OPE between the energy-momentum tensor
and
the Wilson loop in $\mathcal{N} = 4$ SYM theory

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1 Introduction

Operator product expansion (OPE) is an important notion in conformal field theory.  
⇒ OPE extracts the conformal weight of the operator.

The OPE in the 2-dimensional CFT

\[ T(z) A(w) = \left( \text{lower-dimensional operators} \right) + \frac{h A(w)}{(z - w)^2} + \frac{\partial A(w)}{z - w} + \cdots. \]

- An example of the lower-dimensional operator:

\[ T(z) T(w) = \frac{c/2}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w} + \cdots. \]

- An important special case is a primary field, on which the OPE reduces to

\[ T(z) A(w) = \frac{h A(w)}{(z - w)^2} + \frac{\partial A(w)}{z - w} + \cdots. \]
The OPE $T_{\mu\nu}(z)W[C]$ in 4-dimensional Euclidean space

We consider the OPE $T_{\mu\nu}(z)W[C]$ in $\mathcal{N} = 4$ SYM theory in the Euclidean space.

\[ \mathcal{L} = \frac{1}{2G^2} \left[ \frac{1}{2}(F_{\mu\nu})^2 + (D_\mu \phi_i)^2 + ([\phi_i, \phi_j])^2 + \xi (\partial^\mu A_\mu)^2 \right], \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \]

\[ D_\mu \phi_i = \partial_\mu \phi_i - i[A_\mu, \phi_i], \]

\[ W[C] = \frac{1}{N!} \text{Tr} \text{P exp} \left[ \oint_C du \left\{ iA_\mu(y(u)) \frac{dy^\mu(u)}{du} + \phi_i(y(u)) \theta^i(u) \right\} \right]. \]

- $y_\mu(u)$: coordinate of the Wilson loop
- $u$: an arc length parameter which satisfies $\left| \frac{dy_\mu(u)}{du} \right| = 1.$

$y_\mu(u_0)$: the nearest point on the loop to the point $z_\mu$.

We consider the conformal Ward identity by wrapping the Wilson loop by the enveloping surface of the spheres $S^2(u_0).$
\[
\int_{\mathcal{M}} d^4z \partial^\mu [T_{\mu\nu}(z)W[C]v^\nu(z)]
= \int du_0 \int_{S^2(u_0)} d\Omega [1 - (z_\alpha - y_\alpha(u_0) \frac{d^2 y^\alpha(u_0)}{du^2})] n^\mu T_{\mu\nu}(z)W[C]v^\nu(z).
\]

- \(S^2(u_0)\): a \(S^2\) sphere of a fixed radius \(\epsilon\) which is perpendicular to the loop and has its center at \(y_\mu(u_0)\).
- \(\mathcal{M}\): the region inside the enveloping surface.
- \(v^\mu(z)\): the conformal Killing vector.
  
  \begin{align*}
  \text{Translation:} \quad & v^\nu(z) = \xi^\nu, \\
  \text{Dilatation:} \quad & v^\nu(z) = \lambda z^\nu, \\
  \text{SCT:} \quad & v^\nu(z) = 2z^\nu (b_\alpha z^\alpha) - b^\nu z^2.
  \end{align*}

- \(T_{\mu\nu}(z)\): energy-momentum tensor \(T_{\mu\nu}(z) = \frac{2}{\sqrt{g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}\).

The weakest singularity in the OPE than contribute to the conformal Ward identity is \(\mathcal{O}(z - y(u_0))^{-2}\), since the surface area of \(S^2(u_0)\) is \(4\pi \epsilon^2\).

We expand the OPE \(T_{\mu\nu}(z)W[C]\) as a power series expansion in \(z_\mu - y_\mu(u_0)\).

\[T_{\mu\nu}(z)W[C] = (T_{\mu\nu}(z)W[C])_c + (T_{\mu\nu}(z)W[C])_{vec} + (T_{\mu\nu}(z)W[C])_{sca}.\]

- \((T_{\mu\nu}(z)W[C])_c\): the terms containing \(W[C]\) itself without any insertion of the fields to it.
- \((T_{\mu\nu}(z)W[C])_{vec, sca}\): the insertion of the fields \(A_\mu(y(u_0))\) and \(\phi_i(y(u_0))\) to \(W[C]\), respectively.

<table>
<thead>
<tr>
<th>(T(z)A(w))</th>
<th>lower-\ldots</th>
<th>(\frac{hA(w)}{(z-w)^2})</th>
<th>(\frac{\partial A(w)}{z-w})</th>
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<tbody>
<tr>
<td>(T_{\mu\nu}(z)W[C])</td>
<td>(-)</td>
<td>(T_{\mu\nu}(z)W[C])_c)</td>
<td>(T_{\mu\nu}(z)W[C])_{vec})</td>
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The general form is obtained up to the free parameters 
$q, q', \alpha, \beta, \gamma$ by the following reasonings:

- Dimensional analysis

- Properties of $T_{\mu\nu}(z)$ in conformal field theory
  
  * $T_{\mu}^{\mu}(z) = 0$: conformal invariance of the action.
  
  * $\partial_{\mu}T_{\mu\nu}(z) = 0$: the conservation law of the energy
    and the momentum.

- The Wilson loop possesses no conformal weight and
  only the deformation occurs with respect to translation:

  $$\int_\mathcal{M} d^4z \partial_{\mu}[T_{\mu\nu}(z)W[C]\xi^\nu] = -\int ds \left(\frac{\delta W[C]}{\delta y^\nu(s)}\right)\xi^\nu.$$

  * $s$: general parameterization running over $0 \leq s \leq 2\pi$. The relationship to the arc length parameter is
    $\frac{dy_0}{ds} = \left|\frac{dy_\mu(s)}{ds}\right|$.

  * $\left(\frac{\delta W[C]}{\delta y^\nu(s)}\right)$: deformation of the Wilson loop.

  $$\left(\frac{\delta W[C]}{\delta y^\nu(s)}\right) = \frac{1}{N}TrP\hat{w}_{2\pi,s}[iF_{\nu\alpha}(y(s)) \frac{dy^\alpha(s)}{ds}$$
  
  $$+ \left[\frac{dy_\mu(s)}{ds}\right][\theta_i(s) D_\nu \phi^i(y(s))] \hat{w}_{s,0}$$

  $$- \frac{d}{ds} \left[\frac{1}{N}TrP \hat{w}_{2\pi,s}\{\theta_i(s) \phi^i(y(s)) \frac{dy_\mu(s)}{ds} \frac{dy_\mu(s)}{ds}\right] \hat{w}_{s,0},$$

  $$\hat{w}_{b,a} = \exp[\int_a^b ds \left\{iA_\mu(y(s)) \frac{dy^\mu(s)}{ds} + \left[\frac{dy_\mu(s)}{ds}\right]\phi_i(y(s)) \theta^i(s)\right\}].$$

- We further assume that the same holds true of dilatation:

  $$\int_\mathcal{M} d^4z \partial_{\mu}[T_{\mu\nu}(z)W[C]\lambda z^\nu] = -\int ds \left(\frac{\delta W[C]}{\delta y^\nu(s)}\right)\lambda y^\nu(s).$$
\[
(T_{\mu\nu}(z)W[C])_c = \frac{q}{24\pi^2} \left[ \frac{1}{|z - y(u_0)|^4} \left( g_{\mu\nu} - 2 \left( \frac{dy_\mu(u_0)}{du} \frac{dy_\nu(u_0)}{du} \right) \right) \right] \left( \frac{z_\mu - y_\mu(u_0)}{|z - y(u_0)|^2} \right) \left( \frac{z_\nu - y_\nu(u_0)}{|z - y(u_0)|^2} \right) W[C] \\
+ \frac{1}{24\pi^2} \left[ -2(q + q') \left( \frac{z_\mu - y_\mu(u_0)}{|z - y(u_0)|^6} \right) \left( \frac{z_\nu - y_\nu(u_0)}{|z - y(u_0)|^4} \right) \left( \frac{z_\alpha - y_\alpha(u_0)}{|z - y(u_0)|^4} \right) \right] \left( \frac{d^2g_{\alpha}(u_0)}{du^2} \right) \left( \frac{d^2g_{\alpha}(u_0)}{du^2} \right) \left( \frac{d^2g_{\alpha}(u_0)}{du^2} \right) \\
+ 2q g_{\mu\nu} \left( \frac{z_\alpha - y_\alpha(u_0)}{|z - y(u_0)|^4} \right) \left( \frac{d^2g_{\alpha}(u_0)}{du^2} \right) \left( \frac{d^2g_{\alpha}(u_0)}{du^2} \right) W[C] + \mathcal{O}(z - y(u_0))^{-2},
\]

\[
(T_{\mu\nu}(z)W[C])_{vec} = \frac{1}{N} Tr P_{w_{L,u_0}} \frac{i}{4\pi |z - y(u_0)|^3} \\
\times \left[ -(z_\mu - y_\mu(u_0))F_{\nu\alpha}(y(u_0)) \frac{dy_{\alpha}(u_0)}{du} - (z_\nu - y_\nu(u_0))F_{\mu\alpha}(y(u_0)) \frac{dy_{\alpha}(u_0)}{du} \right] \\
+ g_{\mu\nu}(z_{\alpha} - y_{\alpha}(u_0))F_{\alpha\beta}(y(u_0)) \frac{dy_{\beta}(u_0)}{du} \\
+ (z_{\alpha} - y_{\alpha}(u_0)) [F_{\mu\alpha}(y(u_0)) \frac{dy_{\mu}(u_0)}{du} + F_{\nu\alpha}(y(u_0)) \frac{dy_{\nu}(u_0)}{du}] ] w_{u_0,0} \\\n+ \mathcal{O}(z - y(u_0))^{-1},
\]
\[(T_{\mu\nu}(z)W[C])_{\text{ sca}} = \frac{1}{N} Tr P \int_{u_0}^1 [\theta_i(u_0)\phi^i(y(u_0))]
\frac{1}{24\pi |z-y(u_0)|^3 (2 - \frac{\alpha}{10} + \frac{\beta}{2})}
\times [g_{\mu\nu} - 3(z_{\mu} - y_{\mu}(u_0))(z_{\nu} - y_{\nu}(u_0))]
\frac{dy_{\mu}(u_0)}{du} dy_{\nu}(u_0)
\frac{\theta_i(u_0)\phi^i(y(u_0))}{24\pi |z-y(u_0)|^3} (1 + \frac{\alpha}{10} - \frac{\beta}{2} + \frac{\gamma}{5})
((z_{\mu} - y_{\mu}(u_0)) \frac{d^2y_{\nu}(u_0)}{du^2} + (z_{\nu} - y_{\nu}(u_0)) \frac{d^2y_{\nu}(u_0)}{du^2}) + (1 - \frac{\alpha}{5} + \beta - \frac{\gamma}{5})g_{\mu\nu}(z_{\alpha} - y_{\alpha}(u_0))
\frac{dy^\alpha(u_0)}{du}]
\frac{\theta_i(u_0)}{24\pi |z-y(u_0)|^3} ((z_{\mu} - y_{\mu}(u_0)) D_{\nu}\phi^i(y(u_0)) + (z_{\nu} - y_{\nu}(u_0)) D_{\mu}\phi^i(y(u_0)))
+ (4 - \frac{\alpha}{5}) g_{\mu\nu}(z_{\alpha} - y_{\alpha}(u_0)) D^\alpha\phi^i(y(u_0))
+ (-6 - \frac{3\alpha}{5}) (z_{\mu} - y_{\mu}(u_0))(z_{\nu} - y_{\nu}(u_0))(z_{\alpha} - y_{\alpha}(u_0)) D^\alpha\phi^i(y(u_0))
\frac{dy_{\nu}(u_0)}{du} ((z_{\mu} - y_{\mu}(u_0)) \frac{dy_{\nu}(u_0)}{du})
\frac{\beta dy_{\alpha}(u_0)}{du} (D^\alpha\phi^i(y(u_0))) |(z_{\mu} - y_{\mu}(u_0)) \frac{dy_{\nu}(u_0)}{du}|
[w_{u_0,0} + \mathcal{O}(z-y(u_0))^{-1}].

The Wilson loop possesses no conformal weight also with respect to the special conformal transformation:

\[\int_{\mathcal{M}} d^4z \partial^\mu[T_{\mu\nu}(z)W[C]v^\nu(z)]\]
\[= - \int ds (\frac{\delta W[C]}{\delta y^\nu(s)})[2y^\nu(s)(b_\alpha y^\alpha(s)) - b^\nu(y(s))^2].\]
Summary

• The Wilson loop possesses no conformal weight with respect to the translation, dilatation and special conformal transformation.

• The Wilson loop undergoes only a deformation with respect to these conformal transformation.

Future Problem

• The computation of $T_{\mu\nu}W[C]$ in terms of supergravity in AdS space.

![Diagram of boundary and Wilson loop](image)

• The computation of the circular $\langle W[C] \rangle$.

N. Drukker, D. J. Gross and H. Ooguri, hep-th/9904191

They argued that the expectation value of the circular Wilson loop in $\mathcal{N} = 4$ SYM is

$$\langle W[C_{\text{circ}}] \rangle = \frac{1}{N} L_{N-1}^{1} \left( -\frac{\lambda}{4N} \right) \exp\left( \frac{\lambda}{8N} \right)$$

$$= \frac{2}{\sqrt{\lambda}} I_{1}(\sqrt{\lambda}) + \frac{\lambda}{48N^2} I_{2}(\sqrt{\lambda}) + \frac{\lambda^2}{1280N^4} I_{4}(\sqrt{\lambda}) + \ldots.$$ 

Our result may serve to discuss the gauge invariance of this quantity.