Matrix models and the gravitational interaction

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Contents

1 Introduction

Large-N reduced models are the most powerful candidates for the constructive definition of superstring theory.

IIB matrix model

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

$$S = -\frac{1}{g^2} Tr_{N\times N} \left(\frac{1}{4} \sum_{\mu,\nu=0}^{9} [A_\mu, A_\nu]^2 + \frac{1}{2} \bar{\psi} \sum_{\mu=0}^{9} \Gamma^\mu [A_\mu, \psi] \right),$$

- Dimensional reduction of $\mathcal{N}=1$ 10-dimensional SYM theory to 0 dimension.
  A_μ and ψ are $N \times N$ Hermitian matrices.
  - ⋆ A_μ: 10-dimensional vectors
  - ⋆ ψ: 10-dimensional Majorana-Weyl (i.e. 16-component) spinors
- Matrix regularization of the Schild form of the Green-Schwarz action of the type IIB superstring theory.
- $SU(N)$ gauge symmetry and $SO(9,1)$ Lorentz symmetry $(SO(9,1) \times SU(N))$.
- The $N \times N$ matrices describe the many-body system.
- No free parameter: $A_\mu \to g^{\frac{1}{2}} A_\mu$, $\psi \to g^{\frac{3}{2}} \psi$.

- $\mathcal{N}=2$ **SUSY**:
  This theory must contain spin-2 gravitons if it contains massless particles.

  - ⋆ **homogeneous :** $\delta^{(1)}_{\epsilon} A_\mu = i\bar{\epsilon}\Gamma_\mu \psi, \quad \delta^{(1)}_{\epsilon}\psi = \frac{i}{2}\Gamma^{\mu\nu}[A_\mu, A_\nu]\epsilon.$
  - ⋆ **inhomogeneous :** $\delta^{(2)}_{\xi} A_\mu = 0, \quad \delta^{(2)}_{\xi}\psi = \xi.$
  - ⋆ **We obtain the following commutation relations:**

$$
\begin{aligned}
&(1) \quad [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]A_\mu = [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]\psi = 0,\\
&(2) \quad [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]A_\mu = [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]\psi = 0,\\
&(3) \quad [\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]A_\mu = -i\bar{\epsilon}\Gamma_\mu\xi, \quad [\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]\psi = 0.
\end{aligned}
$$

  This gives a shift of the bosonic variables for $\tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)},\ \tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)})$: $(\alpha, \beta = 1, 2)$

$$
\boxed{
\begin{aligned}
&[\tilde{\delta}^{(\alpha)}_{\epsilon}, \tilde{\delta}^{(\beta)}_{\xi}]\psi = 0,\\
&[\tilde{\delta}^{(\alpha)}_{\epsilon}, \tilde{\delta}^{(\beta)}_{\xi}]A_\mu = -2i\delta^{\alpha\beta}\bar{\epsilon}\Gamma_\mu\xi.
\end{aligned}
}
$$

  This leads us to interpret the eigenvalues of A_μ as the spacetime coordinate.

2 Matrix model based on the $osp(1|32, R)$ super Lie algebra

$osp(1|32, R)$: first mentioned with the relation to the 11-dimensional supergravity.

E. Cremmer, B. Julia, J. Scherk, Phys.Lett.B76:409-412,1978.

$\Rightarrow$ This has attracted a new attention as the unified super Lie algebra for the M-theory.

E. Bergshoeff, A. Van Proeyen, hep-th/0003261

The matrix model based on $osp(1|32, R)$ is a natural extension to IIB matrix model.

L. Smolin, hep-th/0002009,0006137
T. Azuma, S. Iso, H. Kawai and Y. Ohwashi, hep-th/0102168

M. Bagnoud, L, Carlevaro and A. Bilal, hep-th/0201183

- Both the bosonic and the fermionic fields are embedded in one multiplet.
- Extra fields:
  The rank-1(3) fields may be identified with the bosonic vector (spin connection).

  Local Lorentz invariant matrix model
- Relation of the supersymmetry

Definition of $osp(1|32, R)$ super Lie algebra

$$M \in osp(1|32,R) \overset{\text{def}}{\Rightarrow} {}^{T}MG + GM = 0, \text{ where}$$
$$G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}.$$

⇓

$$M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}, \text{ where } {}^{T}m\Gamma^0 + \Gamma^0 m = 0; \text{ i.e. } m \in sp(32),$$
$$m = u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}.$$

2.1 Cubic nongauged model

We start from the cubic action of the supermatrix model.

$$\begin{aligned} S &= \frac{i}{g^2}Tr_{N\times N}\sum_{Q,R=1}^{33}[(\sum_{p=1}^{32} M_p{}^Q[M_Q{}^R, M_R{}^p]) - M_{33}{}^Q[M_Q{}^R, M_R{}^{33}]] \\ &= -\frac{f^{abc}}{2g^2}\sum_{a,b,c=1}^{N^2} Str_{33\times 33}(M_aM_bM_c) \\ &= \frac{i}{g^2}Tr_{N\times N}[m_p{}^q[m_q{}^r, m_r{}^p] - 3i\bar{\psi}^p[m_p{}^q, \psi^q]]. \end{aligned}$$

- Each component of the 33×33 supermatrices is promoted to a large N hermitian matrix.
- $OSp(1|32, R)$ and $U(N)$ symmetries are separated.
  - $*$ $M \to M + [M, (S \otimes 1_{N\times N})]$ for $S \in osp(1|32, R)$,
  - $*$ $M \to M + [M, (1_{33\times 33} \otimes U)]$ for $U \in u(N)$.

Supersymmetry

The SUSY transformation of the $osp(1|32,R)$ is identified with that of the IIB matrix model.

- homogeneous SUSY:
  The SUSY transformation by the supercharge $Q = \begin{pmatrix} 0 & \chi \\ i\bar{\chi} & 0 \end{pmatrix}$.

$$\delta^{(1)}_{\chi} M = [Q, M] = \begin{pmatrix} i(\chi\bar{\psi} - \psi\bar{\chi}) & -m\chi \\ i\bar{\chi}m & 0 \end{pmatrix}.$$

- inhomogeneous SUSY:
  The translation of the fermionic field $\delta^{(2)}_{\epsilon}\psi = \epsilon$.

In order to see the correspondence of the fields with the IIB matrix model, we express the bosonic 32×32 matrices in terms of the 10-dimensional indices ($\mu = 0, \cdots 9$, $\sharp = 10$).

$$\begin{aligned} m \;=\; & W\Gamma^{\sharp} + \frac{1}{2}[A^{(+)}_{\mu}\Gamma^{\mu}(1+\Gamma^{\sharp}) + A^{(-)}_{\mu}\Gamma^{\mu}(1-\Gamma^{\sharp})] + \frac{1}{2!}C_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2} + \\ & + \frac{1}{4!}H_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4\sharp} + \frac{1}{5!}[I^{(+)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}(1+\Gamma^{\sharp}) + I^{(-)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}(1-\Gamma^{\sharp})]. \end{aligned}$$

Identification of the fields

$$\begin{aligned} &\delta^{(1)}_{\chi} A^{(+)}_{\mu} = \frac{i}{16}\bar{\chi}\Gamma_{\mu}(1-\Gamma_{\sharp})\psi = \frac{i}{8}\bar{\chi}_R\Gamma_{\mu}\psi_R, \\ &\delta^{(1)}_{\chi} A^{(-)}_{\mu} = \frac{i}{16}\bar{\chi}\Gamma_{\mu}(1+\Gamma_{\sharp})\psi = \frac{i}{8}\bar{\chi}_L\Gamma_{\mu}\psi_L, \\ &\delta^{(1)}_{\chi}\psi = -m\psi. \end{aligned}$$

Commutation relations

- $[\delta^{(1)}_{\chi}, \delta^{(2)}_{\epsilon}]m = -i(\chi\bar{\epsilon} - \epsilon\bar{\chi}), \quad [\delta^{(1)}_{\chi}, \delta^{(2)}_{\epsilon}]\psi = 0.$

$$[\delta^{(1)}_{\chi_R}, \delta^{(2)}_{\epsilon_R}]A^{(+)}_{\mu} = \frac{i}{8}\bar{\epsilon}_R\Gamma_{\mu}\chi_R, \quad [\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_L}]A^{(+)}_{\mu} = 0,$$
$$[\delta^{(1)}_{\chi_R}, \delta^{(2)}_{\epsilon_R}]A^{(-)}_{\mu} = 0, \qquad [\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_L}]A^{(-)}_{\mu} = \frac{i}{8}\bar{\epsilon}_L\Gamma_{\mu}\chi_L,$$
$$[\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_R}]A^{(\pm)}_{\mu} = [\delta^{(1)}_{\chi_R}, \delta^{(2)}_{\epsilon_L}]A^{(\pm)}_{\mu} = 0.$$

- $[\delta^{(2)}_{\chi}, \delta^{(2)}_{\epsilon}]m = [\delta^{(2)}_{\chi}, \delta^{(2)}_{\epsilon}]\psi = 0$ is trivial.

- $[\delta^{(1)}_{\chi}, \delta^{(1)}_{\epsilon}]m = i[\chi\bar{\epsilon} - \epsilon\bar{\chi}, m], \quad [\delta^{(1)}_{\chi}, \delta^{(1)}_{\epsilon}]\psi = i(\chi\bar{\epsilon} - \epsilon\bar{\chi})\psi.$

  ⋆ $[\delta^{(1)}_{\chi_R}, \delta^{(1)}_{\epsilon_R}]A^{(+)}_{\mu} = \frac{i}{8}\bar{\chi}_R[m, \Gamma_{\mu}]\epsilon_R.$
  In the (r.h.s.), the fields W, $C_{\mu_1\mu_2}$ and $H_{\mu_1\cdots\mu_4}$ survive.
  $\rightarrow$ these fields are integrated out.

  ⋆ $[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]A^{(+)}_{\mu} = -\frac{i}{8}\bar{\chi}_L A^{(+)}_{\nu}\Gamma_{\mu}{}^{\nu}\epsilon_R + \cdots.$
  The fields $A^{(\pm)}_{\mu}$ itself remains in the commutator!

Summary

The $osp(1|32, R)$ cubic matrix model possesses a two-fold structure of the SUSY of the IIB matrix model.

IIB matrix model	bosons A_{μ}	fermions ψ	SUSY parameters
SUSY I	$A^{(+)}_{\mu}$	ψ_R	χ_R, ϵ_R
SUSY II	$A^{(-)}_{\mu}$	ψ_L	χ_L, ϵ_L

2.2 Gauged matrix model for the local Lorentz invariance

Smolin's proposal for the "gauged" model turns out to be essential for the local Lorentz invariant matrix model.

The symmetry of IIB matrix model:

$SO(9,1)$ and $U(N)$ symmetry is decoupled.
The $SO(9,1) \times U(N)$ symmetry is a tensor product of the group. For $\zeta \in so(9,1)$ and $u \in u(N)$,

$$\exp(\zeta \otimes 1 + 1 \otimes u) = e^{\zeta} \otimes e^{u}.$$

The spacetime coordinate is embedded in the eigenvalues of the large N matrices.

$\Rightarrow$ If we are to formulate a matrix model with local Lorentz invariance, the $so(9,1)$ Lorentz symmetry and the $u(N)$ gauge symmetry must be unified.

(*) $\mathcal{A}, \mathcal{B}$ = [The Lie algebras whose bases are $\{a_i\}$ and $\{b_j\}$, respectively.]

- $\mathcal{A} \otimes \mathcal{B}$: The space spanned by the basis $a_i \otimes b_j$. This is not necessarily a closed Lie algebra.
- $\mathcal{A} \check{\otimes} \mathcal{B}$: The smallest Lie algebra that includes $\mathcal{A} \otimes \mathcal{B}$ as a subset.

The gauge group must close with respect to the commutator

$$[a \otimes A, b \otimes B] = \frac{1}{2} \left([a,b] \otimes \{A,B\} + \{a,b\} \otimes [A,B] \right).$$

We identify infinitely large N matrices with differential operator.

The information of spacetime can be embedded to matrices in various ways.

- Twisted Eguchi-Kawai(TEK) model:

  A. Gonzalez-Arroyo and M. Okawa, Phys. Rev. D 27, 2397 (1983).

$$A_\mu \sim \partial_\mu + a_\mu.$$

  The matrices A_μ represent the covariant derivative on the spacetime.

- The IIB matrix model:

$$A_\mu \sim X_\mu.$$

  A_i itself represent the space-time coordinate.

The IIB matrix model with noncommutative background

$$[\hat{p}_\mu, \hat{p}_\nu] = iB_{\mu\nu}, (B_{\mu\nu} = \text{c-numbers})$$

interpolates these two pictures.

H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, hep-th/9908141

$-Tr_{N\times N}\bar{\psi}\Gamma^\mu[A_\mu, \psi]$ reduces to the fermionic action $\int d^dx\bar{\psi}(x)\Gamma^\mu(\partial_\mu\psi(x) + [a_\mu(x), \psi(x)])$ in the flat space in the low-energy limit.

Naive promotion to the matrix model

The local Lorentz invariant action of the fermion in the curved space:

$$\begin{aligned} S_F &= \int d^dx e(x)\bar{\psi}(x) i\Gamma^\mu e_\mu{}^i(x)\left(\partial_i\psi(x) + [A_i(x),\psi(x)] + \frac{1}{4}\Gamma^{\nu\rho}\omega_{i\nu\rho}(x)\psi(x)\right) \\ &\quad \Downarrow \text{(absorb the determinant } e(x) \text{ into the definition} \\ &\qquad \text{of } \Psi(x) \text{ as } \Psi(x) = e^{\frac{1}{2}}(x)\psi(x)\text{)} \\ &= \int d^dx \bar{\Psi}(x) i\Gamma^\mu \left(e_\mu{}^i(x)(\partial_i\Psi(x) + [A_i(x),\Psi(x)]) + \frac{1}{2}e_\nu{}^i(x)\omega_i{}^\nu{}_\mu(x)\Psi(x) \right. \\ &\qquad \left. + e_\mu{}^i(x)e^{\frac{1}{2}}(x)(\partial_i e^{-\frac{1}{2}}(x))\Psi(x)\right) \\ &\quad + \frac{i}{4}\bar{\Psi}(x)\Gamma^{\mu_1\mu_2\mu_3}e_{[\mu_1}{}^i(x)\omega_{i\mu_2\mu_3]}(x)\Psi(x). \\ &\quad \Downarrow \text{(use } \bar{\Psi}(x)\Gamma^\mu\Psi(x) = 0\text{)} \\ &= \int d^dx \bar{\Psi}(x) i\Gamma^\mu (e_\mu{}^i(x)(\partial_i\Psi(x) + [A_i(x),\Psi(x)])) \\ &\quad + \frac{i}{4}\bar{\Psi}(x)\Gamma^{\mu_1\mu_2\mu_3}e_{[\mu_1}{}^i(x)\omega_{i\mu_2\mu_3]}(x)\Psi(x). \end{aligned}$$

The corresponding promotion to a matrix model is

$$S'_F = -\frac{1}{2}Tr\bar{\psi}\Gamma^\mu[A_\mu,\psi] - \frac{i}{2}Tr\bar{\psi}\Gamma^{\mu_1\mu_2\mu_3}\{A_{\mu_1\mu_2\mu_3},\psi\}.$$

When ψ is a Majorana fermion, the following equality holds:

$$S'_F = S''_F = -Tr\bar{\psi}\left(\Gamma^\mu A_\mu + i\Gamma^{\mu_1\mu_2\mu_3}A_{\mu_1\mu_2\mu_3}\right)\psi.$$

It is extremely difficult to retain the hermiticity in the fermion ψ.

Originally, the local Lorentz transformation should be promoted as

$$\delta\psi = \frac{1}{4}\Gamma^{\mu_1\mu_2}\underbrace{\{\varepsilon_{\mu_1\mu_2}, \psi\}}_{\text{hermitian}}.$$

However, it is difficult to build an action invariant under this transformation.

The transformation of the action S''_F entails the term

$$\delta S''_F = -\frac{i}{4} Tr\bar{\psi}\Gamma^{\mu}A_{\mu}\Gamma^{\nu_1\nu_2}\psi\varepsilon_{\nu_1\nu_2} + \cdots.$$

This leads us to abstain the hermiticity of ψ and take the local Lorentz transformation to be

$$\delta\psi = \frac{1}{4}\Gamma^{\mu_1\mu_2}\varepsilon_{\mu_1\mu_2}\psi,$$

instead of $\delta\psi = \frac{1}{4}\Gamma^{\mu_1\mu_2}\{\varepsilon_{\mu_1\mu_2}, \psi\}$.

$$\delta S''_F = -\frac{1}{4} Tr\bar{\psi}[\Gamma^{\mu}A_{\mu} + i\Gamma^{\mu_1\mu_2\mu_3}A_{\mu_1\mu_2\mu_3}, \Gamma^{\nu_1\nu_2}\varepsilon_{\nu_1\nu_2}]\psi.$$

However, this action does not close with respect to the local Lorentz transformation:

$$\begin{aligned}&[\Gamma^{\mu_1\mu_2\mu_3}A_{\mu_1\mu_2\mu_3}, \Gamma^{\nu_1\nu_2}\varepsilon_{\nu_1\nu_2}]\\ =\ &\frac{1}{2}\underbrace{[\Gamma^{\mu_1\mu_2\mu_3}, \Gamma^{\nu_1\nu_2}]}_{\text{rank 3}}\{A_{\mu_1\mu_2\mu_3}, \varepsilon_{\nu_1\nu_2}\} + \frac{1}{2}\underbrace{\{\Gamma^{\mu_1\mu_2\mu_3}, \Gamma^{\nu_1\nu_2}\}}_{\text{rank 1, 5}}[A_{\mu_1\mu_2\mu_3}, \varepsilon_{\nu_1\nu_2}].\end{aligned}$$

We need the terms of all odd ranks in order to formulate a local Lorentz invariant matrix model.

We investigate the gauged matrix model in this point of view.

$u(1|16,16)$ super Lie algebra

- $M \in u(1|16,16) \Rightarrow M^{\dagger}G + GM = 0$, where $G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}$.
- $M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & v \end{pmatrix}$, where $m^{\dagger}\Gamma^0 + \Gamma^0 m = 0$.
- $m = u1 + u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{3!}u_{A_1A_2A_3}\Gamma^{A_1A_2A_3}$ $+\frac{1}{4!}u_{A_1\cdots A_4}\Gamma^{A_1\cdots A_4} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}$.
- $\begin{cases} u_{A_1}, u_{A_1A_2}, u_{A_1\cdots A_5} & \Rightarrow \text{real number} \\ v, u, u_{A_1A_2A_3}, u_{A_1\cdots u_{A_4}} & \Rightarrow \text{pure imaginary} \end{cases}$

$u(1|16,16)$ is the direct sum of the two different representations of $osp(1|32,R)$.

♣ $u(1|16,16) = \mathcal{H} \oplus \mathcal{A}'$, where

$$\mathcal{H} = \{M = \begin{pmatrix} m_h & \psi_h \\ i\bar{\psi}_h & 0 \end{pmatrix} | m_h = u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}, \quad u_{A_1}, u_{A_1A_2}, u_{A_1\cdots A_5}, \psi_h \in \mathcal{R}\},$$

$$\mathcal{A}' = \{M = \begin{pmatrix} m_a & i\psi_a \\ \bar{\psi}_a & iv \end{pmatrix} | m_a = u + \frac{1}{3!}u_{A_1A_2A_3}\Gamma^{A_1A_2A_3} + \frac{1}{4!}u_{A_1\cdots A_4}\Gamma^{A_1\cdots A_4}, \quad u, u_{A_1A_2A_3}, u_{A_1\cdots A_4}, i\psi_a, iv \in (\text{pure imaginary})\}.$$

$gl(1|32,R)$ super Lie algebra

- $M \in gl(1|32,R) \Rightarrow M = \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix}$
- $m = u1 + u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{3!}u_{A_1A_2A_3}\Gamma^{A_1A_2A_3}$ $+\frac{1}{4!}u_{A_1\cdots A_4}\Gamma^{A_1\cdots A_4} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}$.
- $u, \cdots u_{A_1\cdots A_5}, \psi, \phi, v$ are all real numbers.

$gl(1|32,R)$ is the analytical continuation of $u(1|16,16)$, in that

♣ $gl(1|32,R) = \mathcal{H} \oplus \mathcal{A}$, where $\mathcal{A}' = i\mathcal{A}$.

action of the cubic model

$$
\begin{aligned}
I &= \frac{1}{g^2} Tr_{N\times N} \sum_{Q,R=1}^{33} [(\sum_{p=1}^{32} M_p{}^Q M_Q{}^R M_R{}^p) - M_{33}{}^Q M_Q{}^R M_R{}^{33}] \\
&= \frac{1}{g^2} \sum_{a,b,c=1}^{N^2} Str_{33\times 33}(M_a M_b M_c) Tr_{N\times N}(T^a T^b T^c) \\
&= \frac{1}{g^2} Tr_{N\times N}[m_p{}^q m_q{}^r m_r{}^p - 3i\bar{\phi}^p m_p{}^q \psi^q - 3iv\bar{\phi}^p \psi_p - v^3].
\end{aligned}
$$

- Each component of the 33×33 supermatrices is promoted to a large N real matrix.
- No free parameter: $M \to g^{\frac{2}{3}} M$.
- $gl(1|32, R) \otimes gl(N)$ gauge symmetry.

$$
M \to M + [M, (S \otimes U)]
$$

for $S \in osp(1|32, R)$ and $U \in u(N)$.

- The bosonic 32×32 matrices are separated into m_e and m_o in terms of 10-dimensional indices.

$$
\begin{aligned}
m_e &= Z + W\Gamma^\sharp + \frac{1}{2!}(C_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2} + D_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2\sharp}) + \frac{1}{4!}(G_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4} + H_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4\sharp}), \\
m_o &= \frac{1}{2}(A_\mu^{(+)}\Gamma^\mu(1+\Gamma^\sharp) + A_\mu^{(-)}\Gamma^\mu(1-\Gamma^\sharp)) \\
&+ \frac{1}{2\times 3!}(E^{(+)}_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3}(1+\Gamma^\sharp) + E^{(-)}_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3}(1-\Gamma^\sharp)) \\
&+ \frac{1}{5!}(I^{(+)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}(1+\Gamma^\sharp) + I^{(-)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}(1-\Gamma^\sharp)).
\end{aligned}
$$

Wigner Inönü contraction

We consider the hyperboloid in the AdS space whose radius R is sufficiently large. The hyperboloid is approximated by the $R^{9,1}$ flat plane at the "north pole".

AdS space: $x^A x^B \eta_{AB} = -R^2$, with $\eta_{AB} = diag(-1, 1, \cdots, 1, -1)$.

To this end, we alter the action as

$$I = \frac{1}{3} Tr(Str M_t^3) - R^2 Tr(Str M_t).$$

The EOM $\frac{\partial I}{\partial M_t} = M_t^2 - R^2 1_{33\times 33} = 0$ possesses a classical solution $\langle M \rangle = \begin{pmatrix} R\Gamma^\sharp \otimes 1_{N\times N} & 0 \\ 0 & R \otimes 1_{N\times N} \end{pmatrix}$.

$$\begin{aligned} M_t &= (\text{classical solution } \langle M \rangle) \quad + (\text{fluctuation } M) \\ &= \begin{pmatrix} R\Gamma^\sharp \otimes 1_{N\times N} & 0 \\ 0 & R \otimes 1_{N\times N} \end{pmatrix} + \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix}. \end{aligned}$$

The action is expressed in terms of the fluctuation as

$$\begin{aligned} I &= R(tr(m_e^2 \Gamma^\sharp) - v^2 - 2i\bar{\phi}_R \psi_L) + (\frac{1}{3} m_e^3 + tr(m_e m_o^2)) \\ &- i(\bar{\phi}_R(m_e + v)\psi_L + \bar{\phi}_L(m_e + v)\psi_R + \bar{\phi}_L m_o \psi_L + \bar{\phi}_R m_o \psi_R) - \frac{1}{3} v^3. \end{aligned}$$

The fluctuation is rescaled as

- $m_t = R\Gamma^\sharp + m = R\Gamma^\sharp + R^{-\frac{1}{2}} m'_e + R^{\frac{1}{4}} m'_o$,
- $v_t = R + v = R + R^{-\frac{1}{2}} v'$,
- $\psi = \psi_L + \psi_R = R^{-\frac{1}{2}} \psi'_L + R^{\frac{1}{4}} \psi'_R$,
- $\bar{\phi} = \bar{\phi}_L + \bar{\phi}_R = R^{\frac{1}{4}} \bar{\phi}'_L + R^{-\frac{1}{2}} \bar{\phi}'_R$.

We obtain the vanishing effective action by integrating out m'_e, ψ'_L, $\bar{\phi}'_R$ and v' .

$$e^{-W} = \int dm'_e d\psi'_L d\bar{\phi}'_R dv e^{-I},$$

$$\Rightarrow \; W = -\frac{1}{4} tr(\Gamma^\sharp \{m'^2_o + i(\psi'_R \bar{\phi}'_L)\}^2) - \frac{1}{4}(\bar{\phi}'_L \psi_R)^2 + \frac{i}{2}(\bar{\phi}'_L m'^2_o \psi'_R) = 0.$$

2.3 Realization of the curved-space background in the supermatrix model

The curved-space background is a fundamental feature of the general relativity.

It is an important question how we realize the physics of the curved-space background via the matrix model.

Classical equation of motion of IIB matrix model:

$$[A^{\nu}, [A_{\mu}, A_{\nu}]] = 0.$$

This has only a flat non-commutative background as a classical solution.

$$[A_{\mu}, A_{\nu}] = ic_{\mu\nu}1_{N\times N}.$$

$\Rightarrow$ In order to surmount this difficulty, we alter a model so that it incorporates a curved-space classical solution ab initio.

To this end, there are several generalizations of the IIB matrix model to accommodate the curved-space classical solution.

Matrix model with the Chern-Simons term

$$S = \frac{1}{g^2} Tr \left(-\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho} A^\mu A^\nu A^\rho + \frac{1}{2}\bar{\psi}\sigma^\mu [A_\mu, \psi] \right),$$

defined in the 3-dimensional Euclidean space $(\mu, \nu, \rho, \cdots = 1,2,3)$.

Its classical equation of motion

$$[A_\mu, [A_\mu, A_\nu]] + i\alpha\epsilon_{\nu\rho\chi}[A_\rho, A_\chi] = 0$$

accommodates the S^2 fuzzy sphere solution

$$A_\mu = \alpha J_\mu, \text{ where } [J_\mu, J_\nu] = i\epsilon_{\mu\nu\rho} J_\rho.$$

J_μ is an $N \times N$ irreducible representation of the $SU(2)$ Lie algebra.

The radius of the fuzzy sphere is given by the Casimir:

$$Q = A_1^2 + A_2^2 + A_3^2 = R^2 1_{N\times N}, \text{ where } R^2 = \alpha^2 \frac{N^2 - 1}{4}.$$

Matrix model with the tachyonic mass term

$$S = -\frac{1}{g^2}\left(\frac{1}{4}\sum_{\mu,\nu=1}^{3} Tr[A_\mu, A_\nu]^2 + \lambda^2 \sum_{\mu=1}^{3} Tr A_\mu^2 \right).$$

EOM: $[A_\nu, [A_\mu, A_\nu]] + 2\lambda^2 A_\mu = 0.$

This equation of motion also accommodates the fuzzy sphere

$$A_\mu = \lambda J_\mu.$$

It is interesting to consider a similar problem in the supermatrix model.

action of the massive supermatrix model

We add a mass term to the pure (nongauged) cubic action:

$$
\begin{aligned}
S &= Tr\left[str\left(3\mu M^2 - \frac{i}{g^2}M[M,M]\right)\right] \\
&= Tr\left[3\mu\left\{\left(\sum_{p=1}^{32} M_p{}^Q M_Q{}^p\right) - M_{33}{}^Q M_Q{}^{33}\right\}\right. \\
&\qquad \left. -\frac{i}{g^2}\left\{\left(\sum_{p=1}^{32} M_p{}^Q[M_Q{}^R, M_R{}^p]\right) - M_{33}{}^Q[M_Q{}^R, M_R{}^{33}]\right\}\right], \\
&= Tr\left[3\mu(tr(m^2) - 2i\bar{\psi}\psi) - \frac{i}{g^2}\left(m_p{}^q[m_q{}^r, m_r{}^p] - 3i\bar{\psi}^p[m_p{}^q, \psi^q]\right)\right].
\end{aligned}
$$

In order to see the correspondence of the fields with IIB matrix model, we express the bosonic 32×32 matrices in terms of the 10-dimensional indices.
$(\mu, \nu, \cdots = 0, 1, \cdots, 9, \sharp = 10)$.

$$W = m_{\sharp}, \quad A_{\mu} = m_{\mu}, \quad B_{\mu} = m_{\mu\sharp}, \quad C_{\mu_1\mu_2} = m_{\mu_1\mu_2},$$
$$H_{\mu_1\cdots\mu_4} = m_{\mu_1\cdots\mu_4\sharp}, \quad Z_{\mu_1\cdots\mu_5} = m_{\mu_1\cdots\mu_5}.$$

Then, the action is decomposed as

$$\begin{aligned} S &= 96\mu Tr\left(+W^2 + A_{\mu}A^{\mu} - B_{\mu}B^{\mu} - \frac{1}{2}C_{\mu_1\mu_2}C^{\mu_1\mu_2} + \frac{1}{4!}H_{\mu_1\cdots\mu_4}H^{\mu_1\cdots\mu_4} \right. \\ &\qquad \left. + \frac{1}{5!}Z_{\mu_1\cdots\mu_5}Z^{\mu_1\cdots\mu_5} + \frac{i}{16}\bar{\psi}\psi\right) \\ &- 32iTr\left(3C_{\mu_1\mu_2}[B^{\mu_1}, B^{\mu_2}] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3}, C^{\mu_3\mu_1}] + \cdots\right) \end{aligned}$$

- The rank-1 and rank-5 fields has a stable trivial commutative classical solution:

$$W = A_{\mu} = H_{\mu_1\cdots\mu_4} = Z_{\mu_1\cdots\mu_5} = 0.$$

- For the rank-2 tachyonic fields $B_{\mu}, C_{\mu_1\mu_2}$, the trivial solution $B_{\mu} = C_{\mu_1\mu_2} = 0$ is unstable.
$\Rightarrow$ They may incorporate an interesting stable non-commutative solution!

From now on, we set the fermions and the positive-mass bosonic fields to zero:

$$\boxed{\begin{aligned} S &= -96\mu Tr\left(B_\mu B^\mu + \frac{1}{2}C_{\mu_1\mu_2}C^{\mu_1\mu_2}\right) \\ &- 32iTr\left(3C_{\mu_1\mu_2}[B^{\mu_1}, B^{\mu_2}] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3}, C^{\mu_3\mu_1}]\right). \end{aligned}}$$

The equations of motion:

$$\begin{aligned} B_\mu &= -i\mu^{-1}[B^\nu, C_{\mu\nu}], \\ C_{\mu_1\mu_2} &= -i\mu^{-1}([B_{\mu_1}, B_{\mu_2}] + [C_{\mu_1}{}^\rho, C_{\mu_2\rho}]). \end{aligned}$$

We integrate out the rank-2 fields (in 10 dimensions) $C_{\mu_1\mu_2}$ by solving the latter equation of motions:

$$\begin{aligned} C_{\mu_1\mu_2} &= -i\mu^{-1}([B_{\mu_1}, B_{\mu_2}] + \underbrace{[C_{\mu_1}{}^\rho, C_{\mu_2\rho}]}_{=(-i\mu^{-1})^2[[B_{\mu_1},B^\rho]+[C_{\mu_1\chi_1},C^{\rho\chi_1}],[B_{\mu_2},B_\rho]+[C_{\mu_2\chi_2},C_\rho{}^{\chi_2}]]}) + \cdots \\ &= -\underbrace{i\mu^{-1}[B_{\mu_1}, B_{\mu_2}]}_{\mathcal{O}(B^2)\text{ with 1 commutator}} + \underbrace{i\mu^{-3}[[B_{\mu_1}, B_\rho], [B_{\mu_2}, B^\rho]]}_{\mathcal{O}(B^4)\text{ with 3 commutators}} + \mathcal{O}(\mu^{-5}). \end{aligned}$$

Then, the action reduces to

$$\begin{aligned} S = Tr\Big(&-96\mu B_\mu B^\mu - 48\mu^{-1}[B_{\mu_1}, B_{\mu_2}][B^{\mu_1}, B^{\mu_2}] \\ &+ (\text{higher-order commutators of the order } \mathcal{O}(\mu^{-2k+1}) \text{ with } k = 2, 3, \cdots)\Big). \end{aligned}$$

Fuzzy-sphere classical solution

1. $S^2 \times S^2 \times S^2$ fuzzy spheres

   This describes a space formed by the Cartesian product of three fuzzy spheres.

$$\begin{aligned}
&[B_i, B_j] = i\mu r \epsilon_{ijk} B_k, \qquad B_1^2 + B_2^2 + B_3^2 = \mu^2 r^2 \frac{N^2-1}{4}, \quad (i,j,k = 1,2,3)\\
&[B_{i'}, B_{j'}] = i\mu r \epsilon_{i'j'k'} B_{k'}, \qquad B_4^2 + B_5^2 + B_6^2 = \mu^2 r^2 \frac{N^2-1}{4}, \quad (i',j',k' = 4,5,6)\\
&[B_{i''}, B_{j''}] = i\mu r \epsilon_{i''j''k''} B_{k''}, \quad B_7^2 + B_8^2 + B_9^2 = \mu^2 r^2 \frac{N^2-1}{4}, \quad (i'',j'',k'' = 7,8,9)\\
&B_0 = 0, \quad [B_\mu, B_\nu] = 0, \quad (\text{otherwise}).
\end{aligned}$$

   (We consider the Cartesian product of three spheres instead of a single S^2 fuzzy sphere

$$[B_i, B_j] = i\mu r \epsilon_{ijk} B_k \ (\text{for } i,j,k = 1,2,3), \quad B_\mu = 0 \ (\text{for } \mu = 0,4,5,\cdots,9),$$

   because the solution $B_4 = \cdots = B_9 = 0$ is trivially unstable.)

2. S^8 fuzzy sphere

   Generally, the S^{2k} fuzzy sphere is constructed by the n-fold symmetric tensor product of $(2k+1)$-dimensional gamma matrices:

$$B_p^{(8)} = \frac{\mu r}{2}[(\Gamma_p^{(2k)} \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes \Gamma_p^{(2k)})]_{\text{sym}}.$$

$$B_p^{(8)} B_p^{(8)} = \frac{\mu^2 r^2}{4} n(n+2k) 1_{N_k \times N_k}.$$

By solving the equations of motions for B_μ and $C_{\mu\nu}$ simultaneously, the radius parameter for the S^{2k} fuzzy sphere is $r = \frac{1}{2k}$.

Comparison of the classical energy

- **Trivial commutative solution** $B_0 = \cdots = B_9 = 0$:

$$E_{B_\mu=0} = -S_{B_\mu=0} = 0.$$

- $S^2 \times S^2 \times S^2$ **fuzzy spheres** ($N_1 = n+1$):

$$\begin{aligned} E_{S^2} &= -S_{S^2} = -\frac{16\mu}{r} Tr(B_\mu B^\mu) \\ &= -12\mu^3 N_1 (N_1 - 1)(N_1 + 1) \\ &\sim -\mathcal{O}(\mu^3 n^3) = -\mathcal{O}(\mu^3 N_1^3). \end{aligned}$$

- S^8 **fuzzy sphere:**

$$\begin{aligned} E_{S^8} &= -S_{S^8} = -\frac{5}{8}\mu^3 n(n+8) N_4 \\ &\sim -\mathcal{O}(\mu^3 n^{12}) = -\mathcal{O}(\mu^3 N_4^{\frac{6}{5}}), \end{aligned}$$

where the size of the matrices $B_p^{(8)}$ **is**

$$N_4 = \frac{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)}{302400} \sim \mathcal{O}(n^{10}).$$

3 Matrix model with manifest general coordinate invariance

Here, we pursue a matrix model with the local Lorentz invariance without the supermatrix model.

The odd-rank (even-rank) fields of the ten-dimensional tensor are allocated for the matter fields (local Lorentz parameter), respectively.

$$S = Tr_{N\times N}[tr_{32\times 32}V(m^2) + \bar{\psi} m \psi]$$

- $Tr(tr)$: the trace for the $N \times N (32 \times 32)$ matrices.
- m includes all odd-rank gamma matrices in 10 dimensions:

$$\begin{aligned} m &= m_\mu \Gamma^\mu + \frac{i}{3!} m_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3} - \frac{1}{5!} m_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5} \\ &- \frac{i}{7!} m_{\mu_1\cdots\mu_7}\Gamma^{\mu_1\cdots\mu_7} + \frac{1}{9!} m_{\mu_1\cdots\mu_9}\Gamma^{\mu_1\cdots\mu_9}, \end{aligned}$$

where $m_{\mu_1\cdots\mu_{2n-1}}$ are hermitian matrices:

$$m_{\mu_1\cdots\mu_{2n-1}} = \frac{i^{n-1}}{32 \times (2n-1)!} tr(m\Gamma_{\mu_1\cdots\mu_{2n-1}}).$$

m satisfies $\Gamma^0 m^\dagger \Gamma^0 = m$, and the action is hermitian.

We want to identify m with the Dirac operator.

$\Rightarrow$ We introduce $D = [(\text{length})^{-1}]$ as an extension of the Dirac operator.

$$\begin{aligned} m &= \tau^{\frac{1}{2}} D, \text{ where } \tau = [(\text{length})]^2, \\ D &= A_\mu \Gamma^\mu + \frac{i}{3!} A_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3} - \frac{1}{5!} A_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5} \\ &- \frac{i}{7!} A_{\mu_1\cdots\mu_7}\Gamma^{\mu_1\cdots\mu_7} + \frac{1}{9!} A_{\mu_1\cdots\mu_9}\Gamma^{\mu_1\cdots\mu_9}. \end{aligned}$$

τ is not an N-dependent cut-off parameter, but a reference scale ($\sim l_s^2$).

$A_{\mu_1\cdots\mu_{2n-1}} = \frac{i^{2n-1}}{32\times(2n-1)!} tr(D\Gamma_{\mu_1\cdots\mu_{2n-1}})$ are hermitian differential operators.

$\Rightarrow$ They are expanded by the number of the derivatives:

$$A_{\mu_1\cdots\mu_{2n-1}} = a_{\mu_1\cdots\mu_{2n-1}}(x) + \sum_{k=1}^{\infty} \frac{i^k}{2}\{\partial_{i_1}\cdots\partial_{i_k}, \underbrace{a^{(i_1\cdots i_k)}{}_{\mu_1\cdots\mu_{2n-1}}(x)}_{[(\text{length})^{-1+k}]}\}.$$

$a_\mu{}^{(i)}(x)$ is identified with the vielbein $e_\mu{}^i(x)$ in the background metric.

$$\begin{aligned} D &= e^{\frac{1}{2}}(x)\left[ie_\mu{}^i(x)\Gamma^\mu\left(\partial_i + \frac{1}{4}\Gamma^{\nu\rho}\omega_{i\nu\rho}(x)\right)\right]e^{-\frac{1}{2}}(x) \\ &+ (\text{higher-rank terms}) + (\text{higher-derivative terms}). \end{aligned}$$

The potential $V(m^2)$ is generically $V(m^2) \sim \exp(-(m^2)^\alpha)$.
$\Rightarrow$ The damping factor is naturally included in the bosonic term, due to the requirement that the flat-space Dirac operator $m_0 = i\tau^{\frac{1}{2}}\Gamma^\mu\partial_\mu$ should be a classical solution.
$\Rightarrow$ The trace for the infinitely large N matrices is finite.

ψ is a Weyl fermion, but not Majorana.

We need to introduce a damping factor so that the trace should be finite.

$$\psi = (\chi(x) + \sum_{l=1}^{\infty} \underbrace{\chi^{(i_1 \cdots i_l)}(x)}_{[(\text{length})]^l} \partial_{i_1} \cdots \partial_{i_l}) e^{-(\tau D^2)^\alpha}.$$

$\boxed{\text{Local Lorentz invariance}}$

The action is invariant under the local Lorentz transformation:

$$\begin{aligned} \delta m &= [m, \varepsilon], \quad \delta\psi = \varepsilon\psi, \quad \delta\bar{\psi} = -\bar{\psi}\varepsilon, \text{ where} \\ \varepsilon &= -i\varepsilon_\emptyset + \frac{1}{2!}\Gamma^{\mu_1\mu_2}\varepsilon_{\mu_1\mu_2} + \frac{i}{4!}\Gamma^{\mu_1\cdots\mu_4}\varepsilon_{\mu_1\cdots\mu_4} - \frac{1}{6!}\Gamma^{\mu_1\cdots\mu_6}\varepsilon_{\mu_1\cdots\mu_6} \\ &\quad -\frac{i}{8!}\Gamma^{\mu_1\cdots\mu_8}\varepsilon_{\mu_1\cdots\mu_8} + \frac{1}{10!}\Gamma^{\mu_1\cdots\mu_{10}}\varepsilon_{\mu_1\cdots\mu_{10}}. \end{aligned}$$

- All even-rank gamma matrices are necessary for the local Lorentz transformation algebra to close.
- ε satisfies $\Gamma^0\varepsilon^\dagger\Gamma^0 = \varepsilon$, and thus the commutator $\delta m = [m, \varepsilon]$ actually satisfies $\Gamma^0(\delta m)^\dagger\Gamma^0 = \delta m$.

The invariance under the local Lorentz transformation:

$$\delta S = 2Tr[tr(V_S'(m^2)m[m, \varepsilon])] + Tr[tr(\bar{\psi}[m, \varepsilon]\psi)] = 0,$$

when the fields damp fast enough at the infinity.

For a generic $V(m^2)$, the bosonic part of the action reduces to the Einstein gravity

$$S \sim \int \frac{d^d x}{(2\pi\tau)^{\frac{d}{2}}} e(x)(\tau R(x) + \mathcal{O}(\tau^{\frac{3}{2}})),$$

in the classical low-energy limit.

$\boxed{\mathcal{N} = 2 \text{ SUSY}}$

The supersymmetry transformation of the model:

$$\begin{aligned} \delta\psi &= 2V'(m^2)\epsilon, \quad \delta\bar{\psi} = 2\bar{\epsilon}V'(m^2), \\ \delta m &= \epsilon\bar{\psi} + \psi\bar{\epsilon}. \end{aligned}$$

Commutator of the supersymmetry transformation on shell:

$$\begin{aligned} &[\delta_\epsilon, \delta_\xi]m = 2[\xi\bar{\epsilon} - \epsilon\bar{\xi}, V'(m^2)], \\ &[\delta_\epsilon, \delta_\xi]\psi = 2\psi(\bar{\epsilon}m\frac{V'(m^2)}{m^2}\xi - \bar{\xi}m\frac{V'(m^2)}{m^2}\epsilon), \end{aligned}$$

where we have utilized the equation of motion:

$$\frac{\partial S}{\partial\bar{\psi}} = 2m\psi = 0, \quad \frac{\partial S}{\partial\psi} = 2\bar{\psi}m = 0.$$

In order to see the structure of the $\mathcal{N}=2$ supersymmetry, we separate the SUSY parameters into the hermitian and the antihermitian parts as

$$\epsilon = \epsilon_1 + i\epsilon_2, \;\; \xi = \xi_1 + i\xi_2,$$

($\xi_1, \xi_2, \epsilon_1, \epsilon_2$ are Majorana-Weyl fermions.)

The translation is attributed to the quartic term in the Taylor expansion of $V(m) = \sum_{k=1}^{\infty} \frac{a_{2k}}{2k} m^{2k}$:

- Bosonic fields:

  $$[\delta_\epsilon, \delta_\xi] A_\mu = \frac{a_4}{8} (\bar{\xi}_1 \Gamma^i \epsilon_1 + \bar{\xi}_2 \Gamma^i \epsilon_2)[A_i, A_\mu] + \cdots,$$

  The field $a_\mu(x)$ receives the translation and the gauge transformation:

  $$\begin{aligned} [A_i, A_\mu] &= [i\partial_i + a_i(x), i\partial_\mu + a_\mu(x)] + \cdots \\ &= \underbrace{i(\partial_i a_\mu(x))}_{\text{translation}} \; \underbrace{-i(\partial_\mu a_i(x)) + [a_i(x), a_\mu(x)]}_{\text{gauge transformation}} + \cdots. \end{aligned}$$

- Fermionic fields:

  $$[\delta_\epsilon, \delta_\xi]\psi = -2a_4(\bar{\xi}_1 \Gamma^j \epsilon_1 + \bar{\xi}_2 \Gamma^j \epsilon_2)\psi A_j + \cdots.$$

  Each fermionic field is transformed as

  $$[\delta_\epsilon, \delta_\xi]\chi(x) = 0 + \cdots,$$
  $$[\delta_\epsilon, \delta_\xi]\chi^{(i_1 \cdots i_{l+1})}(x) = -2a_4(\bar{\xi}_1 \Gamma^j \epsilon_1 + \bar{\xi}_2 \Gamma^j \epsilon_2)\chi^{(\{i_1 \cdots i_l)}(x)\delta^{i_{l+1}\}j} + \cdots.$$

  (*) $\cdots$ denotes the omission of the non-linear terms of the fields.
  The fermions do not receive the translation.

It is a future problem to overcome this difficulty.

4 Monte Carlo simulation of the fuzzy sphere solution

It is difficult to substantiate the quantum stability of the fuzzy sphere classical solution.

There are several perturbative approaches to the quantum stability: hep-th/0101102,0303120,0303196,0307007,0312241

Here, we address this issue non-perturbatively through the Monte Carlo simulation.

For brevity, we focus on the following 3-dimensional bosonic action.

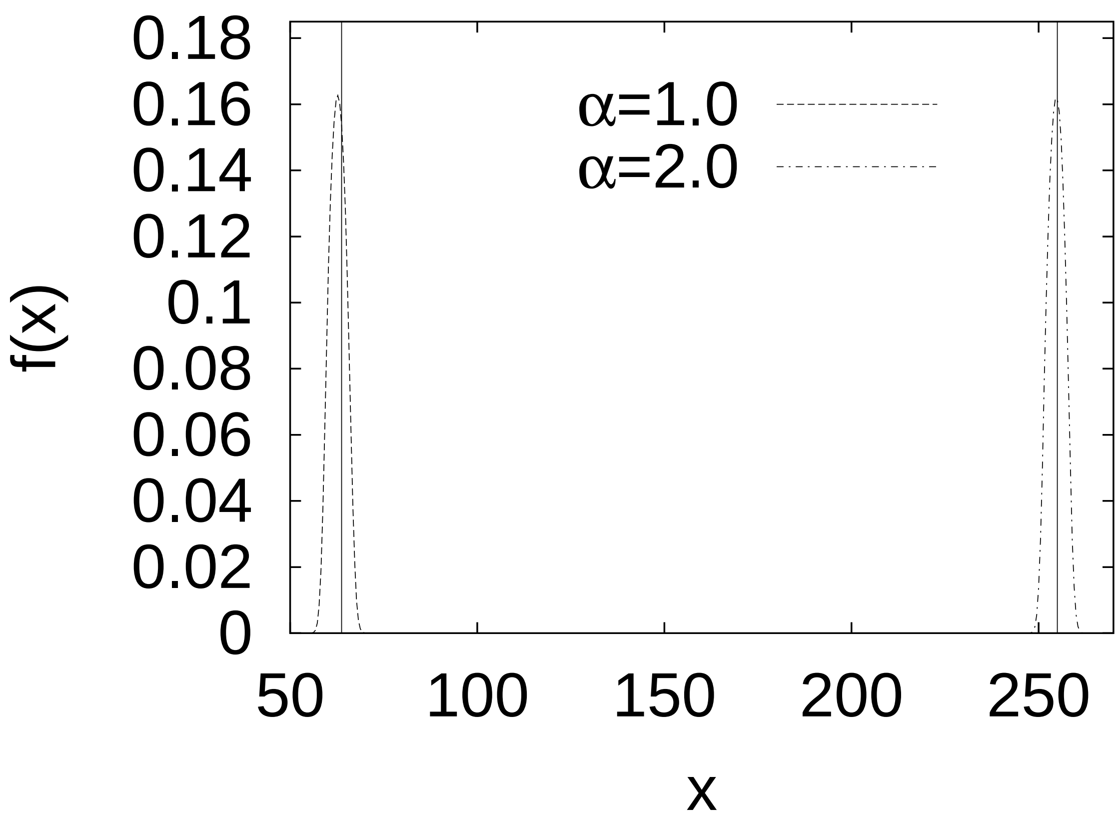
$$S = NTr\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).$$

We start the Monte Carlo simulation from the initial condition

$$A_\mu^{(0)} = \alpha J_\mu,$$

for the $N = 16$, $\alpha = 1.0, 2.0$ case.

We plot the eigenvalue distribution of the Casimir $Q = A_1^2 + A_2^2 + A_3^2$.



The eigenvalues are peaked around

$$R^2 = \frac{1}{4}\alpha^2(N^2 - 1).$$

Nonperturbative stability of the fuzzy spheres!

The stability is ascribed to the small quantum effect at large α.

For the effective action $W = \int dA_\mu e^{-S}$

- Effect of the classical fuzzy sphere: $\mathcal{O}(\alpha^4 N^4)$.
- Effect of the path integral measure: $\mathcal{O}(N^2)$.

The quantum effect is small when $\alpha \gg \mathcal{O}(\frac{1}{\sqrt{N}})$.

$\boxed{\text{Main results}}$

- The first-order phase transition, as we vary the parameter α.

  Lower critical point $(A_\mu^{(0)} = \alpha J_\mu$ start) $\alpha_{cr}^{(l)} \sim \frac{2.1}{\sqrt{N}}$,

  Higher critical point $(A_\mu^{(0)} = 0$ start) $\alpha_{cr}^{(u)} \sim 0.66$.

  ⋆ The Yang-Mills phase $(\alpha < \alpha_{cr})$

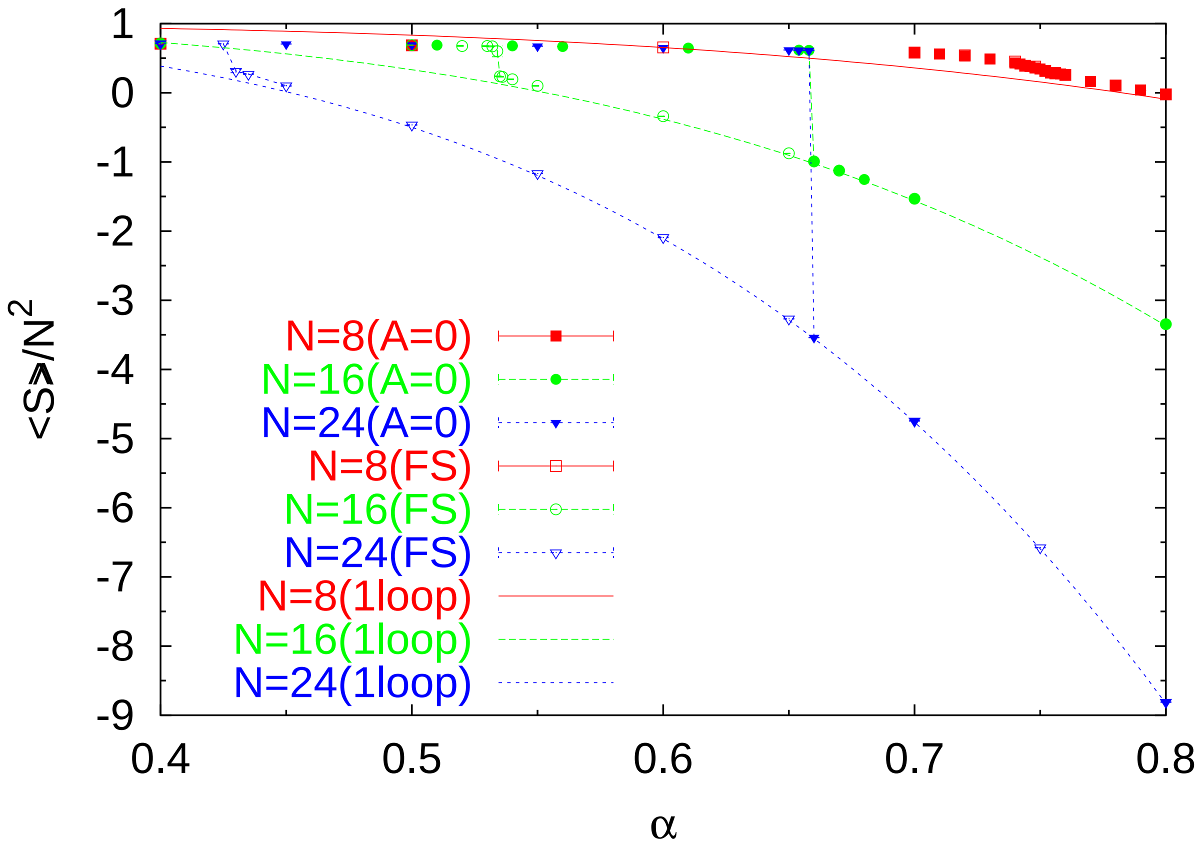
  The fuzzy sphere is unstable.

  ⋆ The fuzzy sphere phase $(\alpha > \alpha_{cr})$

  The fuzzy sphere is stable.

- One-loop dominance:

  The higher-loop effect is suppressed for $N \to \infty$.



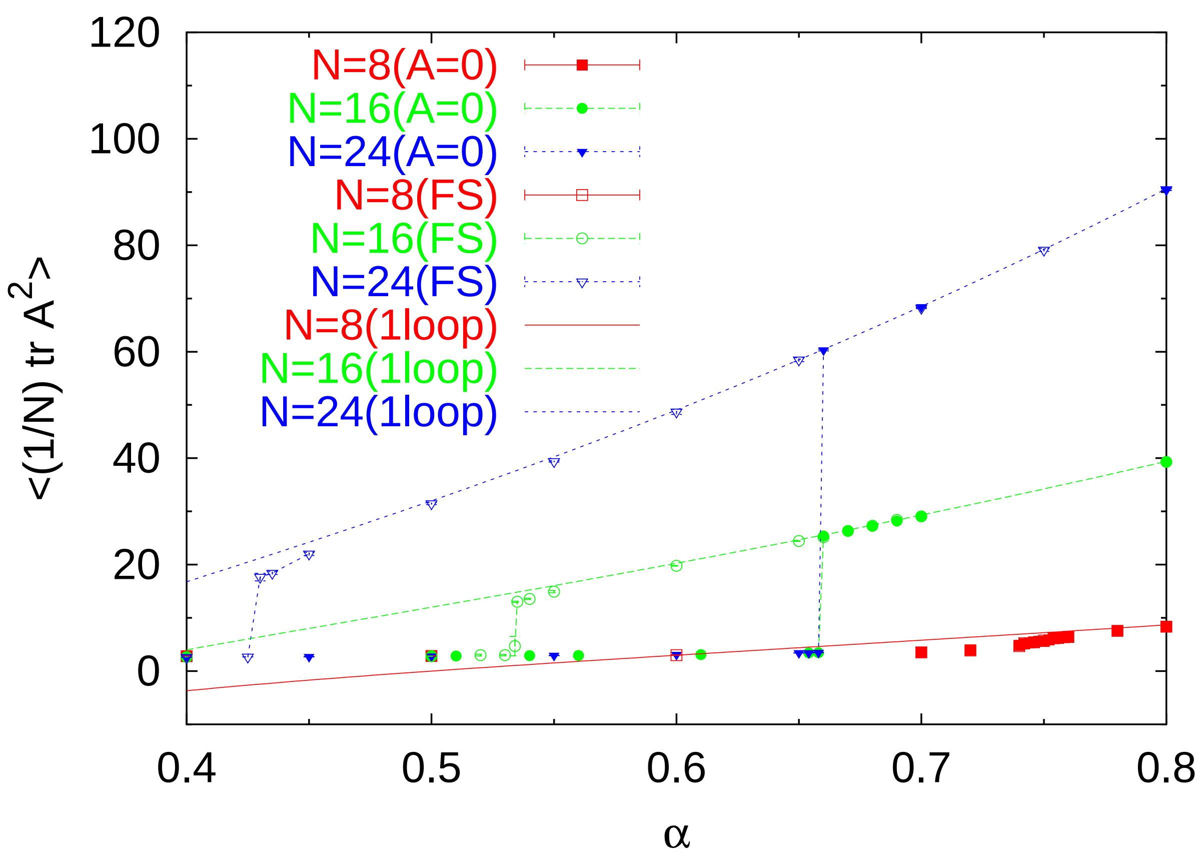


Figure 1: The hysteresis cycle of (Upper) $\langle \boldsymbol{S} \rangle / \boldsymbol{N^2}$ and (Lower) $\langle \frac{1}{N} \boldsymbol{Tr A_\mu^2} \rangle$.

5 Conclusion

In this thesis, we have discussed several works concerning the relation between the gravitational interaction and the large-N reduced model:

Future direction:

- **Pursuit of the relation to the supergravity:**
  The matrix model that reduces to the (type IIB) supergravity in the low-energy limit.
- **Description of the more general curved-space background.**
- **Dynamical generation of the spacetime and the gauge group.**