Curved-space classical solution of a massive supermatrix model

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1 Introduction

Large-$N$ reduced models are the most powerful candidates for the constructive definition of superstring theory.

**IIB matrix model**

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

$$S = \frac{1}{g^2} Tr_{N\times N} \left( \frac{1}{4} \sum_{\mu,\nu=0}^{9} [A_{\mu}, A_{\nu}]^2 + \frac{1}{2} \bar{\psi} \sum_{\mu=0}^{9} \Gamma^\mu [A_{\mu}, \psi] \right),$$

(where $Z = \int dA d\bar{\psi} e^{+S}$).

- Dimensional reduction of $\mathcal{N} = 1$ 10-dimensional SYM theory to 0 dimension.
  $A_{\mu}$ and $\psi$ are $N \times N$ Hermitian matrices.
  * $A_{\mu}$: 10-dimensional vectors
  * $\psi$: 10-dimensional Majorana-Weyl (i.e. 16-component) spinors

- Matrix regularization of the Schild form of the Green-Schwarz action of the type IIB superstring theory.

- $SU(N)$ gauge symmetry and $SO(9,1)$ Lorentz symmetry ($SO(9,1) \times U(N)$).

- The $N \times N$ matrices describe the many-body system.

- No free parameter: $A_{\mu} \rightarrow g^2 A_{\mu}$, $\psi \rightarrow g^3 \psi$. 
• \( \mathcal{N} = 2 \) SUSY: This theory must contain spin-2 gravitons if it contains massless particles.

* homogeneous: \( \delta^{(1)}_\epsilon A_\mu = i\epsilon \Gamma_\mu \psi, \quad \delta^{(1)}_\epsilon \psi = \frac{i}{2}\Gamma^{\mu\nu}[A_\mu, A_\nu]\epsilon. \)

* inhomogeneous: \( \delta^{(2)}_\xi A_\mu = 0, \quad \delta^{(2)}_\xi \psi = \xi. \)

* We obtain the following commutation relations:

\[
\begin{align*}
(1) \quad [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}] A_\mu &= [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}] \psi = 0, \\
(2) \quad [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}] A_\mu &= [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}] \psi = 0, \\
(3) \quad [\delta^{(1)}_\epsilon, \delta^{(2)}_\xi] A_\mu &= -i\epsilon \Gamma_\mu \xi, \quad [\delta^{(1)}_\epsilon, \delta^{(2)}_\xi] \psi = 0.
\end{align*}
\]

This gives a shift of the bosonic variables for \( \tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)}, \quad \tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)}): (\alpha, \beta = 1, 2) \)

\[
\begin{align*}
[\tilde{\delta}^{(\alpha)}_\epsilon, \tilde{\delta}^{(\beta)}_\xi] \psi &= 0, \\
[\tilde{\delta}^{(\alpha)}_\epsilon, \tilde{\delta}^{(\beta)}_\xi] A_\mu &= -2i\delta^{\alpha\beta} \epsilon \Gamma_\mu \xi.
\end{align*}
\]

\( \Rightarrow \) Therefore, the eigenvalues of the bosonic large-\( N \) matrices \( A_\mu \) represent the spacetime coordinates.
If the large-$N$ reduced models are to be an authentic framework to unify all interactions in nature \ldots,

\[\downarrow\]

[Q] How can we express the gravitational interaction more manifestly in terms of a large-$N$ reduced model?

1. IIB matrix model itself is an eligible framework to describe the gravity.

- **General coordinate invariance**
  

  The general coordinate invariance is interpreted as the permutation $S_N$ invariance of the eigenvalues of the large $N$ matrices.

  \[
x^i \rightarrow x^{\sigma(i)} \text{ for } \sigma \in S_N, \\
  \downarrow
  x \rightarrow \xi(x) \text{ such that } \xi(x^i) = x^{\sigma(i)}.
  \]
The computation of the one-loop effective Lagrangian reveals the graviton and dilaton exchange in IIB matrix model.

\[
A_\mu = \begin{pmatrix}
x_\mu 1_{n_1} + a_\mu^{(1)} \\
y_\mu 1_{n_2} + a_\mu^{(2)}
\end{pmatrix}
\]

\[
W_{\text{eff}} \sim -\frac{12}{(x-y)^8} \text{Tr}_{n_1 \times n_1} (f^{(1)}_{\mu \rho} f^{(1)}_{\rho \nu}) \text{Tr}_{n_2 \times n_2} (f^{(2)}_{\mu \sigma} f^{(2)}_{\sigma \nu}) \frac{1}{(x-y)^8}
\]

graviton exchange

\[
+ \frac{3}{2(x-y)^8} \text{Tr}_{n_1 \times n_1} (f^{(1)}_{\mu \nu} f^{(1)}_{\mu \nu}) \text{Tr}_{n_2 \times n_2} (f^{(2)}_{\rho \sigma} f^{(2)}_{\rho \sigma}),
\]

dilaton exchange

where \( f^{(1,2)}_{\mu \nu} = i [a^{(1,2)}_\mu, a^{(1,2)}_\nu]. \)
2. A matrix model must incorporate a local Lorentz invariance

We need to enlarge the symmetry of the model.


Symmetry of IIB matrix model is \( SO(9, 1) \times U(N) \): \( so(9, 1) \) Lorentz symmetry and \( u(N) \) gauge symmetry are decoupled.

\[
\exp(\xi \otimes 1 + 1 \otimes u) = e^{\xi} \otimes e^{u}, \quad \text{where} \quad \xi \in so(9, 1), \ u \in u(N).
\]

\( \Rightarrow \) In IIB matrix model, the eigenvalues of the bosonic matrices \( A_{\mu} \) are regarded as the spacetime coordinate.

\( \Rightarrow \) If we are to formulate a matrix model with local Lorentz invariance, the parameters of the Lorentz transformation \( \xi \) must be promoted to (nontrivial) \( u(N) \) matrices.

\( \Rightarrow \) \( so(9, 1) \) Lorentz symmetry and \( u(N) \) gauge symmetry must be unified; i.e. the symmetry is the tensor product of the Lie algebra \( so(9, 1) \otimes u(N) \), rather than \( SO(9, 1) \times U(N) \).

\( \mathcal{A}, \mathcal{B} = \) [Lie algebras whose bases are \( \{a_i\} \) and \( \{b_j\} \), respectively.]

- \( \mathcal{A} \otimes \mathcal{B} \): The space spanned by the basis \( a_i \otimes b_j \). This is not necessarily a closed Lie algebra.

- \( \mathcal{A} \tilde{\otimes} \mathcal{B} \): The smallest Lie algebra that includes \( \mathcal{A} \otimes \mathcal{B} \) as a subset.

The gauge group must close with respect to the commutator

\[
[a \otimes A, b \otimes B] = \frac{1}{2} ([a, b] \otimes \{A, B\} + \{a, b\} \otimes [A, B]).
\]
3. A matrix model must incorporate a classical solution of a curved space.

Classical equation of motion of IIB matrix model:

$$[A^\nu, [A_\mu, A_\nu]] = 0.$$ 

This has only a flat non-commutative background as a classical solution.

$$[A_\mu, A_\nu] = i\epsilon_{\mu\nu}1_{N\times N}.$$ 

⇒ In order to surmount this difficulty, we alter a model so that it incorporates a curved-space classical solution ab initio.

[Example] IIB matrix model with a tachyonic mass term:


$$S = \frac{1}{g^2} Tr \left( \frac{1}{4} [A_a, A_b]^2 + \chi^2 A_a A_a \right),$$

EOM: $$[A_b, [A_a, A_b]] + 2\chi^2 A_a = 0.$$

• $SO(4) \times U(N)$ symmetry.
  $a, b$ runs over 1, 2, 3, 4 in the Euclidean space.

• Classical solutions of compact curved spacetime:

  * $SO(3)$ fuzzy sphere:
    $$[A_i, A_j] = i\chi \epsilon_{ijk} A_k \ (i, j, k = 1, 2, 3), \ A_4 = 0.$$ 

  * two-dimensional fuzzy torus:

  $$A_1 = \frac{r}{2}(U + U^\dagger), \ A_2 = \frac{r}{2i}(U - U^\dagger), \ A_3 = \frac{r}{2}(V + V^\dagger), \ A_4 = \frac{r}{2i}(V - V^\dagger),$$

  $$U = \begin{pmatrix}
  1 \\
  \omega \\
  \omega^2 \\
  \ldots \\
  \omega^{N-1}
  \end{pmatrix}, \ V = \begin{pmatrix}
  0 & 1 & 0 & \ldots & 1 \\
  1 & 0 & 1 & \ldots & 0 \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  1 & 0 & 1 & \ldots & 0
  \end{pmatrix},$$

  $$\omega = e^{i\theta}, \ \theta = \frac{2\pi}{N}, \ UV = e^{i\theta} UV, \ \chi^2 = r^2(1 - \cos \theta).$$
2 Massive supermatrix model

We consider the 3rd way in terms of an $osp(1,32|R)$ supermatrix model.

osp$(1|32,R)$ super Lie algebra

osp$(1|32,R)$: first mentioned with the relation to the 11-dimensional supergravity.


⇒ This has attracted a new attention as the unified super Lie algebra for the M-theory.

E. Bergshoeff, A. Van Proeyen, hep-th/0003261

The matrix model based on $osp(1|32,R)$ is a natural extension to IIB matrix model.

L. Smolin, hep-th/0002009
L. Smolin, hep-th/0006137
T. Azuma, S. Iso, H. Kawai and Y. Ohwashi, hep-th/0102168
M. Bagnoud, L, Carlevaro and A. Bilal, hep-th/0201183

• Extra fields:
  The $32 \times 32$ bosonic part of $osp(1|32,R)$ has the components of rank-$1,2,5$ of the 11-dimensional gamma matrices $u_\alpha, u_{\alpha_1\alpha_2}, u_{\alpha_1...\alpha_5}$.
  ⇒ The rank-1 components $u_\alpha$ can be identified with the bosonic vector of IIB matrix model $A_\mu$. 
• **Realization of the spin connection:**
  The complexification $u(1|16, 16)$ incorporates the rank-3 components.
  ⇒ This may be identified with the spin connection in the supergravity theory.
  ⇒ We can elucidate the inclusion of the gravity more manifestly.

• **Relation of the supersymmetry:**
  $osp(1|32, R)$ cubic supermatrix model (without mass term) nearly has the double structure of the 10-dimensional $\mathcal{N} = 2$ SUSY of IIB matrix model.

  * IIB matrix model: *16-component fermion*
  * $osp(1|32, R)$ model: *32-component fermion*
  
  twice as many fermions!!

The SUSY transformation of the rank-1 component $u_\alpha$ resembles that of the vector field $A_\mu$ of IIB matrix model.
\[ M \in osp(1|32, R) \overset{\text{def}}{=} TMG + GM = 0, \text{ where} \]
\[ G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}. \]

\[ M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}, \text{ where} \ Tm\Gamma^0 + \Gamma^0m = 0; \]
i.e. \( m \in sp(32), \)
\[ m = u_{A_1} \Gamma^{A_1} + \frac{1}{2!} u_{A_1 A_2} \Gamma^{A_1 A_2} + \frac{1}{5!} u_{A_1 \cdots A_5} \Gamma^{A_1 \cdots A_5}. \]

(Proof) We start with the general form \( M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & v \end{pmatrix}. \)

\[
0 = TMG + GM = \begin{pmatrix} Tm & -T\bar{\phi} \\ T\psi & v \end{pmatrix} \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix} + \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} m & \psi \\ i\bar{\psi} & v \end{pmatrix}
\]
\[
= \begin{pmatrix} Tm\Gamma^0 + \Gamma^0m & T\bar{\phi} + \Gamma^0\psi \\ T\psi\Gamma^0 - \bar{\phi} & 2iv \end{pmatrix}.
\]

Therefore, we obtain \( v = 0, \psi = \phi \) and \( Tm\Gamma^0 + \Gamma^0m = 0. \)

We next investigate what ranks of \( m \) survive. Since \( m \in sp(32), \) it follows that \( m = -(\Gamma^0)^{-1}(Tm)\Gamma^0 = +\Gamma^0(Tm)\Gamma^0. \)

\[
\Gamma^0 (T\Gamma^{A_1 \cdots A_k})\Gamma^0 = (-1)^{k-1} (\Gamma^0 (T\Gamma^{A_k})\Gamma^0) \cdots (\Gamma^0 (T\Gamma^{A_1})\Gamma^0)
\]
\[
= (-1)^{k-1} \Gamma^{A_k A_{k-1} \cdots A_1} = (-1)^{k-1} (-1)^{\frac{k(k-1)}{2}} \Gamma^{A_1 A_2 \cdots A_k}
\]
\[
= (-1)^{(k+2)(k-1)/2} \Gamma^{A_1 A_2 \cdots A_k}
\]
\[
= \begin{cases} 
+\Gamma^{A_1 A_2 \cdots A_k} & (k = 1, 2, 5) \\
-\Gamma^{A_1 A_2 \cdots A_k} & (k = 0, 3, 4)
\end{cases}
\]
Action of the massive supermatrix model

We add a mass term to the pure cubic action:

\[
S = Tr \left[ str \left( -3\mu M^2 + \frac{i}{g^2} M[M, M] \right) \right] \\
= Tr \left[ -3\mu \left\{ \sum_{p=1}^{32} M_p^Q M_Q^p \right\} - M_{33}^Q M_{33}^Q \right] \\
+ \frac{i}{g^2} \left\{ \sum_{p=1}^{32} M_p^Q [M_Q^R, M_R^p] \right\} - M_{33}^Q [M_Q^R, M_R^33] \right] , \\
= Tr \left[ 3\mu (-tr(m^2) + 2i\bar{\psi}\psi) + \frac{i}{g^2} (m_p^q [m_q^r, m_r^p] - 3i\bar{\psi}^p [m_p^q, \psi^q]) \right].
\]

- Each component of the $33 \times 33$ supermatrices is promoted to a large $N$ hermitian matrix.
- $osp(1|32, R)$ symmetry and $u(N)$ gauge symmetry are decoupled (i.e. $Osp(1|32, R) \times U(N)$ symmetry).
  * $M \rightarrow M + [M, (S \otimes 1_{N \times N})]$ for $S \in osp(1|32, R)$,
  * $M \rightarrow M + [M, (1_{33\times33} \otimes U)]$ for $U \in u(N)$. 
In order to see the correspondence of the fields with IIB matrix model, we express the bosonic $32 \times 32$ matrices in terms of the 10-dimensional indices. $(\mu, \nu, \cdots = 0, 1, \cdots, 9, \# = 10)$.

\[
W = u_\mu^\dagger, \quad A_\mu = u_\mu, \quad B_\mu = u_\mu^\dagger, \quad C_{\mu_1 \mu_2} = u_{\mu_1 \mu_2}, \\
H_{\mu_1 \cdots \mu_4} = u_{\mu_1 \cdots \mu_4}^\dagger, \quad Z_{\mu_1 \cdots \mu_5} = u_{\mu_1 \cdots \mu_5}.
\]

Then, the action is decomposed as

\[
S = 96 \mu Tr \left( -W^2 - A_\mu A^\mu + B_\mu B^\mu + \frac{1}{2} C_{\mu_1 \mu_2} C^{\mu_1 \mu_2} - \frac{1}{4!} H_{\mu_1 \cdots \mu_4} H^{\mu_1 \cdots \mu_4} \\
- \frac{1}{5!} Z_{\mu_1 \cdots \mu_5} Z^{\mu_1 \cdots \mu_5} + \frac{i}{16} \bar{\psi} \psi \right) \\
+ 32 i Tr \left( -3 C_{\mu_1 \mu_2} [A^{\mu_1}, A^{\mu_2}] + 3 C_{\mu_1 \mu_2} [B^{\mu_1}, B^{\mu_2}] + 6 W [A_\mu, B^\mu] + C_{\mu_1 \mu_2} [C^{\mu_2}_{\mu_3}, C^{\mu_3 \mu_4}] \\
+ \frac{1}{4} B_{\mu_1} H_{\mu_2 \cdots \mu_5} Z^{\mu_1 \cdots \mu_5} - \frac{1}{8} C_{\mu_1 \mu_2} (4 [H^{\mu_1}_{\rho_1 \rho_2 \rho_3}, H^{\mu_2 \rho_1 \rho_2 \rho_3} + [Z^{\mu_1}_{\rho_1 \cdots \rho_4}, Z^{\mu_1 \rho_1 \cdots \rho_4}]) \\
+ \frac{3}{(5!)^2} e^{\mu_1 \cdots \mu_{10}} (-W [Z_{\mu_1 \cdots \mu_5}, Z_{\mu_6 \cdots \mu_{10}}] + 10 A_{\mu_1} [H_{\mu_2 \cdots \mu_5}, Z_{\mu_6 \cdots \mu_{10}}]) \\
+ \frac{200}{(5!)^3} e^{\mu_1 \cdots \mu_{10}} (5 H_{\mu_1 \cdots \mu_4} [Z_{\mu_5 \mu_6 \mu_7}^{\rho}, Z_{\mu_8 \mu_9 \mu_{10}^{\rho}}] + 10 H_{\mu_1 \cdots \mu_4} [H_{\mu_5 \mu_6 \mu_7}^{\rho}, H_{\mu_8 \mu_9 \mu_{10}^{\rho}}] \\
+ 6 H^{\rho \mu_{1} \mu_{2}} [Z_{\mu_3 \mu_4 \mu_5 \rho}, Z_{\mu_6 \cdots \mu_{10}}]) \\
+ 3 Tr \left( \bar{\psi} \Gamma^\mu [W, \psi] + \bar{\psi} \Gamma^{\mu \nu} [A_\mu, \psi] + \bar{\psi} \Gamma^{\mu \nu} [B_\mu, \psi] + \frac{1}{2} \bar{\psi} \Gamma^{\mu \nu \mu_2} [C_{\mu_1 \mu_2}, \psi] \\
+ \frac{1}{4!} \bar{\psi} \Gamma^{\mu_1 \cdots \mu_4 \mu_5} [H_{\mu_1 \cdots \mu_4}, \psi] + \frac{1}{5!} \bar{\psi} \Gamma^{\mu_1 \cdots \mu_5} [Z_{\mu_1 \cdots \mu_5}, \psi] \right). \]

- The rank-1 and rank-5 fields (in 11 dimensions) have a positive mass, while the rank-2 fields are tachyonic.

\[
\frac{\Gamma_{\mu_1 \cdots \mu_5}}{\text{no sum}} = \Gamma_{\mu_1 \cdots \mu_5} \frac{\Gamma_{\mu_1 \cdots \mu_5}}{\text{no sum}} = +132 \times 32, \quad \frac{\Gamma_{\mu_1 \cdots \mu_5}}{\text{no sum}} = -132 \times 32.
\]

- The rank-1 and rank-5 fields has a stable trivial commutative classical solution:

\[
W = A_\mu = H_{\mu_1 \cdots \mu_4} = Z_{\mu_1 \cdots \mu_5} = 0.
\]

- For the rank-2 tachyonic fields $B_\mu, C_{\mu_1 \mu_2}$, the trivial solution $B_\mu = C_{\mu_1 \mu_2} = 0$ is unstable.

$\Rightarrow$ They may incorporate an interesting stable non-commutative solution!
From now on, we set the fermions and the positive-mass bosonic fields to zero:

\[
S = 96\mu Tr \left( B_\mu B^\mu + \frac{1}{2} C_{\mu_1\mu_2} C^{\mu_1\mu_2} \right) \\
+ 32i Tr \left( 3 C_{\mu_1\mu_2} [B^{\mu_1}, B^{\mu_2}] + C_{\mu_1\mu_2} [C^{\mu_2\mu_3}, C^{\mu_3\mu_1}] \right).
\]

The equations of motion:

\[
B_\mu = -i\mu^{-1} [B^\nu, C_{\mu
u}], \\
C_{\mu_1\mu_2} = -i\mu^{-1} ([B_{\mu_1}, B_{\mu_2}] + [C_{\mu_1}^\rho, C_{\mu_2\rho}]).
\]

We integrate out the rank-2 fields (in 10 dimensions) \( C_{\mu_1\mu_2} \) by solving the latter equation of motions iteratively.

\[
C_{\mu_1\mu_2} = -i\mu^{-1} ([B_{\mu_1}, B_{\mu_2}] + [C_{\mu_1}^\rho, C_{\mu_2\rho}] ) + \cdots
\]

\[
= (-i\mu^{-1})^2 [B_{\mu_1}, B_{\rho}] + [C_{\mu_1\chi_1}, C_{\rho\chi_1}] [B_{\mu_2}, B_{\rho}] + [C_{\mu_2\chi_2}, C_{\rho\chi_2}]
\]

\[
- i\mu^{-5} [B_{[\mu_1}, B_{\rho]}, [B_{\mu_2}, B_{\chi}], [B^\rho, B^\chi]]
\]

\( \mathcal{O}(B^2) \) with 1 commutator

\( \mathcal{O}(B^4) \) with 3 commutators

\[
- 2i\mu^{-7} [[B_{[\mu_1}, B_{[\rho}}, [B_{\mu_2}, B_{\chi]}, [B_{\rho]}, B^\chi] ]]
\]

\( \mathcal{O}(B^6) \) with 5 commutators

\[
+ i\mu^{-7} [[B_{[\mu_1}, B_{\chi_1]}, [B_{\rho}, B^{\chi_1}], [[B_{\mu_2}, B_{\chi_2}], [B^\rho, B^{\chi_2}]]]
\]

\( \mathcal{O}(B^8) \) with 7 commutators

\[
- 2i\mu^{-7} [[B_{[\mu_1}, B_{\rho}], [B^\rho, B_{\chi}], [B_{\mu_2}, B_{\sigma}], [B^\chi, B^\sigma]]] + \mathcal{O}(\mu^{-9}).(*)
\]

Then, the action reduces to

\[
S = Tr \left( 96\mu B_\mu B^\mu + 48\mu^{-1} [B_{\mu_1}, B_{\mu_2}] [B^{\mu_1}, B^{\mu_2}] \\
+ (\text{higher-order commutators of the order } \mathcal{O}(\mu^{-2k+1}) \text{ with } k = 2, 3, \ldots) \right).
\]

We consider the classical solution of the equation of motion \( B_\mu = -i\mu^{-1} [B^\nu, C_{\mu
u}] \) with \( C_{\mu_1\mu_2} \) substituted for \((*)\).
1. **SO(3) × SO(3) × SO(3) fuzzy spheres**

This describes a space formed by the Cartesian product of three fuzzy spheres.

\[
[B_i, B_j] = i \mu r \epsilon_{ijk} B_k, \quad B_i^2 + B_j^2 + B_k^2 = \mu^2 r^2 \frac{N^2 - 1}{4}, \quad (i, j, k = 1, 2, 3)
\]

\[
[B_{i'}, B_{j'}] = i \mu r \epsilon_{i'j'k'} B_{k'}, \quad B_{i'}^2 + B_{j'}^2 + B_{k'}^2 = \mu^2 r^2 \frac{N^2 - 1}{4}, \quad (i', j', k' = 4, 5, 6)
\]

\[
[B_{i''}, B_{j''}] = i \mu r \epsilon_{i''j''k''} B_{k''}, \quad B_{i''}^2 + B_{j''}^2 + B_{k''}^2 = \mu^2 r^2 \frac{N^2 - 1}{4}, \quad (i'', j'', k'' = 7, 8, 9)
\]

\[B_0 = 0, \quad |B_{\mu}, B_{\nu}| = 0, \quad \text{(otherwise)}.\]

(We consider the Cartesian product of three spheres instead of a single SO(3) fuzzy sphere)

\[
[B_i, B_j] = i \mu r \epsilon_{ijk} B_k \quad \text{(for } i, j, k = 1, 2, 3), \quad B_\mu = 0 \quad \text{(for } \mu = 0, 4, 5, \ldots, 9),
\]

because the solution \(B_4 = \cdots = B_9 = 0\) is trivially unstable.

2. **SO(9) fuzzy sphere**

Generally, the SO(2k + 1) fuzzy sphere \((S^{2k} \text{ fuzzy sphere})\) is constructed by the \(n\)-fold symmetric tensor product of \((2k + 1)\)-dimensional gamma matrices.

We should answer the following two questions about these solutions:

1. Are these solutions not perturbed by the infinite tower of the higher-order commutator?
2. Which solution is energetically favored?
Properties of the fuzzy $2k$-sphere

S. Ramgoolam, hep-th/0105006

Y. Kimura, hep-th/0301055

The $SO(2k + 1)$ fuzzy sphere ($S^{2k}$ fuzzy sphere) is constructed by the $n$-fold symmetric tensor product of $(2k + 1)$-dimensional gamma matrices:

$$B_p^{SO(2k+1)} = \frac{\mu r}{2}[(\Gamma_p^{(2k)} \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes \Gamma_p^{(2k)})]_{\text{sym}}.$$ 

$p$ runs over $1, 2, \cdots, 2k + 1$ in the $(2k + 1)$-dimensional Euclidean space.

The commutation and self-duality relation

$$(B_{pq}^{SO(2k+1)} = [B_p^{SO(2k+1)}, B_q^{SO(2k+1)}]):$$

\[\begin{align*}
\heartsuit \ & B_p^{SO(2k+1)} B_p^{SO(2k+1)} = \frac{\mu^2 r^2}{4} n(n + 2k)1_{N_k \times N_k}, \\
\heartsuit \ & B_{pq}^{SO(2k+1)} B_{pq}^{SO(2k+1)} = -\frac{\mu^2 r^2}{4} 8kn(n + 2k), \\
\clubsuit \ & [B_{pq}^{SO(2k+1)}, B_s^{SO(2k+1)}] = \mu^2 r^2 (-\delta_{ps} B_q^{SO(2k+1)} + \delta_{qs} B_p^{SO(2k+1)})1_{N_k \times N_k}, \\
\clubsuit \ & [B_{pq}^{SO(2k+1)}, B_{st}^{SO(2k+1)}] = \mu^2 r^2 (\delta_{qs} B_{pt}^{SO(2k+1)} + \delta_{pt} B_{qs}^{SO(2k+1)}) \\
& \quad - \delta_{ps} B_{qt}^{SO(2k+1)} - \delta_{qt} B_{ps}^{SO(2k+1)}, \\
\diamondsuit \ & \epsilon_{p_1 \cdots p_{2k+1}} B_{p_1}^{SO(2k+1)} B_{p_2}^{SO(2k+1)} \cdots B_{p_{2k}}^{SO(2k+1)} = \left(\frac{\mu r}{2}\right)^{2k-1} m_k B_{p_{2k+1}}^{SO(2k+1)}, \\
& m_1 = 2i, \ m_2 = 8(n + 2), \ m_3 = -48i(n + 2)(n + 4), \\
& m_4 = -384(n + 2)(n + 4)(n + 6), \text{ more generally,} \\
& m_k = -2(-i)^k \left(\prod_{p=2}^k 2p \times (n + 2p - 2)\right) \text{ (for } k \geq 2).}
\]
• size of the matrix:
The size of the matrix \( N_k \) for the \( SO(2k + 1) \) fuzzy sphere:

\[
N_2 = \frac{(n + 1)(n + 2)(n + 3)}{6} (= 4 [\text{for } n = 1]),
\]
\[
N_3 = \frac{(n + 1)(n + 2)(n + 3)^3(n + 4)(n + 5)}{360} (= 8 [\text{for } n = 1]),
\]
\[
N_4 = \frac{(n + 1)(n + 2)(n + 3)^2(n + 4)^2(n + 5)^2(n + 6)(n + 7)}{302400} (= 16 [\text{for } n = 1]).
\]

Unlike the \( SO(3) \) fuzzy sphere, the \( SO(5, 7, 9, \cdots) \) sphere cannot be realized for all \( N = 2, 3, 4, \cdots \).

\( N_4 = 16(n = 1), 126(n = 2), 672(n = 3), 2772(n = 4), \cdots \).

• special case \( k = 1 \):

This definition is identical to the \( SO(3) \) Lie algebra:

1. This is effectively a matrix acting on the symmetrized \( N = (n + 1) \)-dimensional irreducible representation of \( so(3) \) Lie algebra, not on the original \( 2^n \)-dimensional space.

2. The radius of the fuzzy sphere is (from (\( \heartsuit \)))

\[
B_i^{SO(3)} B_i^{SO(3)} = \frac{\mu^2 r^2}{4} n(n + 2) = (\mu r)^2 \frac{N^2 - 1}{4}, \quad \text{where}
\]

\[
\frac{N^2 - 1}{4}
\]

is the Casimir of \( so(3) \).

3. \( \Gamma_i^{(2)} \) are identical to the Pauli matrices \( \sigma_i \).

4. Self-duality condition (\( \bowtie \)) is trivially identical to the commutation relation

\[
[B_i^{SO(3)}, B_j^{SO(3)}] = i\mu r \epsilon_{ijk} B_k^{SO(3)}.
\]
Effect of the higher-order commutators

We start with the ansatz for the rank-2 fields $C_{pq}^{SO(2k+1)}$ for the $SO(2k + 1)$ fuzzy spheres:

$$C_{pq}^{SO(2k+1)} = -i\mu^{-1}f(r)B_{pq}^{SO(2k+1)}.$$ 

The equation of motion for $C_{pq}^{SO(2k+1)}$ reduces to

$$C_{pq}^{SO(2k+1)} = -i\mu^{-1}([B_{p}^{SO(2k+1)}, B_{q}^{SO(2k+1)}] + [C_{pr}^{SO(2k+1)}, C_{qr}^{SO(2k+1)}])$$

$$\downarrow$$

$$-\frac{i}{\mu}B_{pq}^{SO(2k+1)}(-f(r) + 1 + (2k - 1)r^2f^2(r)) = 0.$$ 

$f(r)$ is determined as

$$f_{\pm}(r) = \frac{1 \pm \sqrt{1 - 4(2k - 1)r^2}}{2(2k - 1)r^2}.$$ 

The equation of motion for $B_{p}^{SO(2k+1)}$ leads to

$$B_{p}^{SO(2k+1)}(1 - 2k r^2 f_{\pm}(r)) = 0.$$ 

$$\uparrow$$

$$\sqrt{1 - 4(2k - 1)r^2} = \pm \frac{k - 1}{k}.$$ 

- $1 - 2k r^2 f_{-}(r) = 0$ (i.e. $\sqrt{1 - 4(2k - 1)r^2} = -\frac{k-1}{k}$) has no solution (except for $k = 1$, in which this is identical to $1 - 2k r^2 f_{+}(r) = 0$).

- $1 - 2k r^2 f_{+}(r) = 0$ (i.e. $\sqrt{1 - 4(2k - 1)r^2} = +\frac{k-1}{k}$) does have a solution $r = \frac{1}{2k}$.

The existence of the solution $r(> 0)$ indicates that the radius of the fuzzy sphere is not much perturbed by the infinite tower of the high-order commutators.
Comparison of the classical energy

- Trivial commutative solution $B_0 = \cdots = B_9 = 0$:
  \[ E_{B_\mu=0} = -S_{B_\mu=0} = 0. \]

- $SO(3) \times SO(3) \times SO(3)$ fuzzy spheres ($N_1 = n + 1$):
  \[ E_{SO(3)^3} = -S_{SO(3)^3} = -\frac{16\mu}{r_{SO(3)^2}} Tr(B_\mu B^\mu) \]
  \[ = -12\mu^3 N_1 (N_1 - 1)(N_1 + 1) \]
  \[ \sim -\mathcal{O}(\mu^3 n^3) = -\mathcal{O}(\mu^3 N_1^3). \]

- $SO(9)$ fuzzy sphere:
  \[ E_{SO(9)} = -S_{SO(9)} = -\frac{5}{8} \mu^3 n(n + 8) N_4 \]
  \[ \sim -\mathcal{O}(\mu^3 n^{12}) = -\mathcal{O}(\mu^3 N_4^{5/2}), \]

where the size of the matrices $B_{SO(9)}^p$ is
\[ N_4 = \frac{(n + 1)(n + 2)(n + 3)^2(n + 4)^2(n + 5)^2(n + 6)(n + 7)}{302400} \sim \mathcal{O}(n^{10}). \]
3 Summary

- We have investigated a massive supermatrix model to seek a curved-space classical solution.
- We have found the triple $SO(3) \times SO(3) \times SO(3)$ and the single $SO(9)$ fuzzy-sphere solutions.
  * These solutions are not perturbed by the infinite tower of the higher-order commutators.
  * We have compared the classical energy.

**Future problems**

- Other classical solutions such as $SO(3) \times SO(6)$ fuzzy sphere, fuzzy torus · · ·
- Quantum fluctuation of the fuzzy-sphere solution, especially for the higher-dimensional $S^{2k}$ spheres.

  T. Azuma, S. Bal and M. Bagnoud, work in progress.
Notations on the supermatrices

The vectors and supermatrices are defined by

\[ v = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \\ b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad \begin{cases} \{\eta_i\} : \text{fermions} \\ \{b_j\} : \text{bosons} \end{cases}, \]

\[ M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}, \quad \begin{cases} a(d) : \quad m \times m(n \times n) \\ \beta(\gamma) : \quad m \times n(n \times m) \end{cases} \text{ bosonic matrices} \]

\[ \text{fermionic matrices} \]

**Transpose**

- The transpose of the vector is defined by

\[ T_v = T \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = (\eta_1, \ldots, \eta_m, b_1, \ldots, b_n). \]
• The transpose of the supermatrix is defined so that $T M$ satisfies $T(Mv) = T(v^T M)$.

\[
\Rightarrow T M = T \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} T a & -T \gamma \\ T \beta & T d \end{pmatrix}.
\]

(Proof) We verify that this is well-defined by going back to the guiding principle $T(Mv) = T(v^T M)$.

(L.H.S.) \[ T(Mv) = T \begin{pmatrix} a \eta + \beta b \\ \gamma \eta + db \end{pmatrix} = (T^T a + T b^T \beta, -T^T \gamma + T b^T d), \]

(R.H.S.) \[ (T^T, T b) \begin{pmatrix} T a \\ T \beta \\ -T \gamma \\ T d \end{pmatrix} = (T^T a + T b^T \beta, -T^T \gamma + T b^T d). \]

• The transpose of the transverse vector $y = (T \eta, T b)$ is defined so that $T(y M) = T(M^T y)$:

\[
\Rightarrow T y = T(T \eta, T b) = \begin{pmatrix} -\eta \\ b \end{pmatrix}.
\]

(Proof) This can be again confirmed by comparing the both hand sides:

(L.H.S.) \[ T(y M) = T(T \eta a + T b \gamma, T \eta \beta + T bd) = \begin{pmatrix} -T(T \eta a) - T(T b \gamma) \\ T(T \eta \beta) + T(T bd) \end{pmatrix} \]

\[ = \begin{pmatrix} -T a \eta - T \gamma b \\ -T \beta \eta + T db \end{pmatrix}, \]

(R.H.S.) \[ T M^T y = \begin{pmatrix} T a \\ T \beta \\ -T \gamma \\ T d \end{pmatrix} \begin{pmatrix} -\eta \\ b \end{pmatrix} = \begin{pmatrix} -T a \eta - T \gamma b \\ -T \beta \eta + T \gamma b \end{pmatrix}. \]

[Remark]: The transpose of the transpose of the vector or supermatrix does not go back to the original one:

\[ T(T(a \begin{pmatrix} \beta \\ \gamma \\ d \end{pmatrix}) = T \begin{pmatrix} T a & -T \gamma \\ T \beta & T d \end{pmatrix} = \begin{pmatrix} a & -\beta \\ -\gamma & d \end{pmatrix}, \]

\[ T(T(\eta \begin{pmatrix} b \end{pmatrix})) = T(T \eta, T b) = \begin{pmatrix} -\eta \\ b \end{pmatrix}. \]
**Hermitian Conjugate**

We settle the complex conjugate of the fermionic numbers $\alpha$ and $\beta$ as

$$(\alpha\beta)\dagger = (\beta)\dagger(\alpha)\dagger.$$  

- We first define the Hermitian conjugate of the vector as

$$v\dagger = \begin{pmatrix} \eta \\ b \end{pmatrix} = (\eta\dagger, b\dagger).$$

- $M\dagger$ is defined so that this satisfies $(Mv)\dagger = v\dagger M\dagger$:

$$M\dagger = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} a\dagger & \gamma\dagger \\ \beta\dagger & d\dagger \end{pmatrix}.$$  

- $y\dagger = (^T\eta, ^Tb)\dagger$ is defined so that $(yM)\dagger = M\dagger y\dagger$:

$$y\dagger = (^T\eta, ^Tb)\dagger = \begin{pmatrix} (^T\eta)^\dagger \\ (^Tb)^\dagger \end{pmatrix}.$$
The complex conjugate is defined so that the supermatrices and the vectors satisfy \((M v)^* = M^* v^*:\)

\[
\begin{align*}
  v^* &= (T v)^\dagger = \begin{pmatrix} \eta \\ b \end{pmatrix}^* = \begin{pmatrix} \eta^* \\ b^* \end{pmatrix}, \\
  M^* &= (T M)^\dagger = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}^* = \begin{pmatrix} a^* & \beta^* \\ -\gamma^* & d^* \end{pmatrix}, \\
  y^* &= (T y)^\dagger = (\eta, b)^* = (-\eta^*, b^*). 
\end{align*}
\]

[Prop] (1) \(T M = (M^*)^\dagger\), (2) \(M^\dagger = T(M^*)\), (3) \((M^*)^* = M\).

A supermatrix \(M\) is real if

\(M\) is a mapping from a real vector to a real vector.

i.e. \(M\) satisfies \(M^* = M\):

\[
\begin{align*}
a^* &= a, \quad \beta^* = \beta, \quad d^* = d, \quad \gamma^* = -\gamma.
\end{align*}
\]