Curved-space classical solution of a massive supermatrix model

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[1] Some mistakes have been found after the JPS meeting, and I have corrected these mistakes in this slide.

1 Introduction

Constructive definition of superstring theory

A large N reduced model has been proposed as a nonperturbative formulation of superstring theory.

IIB matrix model

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

For a review, hep-th/9908038

$$S = \frac{1}{g^2} Tr_{N\times N}\left(\frac{1}{4}\sum_{\mu,\nu=0}^{9}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\psi}\sum_{\mu=0}^{9}\Gamma^\mu[A_\mu,\psi]\right),$$

$$(\text{ where } Z = \int dA d\psi e^{+S}).$$

- A_μ and ψ are $N \times N$ Hermitian matrices.
  - * A_μ: 10-dimensional vectors
  - * ψ: 10-dimensional Majorana-Weyl (i.e. 16-component) spinors
- This model possesses $SU(N)$ gauge symmetry and $SO(9,1)$ Lorentz symmetry.
- $\mathcal{N} = 2$ SUSY: This theory must contain spin-2 gravitons if it contains massless particles.
  - * homogeneous : $\delta_\epsilon^{(1)} A_a = i\bar{\epsilon}\Gamma_a\psi, \quad \delta_\epsilon^{(1)}\psi = \frac{i}{2}\Gamma^{ab}[A_a, A_b]\epsilon.$
  - * inhomogeneous : $\delta_\xi^{(2)} A_a = 0, \quad \delta_\xi^{(2)}\psi = \xi.$

  This gives a shift of the bosonic variables for
  $\tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)},\ \tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)})$: $(\alpha, \beta = 1, 2)$

$$[\tilde{\delta}_\epsilon^{(\alpha)}, \tilde{\delta}_\xi^{(\beta)}]\psi = 0,$$
$$[\tilde{\delta}_\epsilon^{(\alpha)}, \tilde{\delta}_\xi^{(\beta)}]A_\mu = -2i\delta^{\alpha\beta}\bar{\epsilon}\Gamma_\mu\xi.$$

Can we describe the curved spacetime by a large N reduced model?

Classical equation of motion of IIB matrix model:

$$[A^\nu, [A_\mu, A_\nu]] = 0.$$

This has only a flat noncommutative background as a classical solution:

$$[A_\mu, A_\nu] = ic_{\mu\nu} 1_{N\times N}.$$

$\Rightarrow$ Some alteration of the action may be necessary in order to surmount this difficulty.

[Example] IIB matrix model with a tachyonic mass term:

Y. Kimura, Prog. Theor. Phys. 106 (2001) 445, [hep-th/0103192].

$$S = \frac{1}{g^2} Tr\left(\frac{1}{4}[A_a, A_b]^2 + \lambda^2 A_a A_a\right),$$
$$\text{EOM: } [A_b, [A_a, A_b]] + 2\lambda^2 A_a = 0.$$

- $SO(4)$ invariance and $u(N)$ gauge symmetry. a, b runs over $1, 2, 3, 4$ in the Euclidean space.
- Classical solution of compact curved spacetime:
  - * $SO(3)$ fuzzy sphere $[A_i, A_j] = i\lambda\epsilon_{ijk}A_k$ $(i, j, k = 1, 2, 3)$, $A_4 = 0$.
  - * Fuzzy torus

2 Massive supermatrix model

It is interesting to consider a similar problem in the supermatrix model.

$osp(1|32, R)$ super Lie algebra

- $M \in osp(1|32,R) \Rightarrow {}^T MG + GM = 0$,
  where $G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}$.
- $M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}$, where ${}^T m\Gamma^0 + \Gamma^0 m = 0$ $(m \in sp(32))$.
- $m = u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}$, where
  $u_A = \frac{1}{32}tr(m\Gamma_A)$, $u_{A_1A_2} = -\frac{1}{32}tr(m\Gamma_{A_1A_2})$, $u_{A_1\cdots A_5} = \frac{1}{32}tr(m\Gamma_{A_1\cdots A_5})$.

action of the massive supermatrix model

We add a mass term to the pure cubic action:

$$\begin{aligned} S &= Tr\left[str\left(-3\mu M^2 + \frac{i}{g^2}M[M,M]\right)\right] \\ &= Tr\left[-3\mu\left\{\left(\sum_{p=1}^{32} M_p{}^Q M_Q{}^p\right) - M_{33}{}^Q M_Q{}^{33}\right\}\right. \\ &\qquad \left. + \frac{i}{g^2}\left\{\left(\sum_{p=1}^{32} M_p{}^Q[M_Q{}^R, M_R{}^p]\right) - M_{33}{}^Q[M_Q{}^R, M_R{}^{33}]\right\}\right], \\ &= Tr\left[3\mu(-tr(m^2) + 2i\bar{\psi}\psi) + \frac{i}{g^2}\left(m_p{}^q[m_q{}^r, m_r{}^p] - 3i\bar{\psi}^p[m_p{}^q, \psi^q]\right)\right]. \end{aligned}$$

- Each component of the 33×33 supermatrices is promoted to a large N hermitian matrix.
- $osp(1|32, R)$ symmetry and $u(N)$ gauge symmetry.

In order to see the correspondence of the fields with IIB matrix model, we express the bosonic 32×32 matrices in terms of the 10-dimensional indices.
$(\mu, \nu, \cdots = 0, 1, \cdots, 9,\ \sharp = 10)$.

$$W = m_{\sharp}, \quad A_{\mu} = m_{\mu}, \quad B_{\mu} = m_{\mu\sharp}, \quad C_{\mu_1\mu_2} = m_{\mu_1\mu_2},$$
$$H_{\mu_1\cdots\mu_4} = m_{\mu_1\cdots\mu_4\sharp}, \quad Z_{\mu_1\cdots\mu_5} = m_{\mu_1\cdots\mu_5}.$$

Then, the action is decomposed as

$$\begin{aligned}
S \;=\;& 96\mu Tr \Big(-W^2 - A_{\mu}A^{\mu} + B_{\mu}B^{\mu} + \frac{1}{2}C_{\mu_1\mu_2}C^{\mu_1\mu_2} - \frac{1}{4!}H_{\mu_1\cdots\mu_4}H^{\mu_1\cdots\mu_4} \\
& \quad - \frac{1}{5!}Z_{\mu_1\cdots\mu_5}Z^{\mu_1\cdots\mu_5} + \frac{i}{16}\bar{\psi}\psi \Big) \\
+\;& 32iTr \Big(-3C_{\mu_1\mu_2}[A^{\mu_1}, A^{\mu_2}] + 3C_{\mu_1\mu_2}[B^{\mu_1}, B^{\mu_2}] + 6W[A_{\mu}, B^{\mu}] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3}, C^{\mu_3\mu_1}] \\
& \quad + \frac{1}{4}B_{\mu_1}[H_{\mu_2\cdots\mu_5}, Z^{\mu_1\cdots\mu_5}] - \frac{1}{8}C_{\mu_1\mu_2}(4[H^{\mu_1}{}_{\rho_1\rho_2\rho_3}, H^{\mu_2\rho_1\rho_2\rho_3}] + [Z^{\mu_1}{}_{\rho_1\cdots\rho_4}, Z^{\mu_1\rho_1\cdots\rho_4}]) \\
& \quad + \frac{3}{(5!)^2}\epsilon^{\mu_1\cdots\mu_{10}\sharp}(-W[Z_{\mu_1\cdots\mu_5}, Z_{\mu_6\cdots\mu_{10}}] + 10A_{\mu_1}[H_{\mu_2\cdots\mu_5}, Z_{\mu_6\cdots\mu_{10}}]) \\
& \quad + \frac{200}{(5!)^3}\epsilon^{\mu_1\cdots\mu_{10}\sharp}(5H_{\mu_1\cdots\mu_4}[Z_{\mu_5\mu_6\mu_7}{}^{\rho\chi}, Z_{\mu_8\mu_9\mu_{10}\rho\chi}] + 10H_{\mu_1\cdots\mu_4}[H_{\mu_5\mu_6\mu_7}{}^{\rho}, H_{\mu_8\mu_9\mu_{10}\rho}] \\
& \qquad + 6H^{\rho\chi}{}_{\mu_1\mu_2}[Z_{\mu_3\mu_4\mu_5\rho\chi}, Z_{\mu_6\cdots\mu_{10}}]) \Big) \\
+\;& 3Tr \Big(\bar{\psi}\Gamma^{\sharp}[W, \psi] + \bar{\psi}\Gamma^{\mu}[A_{\mu}, \psi] + \bar{\psi}\Gamma^{\mu\sharp}[B_{\mu}, \psi] + \frac{1}{2!}\bar{\psi}\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2}, \psi] \\
& \quad + \frac{1}{4!}\bar{\psi}\Gamma^{\mu_1\cdots\mu_4\sharp}[H_{\mu_1\cdots\mu_4}, \psi] + \frac{1}{5!}\bar{\psi}\Gamma^{\mu_1\cdots\mu_5}[Z_{\mu_1\cdots\mu_5}, \psi] \Big).
\end{aligned}$$

- The rank-1 and rank-5 fields (in 11 dimensions) have a positive mass, while the rank-2 fields are tachyonic.

$$\underbrace{\Gamma_A\Gamma^A}_{\text{no sum}} = \underbrace{\Gamma_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}}_{\text{no sum}} = +1_{32\times 32}, \quad \underbrace{\Gamma_{A_1A_2}\Gamma^{A_1A_2}}_{\text{no sum}} = -1_{32\times 32}.$$

- The rank-1 and rank-5 fields has a stable trivial commutative classical solution:

$$W = A_{\mu} = H_{\mu_1\cdots\mu_4} = Z_{\mu_1\cdots\mu_5} = 0.$$

- For the rank-2 tachyonic fields $B_{\mu}, C_{\mu_1\mu_2}$, the trivial solution $B_{\mu} = C_{\mu_1\mu_2} = 0$ is unstable.
$\Rightarrow$ They may incorporate an interesting stable non-commutative solution!

From now on, we set the fermions and the positive-mass bosonic fields to zero:

$$\begin{aligned} S &= 96\mu Tr\left(B_\mu B^\mu + \frac{1}{2}C_{\mu_1\mu_2}C^{\mu_1\mu_2}\right) \\ &+ 32iTr\left(3C_{\mu_1\mu_2}[B^{\mu_1}, B^{\mu_2}] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3}, C^{\mu_3\mu_1}]\right). \end{aligned}$$

The equations of motion:

$$\begin{aligned} B_\mu &= -i\mu^{-1}[B^\nu, C_{\mu\nu}], \\ C_{\mu_1\mu_2} &= -i\mu^{-1}([B_{\mu_1}, B_{\mu_2}] + [C_{\mu_1}{}^\rho, C_{\mu_2\rho}]). \end{aligned}$$

We integrate out the rank-2 fields (in 10 dimensions) $C_{\mu_1\mu_2}$ by solving the latter equation of motions iteratively.

$$\begin{aligned} C_{\mu_1\mu_2} &= -i\mu^{-1}([B_{\mu_1}, B_{\mu_2}] + \underbrace{[C_{\mu_1}{}^\rho, C_{\mu_2\rho}]}_{=(-i\mu^{-1})^2[[B_{\mu_1},B^\rho]+[C_{\mu_1\chi_1},C^{\rho\chi_1}],[B_{\mu_2},B_\rho]+[C_{\mu_2\chi_2},C_\rho{}^{\chi_2}]]}) + \cdots \\ &= -\underbrace{i\mu^{-1}[B_{\mu_1}, B_{\mu_2}]}_{\mathcal{O}(B^2)\text{ with 1 commutator}} + \underbrace{i\mu^{-3}[[B_{\mu_1}, B_\rho], [B_{\mu_2}, B^\rho]]}_{\mathcal{O}(B^4)\text{ with 3 commutators}} \\ &- \underbrace{2i\mu^{-5}[[B_{[\mu_1}, B_\rho], [[B_{\mu_2]}, B_\chi], [B^\rho, B^\chi]]]}_{\mathcal{O}(B^6)\text{ with 5 commutators}} \\ &+ i\mu^{-7}[[[B_{\mu_1}, B_{\chi_1}], [B_\rho, B^{\chi_1}]], [[B_{\mu_2}, B_{\chi_2}], [B^\rho, B^{\chi_2}]]] \\ &+ 2i\mu^{-7}[[B_{[\mu_1}, B_\rho], [[B_{\mu_2]}, B_\chi], [[B^\rho, B_\sigma], [B^\chi, B^\sigma]]]] \\ &- \underbrace{2i\mu^{-7}[[B_{[\mu_1}, B_\rho], [[B^\rho, B_\chi], [[B_{\mu_2]}, B_\sigma], [B^\chi, B^\sigma]]]]}_{\mathcal{O}(B^8)\text{ with 7 commutators}} \\ &+ 2i\mu^{-9}[[B_{[\mu_1}, B_\rho], [[B_{\mu_2]}, B_\chi], [[B^\chi, B_\sigma], [[B^\rho, B_\alpha], [B^\sigma, B^\alpha]]]]] \\ &- 2i\mu^{-9}[[B_{[\mu_1}, B_\rho], [[B^\rho, B_\chi], [[B^\chi, B_\sigma], [[B_{\mu_2]}, B_\alpha], [B^\sigma, B^\alpha]]]]] \\ &- 2i\mu^{-9}[[B_{[\mu_1}, B_\rho], [[B_{\mu_2]}, B_\chi], [[B^\rho, B_\sigma], [[B^\chi, B_\alpha], [B^\sigma, B^\alpha]]]]] \\ &+ 2i\mu^{-9}[[B_{[\mu_1}, B_\rho], [[B^\rho, B_\chi], [[B_{\mu_2]}, B_\sigma], [[B^\chi, B_\alpha], [B^\sigma, B^\alpha]]]]] \\ &+ 2i\mu^{-9}[[[B_{[\mu_1}, B_{\chi_1}], [B_\rho, B^{\chi_1}]], [[B^\rho, B_{\chi_2}], [[B_{\mu_2]}, B_\sigma], [B^{\chi_2}, B^\sigma]]]] \\ &- 2i\mu^{-9}[[[B_{[\mu_1}, B_{\chi_1}], [B_\rho, B^{\chi_1}]], [[B_{\mu_2]}, B_{\chi_2}], [[B^\rho, B_\sigma], [B^{\chi_2}, B^\sigma]]]] \\ &- \underbrace{2i\mu^{-9}[[B_{[\mu_1}, B_\rho], [[[B_{\mu_2]}, B_{\sigma_1}], [B_\chi, B^{\sigma_1}]], [[B^\rho, B_{\sigma_2}], [B^\chi, B^{\sigma_2}]]]]}_{\mathcal{O}(B^{10})\text{ with 9 commutators}} + \mathcal{O}(\mu^{-11}). (\star) \end{aligned}$$

Then, the action reduces to

$$\begin{aligned} S &= Tr\left(96\mu B_\mu B^\mu + 48\mu^{-1}[B_{\mu_1}, B_{\mu_2}][B^{\mu_1}, B^{\mu_2}]\right. \\ &\quad \left. + (\text{higher-order commutators of the order } \mathcal{O}(\mu^{-2k+1}) \text{ with } k = 2, 3, \cdots)\right). \end{aligned}$$

We consider the classical solution of the equation of motion $B_\mu = -i\mu^{-1}[B^\nu, C_{\mu\nu}]$ with $C_{\mu_1\mu_2}$ substituted for $(\star)$.

Fuzzy-sphere classical solution

1. **$SO(3) \times SO(3) \times SO(3)$ fuzzy spheres**

   This describes a space formed by the Cartesian product of three fuzzy spheres.

$$[B_i, B_j] = i\mu r \epsilon_{ijk} B_k, \qquad B_1^2 + B_2^2 + B_3^2 = \mu^2 r^2 \frac{N^2-1}{4}, \quad (i,j,k = 1,2,3)$$
$$[B_{i'}, B_{j'}] = i\mu r \epsilon_{i'j'k'} B_{k'}, \qquad B_4^2 + B_5^2 + B_6^2 = \mu^2 r^2 \frac{N^2-1}{4}, \quad (i',j',k' = 4,5,6)$$
$$[B_{i''}, B_{j''}] = i\mu r \epsilon_{i''j''k''} B_{k''}, \quad B_7^2 + B_8^2 + B_9^2 = \mu^2 r^2 \frac{N^2-1}{4}, \quad (i'',j'',k'' = 7,8,9)$$
$$B_0 = 0, \quad [B_\mu, B_\nu] = 0, \quad \text{(otherwise)}.$$

   (We consider the Cartesian product of three spheres instead of a single $SO(3)$ fuzzy sphere

$$[B_i, B_j] = i\mu r \epsilon_{ijk} B_k \ (\text{for } i,j,k = 1,2,3), \quad B_\mu = 0 \ (\text{for } \mu = 0,4,5,\cdots,9),$$

   because the solution $B_4 = \cdots = B_9 = 0$ is trivially unstable.)

2. **$SO(9)$ fuzzy sphere**

   Generally, the $SO(2k+1)$ fuzzy sphere (S^{2k} fuzzy sphere) is constructed by the n-fold symmetric tensor product of $(2k+1)$-dimensional gamma matrices:

$$B_p^{SO(2k+1)} = \frac{\mu r}{2} [(\Gamma_p^{(2k)} \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes \Gamma_p^{(2k)})]_{\text{sym}}.$$
$$B_p^{SO(2k+1)} B_p^{SO(2k+1)} = \frac{\mu^2 r^2}{4} n(n+2k) 1_{N_k \times N_k}.$$

We should answer the following two questions about this solution:

1. **Is this solution not perturbed by the infinite tower of the higher-order commutator?**
2. **Which solution is energetically favored?**

Effect of the higher-order commutators

We start with the ansatz for the rank-2 fields $C_{pq}^{SO(2k+1)}$ for the $SO(2k+1)$ fuzzy spheres:

$$C_{pq}^{SO(2k+1)} = -i\mu^{-1} f(r) B_{pq}^{SO(2k+1)}.$$

⇓

The equation of motion for $C_{pq}^{SO(2k+1)}$ reduces to

$$C_{pq}^{SO(2k+1)} = -i\mu^{-1}([B_p^{SO(2k+1)}, B_q^{SO(2k+1)}] + [C_{pr}^{SO(2k+1)}, C_{qr}^{SO(2k+1)}])$$

$$\Downarrow$$

$$\frac{-i}{\mu} B_{pq}^{SO(2k+1)}(-f(r) + 1 + (2k-1)r^2 f^2(r)) = 0.$$

$f(r)$ is determined as

$$f_\pm(r) = \frac{1 \pm \sqrt{1-4(2k-1)r^2}}{2(2k-1)r^2}.$$

⇓

The equation of motion for $B_p^{SO(2k+1)}$ leads to

$$B_p^{SO(2k+1)}(1 - 2kr^2 f_\pm(r)) = 0.$$

$$\Updownarrow$$

$$\sqrt{1-4(2k-1)r^2} = \pm\frac{k-1}{k}.$$

- $1 - 2kr^2 f_-(r) = 0$ (i.e. $\sqrt{1-4(2k-1)r^2} = -\frac{k-1}{k}$) has no solution (except for $k=1$, in which this is identical to $1-2kr^2 f_+(r) = 0$).
- $1 - 2kr^2 f_+(r) = 0$ (i.e. $\sqrt{1-4(2k-1)r^2} = +\frac{k-1}{k}$) does have a solution $r = \frac{1}{2k}$

The existence of the solution $r(>0)$ indicates that the radius of the fuzzy sphere is not much perturbed by the infinite tower of the high-order commutators.

Comparison of the classical energy

- Trivial commutative solution $B_0 = \cdots = B_9 = 0$:

$$E_{B_\mu = 0} = -S_{B_\mu = 0} = 0.$$

- $SO(3) \times SO(3) \times SO(3)$ fuzzy spheres ($N_1 = n + 1$):

$$\begin{aligned} E_{SO(3)^3} &= -S_{SO(3)^3} = -\frac{16\mu}{r_{SO(3)^2}} Tr(B_\mu B^\mu) \\ &= -12\mu^3 N_1 (N_1 - 1)(N_1 + 1) \\ &\sim -\mathcal{O}(\mu^3 n^3) = -\mathcal{O}(\mu^3 N_1^3). \end{aligned}$$

- $SO(9)$ fuzzy sphere:

$$\begin{aligned} E_{SO(9)} &= -S_{SO(9)} = -\frac{5}{8}\mu^3 n(n+8) N_4 \\ &\sim -\mathcal{O}(\mu^3 n^{12}) = -\mathcal{O}(\mu^3 N_4^{\frac{6}{5}}), \end{aligned}$$

where the size of the matrices $B_p^{SO(9)}$ is

$$N_4 = \frac{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)}{302400} \sim \mathcal{O}(n^{10}).$$

3 Summary

- We have investigated a massive supermatrix model to seek a curved-space classical solution.
- We have found the triple $SO(3) \times SO(3) \times SO(3)$ and the single $SO(9)$ fuzzy-sphere solutions.
  - * These solutions are not perturbed by the infinite tower of the higher-order commutators.
  - * We have compared the classical energy.

Future problems

- Other classical solutions such as $SO(3) \times SO(6)$ fuzzy sphere, fuzzy torus $\cdots$.
- Relation to the BMN matrix model
  D. Berenstein, J. M. Maldacena and H. Nastase, [hep-th/0202021]
- Structure of the $\mathcal{N} = 2$ supersymmetry.

Properties of the fuzzy $2k$-sphere

The $SO(2k+1)$ fuzzy sphere (S^{2k} fuzzy sphere) is constructed by the n-fold symmetric tensor product of $(2k+1)$-dimensional gamma matrices:

$$B_p^{SO(2k+1)} = \frac{\mu r}{2}[(\Gamma_p^{(2k)} \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes \Gamma_p^{(2k)})]_{\text{sym}}.$$

p runs over $1, 2, \cdots, 2k+1$ in the $(2k+1)$-dimensional Euclidean space.

The commutation and self-duality relation ($B_{pq}^{SO(2k+1)} = [B_p^{SO(2k+1)}, B_q^{SO(2k+1)}]$):

$$\heartsuit \quad B_p^{SO(2k+1)} B_p^{SO(2k+1)} = \frac{\mu^2 r^2}{4} n(n+2k) 1_{N_k \times N_k},$$

$$\heartsuit \quad B_{pq}^{SO(2k+1)} B_{pq}^{SO(2k+1)} = -(\frac{\mu r}{2})^4 8kn(n+2k),$$

$$\clubsuit \quad [B_{pq}^{SO(2k+1)}, B_s^{SO(2k+1)}] = \mu^2 r^2 (-\delta_{ps} B_q^{SO(2k+1)} + \delta_{qs} B_p^{SO(2k+1)}) 1_{N_k \times N_k},$$

$$\clubsuit \quad [B_{pq}^{SO(2k+1)}, B_{st}^{SO(2k+1)}] = \mu^2 r^2 (\delta_{qs} B_{pt}^{SO(2k+1)} + \delta_{pt} B_{qs}^{SO(2k+1)} - \delta_{ps} B_{qt}^{SO(2k+1)} - \delta_{qt} B_{ps}^{SO(2k+1)}),$$

$$\diamondsuit \quad \epsilon_{p_1 \cdots p_{2k+1}} B_{p_1}^{SO(2k+1)} B_{p_2}^{SO(2k+1)} \cdots B_{p_{2k}}^{SO(2k+1)} = \left(\frac{\mu r}{2}\right)^{2k-1} m_k B_{p_{2k+1}}^{SO(2k+1)},$$

$$m_1 = 2i, \quad m_2 = 8(n+2), \quad m_3 = -48i(n+2)(n+4),$$

$$m_4 = -384(n+2)(n+4)(n+6).$$

For $k = 1$, this definition is identical to the $SO(3)$ Lie algebra:

1. This is effectively a matrix acting on the symmetrized $N = (n+1)$-dimensional irreducible representation of $so(3)$ Lie algebra, not on the original 2^n-dimensional space.
2. The radius of the fuzzy sphere is (from ($\heartsuit$)) $B_i^{SO(3)} B_i^{SO(3)} = \frac{\mu^2 r^2}{4} n(n+2) = (\mu r)^2 \frac{N^2-1}{4}$, where $\frac{N^2-1}{4}$ is the Casimir of $so(3)$.
3. $\Gamma_i^{(2)}$ are identical to the Pauli matrices σ_i.
4. Self-duality condition ($\diamondsuit$) is trivially identical to the commutation relation $[B_i^{SO(3)}, B_j^{SO(3)}] = i\mu r \epsilon_{ijk} B_k^{SO(3)}$.

Computation of m_2, m_3 and m_4

In this appendix, we give the derivation of the coefficients m_k in the self-duality relation for the $SO(2k+1)$ fuzzy sphere. In this appendix, we define the $2^k \times 2^k$ gamma matrices in the $2k$-dimensional Euclidean space $\Gamma^{(2k)}_p$ by the following recursive relation:

$$\Gamma^{(2k+2)}_p = \Gamma^{(2k)}_p \otimes \sigma_2 = \begin{pmatrix} 0 & -i\Gamma^{(2k)}_p \\ i\Gamma^{(2k)}_p & 0 \end{pmatrix}, \quad \Gamma^{(2k+2)}_{2k+2} = \mathbf{1}_{2^k\times 2^k} \otimes \sigma_1 = \begin{pmatrix} 0 & \mathbf{1}_{2^k\times 2^k} \\ \mathbf{1}_{2^k\times 2^k} & 0 \end{pmatrix},$$
$$\Gamma^{(2k+2)}_{2k+3} = \mathbf{1}_{2^k\times 2^k} \otimes \sigma_3 = \begin{pmatrix} \mathbf{1}_{2^k\times 2^k} & 0 \\ 0 & -\mathbf{1}_{2^k\times 2^k} \end{pmatrix}, \tag{1}$$

where the index p runs over $p = 1, 2, \cdots, 2k+1$. The 2-dimensional gamma matrices are identical to the Pauli matrices: $\Gamma^{(2)}_i = \sigma_i$. Under this notation, we obtain

$$\sigma_1\sigma_2 = i\sigma_3, \quad \Gamma^{(4)}_1\Gamma^{(4)}_2\Gamma^{(4)}_3\Gamma^{(4)}_4 = \Gamma^{(4)}_5, \quad \Gamma^{(6)}_1\Gamma^{(6)}_2\cdots\Gamma^{(6)}_6 = -i\Gamma^{(6)}_7, \quad \Gamma^{(8)}_1\Gamma^{(8)}_2\cdots\Gamma^{(8)}_8 = -\Gamma^{(8)}_9. \tag{2}$$

It is trivial that $m_1 = 2i$ for the $SO(3)$ fuzzy sphere. Then, we start with the coefficient m_2. In this appendix, we set $\frac{\mu r}{2} = 1$ and omit "sym", which indicates that the tensor product is restricted to the fully symmetric subspace.

3.1 Computation of m_2

We first perform the computation of m_2 for the $SO(5)$ fuzzy sphere. We frequently utilize the following identity for the symmetric tensor product:

$$\sum_{i=1}^{3}(\sigma_i \otimes \sigma_i) = (\mathbf{1}_{2\times 2} \otimes \mathbf{1}_{2\times 2}). \tag{3}$$

Now, we consider the case in which $n = 2$ for brevity; i.e. the $SO(5)$ fuzzy sphere is described by the 2-fold symmetric tensor products as

$$B^{SO(5)}_p = [(\Gamma^{(4)}_p \otimes \mathbf{1}_{4\times 4}) + (\mathbf{1}_{4\times 4} \otimes \Gamma^{(4)}_p)]. \tag{4}$$

Then, the left-hand side is

$$\begin{aligned} & \epsilon_{p_1\cdots p_4 5} B^{SO(5)}_{p_1} B^{SO(5)}_{p_2} B^{SO(5)}_{p_3} B^{SO(5)}_{p_4} \\ = \; & \epsilon_{p_1\cdots p_4 5}[(\Gamma^{(4)}_{p_1\cdots p_4} \otimes \mathbf{1}_{4\times 4}) + (\mathbf{1}_{4\times 4} \otimes \Gamma^{(4)}_{p_1\cdots p_4}) + 2(\Gamma^{(4)}_{p_1p_2} \otimes \Gamma^{(4)}_{p_3p_4})]. \end{aligned} \tag{5}$$

We do not lose any generality if we set $p_5 = 5$, and the indices $p_1, \cdots, p_4$ run over $1, 2, 3, 4$. The first two terms give $4! = 24$ of $(\Gamma_{1234} \otimes \mathbf{1}_{4\times 4}) + (\mathbf{1}_{4\times 4} \otimes \Gamma_{1234})$, to constitute $24B^{SO(5)}_5$. On the other hand, the third term is computed as

$$\begin{aligned} & 2\epsilon_{p_1\cdots p_4 5}(\Gamma^{(4)}_{p_1p_2} \otimes \Gamma^{(4)}_{p_3p_4}) = 4\epsilon_{ijk}[(\Gamma^{(4)}_{ij} \otimes \Gamma^{(4)}_{k4}) + (\Gamma^{(4)}_{k4} \otimes \Gamma^{(4)}_{ij})] \\ = \; & -8[(\Gamma^{(4)}_{k45} \otimes \Gamma^{(4)}_{k4}) + (\Gamma^{(4)}_{k4} \otimes \Gamma^{(4)}_{k45})] \\ = \; & -8[(\sigma_k \otimes (-i\mathbf{1}_{2\times 2})) \otimes (\sigma_k \otimes (-i\sigma_3)) + (\sigma_k \otimes (-i\sigma_3)) \otimes (\sigma_k \otimes (-i\mathbf{1}_{2\times 2}))] \\ = \; & 8[(\mathbf{1}_{2\times 2} \otimes \sigma_3) \otimes (\mathbf{1}_{2\times 2} \otimes \mathbf{1}_{2\times 2}) + (\mathbf{1}_{2\times 2} \otimes \mathbf{1}_{2\times 2}) \otimes (\mathbf{1}_{2\times 2} \otimes \sigma_3)] = 8B^{SO(5)}_5. \end{aligned} \tag{6}$$

By the same token, this kind of contribution makes $8(n-1)B^{SO(5)}_5$ for any n. Altogether, we have $m_2 = 8(n+2)$.

3.2 Computation of m_3

The computation of m_3 for the $SO(7)$ fuzzy sphere goes in the similar way. In this computation, we utilize the formulae

$$\sum_{l=1}^{5}(\Gamma^{(4)}_l \otimes \Gamma^{(4)}_l) = (\mathbf{1}_{4\times 4}\otimes \mathbf{1}_{4\times 4}), \quad \sum_{l_1,l_2=1}^{5}(\Gamma^{(4)}_{l_1l_2}\otimes\Gamma^{(4)}_{l_1l_2}) = -4(\mathbf{1}_{4\times 4}\otimes \mathbf{1}_{4\times 4}). \tag{7}$$

Now, we set $p_7 = 7$ without loss of generality, and consider the 3-fold tensor product. The left-hand side is now

$$\begin{aligned} & \epsilon_{p_1\cdots p_6 7} B^{SO(7)}_{p_1} B^{SO(7)}_{p_2}\cdots B^{SO(7)}_{p_6} \\ = \; & \epsilon_{p_1\cdots p_6 7}[\{(\Gamma^{(6)}_{p_1\cdots p_6}\otimes \mathbf{1}_{8\times 8}\otimes \mathbf{1}_{8\times 8}) + (\mathbf{1}_{8\times 8}\otimes\Gamma^{(6)}_{p_1\cdots p_6}\otimes \mathbf{1}_{8\times 8}) + (\mathbf{1}_{8\times 8}\otimes \mathbf{1}_{8\times 8}\otimes\Gamma^{(6)}_{p_1\cdots p_6})\} \end{aligned} \tag{8}$$

$$+3\{(\Gamma^{(6)}_{p_1\cdots p_4}\otimes\Gamma^{(6)}_{p_5p_6}\otimes \mathbf{1}_{8\times 8}) + (5 \text{ other permutations of this kind})\} \tag{9}$$

$$+6(\Gamma^{(6)}_{p_1p_2}\otimes\Gamma^{(6)}_{p_3p_4}\otimes\Gamma^{(6)}_{p_5p_6})]. \tag{10}$$

- We first consider the contribution of (8). Since there are $6! = 720$ ways to contract the indices $p_1,\cdots,p_6$, this gives

  $$-720i[(\Gamma^{(6)}_7\otimes \mathbf{1}_{8\times 8}\otimes \mathbf{1}_{8\times 8}) + (\mathbf{1}_{8\times 8}\otimes\Gamma^{(6)}_7\otimes \mathbf{1}_{8\times 8}) + (\mathbf{1}_{8\times 8}\otimes \mathbf{1}_{8\times 8}\otimes\Gamma^{(6)}_7)] = -720iB^{SO(7)}_7.$$

- We then go on to the contribution of (9):

  $$\begin{aligned} & \epsilon_{p_1\cdots p_6 7}(\Gamma^{(6)}_{p_1\cdots p_4}\otimes\Gamma^{(6)}_{p_5p_6}\otimes \mathbf{1}_{8\times 8}) \\ = \; & \epsilon_{l_1\cdots l_5 67}[4(\Gamma^{(6)}_{l_1\cdots l_3 6}\otimes\Gamma^{(6)}_{l_4l_5}\otimes \mathbf{1}_{8\times 8}) + 2(\Gamma^{(6)}_{l_1\cdots l_4}\otimes\Gamma^{(6)}_{l_5 6}\otimes \mathbf{1}_{8\times 8})] \\ = \; & (4!)i[(\Gamma^{(6)}_{l_4l_5 7}\otimes\Gamma^{(6)}_{l_4l_5}\otimes \mathbf{1}_{8\times 8}) + 2(\Gamma^{(6)}_{l_5 67}\otimes\Gamma^{(6)}_{l_5 6}\otimes \mathbf{1}_{8\times 8})] \\ = \; & (4!)i[((\Gamma^{(4)}_{l_4l_5}\otimes\sigma_3)\otimes(\Gamma^{(4)}_{l_4l_5}\otimes \mathbf{1}_{2\times 2})\otimes \mathbf{1}_{8\times 8}) - 2((\Gamma^{(4)}_{l_5}\otimes \mathbf{1}_{2\times 2})\otimes(\Gamma^{(4)}_{l_5}\otimes\sigma_3)\otimes \mathbf{1}_{8\times 8})] \\ = \; & -(4!)i[4((\mathbf{1}_{4\times 4}\otimes\sigma_3)\otimes \mathbf{1}_{8\times 8}\otimes \mathbf{1}_{8\times 8}) + 2(\mathbf{1}_{8\times 8}\otimes(\mathbf{1}_{4\times 4}\otimes\sigma_3)\otimes \mathbf{1}_{8\times 8})], \end{aligned}$$

  where the indices $l_1, l_2, \cdots$ run over $1, 2, \cdots, 5$ and we have utilized the formulae (7). Summing up all 6 permutations, we obtain $-864iB^{SO(7)}_7$. When we extend this argument for the general n-fold tensor product, the result is $-432i(n-1)B^{SO(7)}_7$.

- Lastly, we investigate the terms (10):

  $$\begin{aligned} & 6\epsilon_{p_1\cdots p_6 7}(\Gamma^{(6)}_{p_1p_2}\otimes\Gamma^{(6)}_{p_3p_4}\otimes\Gamma^{(6)}_{p_5p_6}) \\ = \; & 12\epsilon_{l_1\cdots l_5 67}[(\Gamma^{(6)}_{l_1l_2}\otimes\Gamma^{(6)}_{l_3l_4}\otimes\Gamma^{(6)}_{l_5 6}) + (2 \text{ other permutations})] \\ = \; & -12(2!)i[(\Gamma^{(6)}_{l_3l_4l_5 67}\otimes\Gamma^{(6)}_{l_3l_4}\otimes\Gamma^{(6)}_{l_5 6}) + (\text{perm.})] \\ = \; & 24i[((\Gamma^{(4)}_{l_3l_4l_5}\otimes \mathbf{1}_{2\times 2})\otimes(\Gamma^{(4)}_{l_3l_4}\otimes \mathbf{1}_{2\times 2})\otimes(\Gamma^{(4)}_{l_5}\otimes\sigma_3)) + (\text{perm.})] \\ = \; & 24i[((\Gamma^{(4)}_{l_3l_4}\Gamma^{(4)}_{l_5}\otimes \mathbf{1}_{2\times 2})\otimes(\Gamma^{(4)}_{l_3l_4}\otimes \mathbf{1}_{2\times 2})\otimes(\Gamma^{(4)}_{l_5}\otimes \mathbf{1}_{2\times 2}) \\ & \quad +2(\Gamma^{(4)}_{i_4}\otimes \mathbf{1}_{2\times 2})\otimes((\Gamma^{(4)}_{i_3}\Gamma^{(4)}_{i_4} - \delta_{i_3i_4}\mathbf{1}_{4\times 4})\otimes \mathbf{1}_{2\times 2})\otimes(\Gamma^{(4)}_{i_3}\otimes\sigma_3)) + (\text{perm.})] \\ = \; & -96i[(\mathbf{1}_{8\times 8}\otimes \mathbf{1}_{8\times 8}\otimes(\mathbf{1}_{4\times 4}\otimes\sigma_3)) + (\mathbf{1}_{8\times 8}\otimes(\mathbf{1}_{4\times 4}\otimes\sigma_3)\otimes \mathbf{1}_{8\times 8}) \\ & \quad +((\mathbf{1}_{4\times 4}\otimes\sigma_3)\otimes \mathbf{1}_{8\times 8}\otimes \mathbf{1}_{8\times 8})] = -96iB^{SO(7)}_7. \end{aligned}$$

  For the general n-fold symmetric tensor product, we obtain $-48i(n-1)(n-2)B^{SO(7)}_7$.

We sum up all the contribution of (8), (9) and (10) to obtain $m_3 = -48i(n+2)(n+4)$.

3.3 Computation of m_4

We next go on to the coefficient m_4 for the $SO(9)$ fuzzy sphere. We repeat the same procedure, but the computation is rather complicated. We exploit the following formulae here:

$$(\Gamma^{(6)}_{l} \otimes \Gamma^{(6)}_{l}) = (\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}), \quad (11)$$
$$(\Gamma^{(6)}_{l_1 l_2} \otimes \Gamma^{(6)}_{l_1 l_2}) = -6(\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}), \quad (12)$$
$$(\Gamma^{(6)}_{l_1 l_2 l_3} \otimes \Gamma^{(6)}_{l_1 l_2 l_3}) = -18(\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}), \quad (13)$$
$$(\Gamma^{(6)}_{l_1 l_2} \otimes \Gamma^{(6)}_{l_3} \otimes \Gamma^{(6)}_{l_1 l_2 l_3}) = -6(\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}), \quad (14)$$
$$(\Gamma^{(6)}_{l_1 l_2} \otimes \Gamma^{(6)}_{l_3 l_4} \otimes \Gamma^{(6)}_{l_1 \cdots l_4}) = 24(\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}), \quad (15)$$
$$(\Gamma^{(6)}_{l_1 l_2} \otimes \Gamma^{(6)}_{l_3 l_4} \otimes \Gamma^{(6)}_{l_5 l_6} \otimes \Gamma^{(6)}_{l_1 \cdots l_6}) = -48(\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}). \quad (16)$$

We set $p_9 = 9$, and the indices $p_1, p_2, \cdots$ and $l_1, l_2, \cdots$ respectively run over $1, 2, \cdots, 8$ and $1, 2, \cdots, 7$. We consider the 4-fold tensor product

$$\begin{aligned}
& \epsilon_{p_1 \cdots p_8 9} B^{SO(9)}_{p_1} B^{SO(9)}_{p_2} \cdots B^{SO(9)}_{p_8} \\
= \ & \epsilon_{p_1 \cdots p_8 9} [((\Gamma^{(8)}_{p_1 \cdots p_8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) + \cdots + (\mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16} \otimes \Gamma^{(8)}_{p_1 \cdots p_8})) \quad (17) \\
& +4((\Gamma^{(8)}_{p_1 \cdots p_6} \otimes \Gamma^{(8)}_{p_7 p_8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) + (11 \text{ other permutations})) \quad (18) \\
& +6((\Gamma^{(8)}_{p_1 \cdots p_4} \otimes \Gamma^{(8)}_{p_5 \cdots p_8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) + (5 \text{ other permuations})) \quad (19) \\
& +12((\Gamma^{(8)}_{p_1 \cdots p_4} \otimes \Gamma^{(8)}_{p_5 p_6} \otimes \Gamma^{(8)}_{p_7 p_8} \otimes \mathbf{1}_{16\times 16}) + (11 \text{ other permutations})) \quad (20) \\
& +24(\Gamma^{(8)}_{p_1 p_2} \otimes \Gamma^{(8)}_{p_3 p_4} \otimes \Gamma^{(8)}_{p_5 p_6} \otimes \Gamma^{(8)}_{p_7 p_8})]. \quad (21)
\end{aligned}$$

- (17) trivially gives $-(8!) B^{SO(9)}_9$.
- The contribution of (18) is computed as follows:

$$\begin{aligned}
& \epsilon_{p_1 \cdots p_8 9} (\Gamma^{(8)}_{p_1 \cdots p_6} \otimes \Gamma^{(8)}_{p_7 p_8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\
= \ & 6(5!)(\Gamma^{(8)}_{l_6 l_7 9} \otimes \Gamma^{(8)}_{l_6 l_7} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) + 2(6!)(\Gamma^{(8)}_{l_7 8 9} \otimes \Gamma^{(8)}_{l_7 8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\
= \ & 6(5!)((\Gamma^{(6)}_{l_6 l_7} \otimes \sigma_3) \otimes (\Gamma^{(6)}_{l_6 l_7} \otimes \mathbf{1}_{2\times 2}) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\
& -2(6!)((\Gamma^{(6)}_{l_7} \otimes \mathbf{1}_{2\times 2}) \otimes (\Gamma^{(6)}_{l_7} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\
= \ & -(6!)[6((\mathbf{1}_{8\times 8} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\
& \quad +2(\mathbf{1}_{16\times 16} \otimes (\mathbf{1}_{8\times 8} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16})].
\end{aligned}$$

  We sum up all 12 permutations to obtain $-69120 B^{SO(9)}_9$ ($-23040(n-1) B^{SO(9)}_9$ for the general n-fold tensor product).

- We go on to the contribution of (19):

$$\begin{aligned}
& \epsilon_{p_1 \cdots p_8 9} (\Gamma^{(8)}_{p_1 \cdots p_4} \otimes \Gamma^{(8)}_{p_5 \cdots p_8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\
= \ & -4(4!)[(\Gamma^{(8)}_{l_5 l_6 l_7 8 9} \otimes \Gamma^{(8)}_{l_5 l_6 l_7 8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) + (\Gamma^{(8)}_{l_5 l_6 l_7 8} \otimes \Gamma^{(8)}_{l_5 l_6 l_7 8 9} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16})] \\
= \ & 4(4!)[((\Gamma^{(6)}_{l_5 l_6 l_7} \otimes \mathbf{1}_{2\times 2}) \otimes (\Gamma^{(6)}_{l_5 l_6 l_7} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\
& \quad +((\Gamma^{(6)}_{l_5 l_6 l_7} \otimes \sigma_3) \otimes (\Gamma^{(6)}_{l_5 l_6 l_7} \otimes \mathbf{1}_{2\times 2}) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16})] \\
= \ & -72(4!)[((\mathbf{1}_{8\times 8} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\
& \quad +(\mathbf{1}_{16\times 16} \otimes (\mathbf{1}_{8\times 8} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16})].
\end{aligned}$$

Therefore, when we sum all the permutations, (19) gives $-31104B_9^{SO(9)}$ ($-10368(n-1)B_9^{SO(9)}$ for the general n).

- We next investigate the terms (20). Together with all the permutations, this gives $-41472B_9^{SO(9)}$ ($-6912(n-1)(n-2)B_9^{SO(9)}$ for any n) due to the following considerations:

$$\begin{aligned}
& \epsilon_{p_1\cdots p_8 9}(\Gamma^{(8)}_{p_1\cdots p_4}\otimes\Gamma^{(8)}_{p_5p_6}\otimes\Gamma^{(8)}_{p_7p_8}\otimes\mathbf{1}_{16\times16}) \\
= \; & (4!)[-((\Gamma^{(6)}_{l_4\cdots l_7}\otimes\sigma_3)\otimes(\Gamma^{(6)}_{l_4l_5}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_6l_7}\otimes\mathbf{1}_{2\times2})\otimes\mathbf{1}_{16\times16}) \\
& \quad +2((\Gamma^{(6)}_{l_5l_6l_7}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_7}\otimes\sigma_3)\otimes\mathbf{1}_{16\times16}) \\
& \quad +2((\Gamma^{(6)}_{l_5l_6l_7}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_7}\otimes\sigma_3)\otimes(\Gamma^{(6)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})\otimes\mathbf{1}_{16\times16})] \\
= \; & -(4!)[24((\mathbf{1}_{8\times8}\otimes\sigma_3)\otimes\mathbf{1}_{16\times16}\otimes\mathbf{1}_{16\times16}\otimes\mathbf{1}_{16\times16}) \\
& \quad +12(\mathbf{1}_{16\times16}\otimes((\mathbf{1}_{8\times8}\otimes\sigma_3)\otimes\mathbf{1}_{16\times16}+\mathbf{1}_{16\times16}\otimes(\mathbf{1}_{8\times8}\otimes\sigma_3))\otimes\mathbf{1}_{16\times16})],
\end{aligned}$$

  where we have used the formulae (14) and (15).

- Lastly, (21) gives $-2304B_9^{SO(9)}$ ($-384(n-1)(n-2)(n-3)B_9^{SO(9)}$ for any n):

$$\begin{aligned}
& 24\epsilon_{p_1\cdots p_8 9}(\Gamma^{(8)}_{p_1p_2}\otimes\Gamma^{(8)}_{p_3p_4}\otimes\Gamma^{(8)}_{p_5p_6}\otimes\Gamma^{(8)}_{p_7p_8}) \\
= \; & 48[(\Gamma^{(6)}_{l_1l_2}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_3l_4}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_1\cdots l_6}\otimes\sigma_3) \\
& +(\Gamma^{(6)}_{l_3l_4}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_1\cdots l_6}\otimes\sigma_3)\otimes(\Gamma^{(6)}_{l_1l_2}\otimes\mathbf{1}_{2\times2}) \\
& +(\Gamma^{(6)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_1\cdots l_6}\otimes\sigma_3)\otimes(\Gamma^{(6)}_{l_1l_2}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_3l_4}\otimes\mathbf{1}_{2\times2}) \\
& +(\Gamma^{(6)}_{l_1\cdots l_6}\otimes\sigma_3)\otimes(\Gamma^{(6)}_{l_1l_2}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_3l_4}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(6)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})] = -2304B_9^{SO(9)}.
\end{aligned}$$

  Here, we have exploited the formula (16).

When we sum up the contribution of (17) $\sim$ (21), we obtain $m_4 = -384(n+2)(n+4)(n+6)$.