Numerical Simulation of the large-N reduced model

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Contents

1 Introduction

What is superstring theory?

Promising candidate for the unification of all interaction.

First string boom (1980's)

Understanding of perturbative aspects of superstring theory.

- **The energy of gravity is free from divergence.**
- **Prospect for reproducing standard model ($E_8 \times E_8$ heterotic superstring theory).**

  $\Rightarrow$ **Infinite number of vacua.**

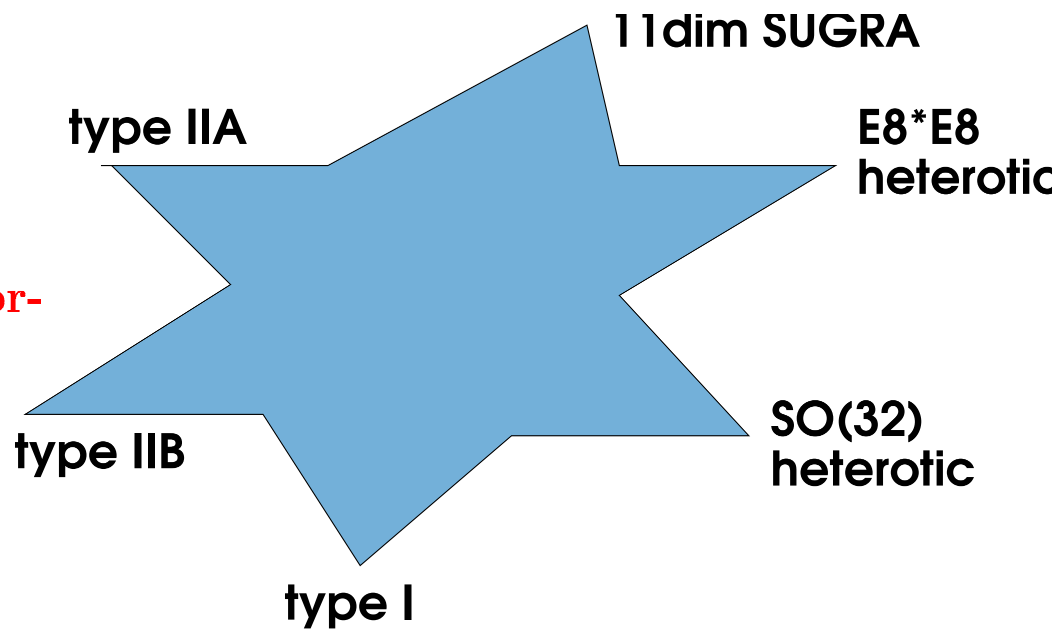
  $\Rightarrow$ **No guiding principle for determining the true vacuum.**

- **Nonperturbative aspects of noncritical string theory**

Second string boom (late 1990's)

Nonperturbative aspects of superstring theory.

- **Discovery of the D-brane**
- **T/S duality of string theory**
- **Proposal of matrix model as a constructive definition (nonperturbative formulation) of superstring theory**



Third string boom (?????)

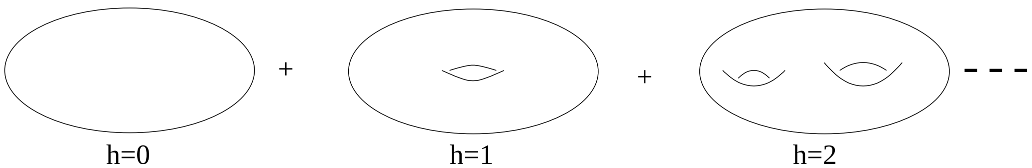
Completion of the constructive definition of superstring theory.

Matrix model

Promising candidate for the constructive definition of superstring theory.

Random triangulation F. David Nucl. Phys. B257 (1985) 543.

Path integral of string theory:



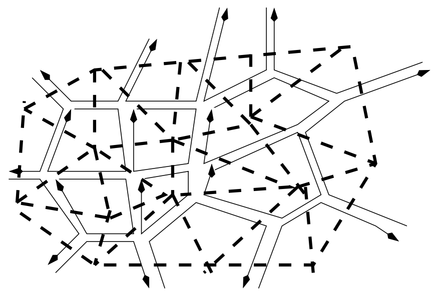
$$Z = \sum_{h=0}^{\infty} \int dg \exp(-\beta A + \gamma\chi).$$

- $A = \frac{1}{8\pi}\int d^2\xi \sqrt{g}$ = (area of world sheet)
- $S_M = \frac{1}{8\pi}\int d^2\xi \sqrt{g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$ is equivalent to A for $D = 0$.
- $\chi = \frac{1}{4\pi}\int d^2\xi \sqrt{g} R = 2(1-h)$ = (Euler character of Riemann surface).

  h = (genus of the world sheet)

Discritization of the worldsheet of string theory into equilateral triangles.

0-dimensional bosonic string theory $\Leftrightarrow$ ϕ^3 one-matrix model $S = \left(\frac{1}{2}\text{tr}\,\phi^2 - \frac{g}{\sqrt{N}}\text{tr}\,\phi^3\right)$.



Studies of noncritical string theory J. Distler and H. Kawai, Nucl. Phys. B321(1989) 509

Quantization of $D \leq 1$ noncritical string theory.

Calculation of string susceptibility:

$$Z \propto A^{\gamma-3}, \quad (\gamma = 2 + \frac{1-h}{12}(D - 25 - \sqrt{(25-D)(1-D)})).$$

Nonperturbative calculation of matrix model E. Brezin and V.A.Kazakov, PLB236 (1990) 144.

Nonperturbative analysis of one-matrix model via orthogonal polynomial method.

String susceptibility $\gamma_h = \frac{-1+5h}{2}$ $(D=0)$ agrees with Distler and Kawai's result.

Important test of the legitimacy of matrix model.

IIB matrix model

The IIB matrix model $\Rightarrow$ promising candidate for the constructive definition of superstring theory.

$$S = N\left(-\frac{1}{4}\text{tr}\,[A_\mu, A_\nu]^2 + \frac{1}{2}\text{tr}\,\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right).$$

- **Dimensional reduction of $\mathcal{N} = 1$ 10-dimensional Super-Yang-Mills (SYM) theory to 0 dimension.**

  A_μ (10-dimensional vector) and ψ (10-dimensional Majorana Weyl spinor) are $N \times N$ matrices .

- **Matrix regularization of the Schild from of the Green-Schwarz action of the type IIB superstring theory.**

- **Many-body system of superstrings.**

- **$\mathcal{N}=2$ supersymmetry:**

  The theory must contain spin-2 graviton if it contains massless particles.

  * **homogeneous:** $\delta^{(1)}_{\epsilon} A_{\mu} = i\bar{\epsilon}\Gamma_{\mu}\psi$, $\delta^{(1)}_{\epsilon}\psi = \frac{i}{2}\Gamma^{\mu\nu}[A_{\mu}, A_{\nu}]\epsilon$.
  * **inhomogeneous:** $\delta^{(2)}_{\xi} A_{\mu} = 0$, $\delta^{(2)}_{\xi}\psi = \xi$.

  Linear combination $\tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)}$, $\tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)})$.

$$[\tilde{\delta}^{(\alpha)}_{\epsilon}, \tilde{\delta}^{(\beta)}_{\xi}]\psi = 0,$$
$$[\tilde{\delta}^{(\alpha)}_{\epsilon}, \tilde{\delta}^{(\beta)}_{\xi}]A_{\mu} = -2i\delta^{\alpha\beta}\bar{\epsilon}\Gamma_{\mu}\xi.$$

  This leads us to interpret the eigenvalues of A_{μ} as the spacetime coordinate.

- **The action of the IIB matrix model does not include the integral.**

  The numerical simulation is easier than that of the quantum field theory.

2 Simulation of matrix models

Rudiment of Monte Carlo simulation

Given $C_0 =$ (initial configuration), we generate series of configurations

$C_0 \to C_1 \to \cdots \to C_n \to C_{n+1} \to \cdots$.

Markov chain: Probability $P(C_{n-1} \to C_n)$ depends only on C_{n-1} and C_n.

$w_n[C] =$ (probability of obtaining C at n-th step).

$$w_n[C] = \sum_{C'} w_{n-1}[C'] P[C' \to C], \quad w_0[C] = \delta_{C,C_0}.$$

Choose $P[C_{n-1} \to C_n]$ such that $w[C] = \lim_{n\to\infty} w_n[C] = e^{-S[C]}$.

- Detailed balance condition: $e^{-S[C]} P(C \to C') = e^{-S[C']} P(C' \to C)$.

- Ergodicity : For any C, C', there is a finite probability of moving from C to C' within finite steps.

Then, $w[C] = \lim_{n\to\infty} w_n[C] = e^{-S[C]}$ is satisfied.

We calculate $\langle \mathcal{O} \rangle$ using a Markov process.

- Thermalization: We have to discard sufficiently many steps in order to achieve equilibrium.
- autocorrelation: Configurations generated by the Markov process are not statistically independent.

Two algorithms to achieve the equilibrium:

- Heat bath algorithm: divide the whole system into subsystems: $C = \{C^{(1)}, C^{(2)}, \cdots, C^{(}$

  For a subsystem $C^{(j)}$, generate new $C'^{(j)}$ with the probability

  $P[C'] \propto \exp(-S[C^{(1)}, \cdots, C'^{(j)}, \cdots, C^{(k)}])$.
- Metropolis algorithm: Generate a trial configuration C'.

  For a uniform random number $x \in [0:1]$, we accept C' when $x < e^{-\Delta S}$

  (where $\Delta S = S[C'] - S[C]$).

(a) Simplest case: quadratic U(N) one-matrix model

$$S = \frac{N}{2} \mathrm{tr}\, \phi^2.$$

To analyze this model via the heat bath algorithm, we rewrite the matrix ϕ as

$$\phi_{ii} = \frac{a_i}{\sqrt{N}}, \quad \begin{cases} \phi_{ij} = \frac{x_{ij}+iy_{ij}}{\sqrt{2N}} \\ \phi_{ji} = \frac{x_{ij}-iy_{ij}}{\sqrt{2N}}, \end{cases} \quad (\text{for } i < j).$$

The N^2 real quantities a_i, x_{ij}, y_{ij} comply with the independent normal Gaussian distribution.

$$S = \frac{1}{2}\sum_{i=1}^{N} a_i^2 + \frac{1}{2}\sum_{i<j}((x_{ij})^2 + (y_{ij})^2).$$

$$Z = \int \prod_{i=1}^{N} da_i \prod_{1\leq i<j\leq N} dx_{ij}dy_{ij} \exp\left(-\frac{1}{2}\sum_{i=1}^{N} a_i^2 - \frac{1}{2}\sum_{1\leq i<j\leq N}((x_{ij})^2 + (y_{ij})^2)\right).$$

a_i, x_{ij}, y_{ij} are updated by the Gaussian random number.

There is no need for thermalization or no autocorrelation.

Feynman rule of this model (use the Gaussian integral $\frac{1}{a} = \frac{\int_{-\infty}^{+\infty} dx x^2 e^{-ax^2/2}}{\int_{-\infty}^{+\infty} dx e^{-ax^2/2}}$):

$$\langle \phi_{ij}\phi_{kl} \rangle = \begin{cases} \frac{1}{N}\langle a_i a_k \rangle = \frac{1}{N}\delta_{ik} & (i=j, k=l) \\ \frac{1}{2N}\langle (x_{ij}+iy_{ij})(x_{kl}+iy_{kl})\rangle = \left\{ \begin{array}{ll} \frac{1}{2N}\langle x_{ij}x_{ij} - y_{ij}y_{ij}\rangle = 0, & (i=k, j=l) \\ \frac{1}{2N}\langle x_{ij}x_{ij} - y_{ij}y_{ji}\rangle = \frac{1}{N}, & (i=l, j=k) \end{array} \right\} & (i \neq j, k \neq l) \end{cases}$$
$$= \frac{1}{N}\delta_{il}\delta_{jk}$$

Some exact results:

$$\langle \frac{1}{N}\mathrm{tr}\, \phi^2 \rangle = \frac{1}{N}\langle \phi_{ij}\phi_{ji}\rangle = \frac{1}{N} \times \frac{1}{N} \times N^2 = 1,$$
$$\langle \frac{1}{N}\mathrm{tr}\, \phi^4 \rangle = \frac{1}{N}\langle \phi_{ij}\phi_{jk}\phi_{kl}\phi_{li}\rangle = 2 + \frac{1}{N^2}.$$

Generation of the uniform random number

We use the congruence method to generate the uniform random number $x \in [0:1]$.

- **We give the random seed z_1,** such as $z_1 = \text{time()}$.
- **We solve the recursion formula $z_{k+1} = a z_k + c \pmod{2^{31}-1}$.**

  The choice $(a, c) = (5^{11}, 0)$ is known to give a good pseudo-random number.
- **The sequence $\{\frac{z_k}{2^{31}-1}\}$ gives a uniform pseudo-random number [0:1].**

Generation of the Gaussian random number

- **We take two uniform random numbers $x, y \in [0:1]$.**
- **We introduce the quantity $r = \sqrt{-a^2 \log x^2}$.**

  This complies with the probability distribution

$$P(r)dr = P(x)\frac{dx}{dr}dr = \frac{2r}{a^2}\exp\left(-\frac{r^2}{a^2}\right)dr.$$

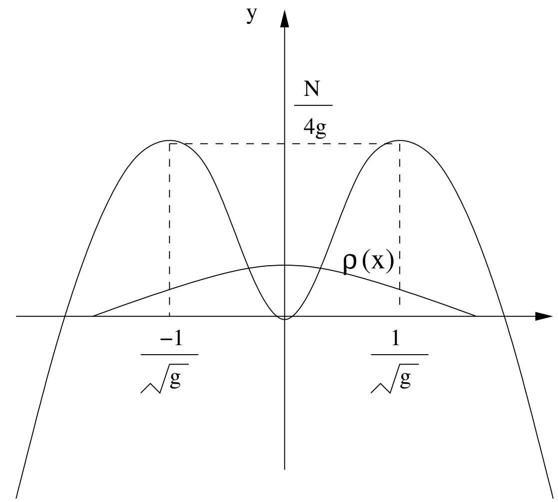
- **We next introduce the quantities**

$$X = r\cos(2\pi y), \quad Y = r\sin(2\pi y).$$

  They comply with the probability distribution

$$P(r)drdy \propto \exp\left(-\frac{1}{a^2}(X^2+Y^2)\right)dXdY.$$

(b) Quartic U(N) one-matrix model



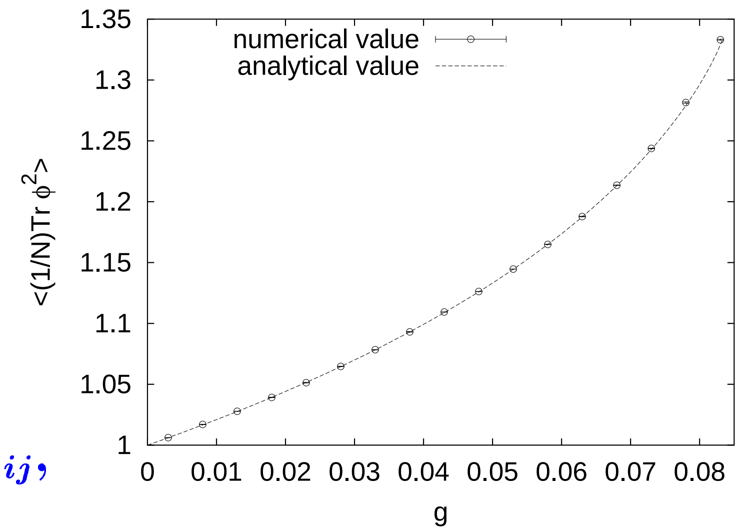
$$S = N\left(\frac{1}{2}\mathrm{tr}\,\phi^2 - \frac{g}{4}\mathrm{tr}\,\phi^4.\right)$$

This action is unbounded below.

Metastability of the origin in the large-N limit.

Auxiliary fields Q (where $\alpha = \sqrt{\frac{g}{2}}$):

$$\begin{aligned}\tilde{S} &= \frac{N}{2}\left(\mathrm{tr}\,\phi^2 + \mathrm{tr}\,Q^2 - 2\mathrm{tr}\,\alpha Q\phi^2\right) \\ &= \frac{N}{2}\mathrm{tr}\,(Q-\alpha\phi^2)^2 + S.\end{aligned}$$



Update Q as

$$Q_{ii} = \frac{a_i}{\sqrt{N}} + \alpha(\phi^2)_{ii},\ Q_{ij} = \frac{x_{ij}+iy_{ij}}{\sqrt{2N}} + \alpha(\phi^2)_{ij},$$

where a_i, x_{ij}, y_{ij} comply with the normal Gaussian distribution.

Dependence of ϕ_{ii}:

$$\tilde{S} = \frac{N}{2}(\phi_{ii})^2 \underbrace{(1 - 2\alpha Q_{ii})}_{=c_i} - N\phi_{ii} \underbrace{(\alpha \sum_{j\neq i}(\phi_{ji}Q_{ij} + Q_{ji}\phi_{ij}))}_{=h_i}.$$

Update of ϕ_{ii}: $\phi_{ii} = \frac{a_i}{\sqrt{Nc_i}} + \frac{h_i}{c_i}$.

Dependence of ϕ_{ij}:

$$\tilde{S} = N \underbrace{(1 - \alpha(Q_{ii} + Q_{jj}))}_{=c_{ij}} |\phi_{ij}|^2 - N(\phi_{ij}h_{ji} + \phi_{ji}h_{ij}), \text{ where}$$

$$h_{ij} = \alpha(\sum_{k\neq j}(\phi_{ik}Q_{kj} + \sum_{k\neq i} Q_{ik}\phi_{kj})).$$

Update of ϕ_{ij}: $\phi_{ij} = \frac{x_{ij}+iy_{ij}}{\sqrt{2Nc_{ij}}} + \frac{h_{ij}}{c_{ij}}$.

Analytical large-N result:

E.Brezin, C.Itzykson, G.Parisi and J.Zuber, Comm. Math. Phys. 59, 35 (1978).

$$\langle \frac{1}{N}\text{tr}\, \phi^2 \rangle = \frac{1}{3}a^2(4 - a^2), \text{ where } a^2 = \frac{2}{1 + \sqrt{1 - 12g}}.$$

Eigenvalue distribution

$$\rho(x) = \frac{1}{2\pi}(-gx^2 - 2ga^2 + 1)\sqrt{4a^2 - x^2}.$$

(c) The bosonic IIB matrix model

T. Hotta, J. Nishimura and A. Tsuchiya hep-th/9811220.

$$S = -\frac{N}{4}\sum_{\mu,\nu=1}^{d} \mathrm{tr}\,[A_\mu, A_\nu]^2 = -\frac{N}{2}\sum_{1\le\mu<\nu\le d} \mathrm{tr}\,\{A_\mu, A_\nu\}^2 + 2N\sum_{1\le\mu<\nu\le d} \mathrm{tr}\,(A_\mu^2 A_\nu^2),$$

defined in the d-dimensional Euclidean space.

Auxiliary field $Q_{\mu\nu}$ (where $G_{\mu\nu} = \{A_\mu, A_\nu\}$):

$$\tilde{S} = \frac{N}{2}\sum_{1\le\mu<\nu\le d} \mathrm{tr}\,\left(Q_{\mu\nu}^2 - 2(Q_{\mu\nu}G_{\mu\nu}) + 4(A_\mu^2 A_\nu^2)\right) = \frac{N}{2}\sum_{1\le\mu<\nu\le d} \mathrm{tr}\,(Q_{\mu\nu} - G_{\mu\nu})^2 + S.$$

Update of $Q_{\mu\nu}$:

$$(Q_{\mu\nu})_{ii} = \frac{a_i}{\sqrt{N}} + (G_{\mu\nu})_{ii},\ (Q_{\mu\nu})_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{2N}} + (G_{\mu\nu})_{ij},$$

Dependence of A_λ:

$$\tilde{S} = -N\mathrm{tr}\,(T_\lambda A_\lambda) + 2N\mathrm{tr}\,(S_\lambda A_\lambda^2) + \cdots, \text{ where}$$

$$S_\lambda = \sum_{\mu\neq\lambda}(A_\mu^2),\ T_\lambda = \sum_{\mu\neq\lambda}(A_\mu Q_{\lambda\mu} + Q_{\lambda\mu}A_\mu).$$

- **Dependence of $(A_\lambda)_{ii}$:**

$$\tilde{S} = 2N(S_\lambda)_{ii}(A_\mu)_{ii}^2 - 4Nh_i(A_\mu)_{ii}, \text{ where}$$
$$h_i = \frac{N}{4}[(T_\lambda)_{ii} - 2\sum_{j\neq i}((S_\lambda)_{ji}(A_\lambda)_{ij} + (S_\lambda)_{ij}(A_\lambda)_{ji})].$$

Update of $(A_\lambda)_{ii}$:

$$(A_\lambda)_{ii} = \frac{a_i}{\sqrt{4N(S_\lambda)_{ii}}} + \frac{h_i}{(S_\lambda)_{ii}}.$$

- **Dependence of $(A_\lambda)_{ij}$:**

$$\tilde{S} = 2Nc_{ij}|(A_\lambda)_{ij}|^2 - 2Nh_{ji}(A_\lambda)_{ij}, \text{ where}$$
$$c_{ij} = (S_\lambda)_{ii} + (S_\lambda)_{jj},$$
$$h_{ij} = \frac{1}{2}(T_\lambda)_{ij} - \sum_{k\neq i}(S_\lambda)_{ik}(A_\lambda)_{kj} - \sum_{k\neq j}(S_\lambda)_{kj}(A_\lambda)_{ik}.$$

Update of $(A_\lambda)_{ij}$:

$$(A_\lambda)_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{4Nh_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$

Exact results derived from Schwinger-Dyson equation:

$$-\langle \frac{1}{N}\mathrm{tr}\,[A_\mu, A_\nu]^2 \rangle = d(1-\frac{1}{N^2}).$$

(Proof)

$$\begin{aligned} 0 &= \int d^d A \sum_{a=1}^{N^2-1}\sum_{\mu=1}^{d} \frac{\partial}{\partial A^a_\mu}[\mathrm{tr}\,(t^a A_\mu) e^{-S}] \\ &= \int d^d A \sum_{a=1}^{N^2-1} [\mathrm{tr}\,(t^a t^a) d e^{-S} + N \mathrm{tr}\,(t^a A_\mu)\mathrm{tr}\,(t^a [A_\nu, [A_\mu, A_\nu]]) e^{-S}] \\ &= \int d^d A \left[d(N^2-1) + N\mathrm{tr}\,[A_\mu, A_\nu]^2 \right] e^{-S} \\ &= \left(\int d^d A e^{-S} \right) \times \left(d(N^2-1) + \frac{N\int d^d A \mathrm{tr}\,([A_\mu, A_\nu]^2) e^{-S}}{\int d^d A e^{-S}} \right). \end{aligned}$$

t^a is the basis of the SU(N) Lie algebra:

$$\mathrm{tr}\,(t^a t^b) = \delta^{ab}, \quad \sum_{a=1}^{N^2-1} (t^a)_{ij} (t^a)_{kl} = \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}.$$

The matrices A_μ are expanded in terms of t^a as $A_\mu = \sum_{a=1}^{N^2-1} A^a_\mu t^a$:

$$\begin{aligned} \sum_{a=1}^{N^2-1} \mathrm{tr}\,(t^a A)\mathrm{tr}\,(t^a B) &= \sum_{a=1}^{N^2-1} A_{ji} B_{lk} (t^a)_{ij} (t^a)_{kl} = A_{ji} B_{lk} (\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}) \\ &= \mathrm{tr}\,(AB) - \frac{1}{N}\mathrm{tr}\,A \mathrm{tr}\,B = \mathrm{tr}\,AB. \end{aligned}$$

Equation of motion

$$\frac{\partial S}{\partial A^a_\mu} = -N\mathrm{tr}\,(t^a [A_\nu, [A_\mu, A_\nu]]).$$

3 3d bosonic Yang-Mills-Chern-Simons model

T. Azuma, S. Bal, K. Nagao and J. Nishimura, hep-th/0401038

Motivation to consider the fuzzy sphere:

- **Relation between the noncommutative field theory and superstring theory.**
- **Prototype of the curved space background of large-N reduced models.**

Yang-Mills-Chern-Simons (YMCS) model $\Rightarrow$ a toy model with fuzzy sphere solutions:

$$S = N\text{tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).$$

- **Defined in the 3-dimensional Euclidean space: $(\mu, \nu, \rho = 1, 2, 3)$.**
- **Classical equation of motion: $[A_\nu, [A_\mu, A_\nu]] - i\alpha\epsilon_{\mu\nu\rho}[A_\nu, A_\rho] = 0$.**
- **Fuzzy S^2 classical solution $A_\mu = X_\mu = \alpha L_\mu$ (where $[L_\mu, L_\nu] = i\epsilon_{\mu\nu\rho}L_\rho$).**

  L_μ = ($N \times N$ irreducible representation of the SU(2) Lie algebra).

  Casimir operator: $Q = A_1^2 + A_2^2 + A_3^2 = R^2 1_N$, where $R^2 = \alpha^2\frac{N^2-1}{4}$.

Monte Carlo simulation of 3d YMCS model

Heat bath algorithm of the 3d YMCS model:

$$\tilde{S} = \sum_{1\leq\mu<\nu\leq 3}\left(\frac{N}{2}\mathrm{tr}\, Q_{\mu\nu}^2 - N\mathrm{tr}\,(Q_{\mu\nu}G_{\mu\nu}) + 2N\mathrm{tr}\,(A_\mu^2 A_\nu^2)\right) + \frac{2i\alpha N}{3}\epsilon_{\mu\nu\rho}\mathrm{tr}\, A_\mu A_\nu A_\rho.$$

Update of $Q_{\mu\nu}$: parallel to $\alpha = 0$ case (in Sec. 2).

Dependence of A_λ:

$$\begin{aligned}\tilde{S} &= -N\mathrm{tr}\,(T_\lambda A_\lambda) + 2N\mathrm{tr}\,(S_\lambda A_\lambda^2) + \cdots, \text{ where} \\ S_\lambda &= \sum_{\mu\neq\lambda} A_\mu^2, \quad T_\lambda = \sum_{\mu\neq\lambda}(A_\mu Q_{\lambda\mu} + Q_{\lambda\mu}A_\mu) \underbrace{-2i\alpha\epsilon_{\lambda\mu\nu}A_\mu A_\nu}_{\text{the only difference!}}.\end{aligned}$$

Update of A_λ: parallel to the $\alpha = 0$ case, except for T_λ.

Initial condition:

$$A_\mu^{(0)} = \begin{cases} X_\mu & \text{(fuzzy sphere start)}, \\ 0 & \text{(zero start)}. \end{cases}$$

Discontinuity:

$$\alpha = \begin{cases} \alpha_{\mathrm{cr}}^{(l)} = \frac{2.1}{\sqrt{N}} & \text{(fuzzy sphere start)} \\ \alpha_{\mathrm{cr}}^{(u)} = 0.66 & \text{(zero start)}. \end{cases}$$

First-order phase transition:

- $\alpha < \alpha_{\rm cr}$: **Yang-Mills phase**

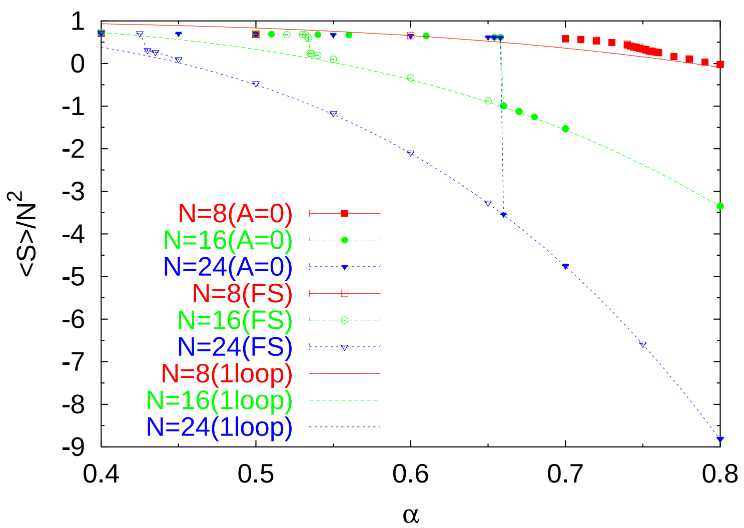
  Strong quantum effects.

  Behavior like $\alpha = 0$ case.

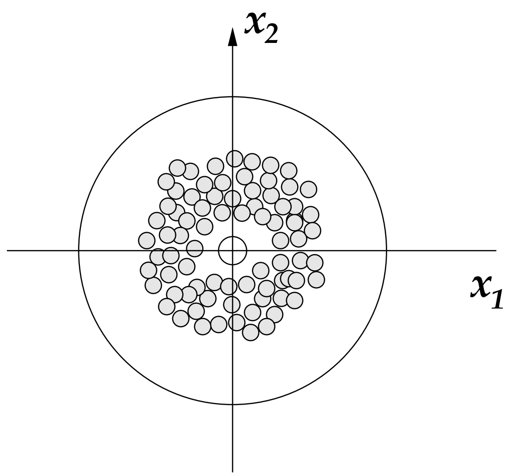
  $\langle \frac{S}{N^2} \rangle \simeq O(1)$, $\langle \frac{1}{N} {\rm tr}\, A_\mu^2 \rangle \simeq O(1)$.

- $\alpha > \alpha_{\rm cr}$: **fuzzy sphere phase**

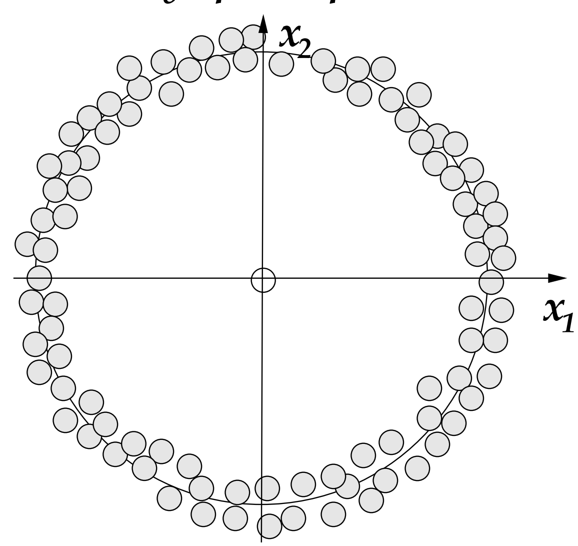
  Fuzzy sphere configuration is stable.



Yang–Mills phase



Fuzzy sphere phase



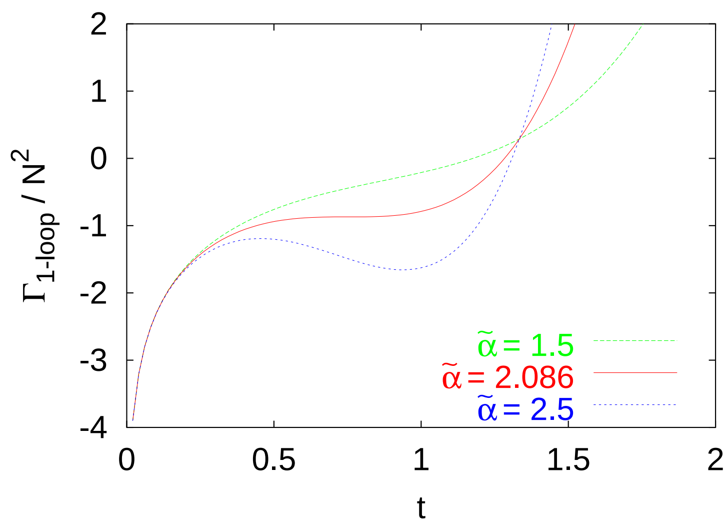
Phase transition from the effective action

The effective action Γ is saturated at the one-loop level.

T. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, hep-th/0303120

Effective action at one loop around $A_\mu = tX_\mu$ (where $\tilde{\alpha} = \alpha\sqrt{N}$).

$$\frac{\Gamma_{1-\mathrm{loop}}}{N^2} \simeq \tilde{\alpha}^4 \left(\frac{t^4}{8} - \frac{t^3}{6}\right) + \log t.$$



The local minimum disappears at $\tilde{\alpha} < \tilde{\alpha}^{(l)}_{\mathrm{cr}} = (\frac{8}{3})^{\frac{3}{4}} = 2.086\cdots$.

Properties of the multi-fuzzy spheres Expansion around k coincide fuzzy spheres $A_\mu = X_\mu + \tilde{A}_\mu$, where

$$X_\mu = \alpha L_\mu^{(n)} \otimes 1_k.$$

Quantum field theory with U(k) gauge group.

Fuzzy sphere is a compact manifold.

It is realized by the finite $N = nk$ matrices.

It facilitates the numerical treatment of the gauge group.

Simulation from zero start $A_\mu^{(0)} = 0$ for $N = 16$, $\alpha = 2.0$.

Metastability of multi-fuzzy-sphere state.

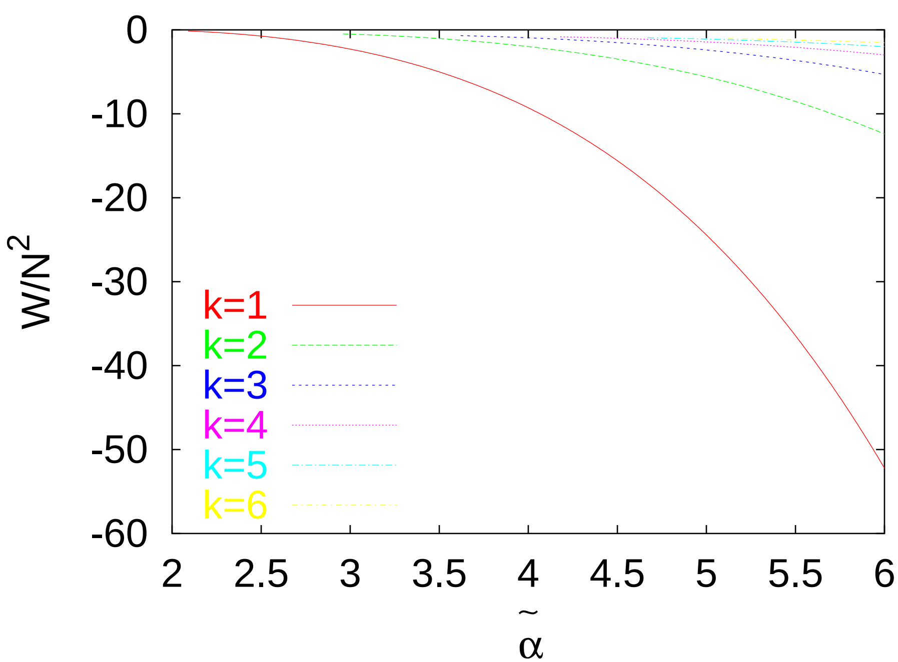
$$\underbrace{A_\mu^{(0)} = 0}_{\text{initial state}} \rightarrow \cdots \rightarrow \underbrace{A_\mu = \alpha \begin{pmatrix} L_\mu^{(6\to5\to4\to3\to2\to1)} & 0 \\ 0 & L_\mu^{(10\to11\to12\to13\to14\to15)} \end{pmatrix}}_{\text{metastable vacuum}} \rightarrow \underbrace{A_\mu = \alpha L_\mu}_{\text{stable vacuum}}$$

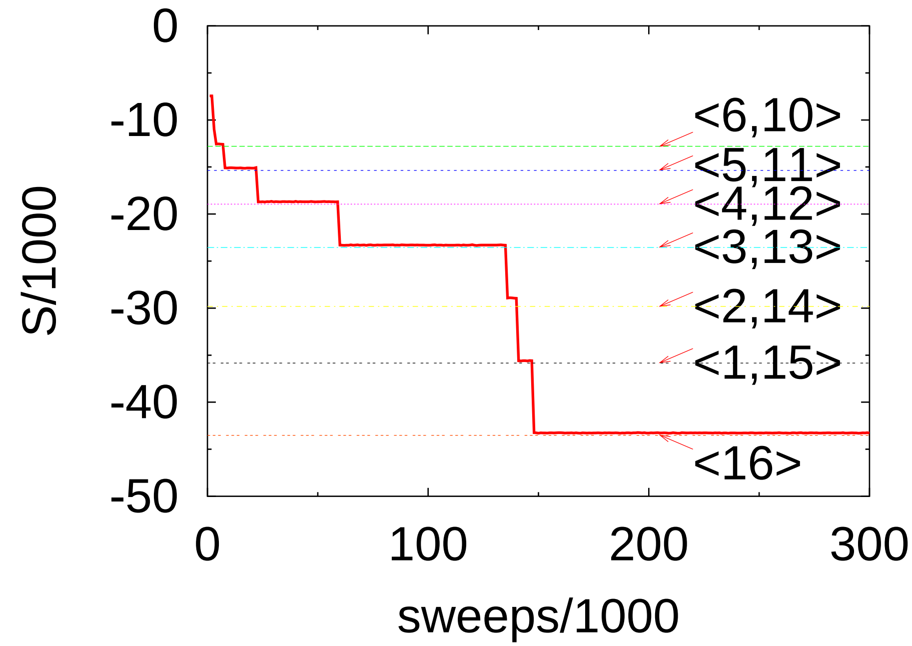
Analytical results

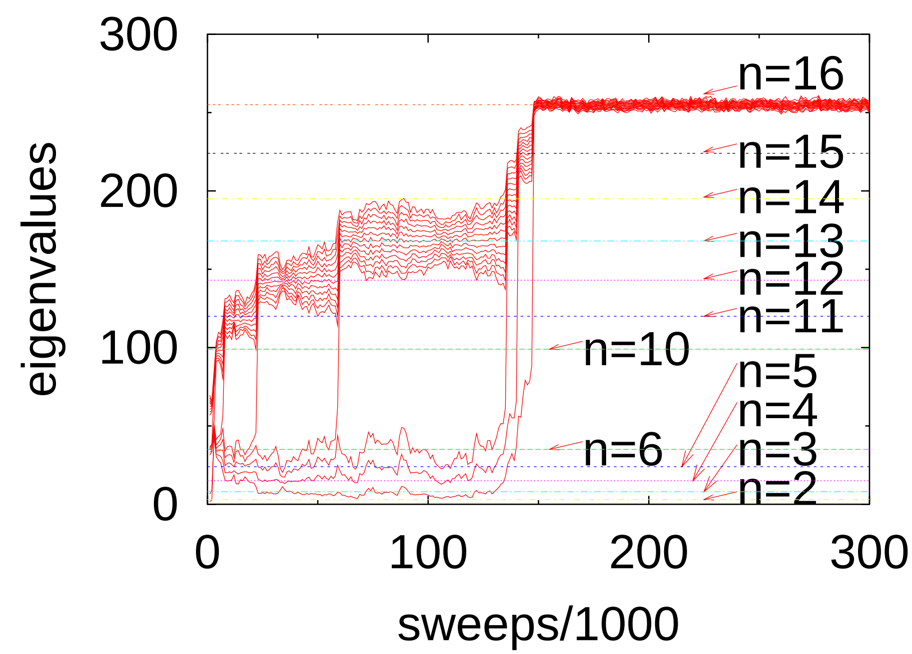
Calculation of the free energy

$$W = -\log \left(\int d\tilde{A} e^{-S} \right).$$

$k = 1$ **has the lowest free energy to all order of perturbation.**







4 Conclusion

We have reviewed the basic technicality of the heat bath algorithm of the large-N reduced model.

The simulation of the IIB matrix model is much easier than the quantum field theory, since the IIB matrix model is the totally reduced model.

We investigated the matrix model with the Chern-Simons term, to deepen the understanding of the fuzzy-sphere background.

Other related works

- **Numerical treatment of the supersymmetric case via the hybrid Monte Carlo simulation.**
- **Extension to the four-dimensional manifolds: fuzzy S^4, CP^2, $S^2 \times S^2$.**
- **3d bosonic massive YMCS model (nontrivial gauge group?)**