Complex Langevin analysis of the spontaneous rotational symmetry breaking in the dimensionally-reduced super-Yang-Mills models

Strings 2018, Jun 27th 2018 (Wed) Konstantinos N. Anagnostopoulos, **Takehiro Azuma**, Yuta Ito, Jun Nishimura, Toshiyuki Okubo and Stratos Kovalkov Papadoudis

1. Introduction

Difficulties in putting complex partition functions on computers.

$$Z=\int dA \exp(-S_0+i\Gamma),\quad Z_0=\int dA e^{-S_0}$$

Sign problem:
The reweighting $\langle \mathscr{O}\rangle=\frac{\langle \mathscr{O}e^{i\Gamma}\rangle_0}{\langle e^{i\Gamma}\rangle_0}$ requires configs. $\exp[O(N^2)]$

$\langle * \rangle_0$ = (V.E.V. for phase-quenched Z_0)

2. The Euclidean IKKT model

Candidate for nonperturbative string theory
[N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115]

$$S=\underbrace{-\frac{N}{4}\mathrm{tr}[A_\mu,A_\nu]^2}_{=S_b}+\underbrace{N\mathrm{tr}\bar\psi_\alpha(\Gamma_\mu)_{\alpha\beta}[A_\mu,\psi_\beta]}_{=S_f}$$

Euclidean case after Wick rotation
$A_0\to iA_{10}, \Gamma^0\to -i\Gamma_{10}$.
⇒Path integral is finite without cutoff.

- A_μ, Ψ_α ⇒N×N Hermitian traceless matrices.
$\mu=1,2,\cdots,D,\quad \alpha,\beta=\begin{cases}1,2,3,4 & (D=6)\\ 1,2,\cdots,16 & (D=10)\end{cases}$
- Originally defined in **D=10**.
We consider the ***simplified D=6 case*** as well.
- Integrating out ψ yields $\det\mathscr{M}$ in D=6 ($\mathrm{Pf}\mathscr{M}$ in D=10)
- Det/Pf$\mathscr{M}$'s *complex phase* ⇒Spontaneous Symmetry Breaking (SSB) of SO(D).

Result of Gaussian Expansion Method (GEM)

[T.Aoyama, J.Nishimura, and T.Okubo, arXiv:1007.0883, J.Nishimura, T.Okubo and F.Sugino, arXiv:1108.1293]

SSB **SO(6)→SO(3)** (In D=10, SO(10)→SO(3))
Dynamical compactification to 3-dim spacetime.

$\lambda_n(\lambda_1\geq\cdots\geq\lambda_D)$: eigenvalues of $T_{\mu\nu}=\frac{1}{N}\mathrm{tr}(A_\mu A_\nu)$

$$\rho_\mu=\frac{\langle\lambda_\mu\rangle}{\sum_{\mu=1}^{6}\langle\lambda_\mu\rangle}=\begin{cases}0.30 & (\mu=1,2,3)\\ 0.035 & (\mu=4,5,6)\end{cases}$$

3. Complex Langevin Method (CLM)

Solve the complex Langevin equation.
[Parisi, Phys.Lett. 131B (1983) 393, Klauder, Phys.Rev. A29 (1984) 2036]

The action S(x) is complex for real x.
x(t) is complexified as x⇒z=x+iy
(S(z) is holomorphic by analytic continuation)

$$\dot z_k^{(\eta)}(t)=\underbrace{-\frac{\partial S}{\partial z_k(t)}}_{\text{drift term}}+\eta_k(t)$$

η_μ: real white noise obeying $\exp\left(-\frac14\int \mathrm{tr}\eta^2(t)dt\right)$

Probability distribution $P(x,y;t)=\left\langle\prod_k\delta(x_k-x_k^{(\eta)}(t))\delta(y_k-y_k^{(\eta)}(t))\right\rangle_\eta$

P(x,y;t) satisfies $\frac{\partial P}{\partial t}=L^\top P$

$$L^\top=\frac{\partial}{\partial x_k}\left\{\mathrm{Re}\left(\frac{\partial S}{\partial z_k}\right)+\frac{\partial}{\partial x_k}\right\}+\frac{\partial}{\partial y_k}\left\{\mathrm{Im}\left(\frac{\partial S}{\partial z_k}\right)\right\}$$

$$\int(Lf(x,y))g(x,y)dxdy=\int f(x,y)(L^\top g(x,y))dxdy$$

$$L=\left\{-\mathrm{Re}\left(\frac{\partial S}{\partial z_k}\right)+\frac{\partial}{\partial x_k}\right\}\frac{\partial}{\partial x_k}+\left\{-\mathrm{Im}\left(\frac{\partial S}{\partial z_k}\right)\right\}\frac{\partial}{\partial y_k}$$

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To justify the CLM, does this actually hold?

$$\int\underbrace{\mathscr{O}(x+iy)}_{\text{holomorphic}}P(x,y;t)dxdy\overset{?}{=}\int\mathscr{O}(x)\rho(x;t)dx$$

$$\frac{\partial\rho(x;t)}{\partial t}=\underbrace{\frac{\partial}{\partial x_k}\left(\frac{\partial S}{\partial x_k}+\frac{\partial}{\partial x_k}\right)}_{=L_0^\top}\rho(x;t)\Rightarrow\rho_{\text{time-indep.}}(x)\propto e^{-S}$$

At t=0, we choose $P(x,y;t=0)=\rho(x;t=0)\delta(y)$

Time evolution at t>0: we define O(z;t) as

$$\frac{\partial}{\partial t}\mathscr{O}(z;t)=\underbrace{\left(\frac{\partial}{\partial z_k}-\frac{\partial S}{\partial z_k}\right)\frac{\partial}{\partial z_k}}_{=\tilde L}\mathscr{O}(z;t)\ [\text{initial condition } \mathscr{O}(z;t=0)=\mathscr{O}(z)]$$

Setting y=0, $\frac{\partial}{\partial t}\mathscr{O}(x;t)=\overbrace{\left(\frac{\partial}{\partial x_k}-\frac{\partial S}{\partial x_k}\right)\frac{\partial}{\partial x_k}}^{=L_0}\mathscr{O}(x;t)$

$[\mathscr{O}(x;t=0)=\mathscr{O}(x)]\quad \int(L_0f(x))g(x)dx=\int f(x)(L_0^\top g(x))dx$

S(z) is holomorphic ⇒ O(z;t) remains holomorphic.

$$Lf(z)=\left\{-\mathrm{Re}\left(\frac{\partial S}{\partial z_k}\right)+\underbrace{\frac{\partial}{\partial x_k}}_{=\partial/\partial z_k}\right\}\underbrace{\frac{\partial f(z)}{\partial x_k}}_{=\partial f(z)/\partial z_k}+\left\{-\mathrm{Im}\left(\frac{\partial S}{\partial z_k}\right)\right\}\underbrace{\frac{\partial f(z)}{\partial y_k}}_{=i\partial f(z)/\partial z_k}$$

f(z)'s holomorphy

$$=\left\{-\left(\frac{\partial S}{\partial z_k}\right)+\frac{\partial}{\partial z_k}\right\}\frac{\partial f(z)}{\partial z_k}=\tilde L f(z)$$

Interpolating function $F(t,\tau)=\int dxdy\,\mathscr{O}(x+iy;\tau)P(x,y;t-\tau)$

$$\frac{\partial F(t,\tau)}{\partial\tau}=\int dxdy\left\{\frac{\partial\mathscr{O}(x+iy;\tau)}{\partial\tau}P(x,y;t-\tau)+\mathscr{O}(x+iy;\tau)\frac{\partial P(x,y;t-\tau)}{\partial\tau}\right\}$$

$$=\int dxdy(\tilde L\mathscr{O}(x+iy;\tau))P(x,y;t-\tau)-\int dxdy\,\mathscr{O}(x+iy;\tau)L^\top P(x,y;t-\tau)$$

integration by part $\overset{?}{=}\int dxdy\{\underbrace{(\tilde L-L)}_{=0}\mathscr{O}(x+iy;\tau)\}P(x,y;t-\tau)=0$

$$\frac{\partial}{\partial\tau}\int dx\,\mathscr{O}(x;\tau)\rho(x;t-\tau)\overset{?}{=}0$$ integration by part w.r.t. real x only (trivial)

Justified when P(x,y;t) damps rapidly
- in the imaginary direction
- around the singularity of the drift term

[G. Aarts, F.A. James, E. Seiler and O. Stamatescu, arXiv:1101.3270, K. Nagata, J. Nishimura and S. Shimasaki, arXiv:1508.02377]

$$F(t,0)=\int dxdy\underbrace{\mathscr{O}(x+iy;0)}_{=\mathscr{O}(x+iy)}P(x,y;t)\overset{?}{=}F(t,t)=\int dxdy\underbrace{\mathscr{O}(x+iy;t)}_{=e^{t\tilde L}\mathscr{O}(x+iy)}\underbrace{P(x,y;0)}_{=\rho(x;0)\delta(y)}$$

$$=\int dx\,\mathscr{O}(x;t)\rho(x;0)\overset{?}{=}\int dx\,\overbrace{\mathscr{O}(x;0)}^{=\mathscr{O}(x)}\rho(x;t)$$ Well-defined at large t?

$$\frac{\partial\mathscr{O}(z;t)}{\partial t}=\tilde L\mathscr{O}(z;t)\Rightarrow\mathscr{O}(z;t)=e^{t\tilde L}\mathscr{O}(z)$$

$$\int dxdy\{e^{\tau\tilde L}\mathscr{O}(z)\}P(x,y;t)=\sum_{n=0}^{+\infty}\frac{\tau^n}{n!}\int dxdy\{\tilde L^n\mathscr{O}(z)\}P(x,y;t)$$

This series should have a finite convergence radius.
Probability of the drift term should fall exponentially.
Look at the drift terms ⇒Get the drift of CLM !
[K. Nagata, J. Nishimura and S. Shimasaki, arXiv:1606.07627]

CLM for the IKKT model

$$\frac{d(A_\mu)_{ij}}{dt}=-\left\{\frac{\partial S_b}{\partial(A_\mu)_{ji}}-c_d\mathrm{Tr}\left(\frac{\partial\mathscr{M}}{\partial(A_\mu)_{ji}}\mathscr{M}^{-1}\right)\right\},\quad c_d=\begin{cases}1 & (D=6\to\det\mathscr{M})\\ \frac12 & (D=10\to\mathrm{Pf}\mathscr{M})\end{cases}$$

A_μ: Hermitian→general complex traceless matrices.
η_μ: Hermitian white noise obeying $\exp\left(-\frac14\int\mathrm{tr}\eta^2(t)dt\right)$

CLM does not work when it encounters…
(1) Excursion problem: A_μ is too far from Hermitian
⇒Gauge Cooling minimizes the Hermitian norm
$\mathscr{N}=\frac{-1}{4N}\sum_{\mu=1}^{D}\mathrm{tr}[(A_\mu-A_\mu^\dagger)^2]$.
(2) Singular drift problem:
The drift term $dS/d(A_\mu)_{ji}$ diverges due to $\mathscr{M}$'s near-zero eigenvalues.

4. Mass deformation

[Y. Ito and J. Nishimura, arXiv:1609.04501]

SO(D) breaking term $\Delta S_b=\frac12 N\varepsilon\sum_{\mu=1}^{D}m_\mu\mathrm{tr}(A_\mu)^2$

Order parameters for SO(D)'s SSB $\lambda_\mu=\mathrm{Re}\left\{\frac1N\mathrm{tr}(A_\mu)^2\right\}$

Fermionic mass term:
$\Delta S_f=Nm_f\mathrm{tr}\left(\bar\psi_\alpha\gamma_{\alpha\beta}\psi_\beta\right),\quad \gamma=\begin{cases}\Gamma_6 & (D=6)\\ \Gamma_8^\dagger\Gamma_9\Gamma_{10} & (D=10)\end{cases}$

Avoids $\mathscr{M}$'s singular eigenvalue distribution

Extrapolation (i) N→∞ ⇒(ii)ε→0 ⇒ (iii) m_f→0

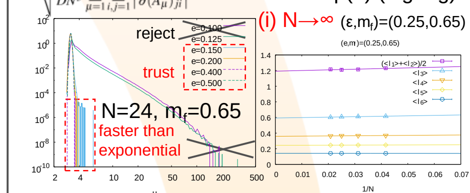
5. Result of D=6

[K.N. Anagnostopoulos, T. Azuma, Y. Ito, J. Nishimura and S.K. Papadoudis, arXiv:1712.07562]

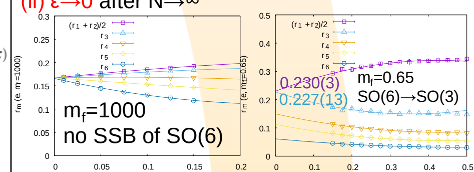
m_μ=(0.5,0.5,1,2,4,8)

$u=\sqrt{\frac{1}{DN^3}\sum_{\mu=1}^{D}\sum_{i,j=1}^{N}\left|\frac{\partial S}{\partial(A_\mu)_{ji}}\right|^2}$'s distribution p(u) (log-log)

(i) N→∞ (ε,m_f)=(0.25,0.65)

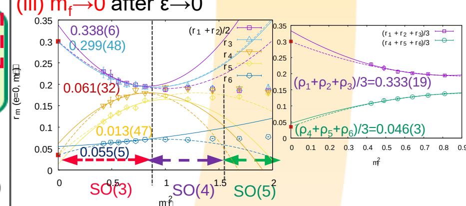


(ii) ε→0 after N→∞



- m_f→∞ : Ψ decouples from A_μ and reduces to the bosonic IKKT, which does not break SO(D).
[T. Hotta, J. Nishimura and A. Tsuchiya, hep-th/9811220]
- The SSB of SO(D) is a physical effect.

(iii) m_f→0 after ε→0



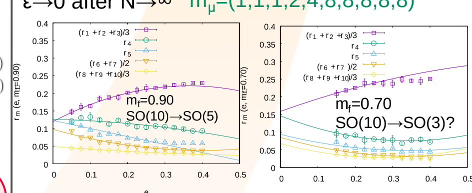
(dotted line: m_f→0 limit fixed to GEM results)

SSB SO(6)→SO(3) Consistent with GEM.

6. Preliminary result of D=10

[K.N. Anagnostopoulos, T. Azuma, Y. Ito, J. Nishimura, T. Okubo and S.K. Papadoudis, work in progress]

ε→0 after N→∞ m_μ=(1,1,1,2,4,8,8,8,8,8)



7. Future Works

- Comparison with GEM for m_f>0.
- Reweighting method [J. Bloch, arXiv:1701.00986]
- Other deformations than the mass deformation
(z=1:original Euclidean, pure imaginary z: fermion det/Pf is real)
[Y. Ito and J. Nishimura, arXiv:1710.07929]

$$N\mathrm{tr}\left(\bar\psi(z\Gamma_D)[A_D,\psi]+\sum_{k=1}^{D-1}\bar\psi\Gamma_k[A_k,\psi]\right)$$