Simulating the four-dimensional fuzzy manifolds

hep-th/0405277

Takehiro Azuma
High-energy Accelerator Research Organization (KEK)

KEK Informal seminar
Jun. 10th 2004 11:00 $\sim$ 12:00

collaborated with S. Bal, K. Nagao and J. Nishimura

references: hep-th/0401038,0401120,0405096,0406***

Contents

1 Introduction

Large-N reduced models are the most powerful candidates for the constructive definition of superstring theory.

Matrix models on the homogeneous space

Several alterations of the IIB matrix model have been proposed, to accommodate the curved-space background.

- The matrix model with the Chern-Simons term:
  (hep-th/0101102,0204256,0207115)

These matrix models accommodate the curved-space fuzzy-manifold classical solutions, based on the homogeneous space.

A homogeneous space is realized as G/H:

- $G =$ (a Lie group)
- $H =$ (a closed subgroup of G)

$$S^2 = SU(2)/U(1),\ S^2 \times S^2,\ S^4 = SO(5)/U(2),$$
$$CP^2 = SU(3)/U(2), \cdots.$$

Such curved-space fuzzy-manifold solutions are interesting in the following senses:

- More manifest realization of the curved-space background:
  Essential for an eligible framework for gravity.
- We may get insight into the dynamical generation of the gauge group.

2 The model and its classical solutions

Here, we scrutinize the bosonic matrix model that accommodates the four-dimensional fuzzy manifold.
In the following, we focus on the fuzzy CP^2 manifold.

$$S = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{8}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\sum_{\mu,\nu,\rho=1}^{8} f_{\mu\nu\rho}A_\mu A_\nu A_\rho\right).$$

- Defined in the 8-dimensional Euclidean space: $(\mu, \nu, \cdots = 1, \cdots, 8)$
- A_μ are promoted to the $N \times N$ hermitian matrices.
- $f_{\mu\nu\rho}$ are the structure constant of the SU(3).

$$f_{123} = 1, \;\; f_{458} = f_{678} = \frac{\sqrt{3}}{2},$$
$$f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}.$$

Its equation of motion

$$[A_\nu, [A_\mu, A_\nu]] - i\alpha f_{\mu\nu\rho}[A_\nu, A_\rho] = 0$$

accommodates the following two classical solutions:

(a)fuzzy S^2 sphere

$$A_\mu^{(\mathrm{S}^2)} = \begin{cases} \alpha L_\mu^{(N)}, & (\mu = 1, 2, 3), \\ 0, & (\text{otherwise}). \end{cases}$$

The Casimir $Q = \sum_{\mu=1}^{8} A_\mu^2$ is given by

$$Q = \rho_{\mathrm{S}^2}^2 1_N = \alpha^2 \frac{N^2-1}{4} 1_N.$$

(b)fuzzy CP^2 space

The fuzzy CP^2 space is realized by the $(m,0)$ representation of the SU(3) Lie algebra:

$$A_\mu^{(\mathrm{CP}^2)} = \alpha T_\mu^{(m,0)}.$$

This corresponds to the $SU(3)/U(2)$ homogeneous space.

This space is realized by the symmetric tensor product of the fundamental representation of the SU(3) Lie algebra t_μ:

$$T_\mu^{(m,0)} = \underbrace{(t_\mu \otimes 1_3 \otimes \cdots \otimes 1_3)_{\mathrm{sym}}}_{m\text{-fold}} + (1_3 \otimes t_\mu \otimes \cdots \otimes 1_3)_{\mathrm{sym}} + \cdots + (1_3 \otimes \cdots \otimes 1_3 \otimes t_\mu)_{\mathrm{sym}}, \quad \text{where}$$

$$t_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$t_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad t_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$t_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Here $_{\mathrm{sym}}$ denotes the symmetric tensor product.

For the orthonormal states $|i\rangle$ and $|j\rangle$, the matrix element of A is $(A)_{ij} = \langle i|A|j\rangle$.

The usual tensor product is

$$\langle i_1, i_2|A \otimes B|j_1, j_2\rangle = \langle i_1|A|j_1\rangle\langle i_2|B|j_2\rangle.$$

The two-fold symmetric tensor product is

$$_{\text{sym}}\langle i_1, i_2|A \otimes B|j_1, j_2\rangle_{\text{sym}}, \quad \text{where}$$

$$|j_1, j_2\rangle_{\text{sym}} = \begin{cases} |j_1\rangle|j_2\rangle, & (\text{for } j_1 = j_2), \\ \frac{1}{\sqrt{2}}(|j_1\rangle|j_2\rangle + |j_2\rangle|j_1\rangle), & (\text{for } j_1 \neq j_2). \end{cases}$$

The symmetric state for the three products:

$$|1,1,1\rangle_{\text{sym}} = |1\rangle|1\rangle|1\rangle,$$
$$|1,1,2\rangle_{\text{sym}} = \frac{1}{\sqrt{3}}(|1\rangle|1\rangle|2\rangle + |1\rangle|2\rangle|1\rangle + |2\rangle|1\rangle|1\rangle),$$
$$|1,2,3\rangle_{\text{sym}} = \frac{1}{\sqrt{6}}(|1\rangle|2\rangle|3\rangle + (\text{the other 5 permutations})).$$

For the 3×3 matrices A and B

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

the symmetrized product is defined as

$$(A \otimes B)_{\text{sym}}$$
$$= \left(\begin{array}{c|cccccc} & |j_1=1, j_2=1\rangle_{\text{sym}} & |1,2\rangle_{\text{sym}} & |1,3\rangle_{\text{sym}} & |2,2\rangle_{\text{sym}} & |2,3\rangle_{\text{sym}} & |3,3\rangle_{\text{sym}} \\ \hline _{\text{sym}}\langle i_1=1, i_2=1| & a_{11}b_{11} & \frac{a_{12}b_{11}+a_{11}b_{12}}{\sqrt{2}} & \frac{a_{13}b_{11}+a_{11}b_{13}}{\sqrt{2}} & a_{12}b_{12} & \frac{a_{12}b_{13}+a_{13}b_{12}}{\sqrt{2}} & a_{13}b_{13} \\ _{\text{sym}}\langle 1,2| & C_{12,11} & C_{12,12} & C_{12,13} & C_{12,22} & C_{12,23} & C_{12,33} \\ _{\text{sym}}\langle 1,3| & C_{13,11} & C_{13,12} & C_{13,13} & C_{13,22} & C_{13,23} & C_{13,33} \\ _{\text{sym}}\langle 2,2| & a_{21}b_{21} & \frac{a_{22}b_{21}+a_{21}b_{22}}{\sqrt{2}} & \frac{a_{23}b_{21}+a_{21}b_{23}}{\sqrt{2}} & a_{22}b_{22} & \frac{a_{22}b_{23}+a_{23}b_{22}}{\sqrt{2}} & a_{23}b_{23} \\ _{\text{sym}}\langle 2,3| & C_{23,11} & C_{23,12} & C_{23,13} & C_{23,22} & C_{23,23} & C_{23,33} \\ _{\text{sym}}\langle 3,3| & a_{31}b_{31} & \frac{a_{32}b_{31}+a_{31}b_{32}}{\sqrt{2}} & \frac{a_{33}b_{31}+a_{31}b_{33}}{\sqrt{2}} & a_{32}b_{32} & \frac{a_{32}b_{33}+a_{33}b_{32}}{\sqrt{2}} & a_{33}b_{33} \end{array}\right),$$

$$C_{12,11} = \frac{a_{21}b_{11}+a_{11}b_{21}}{\sqrt{2}}, \quad C_{12,12} = \frac{a_{22}b_{11}+a_{21}b_{12}+a_{12}b_{21}+a_{11}b_{22}}{(\sqrt{2})^2}, \quad C_{12,13} = \frac{a_{23}b_{11}+a_{21}b_{13}+a_{13}b_{21}+a_{11}b_{23}}{(\sqrt{2})^2},$$
$$C_{12,22} = \frac{a_{22}b_{12}+a_{12}b_{22}}{\sqrt{2}}, \quad C_{12,23} = \frac{a_{22}b_{13}+a_{23}b_{12}+a_{12}b_{23}+a_{13}b_{22}}{(\sqrt{2})^2}, \quad C_{12,33} = \frac{a_{23}b_{13}+a_{13}b_{23}}{\sqrt{2}},$$
$$C_{13,11} = \frac{a_{31}b_{11}+a_{11}b_{31}}{\sqrt{2}}, \quad C_{13,12} = \frac{a_{32}b_{11}+a_{31}b_{12}+a_{12}b_{31}+a_{11}b_{32}}{(\sqrt{2})^2}, \quad C_{13,13} = \frac{a_{33}b_{11}+a_{31}b_{13}+a_{13}b_{31}+a_{11}b_{33}}{(\sqrt{2})^2},$$
$$C_{13,22} = \frac{a_{32}b_{12}+a_{12}b_{32}}{\sqrt{2}}, \quad C_{13,23} = \frac{a_{32}b_{13}+a_{33}b_{12}+a_{12}b_{33}+a_{13}b_{32}}{(\sqrt{2})^2}, \quad C_{13,33} = \frac{a_{33}b_{13}+a_{13}b_{33}}{\sqrt{2}},$$
$$C_{23,11} = \frac{a_{31}b_{21}+a_{21}b_{31}}{\sqrt{2}}, \quad C_{23,12} = \frac{a_{32}b_{21}+a_{31}b_{22}+a_{22}b_{31}+a_{21}b_{32}}{(\sqrt{2})^2}, \quad C_{23,13} = \frac{a_{33}b_{21}+a_{31}b_{23}+a_{23}b_{31}+a_{21}b_{33}}{(\sqrt{2})^2},$$
$$C_{23,22} = \frac{a_{32}b_{22}+a_{22}b_{32}}{\sqrt{2}}, \quad C_{23,23} = \frac{a_{32}b_{23}+a_{33}b_{22}+a_{22}b_{33}+a_{23}b_{32}}{(\sqrt{2})^2}, \quad C_{23,33} = \frac{a_{33}b_{23}+a_{23}b_{33}}{\sqrt{2}}.$$

Using this definition, we derive the following formula:

$$\sum_{\mu=1}^{8}(t_\mu \otimes t_\mu) = \frac{1}{12}\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 6 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \overset{\text{sym}}{\to} \frac{1}{12}\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

$$= \frac{1}{3}(1_3 \otimes 1_3)_{\text{sym}}.$$

The Casimir is thus given by

$$\begin{aligned} Q &= \rho^2_{\text{CP}^2} 1_N = \alpha^2 \sum_{\mu=1}^{8} T_\mu^{(m,0)} T_\mu^{(m,0)} \\ &= \alpha^2 \underbrace{(t_\mu^2 \otimes 1_3 \otimes \cdots \otimes 1_3) + \cdots + (1_3 \otimes \cdots \otimes 1_3 \otimes t_\mu^2)}_{m \text{ terms}} \\ &+ \alpha^2 \underbrace{(t_\mu \otimes t_\mu \otimes 1_3 \otimes \cdots \otimes 1_3) + \cdots}_{m(m-1) \text{ terms}} \\ &= \alpha^2 \left(\frac{4m}{3} + \frac{1}{3} m(m-1)\right) 1_N = \alpha^2 \frac{m(m+3)}{3} 1_N, \end{aligned}$$

where we have used $\sum_{\mu=1}^{8}(t_\mu \otimes t_\mu)_{\text{sym}} = \frac{1}{3}(1_3 \otimes 1_3)_{\text{sym}}$ and $\sum_{\mu=1}^{8} t_\mu^2 = \frac{4}{3} 1_3$.

The matrix size of this representation is

$$N = \frac{(m+1)(m+2)}{3}, \ (\text{for } m = 1, 2, 3, \cdots).$$

Thus, this representation is realized for a limited size of the matrices $N = 3, 6, 10, 15, 21, \cdots$.

3 Monte Carlo simulation of the matrix model

We analyze the matrix model through the heat-bath algorithm of the Monte Carlo simulation.

In this sense, our approach is nonperturbative.

Heat bath algorithm of the matrix model

(a) Warm-up: quadratic U(N) one-matrix model

We start with the simplest case – quadratic U(N) one-matrix model:

$$S = \frac{N}{2}\mathrm{tr}\,\phi^2.$$

We analyze this model via the heat bath algorithm. To this end, we rewrite the U(N) matrix ϕ as

$$\phi_{ii} = \frac{a_i}{\sqrt{N}}, \quad \begin{cases} \phi_{ij} = \frac{x_{ij}+iy_{ij}}{\sqrt{2N}} \\ \phi_{ji} = \frac{x_{ij}-iy_{ij}}{\sqrt{2N}}, \end{cases} \quad (\text{for } i < j).$$

The N^2 real quantities a_i, x_{ij}, y_{ij} comply with the independent normal Gaussian distribution.

$$S = \frac{1}{2}\sum_{i=1}^{N} a_i^2 + \frac{1}{2}\sum_{i<j}((x_{ij})^2 + (y_{ij})^2).$$

$$Z = \int \prod_{i=1}^{N} da_i \prod_{1\leq i<j\leq N} dx_{ij}dy_{ij} \exp\left(-\frac{1}{2}\sum_{i=1}^{N} a_i^2 - \frac{1}{2}\sum_{1\leq i<j\leq N}((x_{ij})^2 + (y_{ij})^2)\right).$$

a_i, x_{ij}, y_{ij} are updated by the normal Gaussian random number.

Generation of the uniform random number

We use the congruence method.

- We give the random seed z_1, such as $z_1 = \mathrm{time}()$.
- We solve the recursion formula

$$z_{k+1} = a z_k + c \pmod{2^{31} - 1}.$$

The choice $(a, c) = (5^{11}, 0)$ is known to give a good pseudo-random number.

- The sequence $\{\frac{z_k}{2^{31}-1}\}$ gives a uniform pseudo-random number $[0{:}1]$.

Generation of the Gaussian random number

- We take two uniform random numbers $x, y \in [0:1]$.
- We introduce the quantity $r = \sqrt{-a^2 \log x^2}$. This complies with the probability distribution

$$P(r)dr = P(x)\frac{dx}{dr}dr = \frac{2r}{a^2}\exp\left(-\frac{r^2}{a^2}\right)dr.$$

- We next introduce the quantities

$$X = r\cos(2\pi y), \quad Y = r\sin(2\pi y).$$

They comply with the probability distribution

$$P(r)drdy \propto \exp\left(-\frac{1}{a^2}(X^2 + Y^2)\right)dXdY.$$

(b) The bosonic IIB matrix model

T. Hotta, J. Nishimura and A. Tsuchiya hep-th/9811220.

We investigate the D-dimensional bosonic IIB matrix model via the the heat bath algorithm:

$$S = -\frac{N}{4}\sum_{\mu,\nu=1}^{D} \mathrm{tr}\,[A_\mu, A_\nu]^2 = -\frac{N}{2}\sum_{1\le\mu<\nu\le D} \mathrm{tr}\,\{A_\mu, A_\nu\}^2 + 2N\sum_{\mu<\nu}\mathrm{tr}\,(A_\mu^2 A_\nu^2).$$

This action is equivalent to $\tilde{S}$, after integrating out $Q_{\mu\nu}$ (where $G_{\mu\nu} = \{A_\mu, A_\nu\}$):

$$\begin{aligned}\tilde{S} &= N\sum_{\mu<\nu}\left(\frac{1}{2}\mathrm{tr}\,Q_{\mu\nu}^2 - \mathrm{tr}\,(Q_{\mu\nu}G_{\mu\nu}) + 2\mathrm{tr}\,(A_\mu^2 A_\nu^2)\right) \\ &= \frac{N}{2}\sum_{\mu<\nu}\mathrm{tr}\,(Q_{\mu\nu} - G_{\mu\nu})^2 + S.\end{aligned}$$

Then, $Q_{\mu\nu}$ is updated as

$$(Q_{\mu\nu})_{ii} = \frac{a_i}{\sqrt{N}} + (G_{\mu\nu})_{ii},\ (Q_{\mu\nu})_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{2N}} + (G_{\mu\nu})_{ij},$$

We next update A_λ. We extract the dependence of A_λ.

$$\tilde{S} = -N\mathrm{tr}\,(T_\lambda A_\lambda) + 2N\mathrm{tr}\,(S_\lambda A_\lambda^2) + \cdots, \text{ where}$$
$$S_\lambda = \sum_{\mu\neq\lambda}(A_\mu^2),\ T_\lambda = \sum_{\mu\neq\lambda}(A_\mu Q_{\lambda\mu} + Q_{\lambda\mu}A_\mu).$$

- The diagonal part A_λ is updated by extracting the dependence of $(A_\lambda)_{ii}$:

$$\tilde{S} = 2N(S_\lambda)_{ii}(A_\lambda)_{ii}^2 - 4Nh_i(A_\lambda)_{ii}, \text{ where}$$
$$h_i = \frac{N}{4}[(T_\lambda)_{ii} - 2\sum_{j\neq i}((S_\lambda)_{ji}(A_\lambda)_{ij} + (S_\lambda)_{ij}(A_\lambda)_{ji})].$$

Then, $(A_\lambda)_{ii}$ is updated as

$$(A_\lambda)_{ii} = \frac{a_i}{\sqrt{4N(S_\lambda)_{ii}}} + \frac{h_i}{(S_\lambda)_{ii}}.$$

- **The other components $(A_\lambda)_{ij}$ are updated likewise by extracting their dependence:**

$$\tilde{S} = 2Nc_{ij}|(A_\lambda)_{ij}|^2 - 2Nh_{ji}(A_\lambda)_{ij}, \text{ where}$$
$$c_{ij} = (S_\lambda)_{ii} + (S_\lambda)_{jj},$$
$$h_{ij} = \frac{1}{2}(T_\lambda)_{ij} - \sum_{k\neq i}(S_\lambda)_{ik}(A_\lambda)_{kj} - \sum_{k\neq j}(S_\lambda)_{kj}(A_\lambda)_{ik}.$$

Then, $(A_\lambda)_{ij}$ are updated as

$$(A_\lambda)_{ij} = \frac{x_{ij} + iy_{ij}}{\sqrt{4Nh_{ij}}} + \frac{h_{ij}}{c_{ij}}.$$

(c) Addition of the Chern-Simons term

The Chern-Simons term is *linear* with respect to *each* A_μ.

The algorithm is similar for the following actions:

$$\mathrm{S}^2 \quad : S_{\mathrm{S}^2} = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{3}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right),$$
$$\mathrm{S}^4 \quad : S_{\mathrm{S}^4} = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{5}[A_\mu, A_\nu]^2 - \frac{\lambda}{5}\epsilon_{\mu_1\cdots\mu_5}A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}A_{\mu_5}\right),$$
$$\mathrm{CP}^2 \quad : S_{\mathrm{CP}^2} = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{8}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\sum_{\mu,\nu=1}^{8}f_{\mu\nu\rho}A_\mu A_\nu A_\rho\right),$$
$$\mathrm{S}^2\times\mathrm{S}^2 \quad : S_{\mathrm{S}^2\times\mathrm{S}^2} = N\mathrm{tr}\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{6}[A_\mu, A_\nu]^2 + \frac{2i}{3}f^{(\mathrm{S}^2\times\mathrm{S}^2)}_{\mu\nu\rho}A_\mu A_\nu A_\rho\right),$$

where the structure constant for the $\mathrm{S}^2 \times \mathrm{S}^2$ model is

$$f^{(\mathrm{S}^2\times\mathrm{S}^2)}_{\mu\nu\rho} = \begin{cases} \alpha_1\epsilon_{\mu\nu\rho}, & (\text{for } \mu,\nu,\rho = 1,2,3), \\ \alpha_2\epsilon_{\mu\nu\rho}, & (\text{for } \mu,\nu,\rho = 4,5,6), \\ 0, & (\text{otherwise}). \end{cases}$$

We have only to replace T_ρ as

$$T_\rho = \sum_{\mu\neq\lambda}(A_\mu Q_{\lambda\mu} + Q_{\lambda\mu}A_\mu) + T_\rho^{(\mathrm{CS})}, \text{ where}$$
$$T_\rho^{(\mathrm{CS})} = \begin{cases} -2i\alpha\sum_{\mu,\nu=1}^{3}\epsilon_{\rho\mu\nu}A_\mu A_\nu, & (\text{for } S^2), \\ -2i\alpha\sum_{\mu,\nu=1}^{8}f_{\rho\mu\nu}A_\mu A_\nu, & (\text{for } CP^2), \\ -2i\sum_{\mu,\nu=1}^{6}f^{(S^2\times S^2)}_{\rho\mu\nu}A_\mu A_\nu, & (\text{for } S^2\times S^2), \\ \lambda\sum_{\mu_1\cdots\mu_4=1}^{5}\epsilon_{\rho\mu_1\cdots\mu_4}A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}, & (\text{for } S^4). \end{cases}$$

4 Fuzzy CP^2 classical solution

We start from the fuzzy CP^2 initial condition:

$$A_\mu^{(0)} = A_\mu^{(CP^2)}.$$

To see the behavior of this solution, we discuss the following observables:

- The action S.
- The spacetime extent $\frac{1}{N}\mathrm{tr}\sum_{\mu=1}^{8} A_\mu^2$.

Here, we introduce the rescaled parameter

$$\bar{\alpha} = \alpha N^{\frac{1}{4}}.$$

first-order phase transition

We have a first-order phase transition, at the critical point

$$\bar{\alpha} = \bar{\alpha}_{cr}^{(CP^2)} (= \alpha_{cr}^{(CP^2)} N^{\frac{1}{4}} \simeq 2.3).$$

- $\alpha < \alpha_{cr}^{(CP^2)}$: the effect of the Chern-Simons term is negated, and we see the following behavior typical of the pure Yang-Mills model:

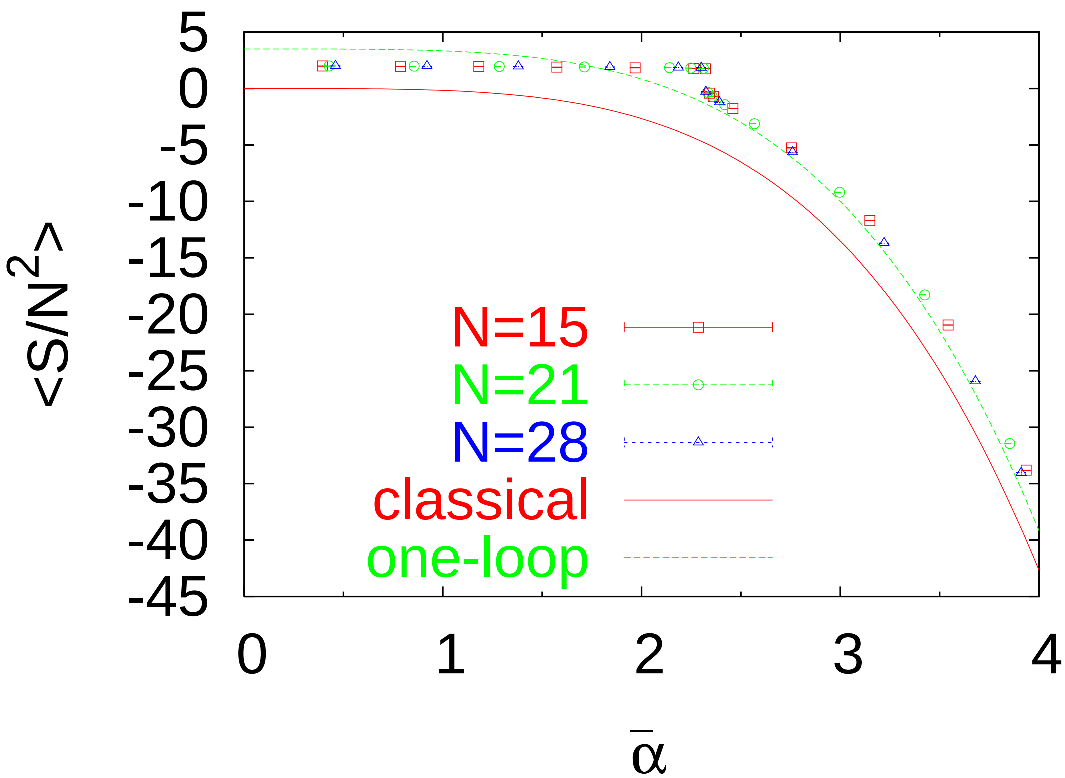
$$\frac{1}{N^2}\langle S\rangle \simeq O(1), \ \langle\frac{1}{N}\mathrm{tr}A_\mu^2\rangle \simeq O(1).$$

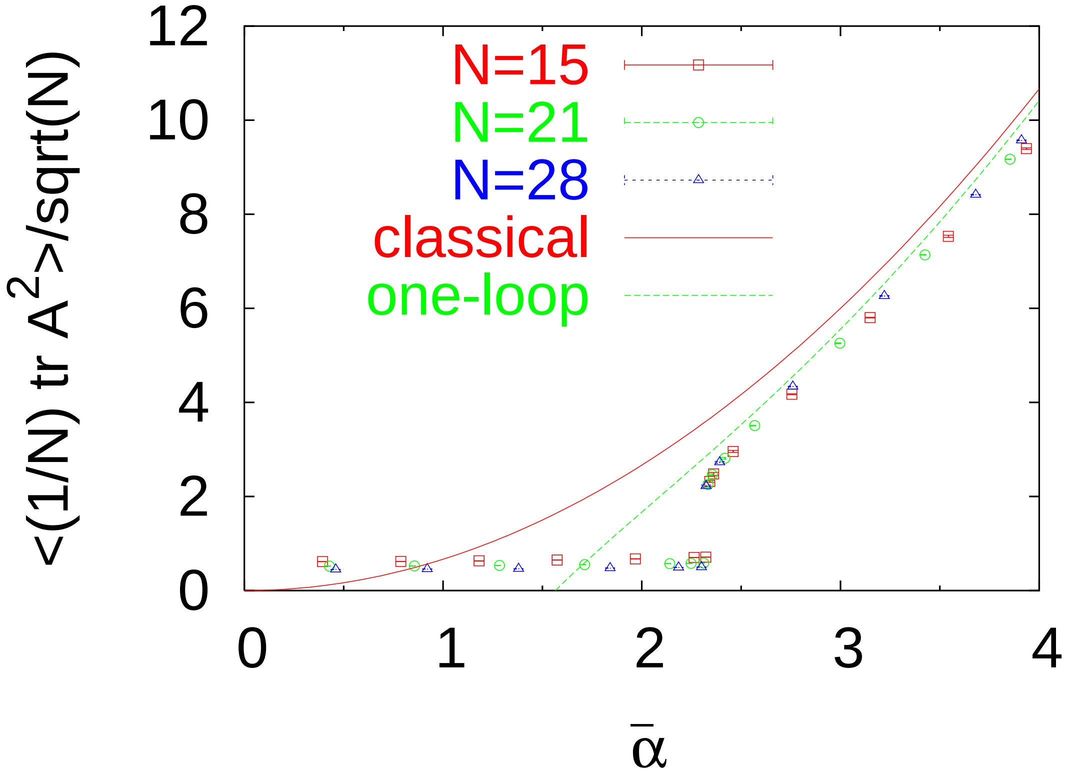
- $\alpha > \alpha_{cr}^{(CP^2)}$: the fuzzy CP^2 is metastable.

one-loop dominance

The numerical results are close to the one-loop result at $\alpha > \alpha_{cr}^{(CP^2)}$:

$$\frac{1}{N^2}\langle S\rangle \simeq -\frac{\bar{\alpha}^4}{6} + \frac{7}{2}, \quad \frac{1}{\sqrt{N}}\langle\frac{1}{N}\mathrm{tr}\sum_{\mu=1}^{8} A_\mu^2\rangle \simeq \frac{2\bar{\alpha}^2}{3} - \frac{4}{\bar{\alpha}^2}.$$

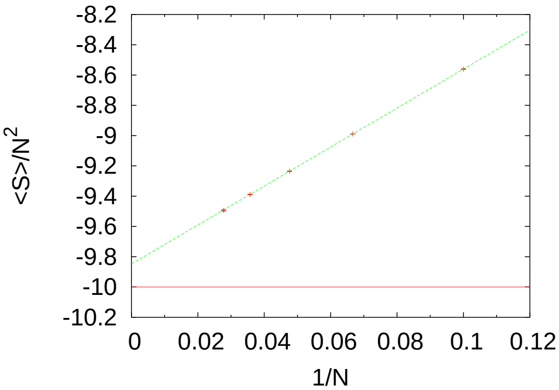

5
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
0
1
2
3
4
<S/N2>
ᾱ
N=15
N=21
N=28
classical
one-loop

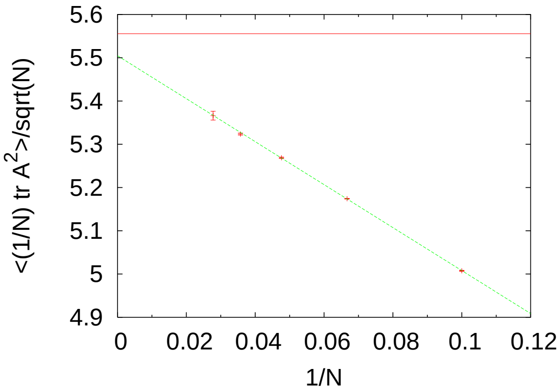

12
10
8
6
4
2
0
0
1
2
3
4
<(1/N) tr A2>/sqrt(N)
ᾱ
N=15
N=21
N=28
classical
one-loop

finite-N effect

We extrapolate the **finite-N effect**, by plotting these observables **against $\frac{1}{N}$**:

- $N = 10, 15, 21, 28, 36$ $(m = 3, 4, 5, 6, 7)$.
- $\bar{\alpha} = 3.0$ is fixed.





- The finite-N effects are of the order $O(\frac{1}{N})$.
- We have a deviation from the one-loop calculation at **large N**.

Since the deviation is rather small, we nevertheless regard this system **as retaining the "one-loop dominance"**.

In fact, the three-dimensional model with fuzzy S^2 classical solution (scrutinized in hep-th/0401038) also has the same deviation.

The critical point $\bar{\alpha}_{\rm cr}^{(\rm CP^2)} \simeq 2.3$ is consistent with the one-loop calculation.

We start with the one-loop effective action around $A_\mu = \beta T_\mu^{(m,0)}$ at large N.

$$W_{\rm CP^2} \simeq N^2 \left(\frac{2}{3k} \left(\frac{3\bar{\beta}^4}{4} - \bar{\alpha}\bar{\beta}^3 \right) + 6 \log \bar{\beta} + (\text{const.}) \right).$$

This has a minimum at $\frac{\partial W_{\rm CP^2}}{\partial \bar{\beta}} = 0$, namely

$$f(\bar{\beta}) = (\bar{\beta}^4 - \bar{\alpha}\bar{\beta}^3) + 3 = 0.$$

$f(\bar{\beta})$ has a minimum at $\bar{\beta}_{\rm min} = \frac{3}{4}\bar{\alpha}$.

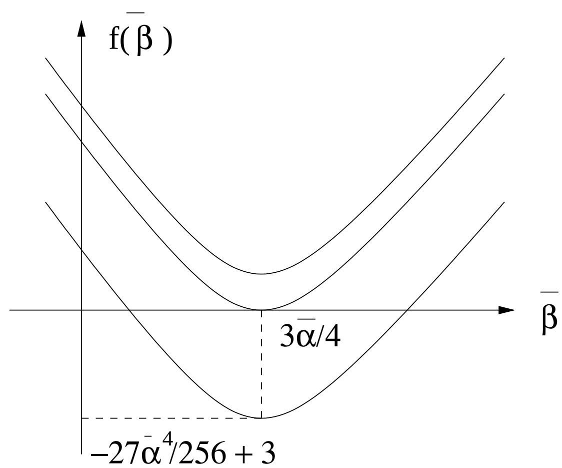
At this critical point, we have

$$f(\bar{\beta}_{\rm min}) \simeq -\frac{1}{3}(\frac{3}{4})^4 \bar{\alpha}^4 + 3 = 0.$$

Then, the critical point is determined as

$$\bar{\alpha}_{\rm cr}^{(\rm CP^2)} = \frac{4}{\sqrt{3}} \simeq 2.3094011 \cdots.$$

This is consistent with the numerical observation.



5 Fuzzy S^2 classical solution

We next start the simulation from the fuzzy S^2 initial condition:

$$A_\mu^{(0)} = A_\mu^{(S^2)}.$$

We plot the observables against the rescaled parameter

$$\tilde{\alpha} = \alpha N^{\frac{1}{2}}.$$

first-order phase transition

We have a first-order phase transition, at the critical point

$$\tilde{\alpha} = \tilde{\alpha}_{\rm cr}^{(S^2)} (= \alpha_{\rm cr}^{(S^2)} N^{\frac{1}{2}} \simeq 3.2).$$

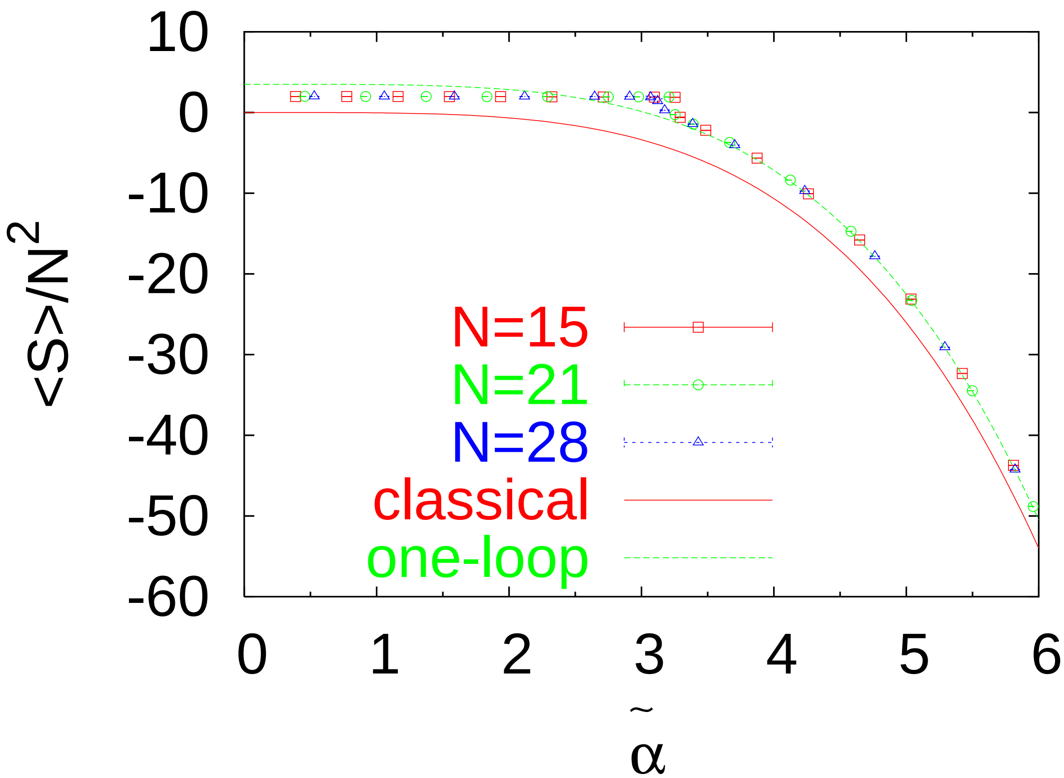
- $\alpha < \alpha_{\rm cr}^{(S^2)}$: The behavior is similar to the pure Yang-Mills model.
- $\alpha > \alpha_{\rm cr}^{(S^2)}$: the fuzzy S^2 is stable.

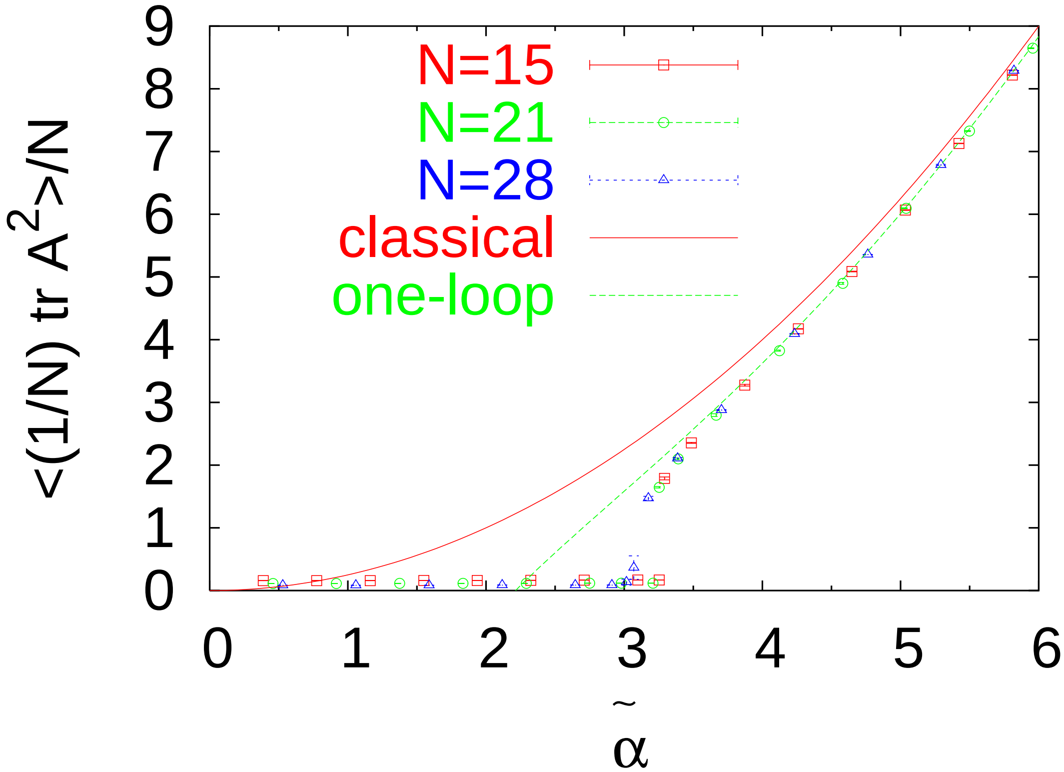
one-loop dominance

The numerical results are close to the one-loop result at $\alpha > \alpha_{\rm cr}^{(S^2)}$:

$$\frac{1}{N^2}\langle S \rangle \simeq -\frac{\tilde{\alpha}^4}{24} + \frac{7}{2},$$

$$\frac{1}{N}\langle \frac{1}{N} \text{tr} \sum_{\mu=1}^{8} A_\mu^2 \rangle \simeq \frac{\tilde{\alpha}^2}{4} - \frac{6}{\tilde{\alpha}^2}.$$

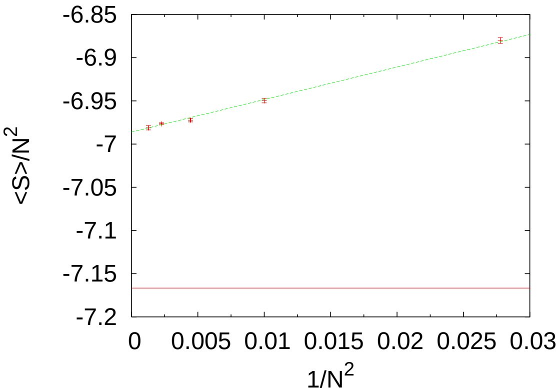

10
0
-10
-20
-30
-40
-50
-60
0
1
2
3
4
5
6
<S>/N^2
α̃
N=15
N=21
N=28
classical
one-loop

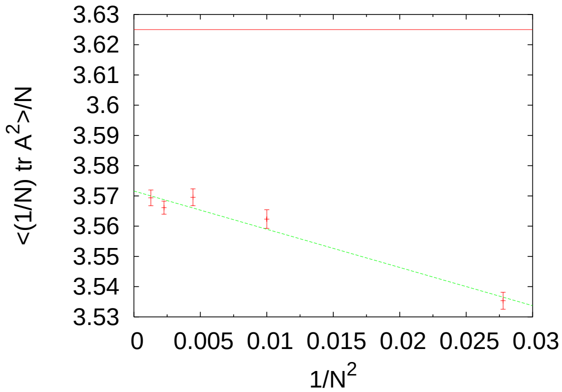

9
8
7
6
5
4
3
2
1
0
0
1
2
3
4
5
6
<(1/N) tr A^2>/N
α̃
N=15
N=21
N=28
classical
one-loop

finite-N effect

We extrapolate the finite-N effect, by plotting these observables against $\frac{1}{N^2}$:

- $N = 6, 10, 15, 21, 28$.
- $\tilde{\alpha} = 4.0$ is fixed.





For the fuzzy S^2 classical solution, we likewise see the nonperturbative deviation from the one loop at large N.

The critical point is derived from the one-loop effective action as

$$\tilde{\alpha}_{\mathrm{cr}}^{(\mathrm{S}^2)} = \sqrt{\frac{32}{3}} \simeq 3.2659863 \cdots .$$

6 Dynamical generation of the gauge group

We discuss the k ($k \geq 2$) coincident fuzzy manifolds (multi fuzzy CP^2), to see the dynamical generation of the gauge group.

The expansion around the k coincide fuzzy manifolds gives rise to the U(k) gauge group.

fuzzy CP^2 space

We define the k coincident fuzzy CP^2 manifolds as

$$A_\mu^{(k,\mathrm{CP}^2)} = \alpha T_\mu^{(m,0)} \otimes 1_k$$

The size of the matrix is $N = \frac{k(m+1)(m+2)}{2}$.

We launch a simulation for $k = 2, m = 3, 4, 5$ ($N = 20, 30, 42$), starting from $A_\mu^{(0)} = A_\mu^{(k=2,\mathrm{CP}^2)}$.

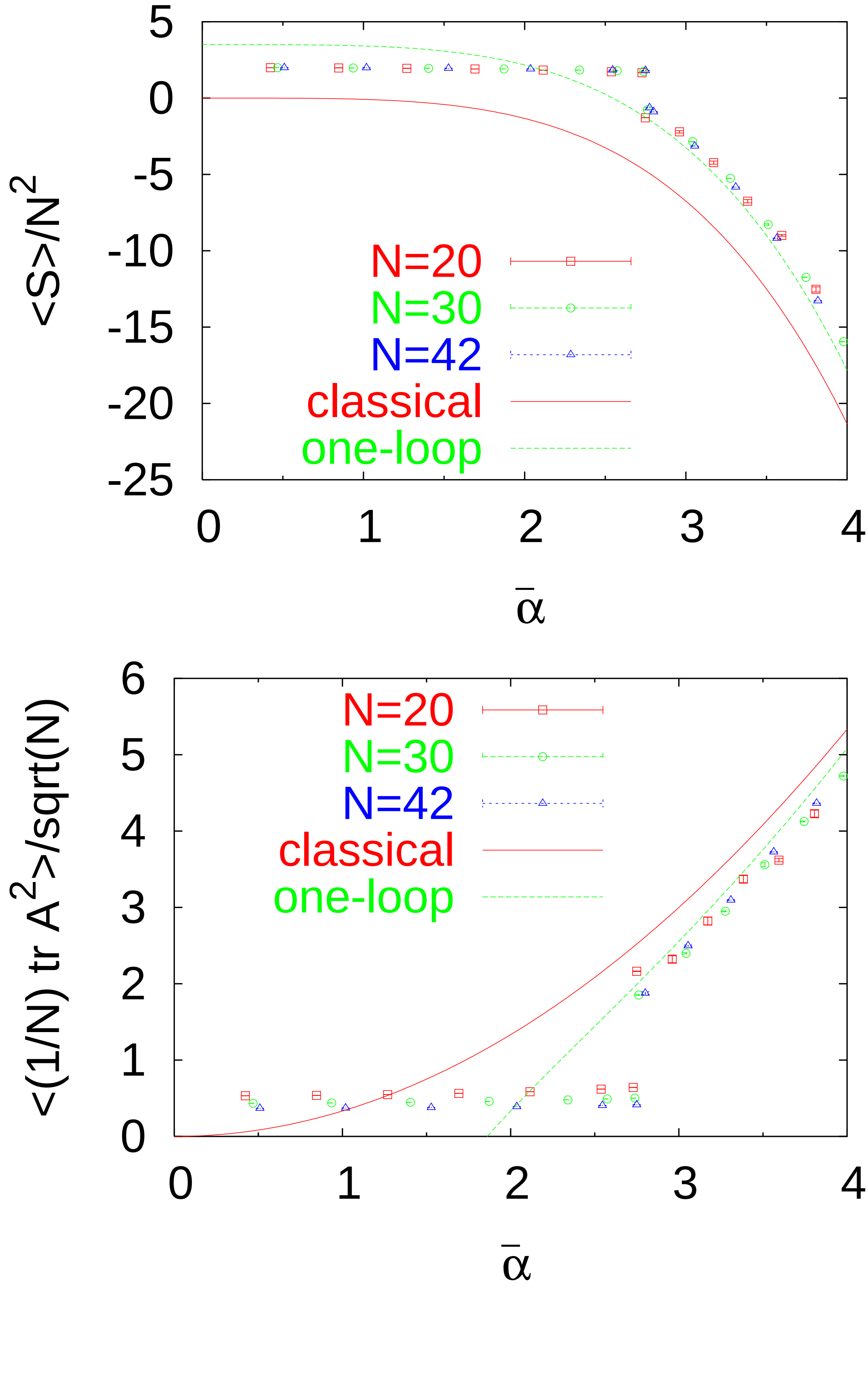
Before this multi fuzzy CP^2's decay, the system has the first-order phase transition at

$$\bar{\alpha}_{\mathrm{cr}}^{(k=2,\mathrm{CP}^2)} \simeq 2.7.$$

At $\alpha > \alpha_{\mathrm{cr}}^{(k=2,\mathrm{CP}^2)}$, the system retains the one-loop dominance, in which the observables are close to the one-loop results.

$$\frac{1}{N^2}\langle S \rangle \simeq -\frac{\bar{\alpha}^4}{6k} + \frac{7}{2}.$$

$$\frac{1}{\sqrt{N}}\langle \frac{1}{N}\mathrm{tr}\, A_\mu^2 \rangle \simeq \frac{2\bar{\alpha}^2}{3k} - \frac{4}{\bar{\alpha}^2}.$$



<S>/N^2
N=20
N=30
N=42
classical
one-loop
ᾱ
<(1/N) tr A^2>/sqrt(N)
N=20
N=30
N=42
classical
one-loop
ᾱ

The critical point agrees with the one-loop calculation

$$\bar{\alpha}_{\text{cr}}^{(k,\text{CP}^2)} = \frac{4}{\sqrt{3}} k^{\frac{1}{4}} \simeq 2.3094011 k^{\frac{1}{4}}.$$

We discuss the stability of the multi fuzzy CP^2 from the one-loop effective action.

This classical solution retains the one-loop dominance.

Thus, we discuss the stability of the k coincident fuzzy CP^2's via the one-loop effective action:

$$W_{k,\text{CP}^2} = N^2 \left(-\frac{\bar{\alpha}^4}{6k} + 6 \log \bar{\alpha} + 3 \log \frac{N^{\frac{3}{2}}}{k} \right).$$

When the single fuzzy CP^2 $(k = 1)$ is more stable than the the multi $(k \geq 2)$ fuzzy CP^2's, we obtain

$$W_{k=1,\text{CP}^2} < W_{k,\text{CP}^2} \Rightarrow \bar{\alpha} > \bar{\alpha}_{k,\text{CP}^2} = \left(\frac{18}{1 - \frac{1}{k}} \log k \right)^{\frac{1}{4}}.$$

Since we always have $\bar{\alpha}_{k,\text{CP}^2} < \bar{\alpha}_{\text{cr}}^{(k,\text{CP}^2)}$ when the fuzzy CP^2 is stable, we have

$$\bar{\alpha}_{k,\text{CP}^2} < \bar{\alpha}_{\text{cr}}^{(k,\text{CP}^2)} < \bar{\alpha}.$$

Therefore, the single $(k = 1)$ multi fuzzy CP^2 is always the most stable.

This leads to the dynamical generation of the U(1) gauge group.

fuzzy S^2 space

We likewise discuss the k coincident fuzzy S^2 spaces (multi fuzzy S^2):

$$A_\mu^{(k,S^2)} = \begin{cases} \alpha L_\mu^{(n)} \otimes 1_k, & (\text{for } \mu = 1,2,3), \\ 0, & (\text{otherwise}). \end{cases}$$

The size of the matrices is $N = nk$.
This system likewise retains the one-loop dominance.

The critical point is calculated as

$$\tilde{\alpha}_{\rm cr}^{(k,S^2)} = \sqrt{\frac{32k}{3}} \simeq 3.2659863\sqrt{k}.$$

The multi fuzzy S^2 retains metastability for $\tilde{\alpha} > \tilde{\alpha}_{\rm cr}^{(k,S^2)}$.
The one-loop effective action at large N is

$$W_{k,S^2} \simeq N^2 \left(-\frac{\tilde{\alpha}^4}{24k^2} + 6 \log \tilde{\alpha} + 6 \log \frac{N}{k} \right).$$

If the single ($k = 1$) fuzzy S^2 is more stable than the multi ($k \geq 2$) fuzzy S^2's, we have $W_{k=1,S^2} < W_{k,S^2}$:

$$\tilde{\alpha} > \tilde{\alpha}_{k,S^2} = \left(\frac{144 \log k}{1 - \frac{1}{k^2}} \right)^{\frac{1}{4}}.$$

Since $\tilde{\alpha}_{k,S^2} < \tilde{\alpha}_{\rm cr}^{(k,S^2)}$, we have in the fuzzy S^2 phase

$$\tilde{\alpha}_{k,S^2} < \tilde{\alpha}_{\rm cr}^{(k,S^2)} < \tilde{\alpha}.$$

Therefore, the single ($k = 1$) multi fuzzy S^2 is always the most stable, which leads to the dynamical generation of the U(1) gauge group.

7 Fuzzy CP^2 versus S^2 — which is the true vacuum?

We determine which is the true vacuum, according to the one-loop dominance.

The one-loop effective action around the fuzzy CP^2 and S^2 is

$$\begin{aligned} W_{CP^2} &= -\frac{m(m+3)}{12}\alpha^4 N^2 + 3\sum_{c=1}^{m}(c+1)^3 \log[N\alpha^2 c(c+2)] \\ &\simeq N^2\left(-\frac{\alpha^4 N}{6} + 6\log\alpha + 6\log N\right), \\ W_{S^2} &= -\frac{1}{24}\alpha^4 N^2(N^2-1) + 3\sum_{l=1}^{N-1}(2l+1)\log[N\alpha^2 l(l+1)] \\ &\simeq N^2\left(-\frac{\alpha^4 N^2}{24} + 6\log\alpha + 9\log N\right). \end{aligned}$$

The difference is calculated (at large N) as

$$\Delta = W_{S^2} - W_{CP^2} = N^2\left\{\alpha^4\left(-\frac{N^2}{24}+\frac{N}{6}\right) + 3\log N\right\}.$$

- The classical effect is $O(N^4)$.
- Whereas, the one-loop quantum effect is $O(N^2\log N)$.

Therefore, $\Delta < 0$, namely $W_{S^2} < W_{CP^2}$.

The fuzzy S^2 is the true vacuum, and the fuzzy CP^2 is a metastable state.

Nevertheless, the fuzzy CP^2 state retains a very strong metastability.

We start from the initial condition $A_\mu^{(0)} = A_\mu^{(\mathrm{CP}^2)}$, for $N = 10(m = 3), \alpha = 1.4$ (in which the fuzzy CP^2 is metastable).

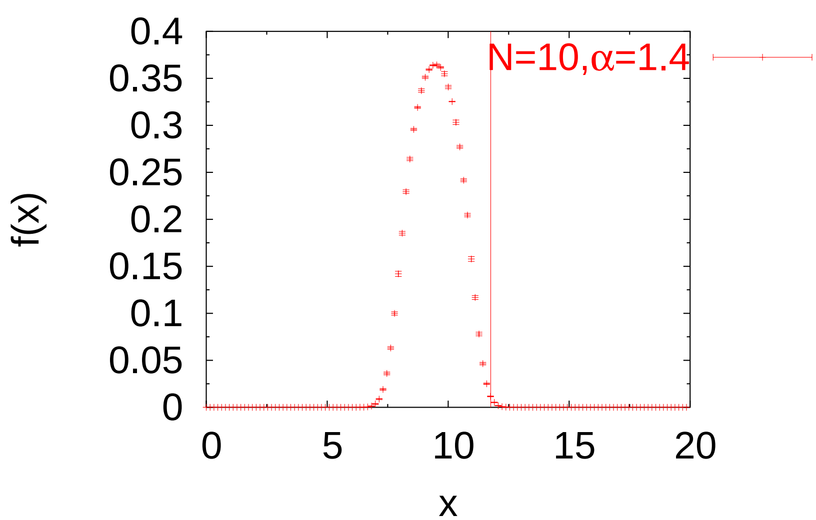
The initial fuzzy CP^2 state endures the 5×10^7 sweeps.

The eigenvalue distribution $f(x)$ of the Casimir Q is defined as

$$f(x) = \frac{1}{N}\sum_{j=1}^{N} \langle \delta(x - \lambda_j) \rangle,$$

where $\{\lambda_j\} =$ (eigenvalues of Q).

Measured after the 5×10^7 sweep, $f(x)$ is plotted below:



Here, the radius of the fuzzy CP^2 space is

$$\rho^2_{\mathrm{CP}^2} = \alpha^2 \frac{m(m+3)}{3} = 11.76.$$

8 Conclusion

In this talk, we have discussed the bosonic matrix model that incorporates the four-dimensional fuzzy CP^2 space.

- **The true vacuum of this matrix model is not the fuzzy CP^2 but the fuzzy S^2.**
- **The fuzzy CP^2 is realized as a metastable state.**
- **Both of these solutions have the one-loop dominance, with a small deviation at large N.**
- **The k ($k \geq 2$) coincident fuzzy spaces are always unstable both for the fuzzy CP^2 and S^2.**
  This leads to the dynamical generation of the U(1) gauge group.

Future works:

- **The investigation of the supersymmetric system:**
  We expect that the four-dimensional fuzzy manifold might be the true vacuum due to the supersymmetry.