Curved-space classical solution of a massive supermatrix model

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Contents

1 Introduction

Large-N reduced models are the most powerful candidates for the constructive definition of superstring theory.

IIB matrix model

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

$$S = \frac{1}{g^2} Tr_{N\times N}\left(\frac{1}{4}\sum_{\mu,\nu=0}^{9}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\psi}\sum_{\mu=0}^{9}\Gamma^\mu[A_\mu,\psi]\right),$$
$$(\text{ where } Z = \int dA d\psi e^{+S}).$$

- Dimensional reduction of $\mathcal{N}=1$ 10-dimensional SYM theory to 0 dimension.
  A_μ and ψ are $N\times N$ Hermitian matrices.
  - * A_μ: 10-dimensional vectors
  - * ψ: 10-dimensional Majorana-Weyl (i.e. 16-component) spinors
- Matrix regularization of the Schild form of the Green-Schwarz action of the type IIB superstring theory.
- $SU(N)$ gauge symmetry and $SO(9,1)$ Lorentz symmetry $(SO(9,1)\times U(N))$.
- The $N\times N$ matrices describe the many-body system.
- No free parameter: $A_\mu \to g^{\frac{1}{2}}A_\mu$, $\psi \to g^{\frac{3}{2}}\psi$.

- $\mathcal{N}=2$ SUSY: This theory must contain spin-2 gravitons if it contains massless particles.
  - * homogeneous : $\delta^{(1)}_\epsilon A_\mu = i\bar\epsilon\Gamma_\mu\psi, \quad \delta^{(1)}_\epsilon\psi = \frac{i}{2}\Gamma^{\mu\nu}[A_\mu, A_\nu]\epsilon.$
  - * inhomogeneous : $\delta^{(2)}_\xi A_\mu = 0, \quad \delta^{(2)}_\xi\psi = \xi.$
  - * We obtain the following commutation relations:

$$\begin{aligned}
&(1)\ \ [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]A_\mu = [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]\psi = 0,\\
&(2)\ \ [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]A_\mu = [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]\psi = 0,\\
&(3)\ \ [\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]A_\mu = -i\bar\epsilon\Gamma_\mu\xi, \quad [\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]\psi = 0.
\end{aligned}$$

This gives a shift of the bosonic variables for
$\tilde\delta^{(1)} = \delta^{(1)} + \delta^{(2)}, \ \tilde\delta^{(2)} = i(\delta^{(1)} - \delta^{(2)})$: $(\alpha, \beta = 1, 2)$

$$\begin{aligned}
&[\tilde\delta^{(\alpha)}_\epsilon, \tilde\delta^{(\beta)}_\xi]\psi = 0,\\
&[\tilde\delta^{(\alpha)}_\epsilon, \tilde\delta^{(\beta)}_\xi]A_\mu = -2i\delta^{\alpha\beta}\bar\epsilon\Gamma_\mu\xi.
\end{aligned}$$

$\Rightarrow$ Therefore, the eigenvalues of the bosonic large-N matrices A_μ represent the spacetime coordinates.

If the large-N reduced models are to be an authentic framework to unify all interactions in nature $\cdots$,

$$\Downarrow$$

[Q] How can we express the gravitational interaction more manifestly in terms of a large-N reduced model?

1. IIB matrix model itself is an eligible framework to describe the gravity.

- **General coordinate invariance**

  S. Iso, H.Kawai. Int. J. Mod. Phys. A 15, 651 (2000) hep-th/9903217

  The general coordinate invariance is interpreted as the permutation $\mathcal{S}_N$ invariance of the eigenvalues of the large N matrices.

$$x^i \to x^{\sigma(i)} \text{ for } \sigma \in \mathcal{S}_N,$$

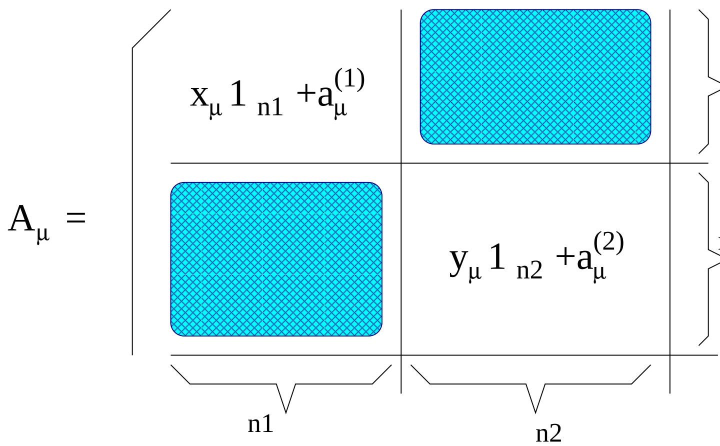
$$\Downarrow$$

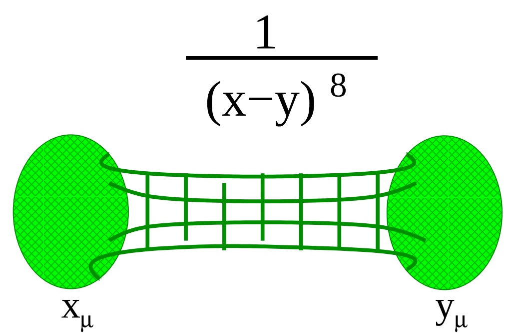
$$x \to \xi(x) \text{ such that } \xi(x^i) = x^{\sigma(i)}.$$

- Graviton and dilaton exchange

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115

The computation of the one-loop effective Lagrangian reveals the graviton and dilaton exchange in IIB matrix model.





$$W_{\text{eff}} \sim -\frac{12}{(x-y)^8} \underbrace{Tr_{n_1\times n_1}(f^{(1)}_{\mu\rho} f^{(1)}_{\rho\nu}) Tr_{n_2\times n_2}(f^{(2)}_{\mu\sigma} f^{(2)}_{\sigma\nu})}_{\text{graviton exchange}}$$

$$+\frac{3}{2(x-y)^8} \underbrace{Tr_{n_1\times n_1}(f^{(1)}_{\mu\nu} f^{(1)}_{\mu\nu}) Tr_{n_2\times n_2}(f^{(2)}_{\rho\sigma} f^{(2)}_{\rho\sigma})}_{\text{dilaton exchange}},$$

where $\quad f^{(1,2)}_{\mu\nu} = i[a^{(1,2)}_\mu, a^{(1,2)}_\nu].$

2. A matrix model must incorporate a local Lorentz invariance

We need to enlarge the symmetry of the model.

T. Azuma and H. Kawai, Phys. Lett B538, 393 (2002) hep-th/0204078

Symmetry of IIB matrix model is $SO(9,1) \times U(N)$: $so(9,1)$ Lorentz symmetry and $u(N)$ gauge symmetry are decoupled.

$$\exp(\xi \otimes 1 + 1 \otimes u) = e^{\xi} \otimes e^{u}, \text{ where } \xi \in so(9,1), \ u \in u(N).$$

⇒ In IIB matrix model, the eigenvalues of the bosonic matrices A_μ are regarded as the spacetime coordinate.

⇒ If we are to formulate a matrix model with local Lorentz invariance, the parameters of the Lorentz transformation ξ must be promoted to (nontrivial) $u(N)$ matrices.

⇒ $so(9,1)$ Lorentz symmetry and $u(N)$ gauge symmetry must be unified; i.e. the symmetry is the tensor product of the Lie algebra $so(9,1)\check{\otimes}u(N)$, rather than $SO(9,1) \times U(N)$.

$\mathcal{A}, \mathcal{B}$ = [Lie algebras whose bases are $\{a_i\}$ and $\{b_j\}$, respectively.]

- $\mathcal{A} \otimes \mathcal{B}$: The space spanned by the basis $a_i \otimes b_j$. This is not necessarily a closed Lie algebra.
- $\mathcal{A} \check{\otimes} \mathcal{B}$: The smallest Lie algebra that includes $\mathcal{A} \otimes \mathcal{B}$ as a subset.

The gauge group must close with respect to the commutator

$$[a \otimes A, b \otimes B] = \frac{1}{2}\left([a,b] \otimes \{A,B\} + \{a,b\} \otimes [A,B]\right).$$

3. A matrix model must incorporate a classical solution of a curved space.

Classical equation of motion of IIB matrix model:

$$[A^\nu, [A_\mu, A_\nu]] = 0.$$

This has only a flat non-commutative background as a classical solution.

$$[A_\mu, A_\nu] = ic_{\mu\nu} 1_{N\times N}.$$

⇒ In order to surmount this difficulty, we alter a model so that it incorporates a curved-space classical solution ab initio.

[Example] IIB matrix model with a tachyonic mass term:

Y. Kimura, Prog. Theor. Phys. 106 (2001) 445, [hep-th/0103192].

$$S = \frac{1}{g^2} Tr \left(\frac{1}{4}[A_a, A_b]^2 + \lambda^2 A_a A_a \right),$$

$$\text{EOM: } [A_b, [A_a, A_b]] + 2\lambda^2 A_a = 0.$$

- $SO(4) \times U(N)$ symmetry.
  a, b runs over $1, 2, 3, 4$ in the Euclidean space.
- Classical solutions of compact curved spacetime:
  - * $SO(3)$ fuzzy sphere:
    $[A_i, A_j] = i\lambda\epsilon_{ijk} A_k \ (i, j, k = 1, 2, 3), \ A_4 = 0.$
  - * two-dimensional fuzzy torus:

$$A_1 = \frac{r}{2}(U + U^\dagger), \ A_2 = \frac{r}{2i}(U - U^\dagger), \ A_3 = \frac{r}{2}(V + V^\dagger), \ A_4 = \frac{r}{2i}(V - V^\dagger),$$

$$U = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix}, \ V = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix},$$

$$\omega = e^{i\theta}, \ \theta = \frac{2\pi}{N}, \ UV = e^{i\theta} VU, \ \lambda^2 = r^2(1 - \cos\theta).$$

2 Massive supermatrix model

We consider the 3rd way in terms of an $osp(1,32|R)$ supermatrix model.

$osp(1|32,R)$ super Lie algebra

$osp(1|32,R)$: first mentioned with the relation to the 11-dimensional supergravity.

E. Cremmer, B. Julia, J. Scherk, Phys.Lett.B76:409-412,1978.

⇒ This has attracted a new attention as the unified super Lie algebra for the M-theory.

E. Bergshoeff, A. Van Proeyen, hep-th/0003261

The matrix model based on $osp(1|32,R)$ is a natural extension to IIB matrix model.

L. Smolin, hep-th/0002009
L. Smolin, hep-th/0006137
T. Azuma, S. Iso, H. Kawai and Y. Ohwashi, hep-th/0102168
M. Bagnoud, L, Carlevaro and A. Bilal, hep-th/0201183

- Extra fields:
  The 32×32 bosonic part of $osp(1|32,R)$ has the components of rank-1,2,5 of the 11-dimensional gamma matrices $u_\alpha, u_{\alpha_1\alpha_2}, u_{\alpha_1\cdots\alpha_5}$.
  ⇒ The rank-1 components u_α can be identified with the bosonic vector of IIB matrix model A_μ.

- **Realization of the spin connection**:
  The complexification $u(1|16,16)$ incorporates the rank-3 components.
  ⇒ This may be identified with the spin connection in the supergravity theory.
  ⇒ We can elucidate the inclusion of the gravity more manifestly.

- **Relation of the supersymmetry**:
  $osp(1|32,R)$ *cubic* supermatrix model (without mass term) nearly has the double structure of the 10-dimensional $\mathcal{N}=2$ SUSY of IIB matrix model.

  * IIB matrix model: 16-component fermion
  * $osp(1|32,R)$ model: $\underbrace{\text{32-component fermion}}_{\text{twice as many fermions!!}}$

  The SUSY transformation of the rank-1 component u_α resembles that of the vector field A_μ of IIB matrix model.

$$M \in osp(1|32,R) \overset{\text{def}}{\Rightarrow} {}^T\!MG + GM = 0, \text{ where } G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}.$$

⇓

$$M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}, \text{ where } {}^T\!m\Gamma^0 + \Gamma^0 m = 0;$$

i.e. $m \in sp(32)$,

$$m = u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}.$$

(Proof) We start with the general form $M = \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix}$.

$$\begin{aligned} 0 &= {}^T\!MG + GM = \begin{pmatrix} {}^T\!m & -{}^T\bar{\phi} \\ {}^T\psi & v \end{pmatrix}\begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix} + \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix} \\ &= \begin{pmatrix} {}^T\!m\Gamma^0 + \Gamma^0 m & {}^T\bar{\phi} + \Gamma^0\psi \\ {}^T\psi\Gamma^0 - \bar{\phi} & 2iv \end{pmatrix}. \end{aligned}$$

Therefore, we obtain $v = 0$, $\psi = \phi$ and ${}^T\!m\Gamma^0 + \Gamma^0 m = 0$.

We next investigate what ranks of m survive. Since $m \in sp(32)$, it follows that $m = -(\Gamma^0)^{-1}({}^T\!m)\Gamma^0 = +\Gamma^0({}^T\!m)\Gamma^0$.

$$\begin{aligned} &\Gamma^0({}^T\Gamma^{A_1\cdots A_k})\Gamma^0 = (-1)^{k-1}(\Gamma^0({}^T\Gamma^{A_k})\Gamma^0)\cdots(\Gamma^0({}^T\Gamma^{A_1})\Gamma^0) \\ &= (-1)^{k-1}\Gamma^{A_kA_{k-1}\cdots A_1} = (-1)^{k-1}(-1)^{\frac{k(k-1)}{2}}\Gamma^{A_1A_2\cdots A_k} \\ &= (-1)^{\frac{(k+2)(k-1)}{2}}\Gamma^{A_1A_2\cdots A_k} \\ &= \begin{cases} +\Gamma^{A_1A_2\cdots A_k} & (k = 1,2,5) \\ -\Gamma^{A_1A_2\cdots A_k} & (k = 0,3,4) \end{cases} \end{aligned}$$

Action of the massive supermatrix model

We add a mass term to the pure cubic action:

$$\begin{aligned} S &= Tr\left[str\left(-3\mu M^2 + \frac{i}{g^2} M[M,M] \right) \right] \\ &= Tr\left[-3\mu \left\{ \left(\sum_{p=1}^{32} M_p{}^Q M_Q{}^p \right) - M_{33}{}^Q M_Q{}^{33} \right\} \right. \\ &\qquad \left. + \frac{i}{g^2} \left\{ \left(\sum_{p=1}^{32} M_p{}^Q [M_Q{}^R, M_R{}^p] \right) - M_{33}{}^Q [M_Q{}^R, M_R{}^{33}] \right\} \right], \\ &= Tr\left[3\mu(-tr(m^2) + 2i\bar{\psi}\psi) + \frac{i}{g^2} \left(m_p{}^q [m_q{}^r, m_r{}^p] - 3i\bar{\psi}^p [m_p{}^q, \psi^q] \right) \right]. \end{aligned}$$

- **Each component of the 33×33 supermatrices is promoted to a large N hermitian matrix.**
- **$osp(1|32, R)$ symmetry and $u(N)$ gauge symmetry are decoupled (i.e. $Osp(1|32, R) \times U(N)$ symmetry).**
  - $M \rightarrow M + [M, (S \otimes 1_{N \times N})]$ for $S \in osp(1|32, R)$,
  - $M \rightarrow M + [M, (1_{33 \times 33} \otimes U)]$ for $U \in u(N)$.

In order to see the correspondence of the fields with IIB matrix model, we express the bosonic 32×32 matrices in terms of the 10-dimensional indices.
$(\mu, \nu, \cdots = 0, 1, \cdots, 9, \sharp = 10)$.

$$W = u_{\sharp}, \quad A_{\mu} = u_{\mu}, \quad B_{\mu} = u_{\mu\sharp}, \quad C_{\mu_1\mu_2} = u_{\mu_1\mu_2},$$
$$H_{\mu_1\cdots\mu_4} = u_{\mu_1\cdots\mu_4\sharp}, \quad Z_{\mu_1\cdots\mu_5} = u_{\mu_1\cdots\mu_5}.$$

Then, the action is decomposed as

$$\begin{aligned}
S &= 96\mu Tr\left(-W^2 - A_{\mu}A^{\mu} + B_{\mu}B^{\mu} + \frac{1}{2}C_{\mu_1\mu_2}C^{\mu_1\mu_2} - \frac{1}{4!}H_{\mu_1\cdots\mu_4}H^{\mu_1\cdots\mu_4}\right. \\
&\quad \left. -\frac{1}{5!}Z_{\mu_1\cdots\mu_5}Z^{\mu_1\cdots\mu_5} + \frac{i}{16}\bar{\psi}\psi\right) \\
&+ 32iTr\left(-3C_{\mu_1\mu_2}[A^{\mu_1}, A^{\mu_2}] + 3C_{\mu_1\mu_2}[B^{\mu_1}, B^{\mu_2}] + 6W[A_{\mu}, B^{\mu}] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3}, C^{\mu_3\mu_1}]\right. \\
&\quad +\frac{1}{4}B_{\mu_1}[H_{\mu_2\cdots\mu_5}, Z^{\mu_1\cdots\mu_5}] - \frac{1}{8}C_{\mu_1\mu_2}(4[H^{\mu_1}{}_{\rho_1\rho_2\rho_3}, H^{\mu_2\rho_1\rho_2\rho_3}] + [Z^{\mu_1}{}_{\rho_1\cdots\rho_4}, Z^{\mu_1\rho_1\cdots\rho_4}]) \\
&\quad +\frac{3}{(5!)^2}\epsilon^{\mu_1\cdots\mu_{10}\sharp}(-W[Z_{\mu_1\cdots\mu_5}, Z_{\mu_6\cdots\mu_{10}}] + 10A_{\mu_1}[H_{\mu_2\cdots\mu_5}, Z_{\mu_6\cdots\mu_{10}}]) \\
&\quad +\frac{200}{(5!)^3}\epsilon^{\mu_1\cdots\mu_{10}\sharp}(5H_{\mu_1\cdots\mu_4}[Z_{\mu_5\mu_6\mu_7}{}^{\rho\chi}, Z_{\mu_8\mu_9\mu_{10}\rho\chi}] + 10H_{\mu_1\cdots\mu_4}[H_{\mu_5\mu_6\mu_7}{}^{\rho}, H_{\mu_8\mu_9\mu_{10}\rho}] \\
&\quad\quad \left. +6H^{\rho\chi}{}_{\mu_1\mu_2}[Z_{\mu_3\mu_4\mu_5\rho\chi}, Z_{\mu_6\cdots\mu_{10}}])\right) \\
&+ 3Tr\left(\bar{\psi}\Gamma^{\sharp}[W, \psi] + \bar{\psi}\Gamma^{\mu}[A_{\mu}, \psi] + \bar{\psi}\Gamma^{\mu\sharp}[B_{\mu}, \psi] + \frac{1}{2!}\bar{\psi}\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2}, \psi]\right. \\
&\quad \left. + \frac{1}{4!}\bar{\psi}\Gamma^{\mu_1\cdots\mu_4\sharp}[H_{\mu_1\cdots\mu_4}, \psi] + \frac{1}{5!}\bar{\psi}\Gamma^{\mu_1\cdots\mu_5}[Z_{\mu_1\cdots\mu_5}, \psi]\right).
\end{aligned}$$

- The rank-1 and rank-5 fields (in 11 dimensions) have a positive mass, while the rank-2 fields are tachyonic.

$$\underbrace{\Gamma_A\Gamma^A}_{\text{no sum}} = \underbrace{\Gamma_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}}_{\text{no sum}} = +1_{32\times 32}, \quad \underbrace{\Gamma_{A_1A_2}\Gamma^{A_1A_2}}_{\text{no sum}} = -1_{32\times 32}.$$

- The rank-1 and rank-5 fields has a stable trivial commutative classical solution:

$$W = A_{\mu} = H_{\mu_1\cdots\mu_4} = Z_{\mu_1\cdots\mu_5} = 0.$$

- For the rank-2 tachyonic fields $B_{\mu}, C_{\mu_1\mu_2}$, the trivial solution $B_{\mu} = C_{\mu_1\mu_2} = 0$ is unstable.
$\Rightarrow$ They may incorporate an interesting stable non-commutative solution!

From now on, we set the fermions and the positive-mass bosonic fields to zero:

$$\begin{aligned} S &= 96\mu Tr\left(B_\mu B^\mu + \frac{1}{2}C_{\mu_1\mu_2}C^{\mu_1\mu_2}\right) \\ &+ 32iTr\left(3C_{\mu_1\mu_2}[B^{\mu_1}, B^{\mu_2}] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3}, C^{\mu_3\mu_1}]\right). \end{aligned}$$

The equations of motion:

$$\begin{aligned} B_\mu &= -i\mu^{-1}[B^\nu, C_{\mu\nu}], \\ C_{\mu_1\mu_2} &= -i\mu^{-1}([B_{\mu_1}, B_{\mu_2}] + [C_{\mu_1}{}^\rho, C_{\mu_2\rho}]). \end{aligned}$$

We integrate out the rank-2 fields (in 10 dimensions) $C_{\mu_1\mu_2}$ by solving the latter equation of motions iteratively.

$$\begin{aligned} C_{\mu_1\mu_2} &= -i\mu^{-1}\Big([B_{\mu_1}, B_{\mu_2}] + \underbrace{[C_{\mu_1}{}^\rho, C_{\mu_2\rho}]}_{=(-i\mu^{-1})^2[[B_{\mu_1},B^\rho]+[C_{\mu_1\chi_1},C^{\rho\chi_1}],[B_{\mu_2},B_\rho]+[C_{\mu_2\chi_2},C_\rho{}^{\chi_2}]]}\Big) + \cdots \\ &= -\underbrace{i\mu^{-1}[B_{\mu_1}, B_{\mu_2}]}_{\mathcal{O}(B^2)\text{ with 1 commutator}} + \underbrace{i\mu^{-3}[[B_{\mu_1}, B_\rho], [B_{\mu_2}, B^\rho]]}_{\mathcal{O}(B^4)\text{ with 3 commutators}} \\ &- \underbrace{2i\mu^{-5}[[B_{[\mu_1}, B_\rho], [[B_{\mu_2]}, B_\chi], [B^\rho, B^\chi]]]}_{\mathcal{O}(B^6)\text{ with 5 commutators}} \\ &+ i\mu^{-7}[[[B_{\mu_1}, B_{\chi_1}], [B_\rho, B^{\chi_1}]], [[B_{\mu_2}, B_{\chi_2}], [B^\rho, B^{\chi_2}]]] \\ &+ 2i\mu^{-7}[[B_{[\mu_1}, B_\rho], [[B_{\mu_2]}, B_\chi], [[B^\rho, B_\sigma], [B^\chi, B^\sigma]]]] \\ &- \underbrace{2i\mu^{-7}[[B_{[\mu_1}, B_\rho], [[B^\rho, B_\chi], [[B_{\mu_2]}, B_\sigma], [B^\chi, B^\sigma]]]]}_{\mathcal{O}(B^8)\text{ with 7 commutators}} + \mathcal{O}(\mu^{-9}).(\star) \end{aligned}$$

Then, the action reduces to

$$\begin{aligned} S = Tr\Big(&96\mu B_\mu B^\mu + 48\mu^{-1}[B_{\mu_1}, B_{\mu_2}][B^{\mu_1}, B^{\mu_2}] \\ &+(\text{higher-order commutators of the order } \mathcal{O}(\mu^{-2k+1}) \text{ with } k = 2, 3, \cdots)\Big). \end{aligned}$$

We consider the classical solution of the equation of motion $B_\mu = -i\mu^{-1}[B^\nu, C_{\mu\nu}]$ with $C_{\mu_1\mu_2}$ substituted for $(\star)$.

Fuzzy-sphere classical solution

1. **$SO(3) \times SO(3) \times SO(3)$ fuzzy spheres**

   This describes a space formed by the Cartesian product of three fuzzy spheres.

$$
\begin{aligned}
&[B_i, B_j] = i\mu r \epsilon_{ijk} B_k, && B_1^2 + B_2^2 + B_3^2 = \mu^2 r^2 \frac{N^2-1}{4}, \quad (i,j,k = 1,2,3) \\
&[B_{i'}, B_{j'}] = i\mu r \epsilon_{i'j'k'} B_{k'}, && B_4^2 + B_5^2 + B_6^2 = \mu^2 r^2 \frac{N^2-1}{4}, \quad (i',j',k' = 4,5,6) \\
&[B_{i''}, B_{j''}] = i\mu r \epsilon_{i''j''k''} B_{k''}, && B_7^2 + B_8^2 + B_9^2 = \mu^2 r^2 \frac{N^2-1}{4}, \quad (i'',j'',k'' = 7,8,9) \\
&B_0 = 0, \quad [B_\mu, B_\nu] = 0, \quad (\text{otherwise}).
\end{aligned}
$$

   (We consider the Cartesian product of three spheres instead of a single $SO(3)$ fuzzy sphere

$$[B_i, B_j] = i\mu r \epsilon_{ijk} B_k \ (\text{for } i,j,k = 1,2,3), \quad B_\mu = 0 \ (\text{for } \mu = 0,4,5,\cdots,9),$$

   because the solution $B_4 = \cdots = B_9 = 0$ is trivially unstable.)

2. **$SO(9)$ fuzzy sphere**

   Generally, the $SO(2k+1)$ fuzzy sphere (S^{2k} fuzzy sphere) is constructed by the n-fold symmetric tensor product of $(2k+1)$-dimensional gamma matrices.

We should answer the following two questions about these solutions:

1. Are these solutions not perturbed by the infinite tower of the higher-order commutator?
2. Which solution is energetically favored?

Properties of the fuzzy $2k$-sphere

S. Ramgoolam, hep-th/0105006

Y. Kimura, hep-th/0301055

The $SO(2k+1)$ fuzzy sphere (S^{2k} fuzzy sphere) is constructed by the n-fold symmetric tensor product of $(2k+1)$-dimensional gamma matrices:

$$B_p^{SO(2k+1)} = \frac{\mu r}{2}[(\Gamma_p^{(2k)} \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes \Gamma_p^{(2k)})]_{\mathrm{sym}}.$$

p runs over $1, 2, \cdots, 2k+1$ in the $(2k+1)$-dimensional Euclidean space.

The commutation and self-duality relation ($B_{pq}^{SO(2k+1)} = [B_p^{SO(2k+1)}, B_q^{SO(2k+1)}]$):

♡ $B_p^{SO(2k+1)} B_p^{SO(2k+1)} = \frac{\mu^2 r^2}{4} n(n+2k) 1_{N_k \times N_k}$,

♡ $B_{pq}^{SO(2k+1)} B_{pq}^{SO(2k+1)} = -(\frac{\mu r}{2})^4 8kn(n+2k)$,

♣ $[B_{pq}^{SO(2k+1)}, B_s^{SO(2k+1)}] = \mu^2 r^2(-\delta_{ps} B_q^{SO(2k+1)} + \delta_{qs} B_p^{SO(2k+1)}) 1_{N_k \times N_k}$,

♣ $[B_{pq}^{SO(2k+1)}, B_{st}^{SO(2k+1)}] = \mu^2 r^2(\delta_{qs} B_{pt}^{SO(2k+1)} + \delta_{pt} B_{qs}^{SO(2k+1)} - \delta_{ps} B_{qt}^{SO(2k+1)} - \delta_{qt} B_{ps}^{SO(2k+1)})$,

◇ $\epsilon_{p_1 \cdots p_{2k+1}} B_{p_1}^{SO(2k+1)} B_{p_2}^{SO(2k+1)} \cdots B_{p_{2k}}^{SO(2k+1)} = \left(\frac{\mu r}{2}\right)^{2k-1} m_k B_{p_{2k+1}}^{SO(2k+1)}$,

$m_1 = 2i, \quad m_2 = 8(n+2), \quad m_3 = -48i(n+2)(n+4)$,

$m_4 = -384(n+2)(n+4)(n+6)$, more generally,

$m_k = -2(-i)^k \left(\prod_{p=2}^{k} 2p \times (n+2p-2)\right)$ (for $k \geq 2$).

- size of the matrix:
The size of the matrix N_k for the $SO(2k+1)$ fuzzy sphere:

$$N_2 = \frac{(n+1)(n+2)(n+3)}{6} (= 4[\text{ for } n=1]),$$
$$N_3 = \frac{(n+1)(n+2)(n+3)^3(n+4)(n+5)}{360} (= 8[\text{ for } n=1]),$$
$$N_4 = \frac{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)}{302400}$$
$$(= 16[\text{ for } n=1]).$$

Unlike the $SO(3)$ fuzzy sphere, the $SO(5,7,9,\cdots)$ sphere cannot be realized for all $N = 2,3,4,\cdots$.
$N_4 = 16(n=1), 126(n=2), 672(n=3), 2772(n=4), \cdots$.

- special case $k=1$:
This definition is identical to the $SO(3)$ Lie algebra:

1. This is effectively a matrix acting on the symmetrized $N=(n+1)$-dimensional irreducible representation of $so(3)$ Lie algebra, not on the original 2^n-dimensional space.
2. The radius of the fuzzy sphere is (from (♡)) $B_i^{SO(3)} B_i^{SO(3)} = \frac{\mu^2 r^2}{4} n(n+2) = (\mu r)^2 \frac{N^2-1}{4}$, where $\frac{N^2-1}{4}$ is the Casimir of $so(3)$.
3. $\Gamma_i^{(2)}$ are identical to the Pauli matrices σ_i.
4. Self-duality condition (◇) is trivially identical to the commutation relation $[B_i^{SO(3)}, B_j^{SO(3)}] = i\mu r \epsilon_{ijk} B_k^{SO(3)}$.

Effect of the higher-order commutators

We start with the ansatz for the rank-2 fields $C_{pq}^{SO(2k+1)}$ for the $SO(2k+1)$ fuzzy spheres:

$$C_{pq}^{SO(2k+1)} = -i\mu^{-1} f(r) B_{pq}^{SO(2k+1)}.$$

$\Downarrow$

The equation of motion for $C_{pq}^{SO(2k+1)}$ reduces to

$$C_{pq}^{SO(2k+1)} = -i\mu^{-1}([B_p^{SO(2k+1)}, B_q^{SO(2k+1)}] + [C_{pr}^{SO(2k+1)}, C_{qr}^{SO(2k+1)}])$$

$$\Downarrow$$

$$\frac{-i}{\mu} B_{pq}^{SO(2k+1)}(-f(r) + 1 + (2k-1)r^2 f^2(r)) = 0.$$

$f(r)$ is determined as

$$f_{\pm}(r) = \frac{1 \pm \sqrt{1-4(2k-1)r^2}}{2(2k-1)r^2}.$$

$\Downarrow$

The equation of motion for $B_p^{SO(2k+1)}$ leads to

$$B_p^{SO(2k+1)}(1 - 2kr^2 f_{\pm}(r)) = 0.$$

$$\Updownarrow$$

$$\sqrt{1-4(2k-1)r^2} = \pm\frac{k-1}{k}.$$

- $1 - 2kr^2 f_{-}(r) = 0$ (i.e. $\sqrt{1-4(2k-1)r^2} = -\frac{k-1}{k}$) has no solution (except for $k = 1$, in which this is identical to $1 - 2kr^2 f_{+}(r) = 0$).
- $1 - 2kr^2 f_{+}(r) = 0$ (i.e. $\sqrt{1-4(2k-1)r^2} = +\frac{k-1}{k}$) does have a solution $r = \frac{1}{2k}$

The existence of the solution $r(> 0)$ indicates that the radius of the fuzzy sphere is not much perturbed by the infinite tower of the high-order commutators.

Comparison of the classical energy

- Trivial commutative solution $B_0 = \cdots = B_9 = 0$:

$$E_{B_\mu=0} = -S_{B_\mu=0} = 0.$$

- $SO(3)\times SO(3)\times SO(3)$ fuzzy spheres ($N_1 = n+1$):

$$\begin{aligned} E_{SO(3)^3} &= -S_{SO(3)^3} = -\frac{16\mu}{r_{SO(3)^2}} Tr(B_\mu B^\mu) \\ &= -12\mu^3 N_1 (N_1-1)(N_1+1) \\ &\sim -\mathcal{O}(\mu^3 n^3) = -\mathcal{O}(\mu^3 N_1^3). \end{aligned}$$

- $SO(9)$ fuzzy sphere:

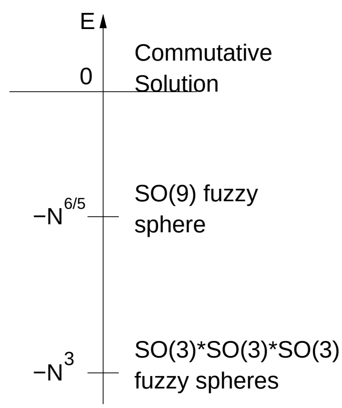
$$\begin{aligned} E_{SO(9)} &= -S_{SO(9)} = -\frac{5}{8}\mu^3 n(n+8) N_4 \\ &\sim -\mathcal{O}(\mu^3 n^{12}) = -\mathcal{O}(\mu^3 N_4^{\frac{6}{5}}), \end{aligned}$$

where the size of the matrices $B_p^{SO(9)}$ is

$$N_4 = \frac{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)}{302400} \sim \mathcal{O}(n^{10}).$$

When the size of the matrices are the same...

$(N_1 = N_4)$



3 Summary

- **We have investigated a massive supermatrix model to seek a curved-space classical solution.**
- **We have found the triple $SO(3) \times SO(3) \times SO(3)$ and the single $SO(9)$ fuzzy-sphere solutions.**
  - * **These solutions are not perturbed by the infinite tower of the higher-order commutators.**
  - * **We have compared the classical energy.**

Future problems

- **Other classical solutions such as $SO(3) \times SO(6)$ fuzzy sphere, fuzzy torus $\cdots$.**
- **Quantum fluctuation of the fuzzy-sphere solution, especially for the higher-dimensional S^{2k} spheres.**

T.Azuma, S. Bal and M. Bagnoud, work in progress.

Notations on the supermatrices

The vectors and supermatrices are defined by

$$v = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \\ b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad \begin{pmatrix} \{\eta_i\} : & \text{fermions} \\ \{b_j\} : & \text{bosons} \end{pmatrix},$$

$$M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}, \quad \begin{pmatrix} a(d) : & m \times m(n \times n) \\ & \text{bosonic matrices} \\ \beta(\gamma) : & m \times n(n \times m) \\ & \text{fermionic matrices} \end{pmatrix}.$$

Transpose

- The transpose of the vector is defined by

$$^T v = {}^T \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = (\eta_1, \cdots, \eta_m, b_1, \cdots, b_n).$$

- **The transpose of the supermatrix is defined so that $^T M$ satisfies $^T(Mv) = {}^T v\, {}^T M$.**

$$\Leftrightarrow {}^T M = {}^T \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} {}^T a & -{}^T\gamma \\ {}^T\beta & {}^T d \end{pmatrix}.$$

(Proof) We verify that this is well-defined by going back to the guiding principle $^T(Mv) = {}^T v\, {}^T M$.

$$\begin{aligned} \text{(L.H.S.)} &= {}^T(Mv) = {}^T \begin{pmatrix} a\eta + \beta b \\ \gamma\eta + db \end{pmatrix} = ({}^T\eta\, {}^T a + {}^T b\, {}^T\beta, -{}^T\eta\, {}^T\gamma + {}^T b\, {}^T d), \\ \text{(R.H.S.)} &= ({}^T\eta, {}^T b) \begin{pmatrix} {}^T a & -{}^T\gamma \\ {}^T\beta & {}^T d \end{pmatrix} = ({}^T\eta\, {}^T a + {}^T b\, {}^T\beta, -{}^T\eta\, {}^T\gamma + {}^T b\, {}^T d). \end{aligned}$$

- **The transpose of the transverse vector $y = ({}^T\eta, {}^T b)$ is defined so that $^T(yM) = {}^T M\, {}^T y$:**

$$\Leftrightarrow {}^T y = {}^T({}^T\eta, {}^T b) = \begin{pmatrix} -\eta \\ b \end{pmatrix}.$$

(Proof) This can be again confirmed by comparing the both hand sides:

$$\begin{aligned} \text{(L.H.S.)} &= {}^T(yM) = {}^T({}^T\eta a + {}^T b\gamma, {}^T\eta\beta + {}^T bd) = \begin{pmatrix} -{}^T({}^T\eta a) - {}^T({}^T b\gamma) \\ {}^T({}^T\eta\beta) + {}^T({}^T bd) \end{pmatrix} \\ &= \begin{pmatrix} -{}^T a\eta - {}^T\gamma b \\ -{}^T\beta\eta + {}^T db \end{pmatrix}, \\ \text{(R.H.S.)} &= {}^T M\, {}^T y = \begin{pmatrix} {}^T a & -{}^T\gamma \\ {}^T\beta & {}^T d \end{pmatrix} \begin{pmatrix} -\eta \\ b \end{pmatrix} = \begin{pmatrix} -{}^T a\eta - {}^T\gamma b \\ -{}^T\beta\eta + {}^T\gamma b \end{pmatrix}. \end{aligned}$$

[Remark]: **The transpose of the transpose of the vector or supermatrix does not go back to the original one:**

$$\begin{aligned} &{}^T({}^T \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}) = {}^T \begin{pmatrix} {}^T a & -{}^T\gamma \\ {}^T\beta & {}^T d \end{pmatrix} = \begin{pmatrix} a & -\beta \\ -\gamma & d \end{pmatrix}, \\ &{}^T({}^T \begin{pmatrix} \eta \\ b \end{pmatrix}) = {}^T({}^T\eta, {}^T b) = \begin{pmatrix} -\eta \\ b \end{pmatrix}. \end{aligned}$$

Hermitian Conjugate

We settle the complex conjugate of the fermionic numbers α and β as

$$(\alpha\beta)^\dagger = (\beta)^\dagger(\alpha)^\dagger.$$

- We first define the Hermitian conjugate of the vector as

$$v^\dagger = \begin{pmatrix} \eta \\ b \end{pmatrix}^\dagger = (\eta^\dagger, b^\dagger).$$

- $M^\dagger$ is defined so that this satisfies $(Mv)^\dagger = v^\dagger M^\dagger$:

$$M^\dagger = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}^\dagger = \begin{pmatrix} a^\dagger & \gamma^\dagger \\ \beta^\dagger & d^\dagger \end{pmatrix}.$$

- $y^\dagger = ({}^T\eta, {}^Tb)^\dagger$ is defined so that $(yM)^\dagger = M^\dagger y^\dagger$:

$$y^\dagger = ({}^T\eta, {}^Tb)^\dagger = \begin{pmatrix} ({}^T\eta)^\dagger \\ ({}^Tb)^\dagger \end{pmatrix}.$$

Complex Conjugate

The complex conjugate is defined so that the supermatrices and the vectors satisfy $(Mv)^* = M^* v^*$:

$$
\begin{aligned}
v^* &= ({}^T v)^\dagger = \begin{pmatrix} \eta \\ b \end{pmatrix}^* = \begin{pmatrix} \eta^* \\ b^* \end{pmatrix}, \\
M^* &= ({}^T M)^\dagger = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}^* = \begin{pmatrix} a^* & \beta^* \\ -\gamma^* & d^* \end{pmatrix}, \\
y^* &= ({}^T y)^\dagger = (\eta, b)^* = (-\eta^*, b^*).
\end{aligned}
$$

[Prop] (1) ${}^T M = (M^*)^\dagger$, (2) $M^\dagger = {}^T(M^*)$, (3) $(M^*)^* = M$.

A supermatrix M is real if

M is a mapping from a real vector to a real vector.

i.e. M satisfies $M^* = M$:

$$a^* = a,\ \beta^* = \beta,\ d^* = d,\ \gamma^* = -\gamma.$$