} - g f_{KL}{}^I A^K_\mu \mathcal{M}^{LJ} - g f_{KL}{}^J A^K_\mu \mathcal{M}^{IL} \\ F \;&=\; \mathrm{d}A + g A \wedge A, \qquad H \;=\; \mathrm{d}B - \frac{1}{2} \mathrm{tr}\Big(A \wedge F + \frac{2g}{3} A \wedge A \wedge A \Big) \\ W(\mathcal{M}) \;&=\; a \mathcal{M}^{II'} \mathcal{M}^{JJ'} \mathcal{M}^{KK'} f_{IJK} f_{I'J'K'} + b \mathcal{M}^{II'} \mathcal{M}^{JJ'} L^{KK'} f_{IJK} f_{I'J'K'} \\ & \quad + c \mathcal{M}^{II'} L^{JJ'} L^{KK'} f_{IJK} f_{I'J'K'} + W_0 \end{aligned}$$

What is T_I? ←-- non-abelianized generators of gauge fields $A^I_\mu = (V^i{}_\mu, B_{\mu i})^T$

$$T_I \ni \begin{cases} Z_i : & \text{generators corresponding to } V^i{}_\mu \\ X^i : & \text{generators corresponding to } B_{\mu i} \end{cases}$$

$$[T_I, T_J] \ = \ f_{IJ}{}^K T_K \quad \dashrightarrow \quad \begin{cases} [Z_i, Z_j] &= \ \tau_{ij}{}^k \, Z_k + h_{ijk} \, X^k \\ [X^i, X^j] &= \ 0 \\ [X^i, Z_j] &= \ \tau^i{}_{jk} \, X^k \end{cases}$$

$\tau_{ij}{}^k$: structure constant of twisted torus

h_{ijk}: VEV of NS three-form H_{ijk}

$\tau^l{}_{i'[i}\tau_{jk]}{}^{i'} = 0$ Jacobi

$h_{i'[ij}\tau_{kl]}{}^{i'} = 0$ $\mathrm{d}H_3 = 0$

$\tau^i{}_{ij} = 0$ invariance of $\sqrt{-g}$

- twisted torus is introduced by vielbein $\mathrm{d}y^i \to \eta^a \ = \ \eta^a{}_i(y)\mathrm{d}y^i$:

$$\begin{aligned} \mathcal{G}_{ij}(x,y) \ &= \ g_{ij}(x) \quad \to \quad g_{ab}(x)\, \eta_i{}^a(y)\, \eta^b{}_j(y) \\ g_{ij}\big(\mathrm{d}y^i + V^i{}_\mu \mathrm{d}x^\mu\big)\big(\mathrm{d}y^j + V^j{}_\nu \mathrm{d}x^\nu\big) \quad &\to \quad g_{ab}\big(\eta^a + V^a{}_\mu \mathrm{d}x^\mu\big)\big(\eta^b + V^b{}_\nu \mathrm{d}x^\nu\big) \\ \mathrm{d}\eta^a \ &= \ -\frac{1}{2}\tau_{bc}{}^a \, \eta^b \wedge \eta^c \end{aligned}$$

Possibility of extension of the Lie algebra

$$\boxed{\begin{aligned} [Z_a, Z_b] &= \tau_{ab}{}^c\, Z_c + h_{abc}\, X^c \\ [X^a, X^b] &= 0 \\ [X^a, Z_b] &= \tau^a{}_{bc}\, X^c \end{aligned}}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} [Z_a, Z_b] &= \tau_{ab}{}^c\, Z_c + h_{abc}\, X^c \\ [X^a, X^b] &= Q^{ab}{}_c\, X^c + R^{abc}\, Z_c \\ [X^a, Z_b] &= \tau^a{}_{bc}\, X^c - Q^{ac}{}_b\, Z_c \end{aligned}}$$

Why should we study the additional structure constants $Q^{ab}{}_c$ and R^{abc}?

$$\downarrow$$

Because they are related via T-duality transformations!

$$h_{abc} \xrightarrow{\text{T-dual}} \tau^a{}_{bc} \xrightarrow{\text{T-dual}} Q^{ab}{}_c \xrightarrow{\text{T-dual}} R^{abc}$$

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- Geometries generated by T-duality transformations:

h_{abc}	flat torus w/ three-form flux
$\downarrow$	T-duality transformation
${\tau^a}_{bc}$	nilmanifold w/ non-trivial isometry group
$\downarrow$	T-duality transformation
${Q^{ab}}_c$	T-fold: globally nongeometric, locally geometric, *stringy*
$\downarrow$	T-duality transformation
R^{abc}	doubled geometry: even locally nongeometric, *stringy*

So far, we discussed effective theory compactified on d-dimensional space in the presence of flux.

In order to include the above information, we double the "dimensions of compactified geometry" and study sigma model

$\longrightarrow$ doubled formalism

Glue two local patches of a conventional string background with transition function by

diffeomorphism

and

duality transformations

Let Y^i be fields in sigma model corresponding to coordinates y^i on a torus T^d.

In formulating CFT on T^d,

extra d coordinates $\widetilde{Y}_i$ for a dual torus $\widetilde{T}^d$ are needed

These are conjugate to the winding number,

and are needed to write vertex operators such as $\mathrm{e}^{ik_L \cdot Y_L}$ where $Y_L = Y - \widetilde{Y}$.

$\rightarrow$ Degrees of freedom of a sigma model are doubled from Y^i to $\{Y^i, \widetilde{Y}_i\}$.

Start with a sigma model on a space $\mathcal{M}_d$ with metric g_{ij} and B-field B_{ij}:

$$S_c = \int_\Sigma \Big(\frac{1}{2} g_{ij}\, \mathrm{d}Y^i \wedge * \mathrm{d}Y^j + \frac{1}{2} B_{ij}\, \mathrm{d}Y^i \wedge \mathrm{d}Y^j \Big)$$

This is extended to the action on a doubled space $\mathcal{M}_{2d}$ w/ scalar moduli matrix $\mathcal{M}_{IJ}$:

$$S = \int_\Sigma \frac{1}{4} \mathcal{M}_{IJ}\, \mathrm{d}\mathbb{Y}^I \wedge * \mathrm{d}\mathbb{Y}^J = \int_\Sigma \frac{1}{4} \mathcal{M}_{AB}\, \eta^A \wedge * \eta^B$$

$$\mathcal{M}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik}\, g^{kl}\, B_{lj} & B_{ik}\, g^{kj} \\ -g^{ik}\, B_{kj} & g^{ij} \end{pmatrix} \qquad \mathbb{Y}^I = \begin{pmatrix} Y^i \\ \widetilde{Y}_i \end{pmatrix}$$

$$\eta^A = \eta^A{}_I \mathrm{d}\mathbb{Y}^I, \qquad \mathcal{M}_{IJ} = \eta_I{}^A\, \delta_{AB}\, \eta^B{}_J, \qquad \eta^A{}_I = \begin{pmatrix} e^a{}_i & \mathbf{0} \\ -e_a{}^j\, B_{ji} & e_a{}^i \end{pmatrix}$$

Bianchi identity (Maurer-Cartan eq.): $\mathrm{d}\eta^A = -\frac{1}{2} f_{BC}{}^A\, \eta^B \wedge \eta^C$

Self-duality constraint (to go back to conventional system): $\eta^A = L^{AB} \mathcal{M}_{BC} * \eta^C$

w/ $O(d,d)$ invariant metric $L^{AB} \equiv \langle \eta^A, \eta^B \rangle$

$$\boxed{\mathcal{M}_{IJ} \text{ takes value in coset } \frac{O(d,d)}{O(d) \times O(d)}}$$

This sigma model on doubled space $\mathcal{M}_{2d}$ has

- $O(d,d)$ global symmetry by $g \in O(d,d)$ w/ $g^A{}_C L^{CD} g_D{}^B = L^{AB}$:

$$\begin{aligned}
\mathbb{Y}^I &\to \mathbb{Y}'^I = g^I{}_J \mathbb{Y}^J \\
\eta^A{}_I(\mathbb{Y}) &\to \eta'^A{}_I(\mathbb{Y}') = g^A{}_B \, \eta^B{}_J(\mathbb{Y}') \, g^J{}_I \\
\mathcal{M}_{IJ}(\mathbb{Y}) &\to \mathcal{M}'_{IJ}(\mathbb{Y}') = g_I{}^K \mathcal{M}_{KL}(\mathbb{Y}') g^L{}_J
\end{aligned}$$

Basis vector is kept invariant under the transformation: so-called "active transformation"

- $O(d) \times O(d)$ local symmetry: $\eta^A{}_I(\mathbb{Y}) \to \eta'^A{}_I(\mathbb{Y}) = h^A{}_B(\mathbb{Y}) \, \eta^B{}_I(\mathbb{Y})$

A realization of fractional transformation of $M_{ij} = \mathcal{G}_{ij} + \mathcal{B}_{ij}$:

$$g = \begin{pmatrix} A & \beta \\ \Theta & D \end{pmatrix} : \qquad M \to \big(DM + \Theta\big)\big(\beta M + A\big)^{-1}$$

$$\begin{cases} \Theta : & \text{gauge transformation of B-field } B \to B + \Theta \\ D, A : & \text{ordinary diffeomorphism on } g_{ij} \\ \beta : & \text{non-local transformation w/ mixing } Y^i \text{ and } \widetilde{Y}_i \end{cases}$$ ←-- duality trsf.

T-duality transformation (ex. $d = 3$ case):

$$\rho_i \; = \; \begin{pmatrix} \mathbb{1}_d - T_i & T_i \\ T_i & \mathbb{1}_d - T_i \end{pmatrix} \; \in \; O(d,d;\mathbb{Z})$$

$$T_1 \; = \; \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \qquad T_2 \; = \; \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} \qquad T_3 \; = \; \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

This action exchanges physical coordinates Y^i with dual coordinates $\widetilde{Y}_i$

A role of L: reduction of (co)tangent bundle of doubled space $\mathcal{M}_{2d}$

$$L^{AB} \;=\; \big\langle \eta^A, \eta^B \big\rangle \;=\; \begin{pmatrix} \mathbf{0}_d & \mathbb{1}_d \\ \mathbb{1}_d & \mathbf{0}_d \end{pmatrix}, \quad L^{IJ} \;\equiv\; \big\langle \mathrm{d}\mathbb{Y}^I, \mathrm{d}\mathbb{Y}^J \big\rangle \;=\; \eta^I{}_A \, L^{AB} \, \eta_B{}^J$$

This implies $T^*\mathcal{M}_{2d} = T\mathcal{M}_d \oplus T^*\mathcal{M}_d$ s.t.

$$\big\langle \mathrm{d}Y^i, \mathrm{d}\widetilde{Y}_j \big\rangle \;=\; \delta^i_j \quad \rightarrow \quad \mathrm{d}\widetilde{Y}_i \;=\; \frac{\partial}{\partial Y^i}$$

$$\eta^A \;=\; \eta^A{}_I \, \mathrm{d}\mathbb{Y}^I \;=\; \begin{pmatrix} e^a{}_i \, \mathrm{d}Y^i \\ e_a{}^i \big(\mathrm{d}\widetilde{Y}_i - B_{ij} \, \mathrm{d}Y^j \big) \end{pmatrix} \;=\; \begin{pmatrix} e^a{}_i \, \mathrm{d}Y^i \\ e_a{}^i \big(\frac{\partial}{\partial Y^i} - B_{ij} \, \mathrm{d}Y^j \big) \end{pmatrix}$$

a connection to generalized geometry

Start from a flat three-torus T^3 with B-field given by the following forms:

$$\mathrm{d}s^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2$$

$$H = \mathrm{d}B = 3\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \qquad B = x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y$$

Doubled vielbein and doubled metric are given as

$$\eta^A{}_I = \begin{pmatrix} e^a{}_i & \mathbf{0} \\ -e_a{}^j B_{ji} & e_a{}^i \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -z & y & 1 & 0 & 0 \\ z & 0 & -x & 0 & 1 & 0 \\ -y & x & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\mathcal{M}_{IJ} = \eta_I{}^A \delta_{AB} \eta^B{}_J = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -z & y & 1 & 0 & 0 \\ z & 0 & -x & 0 & 1 & 0 \\ -y & x & 0 & 0 & 0 & 1 \end{array}\right)$$

- Maurer-Cartan eq. of doubled vielbein $\eta^A = \begin{pmatrix} \eta^a \\ \widetilde{\eta}_a \end{pmatrix}$ gives a structure constant $f_{AB}{}^C$:

$$
\begin{aligned}
\mathrm{d}\eta^1 &= 0 & \mathrm{d}\eta^2 &= 0 & \mathrm{d}\eta^3 &= 0 \\
\mathrm{d}\widetilde{\eta}_1 &= 2\eta^2 \wedge \eta^3 & \mathrm{d}\widetilde{\eta}_2 &= 2\eta^3 \wedge \eta^1 & \mathrm{d}\widetilde{\eta}_3 &= 2\eta^2 \wedge \eta^3
\end{aligned}
$$

$$
\therefore \quad \mathrm{d}\eta^a = 0 \qquad \mathrm{d}\widetilde{\eta}_a = -\frac{1}{2} h_{abc}\, \eta^b \wedge \eta^c
$$

$$
[Z_a, Z_b] = h_{abc}\, X^c \qquad h_{123} = -2
$$

- Periodicity of physical coordinates Y^i and dual ones $\widetilde{Y}_i$:

$$
\begin{aligned}
(x, \widetilde{y}, \widetilde{z}) &\sim (x+1, \widetilde{y}+z, \widetilde{z}-y) & \qquad \widetilde{x} &\sim \widetilde{x}+1 \\
(y, \widetilde{z}, \widetilde{x}) &\sim (y+1, \widetilde{z}+x, \widetilde{x}-z) & \qquad \widetilde{y} &\sim \widetilde{y}+1 \\
(z, \widetilde{x}, \widetilde{y}) &\sim (z+1, \widetilde{x}+y, \widetilde{y}-x) & \qquad \widetilde{z} &\sim \widetilde{z}+1
\end{aligned}
$$

Existence of the non-trivial B-field is encoded in the redefinition of the dual coordinates by the actual coordinates, when the latter are shifted.

- Doubled vielbein by T-duality along z-direction:

$$(\eta_\tau)^A{}_I \;=\; (\rho_z)^A{}_B\, \eta^B{}_J (\rho_z)^J{}_I \;=\; \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -y & x & 1 & 0 & 0 & 0 \\ \hdashline 0 & -\widetilde{z} & 0 & 1 & 0 & y \\ \widetilde{z} & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{aligned}(\mathcal{M}_\tau)_{IJ} \;&=\; (\rho_z)_I{}^K \mathcal{M}_{KL} (\rho_z)^L{}_J \;=\; (\eta_\tau)_I{}^A\, \delta_{AB} (\eta_\tau)^B{}_J \\ &=\; \left(\begin{array}{ccc:ccc} 1+y^2+\widetilde{z}^2 & -xy & -y & 0 & \widetilde{z} & -x\widetilde{z} \\ -xy & 1+x^2+\widetilde{z}^2 & x & -\widetilde{z} & 0 & -y\widetilde{z} \\ -y & x & 1 & 0 & 0 & 0 \\ \hdashline 0 & -\widetilde{z} & 0 & 1 & 0 & y \\ \widetilde{z} & 0 & 0 & 0 & 1 & -x \\ -x\widetilde{z} & -y\widetilde{z} & 0 & y & -x & 1+x^2+y^2 \end{array}\right)\end{aligned}$$

Metric and B-field can be read from the doubled metric as

$$\mathrm{d}s^2 \;=\; (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z - y\mathrm{d}x + x\mathrm{d}y)^2$$

$$B \;=\; \widetilde{z}\,\mathrm{d}x \wedge \mathrm{d}y \qquad\qquad H \;=\; \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\widetilde{z}$$

- Maurer-Cartan eq. of doubled vielbein $\eta^A = \begin{pmatrix} \eta^a \\ \widetilde{\eta}_a \end{pmatrix}$ gives a structure constant $f_{AB}{}^C$:

$$
\begin{aligned}
\mathrm{d}\eta^1 &= 0 & \mathrm{d}\eta^2 &= 0 & \mathrm{d}\eta^3 &= 2\eta^1 \wedge \eta^2 \\
\mathrm{d}\widetilde{\eta}_1 &= 2\eta^2 \wedge \widetilde{\eta}_3 & \mathrm{d}\widetilde{\eta}_2 &= 2\widetilde{\eta}_3 \wedge \eta^1 & \mathrm{d}\widetilde{\eta}_3 &= 0
\end{aligned}
$$

$$
\therefore \quad \mathrm{d}\eta^a = -\frac{1}{2}\tau^a{}_{bc}\,\eta^b \wedge \eta^c \qquad \mathrm{d}\widetilde{\eta}_a = -\frac{1}{2}\tau_{ab}{}^c\,\eta^b \wedge \eta^c
$$

$$
[Z_a, Z_b] = \tau_{ab}{}^c\, Z_c \qquad [X^a, Z_b] = \tau^a{}_{bc}\, X^c \qquad \tau^1{}_{23} = -2
$$

a geometry with torsion, similar to half-flat manifold: a connection to generalized geometry

- Periodicity of physical coordinates Y^i and dual ones $\widetilde{Y}_i$:

$$
\begin{aligned}
(x, \widetilde{y}, z) &\sim (x+1, \widetilde{y}+z, z-y) & \widetilde{x} &\sim \widetilde{x}+1 \\
(y, z, \widetilde{x}) &\sim (y+1, z+x, \widetilde{x}-z) & \widetilde{y} &\sim \widetilde{y}+1 \\
z &\sim z+1 & (\widetilde{z}, \widetilde{x}, \widetilde{y}) &\sim (\widetilde{z}+1, \widetilde{x}+\widetilde{y}, \widetilde{y}-\widetilde{x})
\end{aligned}
$$

- Doubled vielbein by T-duality along (y, z)-directions:

$$(\eta_Q)^A{}_I \;=\; (\rho_y \rho_z)^A{}_B \, \eta^B{}_J (\rho_z \rho_y)^J{}_I \;=\; \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \widetilde{z} & 1 & 0 & 0 & 0 & -x \\ -\widetilde{y} & 0 & 1 & 0 & x & 0 \\ \hdashline 0 & 0 & 0 & 1 & -\widetilde{z} & \widetilde{y} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{aligned} (\mathcal{M}_Q)_{IJ} \;&=\; (\rho_y \rho_z)_I{}^K \, \mathcal{M}_{KL} (\rho_z \rho_y)^L{}_J \;=\; (\eta_Q)_I{}^A \, \delta_{AB} (\eta_Q)^B{}_J \\ &=\; \left(\begin{array}{ccc:ccc} 1 + \widetilde{y}^2 + \widetilde{z}^2 & \widetilde{z} & -\widetilde{y} & 0 & -x\widetilde{y} & -x\widetilde{z} \\ \widetilde{z} & 1 & 0 & 0 & 0 & -x \\ -\widetilde{y} & 0 & 1 & 0 & x & 0 \\ \hdashline 0 & 0 & 0 & 1 & -\widetilde{z} & \widetilde{y} \\ -x\widetilde{y} & 0 & x & -\widetilde{z} & 1 + x^2 + \widetilde{z}^2 & -\widetilde{y}\widetilde{z} \\ -x\widetilde{z} & -x & 0 & \widetilde{y} & -\widetilde{y}\widetilde{z} & 1 + x^2 + \widetilde{y}^2 \end{array}\right) \end{aligned}$$

Doubled vielbein η_Q is not a lower block triangular form, even though correct metric and B-field can be read from $\mathcal{M}_Q$.

local $O(3) \times O(3)$ transformation to describe correct form of doubled vielbein

$$
(\eta'_Q)^A{}_I = h^A{}_B (\eta_Q)^B{}_I \equiv \begin{pmatrix} (e_Q)^a{}_i & \mathbf{0}_3 \\ -(e_Q)_a{}^j (B_Q)_{ji} & (e_Q)_a{}^i \end{pmatrix}
$$

$$
= \left(\begin{array}{ccc:ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{\widetilde{z}}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} & 0 & 0 & 0 & 0 \\
-\frac{\widetilde{y}}{\sqrt{1+x^2}} & 0 & \frac{1}{\sqrt{1+x^2}} & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 1 & -\widetilde{z} & \widetilde{y} \\
-\frac{x\widetilde{y}}{\sqrt{1+x^2}} & 0 & \frac{x}{\sqrt{1+x^2}} & 0 & \sqrt{1+x^2} & 0 \\
-\frac{x\widetilde{z}}{\sqrt{1+x^2}} & -\frac{x}{\sqrt{1+x^2}} & 0 & 0 & 0 & \sqrt{1+x^2}
\end{array}\right)
$$

$$
h^A{}_B = \left(\begin{array}{ccc:ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{1+x^2}} & 0 & 0 & 0 & \frac{x}{\sqrt{1+x^2}} \\
0 & 0 & \frac{1}{\sqrt{1+x^2}} & 0 & -\frac{x}{\sqrt{1+x^2}} & 0 \\
\hdashline
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{x}{\sqrt{1+x^2}} & 0 & \frac{1}{\sqrt{1+x^2}} & 0 \\
0 & -\frac{x}{\sqrt{1+x^2}} & 0 & 0 & 0 & \frac{1}{\sqrt{1+x^2}}
\end{array}\right)
$$

- Metric and B-field can be read from the doubled metric as

$$
\begin{aligned}
\mathrm{d}s^2 &= (\mathrm{d}x)^2 + \frac{1}{1+x^2}\Big[\big(\mathrm{d}y + \widetilde{z}\,\mathrm{d}x\big)^2 + \big(\mathrm{d}z - \widetilde{y}\,\mathrm{d}x\big)^2\Big] \\
B &= \frac{x}{1+x^2}\Big[-\widetilde{y}\,\mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}y \wedge \mathrm{d}z + \widetilde{z}\,\mathrm{d}z \wedge \mathrm{d}x\Big] \\
H &= \frac{-1+x^2}{(1+x^2)^2}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - \frac{x}{1+x^2}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\widetilde{y} + \frac{x}{1+x^2}\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}\widetilde{z}
\end{aligned}
$$

Both g_{ij} and B_{ij} are not well-defined functions of x.

They give however a good global description upon using the identification.

- Maurer-Cartan eq. of doubled vielbein $\eta^A = \begin{pmatrix} \eta^a \\ \widetilde{\eta}_a \end{pmatrix}$ gives a structure constant $f_{AB}{}^C$:

$$
\begin{aligned}
\mathrm{d}\eta^1 &= 0 & \mathrm{d}\eta^2 &= 2\widetilde{\eta}_3 \wedge \eta^1 & \mathrm{d}\eta^3 &= 2\eta^1 \wedge \widetilde{\eta}_2 \\
\mathrm{d}\widetilde{\eta}_1 &= 2\widetilde{\eta}_2 \wedge \widetilde{\eta}_3 & \mathrm{d}\widetilde{\eta}_2 &= 0 & \mathrm{d}\widetilde{\eta}_3 &= 0
\end{aligned}
$$

$$
\therefore \quad \mathrm{d}\eta^a = -\frac{1}{2}Q^{ab}{}_c\,\widetilde{\eta}_b \wedge \eta^c \qquad \mathrm{d}\widetilde{\eta}_a = -\frac{1}{2}Q_a{}^{bc}\,\widetilde{\eta}_b \wedge \widetilde{\eta}_c
$$

$$
[X^a, X^b] = Q^{ab}{}_c\,X^c \qquad [Z_a, X^b] = Q_a{}^{bc}\,Z_c \qquad Q^{12}{}_3 = -2
$$

- Periodicity of physical coordinates Y^i and dual ones $\widetilde{Y}_i$:

$$
\begin{aligned}
(x, y, z) &\sim (x+1, y+\widetilde{z}, z-\widetilde{y}) & \qquad \widetilde{x} &\sim \widetilde{x}+1 \\
y &\sim y+1 & (\widetilde{y}, z, \widetilde{x}) &\sim (\widetilde{y}+1, z+x, \widetilde{x}-\widetilde{z}) \\
z &\sim z+1 & (\widetilde{z}, \widetilde{x}, y) &\sim (\widetilde{z}+1, \widetilde{x}+\widetilde{y}, y-x)
\end{aligned}
$$

Identification which shifts the ordinary coordinates Y^i by a dual one $\widetilde{Y}_i$ when identifying a base coordinate $x \sim x+1$.

This means that one should identify properly the B-field B_{ij} and the metric g_{ij}.

- Doubled vielbein by T-duality along (x, y, z)-directions:

$$(\eta_R)^A{}_I = (\rho_z \rho_y \rho_x)^A{}_B \, \eta^B{}_J (\rho_x \rho_y \rho_z)^J{}_I = \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & -\widetilde{z} & \widetilde{y} \\ 0 & 1 & 0 & \widetilde{z} & 0 & -\widetilde{x} \\ 0 & 0 & 1 & -\widetilde{y} & \widetilde{x} & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{aligned} (\mathcal{M}_R)_{IJ} &= (\rho_y \rho_x)_I{}^K \, \mathcal{M}_{KL} (\rho_x \rho_y)^L{}_J = (\eta_R)_I{}^A \, \delta_{AB} (\eta_R)^B{}_J \\ &= \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & -\widetilde{z} & \widetilde{y} \\ 0 & 1 & 0 & \widetilde{z} & 0 & -\widetilde{x} \\ 0 & 0 & 1 & -\widetilde{y} & \widetilde{x} & 0 \\ \hdashline 0 & \widetilde{z} & -\widetilde{y} & 1 + \widetilde{y}^2 + \widetilde{z}^2 & -\widetilde{x}\widetilde{y} & -\widetilde{x}\widetilde{z} \\ -\widetilde{z} & 0 & \widetilde{x} & -\widetilde{x}\widetilde{y} & 1 + \widetilde{x}^2 + \widetilde{z}^2 & -\widetilde{y}\widetilde{z} \\ \widetilde{y} & -\widetilde{x} & 0 & -\widetilde{x}\widetilde{z} & -\widetilde{y}\widetilde{z} & 1 + \widetilde{x}^2 + \widetilde{y}^2 \end{array}\right) \end{aligned}$$

Doubled vielbein η_R is not a lower block triangular form, even though correct metric and B-field can be read from $\mathcal{M}_R$.

local $O(3) \times O(3)$ transformation to describe correct form of doubled vielbein

$$
\begin{aligned}
(\eta'_R)^A{}_I &= h^A{}_B (\eta_R)^B{}_I \equiv \begin{pmatrix} (e_R)^a{}_i & \mathbf{0}_3 \\ -(e_R)_a{}^j (B_R)_{ji} & (e_R)_a{}^i \end{pmatrix} \\
&= \left(\begin{array}{ccc:ccc}
\chi(1+\widetilde{x}^2) & \chi(\widetilde{x}\widetilde{y}+\widetilde{z}) & \chi(\widetilde{z}\widetilde{x}-\widetilde{y}) & 0 & 0 & 0 \\
\chi(\widetilde{x}\widetilde{y}-\widetilde{z}) & \chi(1+\widetilde{y}^2) & \chi(\widetilde{y}\widetilde{z}+\widetilde{x}) & 0 & 0 & 0 \\
\chi(\widetilde{z}\widetilde{x}+\widetilde{y}) & \chi(\widetilde{y}\widetilde{z}-\widetilde{x}) & \chi(1+\widetilde{z}^2) & 0 & 0 & 0 \\
\hdashline
-\chi(\widetilde{y}^2+\widetilde{z}^2) & \chi(\widetilde{x}\widetilde{y}+\widetilde{z}) & \chi(\widetilde{z}\widetilde{x}-\widetilde{y}) & 1 & \widetilde{z} & -\widetilde{y} \\
\chi(\widetilde{x}\widetilde{y}-\widetilde{z}) & -\chi(\widetilde{x}^2+\widetilde{z}^2) & \chi(\widetilde{x}+\widetilde{y}\widetilde{z}) & -\widetilde{z} & 1 & \widetilde{x} \\
\chi(\widetilde{z}\widetilde{x}+\widetilde{y}) & \chi(\widetilde{y}\widetilde{z}-\widetilde{x}) & -\chi(\widetilde{x}^2+\widetilde{y}^2) & \widetilde{y} & -\widetilde{x} & 1
\end{array}\right) \\
h^A{}_B &= \chi \left(\begin{array}{ccc:ccc}
1+\widetilde{x}^2 & \widetilde{x}\widetilde{y}+\widetilde{z} & \widetilde{z}\widetilde{x}-\widetilde{y} & -\widetilde{y}^2-\widetilde{z}^2 & \widetilde{x}\widetilde{y}+\widetilde{z} & \widetilde{z}\widetilde{x}-\widetilde{y} \\
\widetilde{x}\widetilde{y}-\widetilde{z} & 1+\widetilde{y}^2 & \widetilde{y}\widetilde{z}+\widetilde{x} & \widetilde{x}\widetilde{y}-\widetilde{z} & -\widetilde{z}^2-\widetilde{x}^2 & \widetilde{y}\widetilde{z}+\widetilde{x} \\
\widetilde{z}\widetilde{x}+\widetilde{y} & \widetilde{y}\widetilde{z}-\widetilde{x} & 1+\widetilde{z}^2 & \widetilde{z}\widetilde{x}+\widetilde{y} & \widetilde{y}\widetilde{z}-\widetilde{x} & -\widetilde{x}^2-\widetilde{y}^2 \\
\hdashline
-\widetilde{y}^2-\widetilde{z}^2 & \widetilde{x}\widetilde{y}+\widetilde{z} & \widetilde{z}\widetilde{x}-\widetilde{y} & 1+\widetilde{x}^2 & \widetilde{x}\widetilde{y}+\widetilde{z} & \widetilde{z}\widetilde{x}-\widetilde{y} \\
\widetilde{x}\widetilde{y}-\widetilde{z} & -\widetilde{z}^2-\widetilde{x}^2 & \widetilde{y}\widetilde{z}+\widetilde{x} & \widetilde{x}\widetilde{y}-\widetilde{z} & 1+\widetilde{y}^2 & \widetilde{y}\widetilde{z}+\widetilde{x} \\
\widetilde{z}\widetilde{x}+\widetilde{y} & \widetilde{y}\widetilde{z}-\widetilde{x} & -\widetilde{x}^2-\widetilde{y}^2 & \widetilde{z}\widetilde{x}+\widetilde{y} & \widetilde{y}\widetilde{z}-\widetilde{x} & 1+\widetilde{z}^2
\end{array}\right) \\
\chi &= \frac{1}{1+\widetilde{x}^2+\widetilde{y}^2+\widetilde{z}^2}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{1+\widetilde{x}^2+\widetilde{y}^2+\widetilde{z}^2}\Big[(\mathrm{d}x)^2+(\mathrm{d}y)^2+(\mathrm{d}z)^2+\big(\widetilde{x}\,\mathrm{d}x+\widetilde{y}\,\mathrm{d}y+\widetilde{z}\,\mathrm{d}z\big)^2\Big] \\
B &= -\frac{1}{1+\widetilde{x}^2+\widetilde{y}^2+\widetilde{z}^2}\Big[\widetilde{z}\,\mathrm{d}x\wedge\mathrm{d}y+\widetilde{x}\,\mathrm{d}y\wedge\mathrm{d}z+\widetilde{y}\,\mathrm{d}z\wedge\mathrm{d}x\Big] \\
H &= \chi^2\Big[2\widetilde{x}\widetilde{z}\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}\widetilde{x}-\big(\chi^{-1}-2\widetilde{x}^2\big)\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}\widetilde{x}+2\widetilde{x}\widetilde{y}\,\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}\widetilde{x} \\
&\qquad +2\widetilde{y}\widetilde{z}\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}\widetilde{y}+2\widetilde{x}\widetilde{y}\,\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}\widetilde{y}-\big(\chi^{-1}-2\widetilde{y}^2\big)\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}\widetilde{y} \\
&\qquad -\big(\chi^{-1}-2\widetilde{z}^2\big)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}\widetilde{z}+2\widetilde{x}\widetilde{z}\,\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}\widetilde{z}+2\widetilde{y}\widetilde{z}\,\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}\widetilde{z}\Big] \\
\chi &= \frac{1}{1+\widetilde{x}^2+\widetilde{y}^2+\widetilde{z}^2}
\end{aligned}
$$

When the dual coordinates $\widetilde{Y}_i$ are projected out (i.e., $\widetilde{Y}_i \equiv 0$),

the geometry looks reduced to a torus without B-field.

▶ Maurer-Cartan eq. of doubled vielbein $\eta^A = \begin{pmatrix} \eta^a \\ \widetilde{\eta}_a \end{pmatrix}$ gives a structure constant ${f_{AB}}^C$:

$$\begin{aligned} \mathrm{d}\eta^1 &= 2\widetilde{\eta}_2 \wedge \widetilde{\eta}_3 & \mathrm{d}\eta^2 &= 2\widetilde{\eta}_3 \wedge \widetilde{\eta}_1 & \mathrm{d}\eta^3 &= 2\widetilde{\eta}_1 \wedge \widetilde{\eta}_2 \\ \mathrm{d}\widetilde{\eta}_1 &= 0 & \mathrm{d}\widetilde{\eta}_2 &= 0 & \mathrm{d}\widetilde{\eta}_3 &= 0 \end{aligned}$$

$$\therefore \quad \mathrm{d}\eta^a = -\frac{1}{2} R^{abc}\, \widetilde{\eta}_b \wedge \widetilde{\eta}_c \qquad \mathrm{d}\widetilde{\eta}_a = 0$$

$$[X^a, X^b] = R^{abc}\, Z_c \qquad R^{123} = -2$$

- Periodicity of physical coordinates Y^i and dual ones $\widetilde{Y}_i$:

$$
\begin{aligned}
x &\sim x+1 & \qquad (\widetilde{x}, y, z) &\sim (\widetilde{x}+1, y+\widetilde{z}, z-\widetilde{y}) \\
y &\sim y+1 & \qquad (\widetilde{y}, z, x) &\sim (\widetilde{y}+1, z+\widetilde{x}, x-\widetilde{z}) \\
z &\sim z+1 & \qquad (\widetilde{z}, x, y) &\sim (\widetilde{z}+1, x+\widetilde{y}, y-\widetilde{x})
\end{aligned}
$$

Although the naive projection to the base space may seem to yield a flat torus with a trivial B-field, the identifications required on $\widetilde{Y}_i$ have an extreme effect on the field content.

These identifications involve β-transformations related to the shift of a base coordinate Y^i by $\widetilde{Y}_i$.

These identifications entangle the metric and the B-field, without any relation to a geometric action on the base coordinates Y^i, i.e., we obtain a fully nongeometric background.

- Start from scalar moduli matrix
- Introduce dual coordinates induced by B-field
- Extend Lie algebra
- Perform T-duality transformations
- Evaluate geometries

Extend to U-fold endowed with U-duality transformation (hidden symmetry)

Investigate quantum aspects of the doubled sigma model

Discussions

- Find a way to analyze dimensions of moduli spaces
- Find relations between generalized geometry and doubled formalism
- Find application to moduli stabilization, landscape of flux vacua, etc.
- Study non-perturbative effects on generalized/doubled geometries
- Include D-branes (and orientifold planes) into generalized/doubled geometries

C. Albertsson, R.A. Reid-Edwards, TK “D-branes and doubled geometry” to appear...

Appendix

Compactification Ansatz for the ten-dimensional spacetime:

$$\mathcal{M}_{1,9} \;=\; \mathcal{M}_{1,3} \times_{\mathsf{W}} \mathcal{M}_6$$
$$\mathrm{d}s^2_{1,9} \;=\; \mathcal{G}_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N \;=\; \mathrm{e}^{2A(y)} g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \mathcal{G}_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j$$

Maximal symmetry of $\mathcal{M}_{1,3} \to \langle \text{fermions} \rangle = 0$

Supersymmetric vacuum $\leftrightarrow \langle \delta(\text{fermions}) \rangle = 0$

$$\delta \begin{pmatrix} \Psi^1_M \\ \Psi^2_M \end{pmatrix} \;=\; D_M \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix} - \frac{1}{96}\mathrm{e}^{-\phi}\Big(\gamma_M{}^{PQR}\mathcal{H}_{PQR} - 9\gamma^{PQ}\mathcal{H}_{MPQ}\Big)\mathcal{P}\begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix}$$
$$- \sum_n \frac{1}{64 n!}\mathrm{e}^{\frac{5-n}{4}\phi}\Big[(n-1)\gamma_M{}^{N_1\cdots N_n} - n(9-n)\delta_M{}^{N_1}\gamma^{N_2\cdots N_n}\Big]\mathcal{F}_{N_1\cdots N_n}\mathcal{P}_n\begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix}$$

	n	$\mathcal{P}$	$\mathcal{P}_n$
IIA	$0, 2, 4, 6, 8$	γ_{11}	$(\gamma_{11})^{n/2}\sigma^1$
IIB	$1, 5, 9$	$-\sigma^3$	$i\sigma^2$
	$3, 7$		σ^1

Decomposition of Lorentz symmetry:

$$Spin(1,9) \to Spin(1,3) \times Spin(6) = SL(2,\mathbb{C}) \times SU(4)$$
$$\mathbf{16}_1 = (\mathbf{2},\mathbf{4})_1 \oplus (\overline{\mathbf{2}},\overline{\mathbf{4}})_1, \quad \mathbf{16}_2 = (\mathbf{2},\overline{\mathbf{4}})_2 \oplus (\overline{\mathbf{2}},\mathbf{4})_2$$

Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\left\{ \begin{array}{rcl} \epsilon^1_{\mathrm{IIA}} &=& \xi^1_+ \otimes (a\eta^1_+) + \xi^1_- \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\mathrm{IIA}} &=& \xi^2_+ \otimes (\overline{b}\eta^2_-) + \xi^2_- \otimes (b\eta^2_+) \end{array} \right. \qquad \left\{ \begin{array}{rcl} \epsilon^1_{\mathrm{IIB}} &=& \xi^1_+ \otimes (a\eta^1_+) + \xi^1_- \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\mathrm{IIB}} &=& \xi^2_+ \otimes (b\eta^2_+) + \xi^2_- \otimes (\overline{b}\eta^2_-) \end{array} \right.$$

Set $SU(3)$ invariant spinor η^A_+ s.t. $D^{(T)}\eta^A_+ = 0$ $(A = 1,2)$:

spacetime $\mathcal{M}_{1,3}$	compactified space $\mathcal{M}_6$
$\mathcal{N} = 2$: (ξ^1_+, ξ^2_+)	a pair of $SU(3)$ (η^1_+, η^2_+)
$\downarrow$	$\downarrow$
$\mathcal{N} = 1$: $(\xi^1_+ = \xi^2_+ = \xi_+)$	a single $SU(3)$ $(\eta^1_+ = \eta^2_+ = \eta_+)$

back to spinor decompositions

Four-dimensional $\mathcal{N}=1$ Minkowski vacua in type IIA hep-th/0509003

IIA	$a=0$ or $b=0$ (type A)	$a=be^{i\beta}$ (type BC)	
1	$\mathcal{W}_1 = H_3^{(1)} = 0$		
	$F_0^{(1)} = \mp F_2^{(1)} = F_4^{(1)} = \mp F_6^{(1)}$	$F_{2n}^{(1)} = 0$	
8	$\mathcal{W}_2 = F_2^{(8)} = F_4^{(8)} = 0$	generic β	$\beta = 0$
		$\mathrm{Re}\mathcal{W}_2 = \mathrm{e}^{\phi} F_2^{(8)}$ $\mathrm{Im}\mathcal{W}_2 = 0$	$\mathrm{Re}\mathcal{W}_2 = \mathrm{e}^{\phi} F_2^{(8)} + \mathrm{e}^{\phi} F_4^{(8)}$ $\mathrm{Im}\mathcal{W}_2 = 0$
6	$\mathcal{W}_3 = \mp *_6 H_3^{(6)}$	$\mathcal{W}_3 = H_3^{(6)}$	
3	$\overline{\mathcal{W}}_5 = 2\mathcal{W}_4 = \mp 2i H_3^{(3)} = \overline{\partial}\phi$ $\overline{\partial}A = \overline{\partial}a = 0$	$F_2^{(3)} = 2i\overline{\mathcal{W}}_5 = -2i\overline{\partial}A = \frac{2i}{3}\overline{\partial}\phi$ $\mathcal{W}_4 = 0$	

type A	NS-flux only (common to IIA, IIB, heterotic) $\mathcal{W}_1 = \mathcal{W}_2 = 0$, $\mathcal{W}_3 \neq 0$: complex
type BC	RR-flux only $\mathcal{W}_1 = \mathrm{Im}\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$, $\mathrm{Re}\mathcal{W}_2 \neq 0$, $\mathcal{W}_5 \neq 0$: symplectic

For $\mathcal{N}=1$ AdS_4 vacua: hep-th/0403049 hep-th/0407263 hep-th/0412250 hep-th/0502154 hep-th/0609124 , etc..

Four-dimensional $\mathcal{N} = 1$ Minkowski vacua in type IIB hep-th/0509003

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IIB	$a = 0$ or $b = 0$ (type A)	$a = \pm ib$ (type B)	$a = \pm b$ (type C)	(type ABC)
1		$\mathcal{W}_1 = F_3^{(1)} = H_3^{(1)} = 0$		
8		$\mathcal{W}_2 = 0$		
6	$F_3^{(6)} = 0$ $\mathcal{W}_3 = \pm * H_3^{(6)}$	$\mathcal{W}_3 = 0$ $e^{\phi} F_3^{(6)} = \mp * H_3^{(6)}$	$H_3^{(6)} = 0$ $\mathcal{W}_3 = \pm e^{\phi} * F_3^{(6)}$	$(* * *)$
3	$\overline{\mathcal{W}}_5 = 2\mathcal{W}_4 = \mp 2iH_3^{(3)} = 2\bar{\partial}\phi$ $\bar{\partial}A = \bar{\partial}a = 0$	$e^{\phi} F_5^{(3)} = \frac{2i}{3}\overline{\mathcal{W}}_5 = i\mathcal{W}_4$ $= -2i\bar{\partial}A = -4i\bar{\partial}\log a$ $\bar{\partial}\phi = 0$ F: $e^{\phi} F_1^{(3)} = 2e^{\phi} F_5^{(3)}$ $= i\overline{\mathcal{W}}_5 = i\mathcal{W}_4 = i\bar{\partial}\phi$	$e^{\phi} F_3^{(3)} = 2i\overline{\mathcal{W}}_5 = -2i\bar{\partial}A$ $= -4i\bar{\partial}\log a = -i\bar{\partial}\phi$	$(* * *)$

type A	NS-flux only (common to IIA, IIB, heterotic) $\mathrm{d}J \pm iH_3$ is (2,1)-primitive $\mathcal{W}_1 = \mathcal{W}_2 = 0$: complex
type B	both NS- and RR-flux $G_3 = F_3 - ie^{-\phi}H_3 = -i *_6 G_3$ is (2,1)-primitive $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$, $2\mathcal{W}_5 = 3\mathcal{W}_4 = -6\bar{\partial}A$: conformally CY
type C	RR-flux only (S-dual of type A) $\mathrm{d}(e^{-\phi}J) \pm iF_3$ is (2,1)-primitive $\mathcal{W}_1 = \mathcal{W}_2 = 0$: complex
type ABC	(skip detail...)

NS-NS fields in ten-dimensional spacetime are expanded as

$$\begin{aligned}
\phi(x,y) &= \varphi(x) \\
\mathcal{G}_{m\overline{n}}(x,y) &= i v^a(x) (\omega_a)_{m\overline{n}}(y) , \quad \mathcal{G}_{mn}(x,y) \ = \ i \overline{z}^k(x) \left(\frac{(\overline{\chi}_k)_{m\overline{pq}} \Omega^{\overline{pq}}{}_n}{||\Omega||^2} \right) (y) \\
\mathcal{B}_2(x,y) &= B_2(x) + b^a(x) \omega_a(y)
\end{aligned}$$

R-R fields in type IIA are

$$\begin{aligned}
\mathcal{C}_1(x,y) &= C_1^0(x) \\
\mathcal{C}_3(x,y) &= C_1^a(x) \omega_a(y) + \xi^K(x) \alpha_K(y) - \widetilde{\xi}_K(x) \beta^K(y)
\end{aligned}$$

R-R fields in type IIB are

$$\begin{aligned}
\mathcal{C}_0(x,y) &= C_0(x) \\
\mathcal{C}_2(x,y) &= C_2(x) + c^a(x) \omega_a(y) \\
\mathcal{C}_4(x,y) &= V_1^K(x) \alpha_K(y) + \rho_a(x) \widetilde{\omega}^a(y)
\end{aligned}$$

cohomology class	basis	
$H^{(1,1)}$	ω_a	$a = 1, \ldots, h^{(1,1)}$
$H^{(0)} \oplus H^{(1,1)}$	$\omega_A = (1, \omega_a)$	$A = 0, 1, \ldots, h^{(1,1)}$
$H^{(2,2)}$	$\widetilde{\omega}^a$	$a = 1, \ldots, h^{(1,1)}$
$H^{(2,1)}$	χ_k	$k = 1, \ldots, h^{(2,1)}$
$H^{(3)}$	(α_K, β^K)	$K = 0, 1, \ldots, h^{(2,1)}$

Four-dimensional type IIA $\mathcal{N}=2$ ungauged supergravity action of bosonic fields is

$$S^{(4)}_{\rm IIA} = \int_{\mathcal{M}_{1,3}} \Big(-\frac{1}{2} R * \mathbf{1} + \frac{1}{2} {\rm Re}\mathcal{N}_{AB} F^A \wedge F^B + \frac{1}{2} {\rm Im}\mathcal{N}_{AB} F^A \wedge * F^B - G_{a\bar{b}}\, {\rm d}t^a \wedge * {\rm d}\bar{t}^{\bar{b}} - h_{uv}\, {\rm d}q^u \wedge * {\rm d}q^v \Big)$$

gravity multiplet	$(g_{\mu\nu}, C_1^0)$	1
vector multiplet	(C_1^a, v^a, b^a)	$a = 1, \ldots, h^{(1,1)}$
hypermultiplet	$(z^k, \xi^k, \widetilde{\xi}_k)$	$k = 1, \ldots, h^{(2,1)}$
tensor multiplet	$(B_2, \varphi, \xi^0, \widetilde{\xi}_0)$	1

Various functions in the actions:

$$B + iJ = (b^a + iv^a)\,\omega_a = t^a \omega_a \qquad K^{\sf KS} = -\log \Big(\tfrac{4}{3} \int_{\mathcal{M}_6} J \wedge J \wedge J \Big)$$

$$\mathcal{K}_{abc} = \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge \omega_c \qquad \mathcal{K}_{ab} = \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge J = \mathcal{K}_{abc} v^c$$

$$\mathcal{K}_a = \int_{\mathcal{M}_6} \omega_a \wedge J \wedge J = \mathcal{K}_{abc} v^b v^c \qquad \mathcal{K} = \int_{\mathcal{M}_6} J \wedge J \wedge J = \mathcal{K}_{abc} v^a v^b v^c$$

$${\rm Re}\mathcal{N}_{AB} = \begin{pmatrix} -\frac{1}{3}\mathcal{K}_{abc} b^a b^b b^c & \frac{1}{2}\mathcal{K}_{abc} b^b b^c \\ \frac{1}{2}\mathcal{K}_{abc} b^b b^c & -\mathcal{K}_{abc} b^c \end{pmatrix} \qquad {\rm Im}\mathcal{N}_{AB} = -\frac{\mathcal{K}}{6} \begin{pmatrix} 1 + 4G_{ab} b^a b^b & -4G_{ab} b^b \\ -4G_{ab} b^b & 4G_{ab} \end{pmatrix}$$

$$G_{a\bar{b}} = \frac{3}{2} \frac{\int \omega_a \wedge * \omega_b}{\int J \wedge J \wedge J} = \partial_{t^a} \overline{\partial}_{\bar{t}^{\bar{b}}} K^{\sf KS}$$

Four-dimensional type IIB $\mathcal{N}=2$ ungauged supergravity action of bosonic fields is

$$S^{(4)}_{\text{IIB}} = \int_{\mathcal{M}_{1,3}} \Big(-\frac{1}{2} R * \mathbf{1} + \frac{1}{2} \mathrm{Re}\mathcal{M}_{KL} F^K \wedge F^L + \frac{1}{2} \mathrm{Im}\mathcal{M}_{KL} F^K \wedge * F^L - G_{k\bar{l}}\, \mathrm{d}z^k \wedge * \mathrm{d}\bar{z}^{\bar{l}} - h_{pq}\, \mathrm{d}\widetilde{q}^p \wedge * \mathrm{d}\widetilde{q}^q \Big)$$

gravity multiplet	$(g_{\mu\nu}, V_1^0)$	1
vector multiplet	(V_1^k, z^k)	$k = 1, \ldots, h^{(2,1)}$
hypermultiplet	(v^a, b^a, c^a, ρ_a)	$a = 1, \ldots, h^{(1,1)}$
tensor multiplet	(B_2, C_2, φ, C_0)	1

Various functions in the actions:

$$\Omega = \mathcal{Z}^K \alpha_K - \mathcal{F}_K \beta^K \qquad z^k = \mathcal{Z}^K / \mathcal{Z}^0 \qquad \mathcal{F}_{KL} = \partial_L \mathcal{F}_K$$

$$K^{\mathsf{CS}} = -\log \Big(i \int_{\mathcal{M}_6} \Omega \wedge \overline{\Omega} \Big) \qquad G_{k\bar{l}} = - \frac{\int \chi_k \wedge \overline{\chi}_{\bar{l}}}{\int \Omega \wedge \overline{\Omega}} = \partial_{z^k} \overline{\partial}_{\bar{z}^{\bar{l}}} K^{\mathsf{CS}}$$

$$\mathcal{M}_{KL} = \overline{\mathcal{F}}_{KL} + 2i \frac{(\mathrm{Im}\mathcal{F})_{KM} \mathcal{Z}^M (\mathrm{Im}\mathcal{F})_{LN} \mathcal{Z}^N}{\mathcal{Z}^N (\mathrm{Im}\mathcal{F})_{NM} \mathcal{Z}^M}$$

back to CY Moduli

- on a single $SU(3)$:

a real two-form	$J_{ij} = \mp 2i\, \eta_\pm^\dagger \gamma_{ij}\, \eta_\pm$
a complex three-form	$\Omega_{ijk} = -2i\, \eta_-^\dagger \gamma_{ijk}\, \eta_+$

- on a pair of $SU(3)$:

two real vectors	$(v - iv')^i = \eta_+^{1\dagger} \gamma^i\, \eta_-^2$
(J^A, Ω^A)	$J^1 = j + v \wedge v' \quad \Omega^1 = w \wedge (v + iv')$ $J^2 = j - v \wedge v' \quad \Omega^1 = w \wedge (v - iv')$ (j, w): local $SU(2)$-invariant forms

If $\eta_+^1 \neq \eta_+^2$ globally, a pair of $SU(3)$ is reduced to global single $SU(2)$ w/ (j, w, v, v')

If $\eta_+^1 = \eta_+^2$ globally, a pair of $SU(3)$ is reduced to a single global $SU(3)$ w/ (J, Ω)

$$\eta_+^2 \;=\; c_\parallel \eta_+^1 + c_\perp (v + iv')^i \gamma_i\, \eta_-^1 \qquad\qquad |c_\parallel|^2 + |c_\perp|^2 = 1$$

cf.) a pair of $SU(3)$ on $T\mathcal{M}_6$ $\sim$ an $SU(3) \times SU(3)$ on $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$

back to pure spinors