Generalized factorization method for the overlap problem in a matrix model with complex action

Takehiro Azuma

Institute for Fundamental Sciences, Setsunan University

JPS meeting, Sep. 11th 2010, 13:45 ∼ 14:00

Collaboration with K.N. Anagnostopoulos and J. Nishimura

Contents

1 Introduction

Sign problem and overlap problem

Partition function with complex integrand:

$$Z = \int dA \exp\left(-S_0 + i\Gamma\right), \quad Z_0 = \int dA e^{-S_0}.$$

e.g. lattice QCD, large-N reduced models for superstring theory$\cdots$.

- **Sign problem:**

  Standard reweighting method

  $$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\Gamma} \rangle_0}{\langle e^{i\Gamma} \rangle_0}, \text{ where } \langle * \rangle_0 = (\text{ V.E.V. for the phase-quenched model } Z_0).$$

  (Number of configurations required) $\simeq e^{\mathrm{O}(N^2)}$.

- **Overlap problem:**

  Discrepancy of important configurations between phase-quenched partition function Z_0 and full partition function Z.

2 Factorization method

Method to sample important configurations for the full partition function.

We constrain the observables $\Sigma = \{\mathcal{O}_k | k = 1, 2, \cdots, n\}$.

Observables are normalized as $\tilde{\mathcal{O}}_k = \dfrac{\mathcal{O}_k}{\langle \mathcal{O}_k \rangle_0}$,

where $\langle \cdots \rangle_0$=(V.E.V. for the phase-quenched partition function Z_0).

Generalized distribution function $\rho(x_1, \cdots, x_n) = \left\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\rangle$ factorizes as

$$\rho(x_1, \cdots, x_n) = \frac{1}{C} \rho^{(0)}(x_1, \cdots, x_n) \, w(x_1, \cdots, x_n) \;, \text{ where}$$

$$\rho^{(0)}(x_1, \cdots, x_n) = \left\langle \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k) \right\rangle_0, \quad w(x_1, \cdots, x_n) \stackrel{\text{def}}{=} \langle e^{i\Gamma} \rangle_{x_1, \cdots, x_n}.$$

$\langle \cdots \rangle_{x_1, \cdots, x_n}$ = V.E.V. for partition function $Z_{x_1, \cdots, x_n} = \int dA e^{-S_0} \prod_{k=1}^{n} \delta(x_k - \tilde{\mathcal{O}}_k)$.

Evaluation of the observables $\langle \tilde{\mathcal{O}}_k \rangle$:

Peak of the distribution function $\rho(x_1, \cdots, x_n)$ at V = (system size) $\to +\infty$

$\Rightarrow$ solution of the saddle-point equation

$$\frac{d}{dx_k} \left\{ \lim_{V \to +\infty} \frac{1}{V} \log \rho^{(0)}(x_1, \cdots, x_n) \right\} = -\frac{d}{dx_k} \left\{ \lim_{V \to +\infty} \frac{1}{V} \log w(x_1, \cdots, x_n) \right\}$$

3 Simplified IKKT model

Simplified model with spontaneous rotational symmetry breakdown
Nishimura, hep-th/0108070.

$$S = \underbrace{\frac{N}{2}\mathrm{tr}\, A_\mu^2}_{=S_b} \underbrace{-\bar{\psi}_\alpha^f (\Gamma_\mu)_{\alpha\beta} A_\mu \psi_\beta^f}_{=S_f}$$

- A_μ: $N \times N$ hermitian matrices ($\mu = 1, \cdots, 4$)
  $\bar{\psi}_\alpha^f, \psi_\alpha^f$: N-dim vector ($\alpha = 1, 2, f = 1, \cdots, N_f$)
  N_f = (number of flavors)
- SO(4) rotational symmetry.
- No supersymmetry.
- Partition function:
  $$Z = \int dA e^{-S_B} (\det \mathcal{D})^{N_f} = \int dA e^{-S_0} e^{i\Gamma},$$
  where
  $\mathcal{D} = \Gamma_\mu A_\mu$ = ($2N \times 2N$ matrices),
  $S_0 = S_b - \log |\det \mathcal{D}|$.
  Fermion determinant $\det \mathcal{D}$ is complex.
- Gaussian Expansion Method (GEM) up to 9th order: Okubo, Nishimura and Sugino, hep-th/0412194.

Observable for probing dimensionality :
$T_{\mu\nu} = \frac{1}{N}\mathrm{tr}\,(A_\mu A_\nu)$.
λ_n ($n = 1, 2, 3, 4$) : eigenvalues of $T_{\mu\nu}$
($\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \lambda_4$)
Spontaneous breakdown of SO(4) to SO(2) at finite $r \left(= \frac{N_f}{N}\right)$.

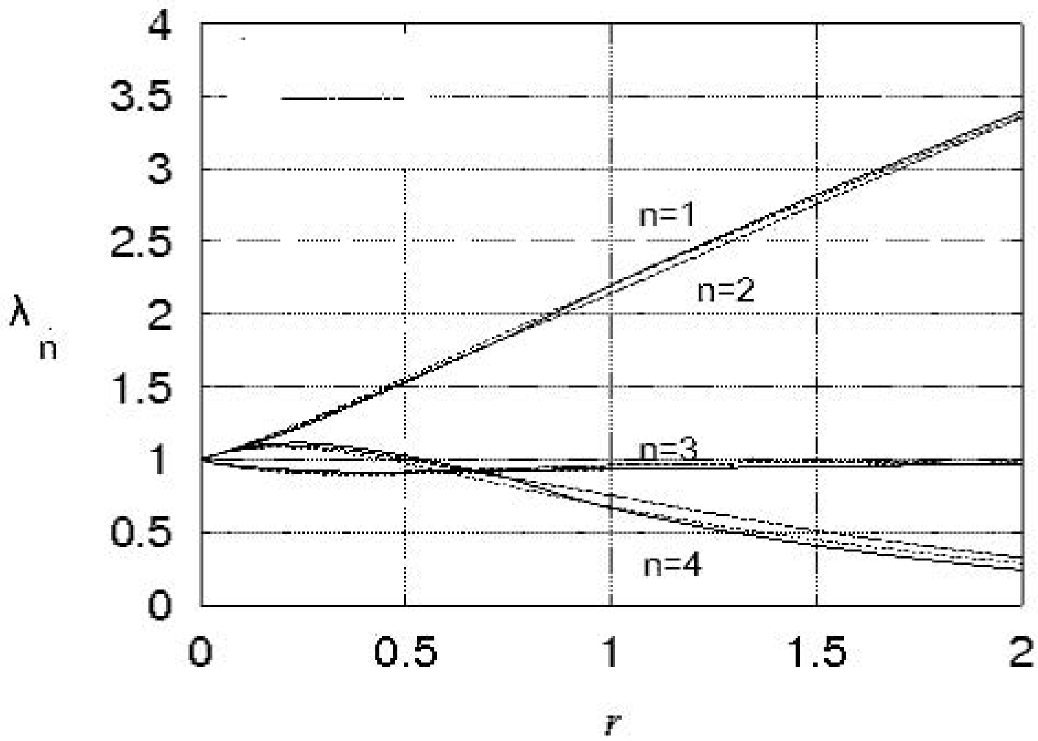
- Choice of the observables to constrain:
  $\Sigma = \{\mathcal{O}_k = \lambda_k | k = 1, 2, 3, 4\}$
  Partition function to simulate:
  $$Z_{x_1,x_2,x_3,x_4} = \int dA e^{-S_0} \prod_{n=1}^{4} \delta(x_n - \tilde{\lambda}_n)$$

SO(2) vacuum

Solutions which satisfy $x_1 = x_2 > 1 > x_3 > x_4$.

Result of GEM up to 9th order at large N:

Okubo, Nishimura and Sugino, hep-th/0412194.



SO(2) ansatz

$$\tilde{\lambda}_n = \frac{\lambda_n}{\langle\lambda_n\rangle_0}, \quad \langle\lambda_n\rangle_0 = 1 + \frac{r}{2} = 1.5 (\text{for } r = 1)$$

for $n = 1, 2, 3, 4$.

$\langle\tilde{\lambda}_{1,2}\rangle = 1.4$, $\langle\tilde{\lambda}_3\rangle = 0.7$, $\langle\tilde{\lambda}_4\rangle = 0.5$ $(r = 1)$.

Minimum of the free energy density $\mathcal{F}(x)$

$$\frac{\partial}{\partial x_n}\rho^{(0)}_{\mathrm{SO}(2)}(x_2, x_3, x_4) = -\frac{\partial}{\partial x_n} w_{\mathrm{SO}(2)}(x_2, x_3, x_4)$$

for $n = 2, 3, 4$,

where

$$\rho^{(0)}_{\mathrm{SO}(2)}(x_2, x_3, x_4) = \rho^{(0)}(x_2, x_2, x_3, x_4) ,$$

$$w_{\mathrm{SO}(2)}(x_2, x_3, x_4) = w(x_2, x_2, x_3, x_4)$$

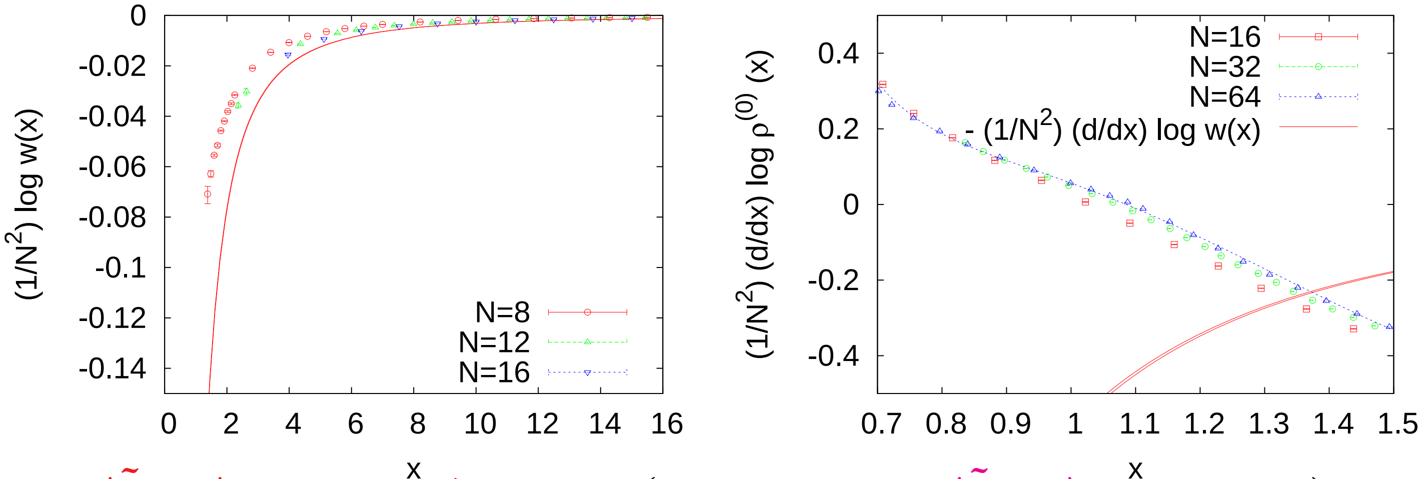
Calculation of $\langle \tilde{\lambda}_n \rangle$ at $r = 1$

Calculation of $\langle \tilde{\lambda}_{n=2} \rangle$ for fixed $\langle \tilde{\lambda}_{n=3} \rangle = 0.7$ and $\langle \tilde{\lambda}_{n=4} \rangle = 0.5$.

$$\frac{1}{N^2}\frac{d}{dx}\log \rho^{(0)}(x) = -\frac{\partial}{\partial x}\frac{1}{N^2}\log w(x),$$

where $\rho^{(0)}(x) = \rho^{(0)}_{\mathrm{SO}(2)}(x, y = 0.7, z = 0.5)$ and $w(x) = w_{\mathrm{SO}(2)}(x, y = 0.7, z = 0.5)$.

Scaling behavior of the phase: $\dfrac{1}{N^2}\log w(x) \sim -\tilde{d}_1 x^{-2} + \tilde{d}_2 x^{-2.5}$.



Numerical Result: $\langle \tilde{\lambda}_{n=2} \rangle = 1.372 \pm 0.002$, (GEM result $\langle \tilde{\lambda}_{n=2} \rangle_{\mathrm{GEM}} = 1.4$).

Similarly, $\langle \tilde{\lambda}_{n=3} \rangle = 0.648 \pm 0.004$, (GEM result $\langle \tilde{\lambda}_{n=3} \rangle_{\mathrm{GEM}} = 0.7$).

$\langle \tilde{\lambda}_{n=4} \rangle = 0.550 \pm 0.002$, (GEM result $\langle \tilde{\lambda}_{n=4} \rangle_{\mathrm{GEM}} = 0.5$).

Is there any more overlap problem?

Observables to constrain: $\Sigma = \{\mathcal{O}_k = \lambda_k | k = 1, 2, 3, 4\}$. **Is this enough?**

Partition function $Z_{\mathcal{O}} = \int dA e^{-S_0} \delta(x - \tilde{\mathcal{O}}) \prod_{n=1}^{4} \delta(x_n - \tilde{\lambda}_n)$

(here we constrain $\Sigma = \{\mathcal{O}, \lambda_1, \cdots, \lambda_4\}$**).**

Peak of the distribution function $\underbrace{\rho(x_1, x_2, x_3, x_4, x)}_{=\rho_{\mathcal{O}}(x)}$ $\Rightarrow$ **solution of the saddle-point equation**

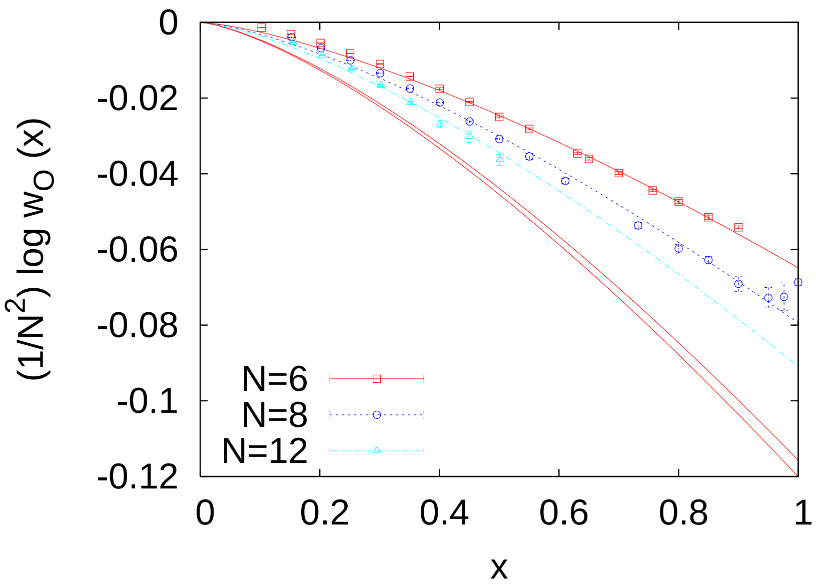
$$\frac{d}{dx}\frac{1}{N^2}\log \underbrace{\rho^{(0)}(x_1, x_2, x_3, x_4, x)}_{=\rho^{(0)}_{\mathcal{O}}(x)} = -\frac{d}{dx}\frac{1}{N^2}\log \underbrace{w(x_1, x_2, x_3, x_4, x)}_{=w_{\mathcal{O}}(x)}.$$

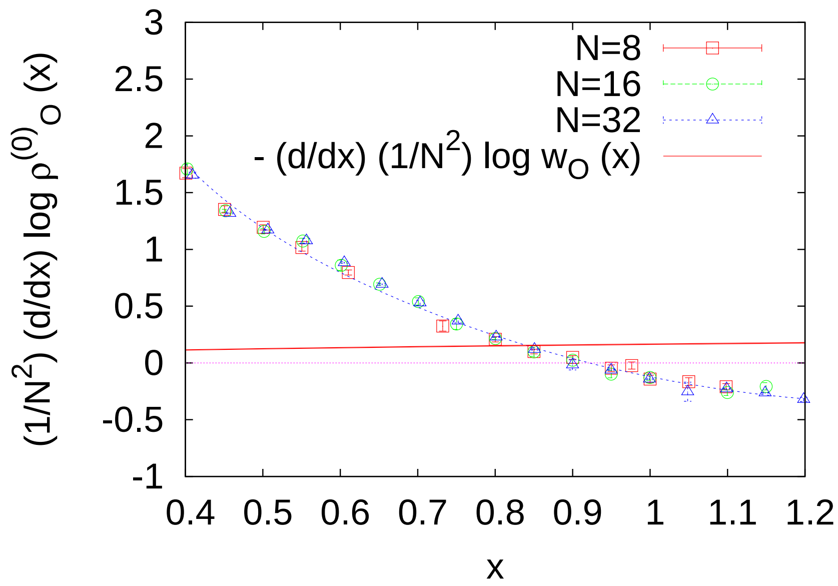
Do the peaks of $\rho^{(0)}_{\mathcal{O}}(x)$ **and** $\rho_{\mathcal{O}}(x)$ **match?**

We consider $\mathcal{O} = -\frac{1}{N}\mathrm{tr}\,[A_\mu, A_\nu]^2$.

Simulation with $\tilde{\lambda}_n$ fixed at GEM result $\langle \tilde{\lambda}_{1,2} \rangle = 1.4$, $\langle \tilde{\lambda}_3 \rangle = 0.7$, $\langle \tilde{\lambda}_4 \rangle = 0.5$ ($r = 1$).

- **Peak of $\rho^{(0)}_{\mathcal{O}}(x)$: Solution of $\frac{1}{N^2}\frac{d}{dx}\log \rho^{(0)}_{\mathcal{O}}(x) = 0 \Rightarrow x \simeq 0.92$.**
- **Peak of $\rho_{\mathcal{O}}(x)$: Solution of $\frac{1}{N^2}\frac{d}{dx}\log \rho^{(0)}_{\mathcal{O}}(x) = -\frac{1}{N^2}\frac{d}{dx}\log w_{\mathcal{O}}(x) \Rightarrow x \simeq 0.84$. Systematic error is around 10%.**





4 Conclusion

Factorization method as a practical approach to the sign problem.

Resolution of the overlap problem.

Monte Carlo simulation of simplified IKKT model

⇒ spontaneous breakdown of SO(4) rotational symmetry.

Good agreement with GEM result.

Future problems

Monte Carlo Simulation of the IKKT model Anagnostopoulos, Aoyama, T. A., Hanada and Nishimura, in progress

Effect of supersymmetry on dynamical generation of spacetime.