Supermatrix Models

hep-th/0102168

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Contents

1 Introduction

constructive definition of superstring theory

Large N reduced models are the most powerful candidate for the constructive definition of superstring theory.

IKKT model

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

For a review, hep-th/9908038

Dimensional reduction of $\mathcal{N} = 1$ 10-dimensional SYM theory to 0 dimension.

Matrix regularization of Green-Schwarz action of type IIB superstring theory.

$$S = -\frac{1}{g^2} Tr_{N\times N}\left(\frac{1}{4}\sum_{i,j=0}^{9}[A_i, A_j]^2 + \frac{1}{2}\bar{\psi}\sum_{i=0}^{9}\Gamma^i[A_i, \psi]\right).$$

- $SO(10) \times SU(N)$ gauge symmetry.
- $\mathcal{N} = 2$ SUSY.
  - * homogeneous : $\delta^{(1)}_{\epsilon} A_i = i\bar{\epsilon}\Gamma_i\psi, \quad \delta^{(1)}_{\epsilon}\psi = \frac{i}{2}\Gamma^{ij}[A_i, A_j]\epsilon.$
  - * inhomogeneous : $\delta^{(2)}_{\xi} A_i = 0, \quad \delta^{(2)}_{\xi}\psi = \xi.$
  - * $[\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]A_i = -i\bar{\epsilon}\Gamma_i\xi, \quad [\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]\psi = 0.$
- The matrices describe the many-body system.
- No free parameter: $A_\mu \to g^{\frac{1}{2}}A_\mu$, $\psi \to g^{\frac{3}{4}}\psi$.

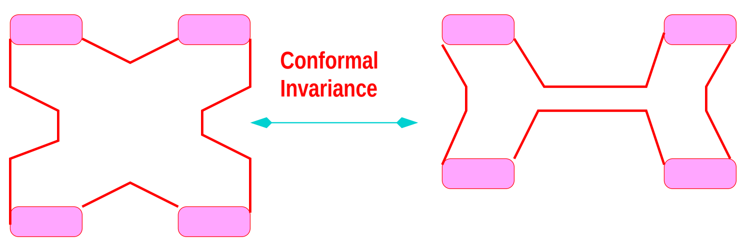
2 $osp(1|32, R)$ cubic matrix model

We investigate a matrix model based on super Lie algebra $osp(1|32, R)$.

L. Smolin, hep-th/0002009

T. Azuma, S. Iso, H. Kawai and Y. Ohwashi, hep-th/0102168

- $OSp(1|32, R)$ is the full symmetry group of $\mathcal{M}$-theory.
- The spacetime is extended to 11 dimensions.
- The theory is described by a cubic action. :
  - * The cubic interaction is the most fundamental one in string theory.



  - * Chern Simons Theory is exactly solvable by means of Jones polynomial.

    E. Witten, *Commun. Math. Phys.* 121 (1989) 351

    The non-perturbative analysis may be exactly performed.

$osp(1|32, R)$ super Lie algebra

- $M \in osp(1|32, R) \Rightarrow {}^T MG + GM = 0$,
  where $G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}$.
- $M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}$, where $m\Gamma^0 + \Gamma^0 m = 0$ $(m \in sp(32))$.
- $m = u_{\mu_1}\Gamma^{\mu_1} + \frac{1}{2!}u_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2} + \frac{1}{5!}u_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}$.

action of the cubic model

$$\begin{aligned} I &= \frac{i}{g^2} Tr_{N\times N} \sum_{Q,R=1}^{33} [(\sum_{p=1}^{32} M_p{}^Q [M_Q{}^R, M_R{}^p]) - M_{33}{}^Q [M_Q{}^R, M_R{}^{33}]] \\ &= -\frac{f^{abc}}{2g^2} \sum_{a,b,c=1}^{N^2} Str_{33\times 33}(M_a M_b M_c) \\ &= \frac{i}{g^2} Tr_{N\times N} [m_p{}^q [m_q{}^r, m_r{}^p] - 3i\bar{\psi}^p [m_p{}^q, \psi^q]]. \end{aligned}$$

- Each component of the 33×33 supermatrices is promoted to a large N hermitian matrix.
- No free parameter: $M \to g^{\frac{2}{3}} M$.
- $OSp(1|32, R) \times U(N)$ gauge symmetry.
  - $M \to M + [M, (S \otimes 1_{N\times N})]$ for $S \in osp(1|32, R)$,
  - $M \to M + [M, (1_{33\times 33} \otimes U)]$ for $U \in u(N)$.

Supersymmetry

The SUSY transformation of the $osp(1|32, R)$ is identified with that of IKKT model.

- homogeneous SUSY:
  The SUSY transformation by the supercharge $Q = \begin{pmatrix} 0 & \chi \\ i\bar{\chi} & 0 \end{pmatrix}$.

$$\delta^{(1)}_{\chi} M = [Q, M] = \begin{pmatrix} i(\chi\bar{\psi} - \psi\bar{\chi}) & -m\chi \\ i\bar{\chi}m & 0 \end{pmatrix}.$$

- inhomogeneous SUSY:
  The translation of the fermionic field $\delta^{(2)}_{\epsilon}\psi = \epsilon$.

In order to see the correspondence of the fields with IKKT model, we express the bosonic 32×32 matrices in terms of the 10-dimensional indices ($i = 0, \cdots 9$, $\sharp = 10$).

$$\begin{aligned} m &= W\Gamma^{\sharp} + \frac{1}{2}[A^{(+)}_i \Gamma^i(1+\Gamma^{\sharp}) + A^{(-)}_i \Gamma^i(1-\Gamma^{\sharp})] + \frac{1}{2!}C_{i_1 i_2}\Gamma^{i_1 i_2} + \\ &\quad + \frac{1}{4!}H_{i_1\cdots i_4}\Gamma^{i_1\cdots i_4\sharp} + \frac{1}{5!}[I^{(+)}_{i_1\cdots i_5}\Gamma^{i_1\cdots i_5}(1+\Gamma^{\sharp}) + I^{(-)}_{i_1\cdots i_5}\Gamma^{i_1\cdots i_5}(1-\Gamma^{\sharp})]. \end{aligned}$$

Identification of the fields

$$\delta^{(1)}_{\chi} A^{(+)}_i = \frac{i}{16}\bar{\chi}\Gamma_i(1-\Gamma_{\sharp})\psi = \frac{i}{8}\bar{\chi}_R\Gamma_i\psi_R,$$
$$\delta^{(1)}_{\chi} A^{(-)}_i = \frac{i}{16}\bar{\chi}\Gamma_i(1+\Gamma_{\sharp})\psi = \frac{i}{8}\bar{\chi}_L\Gamma_i\psi_L,$$
$$\delta^{(1)}_{\chi}\psi = -m\psi.$$

Commutation relations

- $[\delta^{(1)}_{\chi}, \delta^{(2)}_{\epsilon}] m = -i(\chi\bar{\epsilon} - \epsilon\bar{\chi}), \quad [\delta^{(1)}_{\chi}, \delta^{(2)}_{\epsilon}]\psi = 0.$

$$[\delta^{(1)}_{\chi_R}, \delta^{(2)}_{\epsilon_R}] A_i^{(+)} = \frac{i}{8}\bar{\epsilon}_R \Gamma_i \chi_R, \quad [\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_L}] A_i^{(+)} = 0,$$
$$[\delta^{(1)}_{\chi_R}, \delta^{(2)}_{\epsilon_R}] A_i^{(-)} = 0, \qquad [\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_L}] A_i^{(-)} = \frac{i}{8}\bar{\epsilon}_L \Gamma_i \chi_L,$$
$$[\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_R}] A_i^{(\pm)} = [\delta^{(1)}_{\chi_R}, \delta^{(2)}_{\epsilon_L}] A_i^{(\pm)} = 0.$$

- $[\delta^{(2)}_{\chi}, \delta^{(2)}_{\epsilon}] m = [\delta^{(2)}_{\chi}, \delta^{(2)}_{\epsilon}]\psi = 0$ is trivial.
- $[\delta^{(1)}_{\chi}, \delta^{(1)}_{\epsilon}] m = i[\chi\bar{\epsilon} - \epsilon\bar{\chi}, m], \quad [\delta^{(1)}_{\chi}, \delta^{(1)}_{\epsilon}]\psi = i(\chi\bar{\epsilon} - \epsilon\bar{\chi})\psi.$

  * $[\delta^{(1)}_{\chi_R}, \delta^{(1)}_{\epsilon_R}] A_i^{(+)} = \frac{i}{8}\bar{\chi}_R [m, \Gamma_i]\epsilon_R.$
    In the (r.h.s.), the fields W, $C_{i_1 i_2}$ and $H_{i_1 \cdots i_4}$ survive.
    → these fields are integrated out.
  * $[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}] A_i^{(+)} = -\frac{i}{8}\bar{\chi}_L A_j^{(+)} \Gamma_i{}^j \epsilon_R + \cdots.$
    The fields $A_i^{(\pm)}$ itself remains in the commutator!

Summary

The $osp(1|32, R)$ cubic matrix model possesses a two-fold structure of the SUSY of IKKT model.

IKKT model	bosons A_i	fermions ψ	SUSY parameters
SUSY I	$A_i^{(+)}$	ψ_R	χ_R, ϵ_R
SUSY II	$A_i^{(-)}$	ψ_L	χ_L, ϵ_L

3 $gl(1|32,R) \otimes gl(N)$ gauged model

We consider the model whose gauge symmetry is enhanced by altering the direct product of the Lie algebra.

L. Smolin, hep-th/0006137

T. Azuma, S. Iso, H. Kawai and Y. Ohwashi, hep-th/0102168

(*) $\mathcal{A}, \mathcal{B}$ = [The Lie algebras whose bases are $\{a_i\}$ and $\{b_j\}$, respectively.]

- $\mathcal{A} \otimes \mathcal{B}$: The space spanned by the basis $a_i \otimes b_j$. This is not necessarily a closed Lie algebra.
- $\mathcal{A} \check{\otimes} \mathcal{B}$: The smallest Lie algebra that includes $\mathcal{A} \otimes \mathcal{B}$ as a subset.

The gauge symmetry $OSp(1|32,R) \times U(N)$ is enhanced to $osp(1|32,R) \check{\otimes} u(N)$.

- $osp(1|32,R) \otimes u(N)$ is not a closed Lie algebra.
- $osp(1|32,R) \check{\otimes} u(N) = u(1|16,16) \otimes u(N)$.
  $u(1|16,16)$ is the complexification of $osp(1|32,R)$.
- We consider the Lie algebra
  $gl(1|32,R) \check{\otimes} gl(N) = gl(1|32,R) \otimes gl(N)$
  as an analytical continuation of $u(1|16,16) \otimes u(N)$.

$u(1|16,16)$ super Lie algebra

- $M \in u(1|16,16) \Rightarrow M^\dagger G + GM = 0$, where $G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}$.
- $M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & v \end{pmatrix}$, where $m^\dagger \Gamma^0 + \Gamma^0 m = 0$.
- $m = u1 + u_{\mu_1}\Gamma^{\mu_1} + \frac{1}{2!}u_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2} + \frac{1}{3!}u_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3}$
  $+\frac{1}{4!}u_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4} + \frac{1}{5!}u_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}$.
- $\begin{cases} u_{\mu_1}, u_{\mu_1\mu_2}, u_{\mu_1\cdots\mu_5} & \Rightarrow \text{real number} \\ v, u, u_{\mu_1\mu_2\mu_3}, u_{\mu_1\cdots u_{\mu_4}} & \Rightarrow \text{pure imaginary} \end{cases}$

$u(1|16,16)$ is the direct sum of the two different representations of $osp(1|32,R)$.

♣ $u(1|16,16) = \mathcal{H} \oplus \mathcal{A}'$, where

$$\mathcal{H} = \{M = \begin{pmatrix} m_h & \psi_h \\ i\bar{\psi}_h & 0 \end{pmatrix} | m_h = u_{\mu_1}\Gamma^{\mu_1} + \frac{1}{2!}u_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2} + \frac{1}{5!}u_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5},$$
$$u_{\mu_1}, u_{\mu_1\mu_2}, u_{\mu_1\cdots\mu_5}, \psi_h \in \mathcal{R}\},$$
$$\mathcal{A}' = \{M = \begin{pmatrix} m_a & i\psi_a \\ \bar{\psi}_a & iv \end{pmatrix} | m_a = u + \frac{1}{3!}u_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3} + \frac{1}{4!}u_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4},$$
$$u, u_{\mu_1\mu_2\mu_3}, u_{\mu_1\cdots\mu_4}, i\psi_a, iv \in (\text{pure imaginary})\}.$$

$gl(1|32,R)$ super Lie algebra

- $M \in gl(1|32,R) \Rightarrow M = \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix}$
- $m = u1 + u_{\mu_1}\Gamma^{\mu_1} + \frac{1}{2!}u_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2} + \frac{1}{3!}u_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3}$
  $+\frac{1}{4!}u_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4} + \frac{1}{5!}u_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}$.
- $u, \cdots u_{\mu_1\cdots\mu_5}, \psi, \phi, v$ are all real numbers.

$gl(1|32,R)$ is the analytical continuation of $u(1|16,16)$, in that

♣ $gl(1|32,R) = \mathcal{H} \oplus \mathcal{A}$, where $\mathcal{A}' = i\mathcal{A}$.

action of the cubic model

$$
\begin{aligned}
I &= \frac{1}{g^2} Tr_{N\times N} \sum_{Q,R=1}^{33} [(\sum_{p=1}^{32} M_p{}^Q M_Q{}^R M_R{}^p) - M_{33}{}^Q M_Q{}^R M_R{}^{33}] \\
&= \frac{1}{g^2} \sum_{a,b,c=1}^{N^2} Str_{33\times 33}(M_a M_b M_c) Tr_{N\times N}(T^a T^b T^c) \\
&= \frac{1}{g^2} Tr_{N\times N}[m_p{}^q m_q{}^r m_r{}^p - 3i\bar{\phi}^p m_p{}^q \psi^q - 3iv\bar{\phi}^p \psi_p - v^3].
\end{aligned}
$$

- Each component of the 33×33 supermatrices is promoted to a large N real matrix.
- No free parameter: $M \rightarrow g^{\frac{2}{3}} M$.
- $gl(1|32, R) \otimes gl(N)$ gauge symmetry.

$$
M \rightarrow M + [M, (S \otimes U)]
$$
$$
\text{for } S \in osp(1|32, R) \text{ and } U \in u(N).
$$

- The bosonic 32×32 matrices are separated into m_e and m_o in terms of 10-dimensional indices.

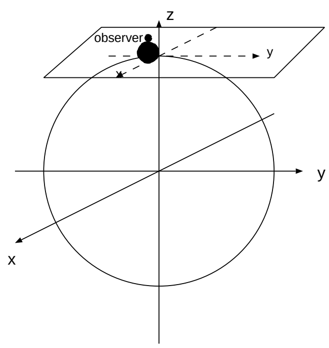
$$
\begin{aligned}
m_e &= Z + W\Gamma^\sharp + \frac{1}{2!}(C_{i_1 i_2}\Gamma^{i_1 i_2} + D_{i_1 i_2}\Gamma^{i_1 i_2 \sharp}) + \frac{1}{4!}(G_{i_1\cdots i_4}\Gamma^{i_1\cdots i_4} + H_{i_1\cdots i_4}\Gamma^{i_1\cdots i_4\sharp}), \\
m_o &= \frac{1}{2}(A_i^{(+)}\Gamma^i(1+\Gamma^\sharp) + A_i^{(-)}\Gamma^i(1-\Gamma^\sharp)) \\
&+ \frac{1}{2\times 3!}(E^{(+)}_{i_1 i_2 i_3}\Gamma^{i_1 i_2 i_3}(1+\Gamma^\sharp) + E^{(-)}_{i_1 i_2 i_3}\Gamma^{i_1 i_2 i_3}(1-\Gamma^\sharp)) \\
&+ \frac{1}{5!}(I^{(+)}_{i_1\cdots i_5}\Gamma^{i_1\cdots i_5}(1+\gamma^\sharp) + I^{(-)}_{i_1\cdots i_5}\Gamma^{i_1\cdots i_5}(1-\Gamma^\sharp)).
\end{aligned}
$$

Wigner Inönü contraction

We consider the hyperboloid in the AdS space whose radius R is sufficiently large. The hyperboloid is approximated by the $R^{9,1}$ flat plane at the "north pole".

AdS space: $x^\mu x^\nu \eta_{\mu\nu} = -R^2$, with $\eta_{\mu\nu} = diag(-1, 1, \cdots, 1, -1)$.

(*)The intuitive image of the Wigner Inönü contraction in the 3-dimensional case.



♣ The Lorentz transformation in the 11-dimensional space $(\mu, \nu = 0, 1, \cdots, 9, \sharp)$:

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}.$$

♣ We consider the algebra in the plane perpendicular to the $x^\sharp$ direction.

- Translation:
  P_i =(The translation in the direction of x_i) $= \frac{1}{R} M_{\sharp i} = \frac{1}{R} \Gamma_{\sharp i}$.
- Lorentz transformation:
  M_{ij} =(The Lorentz transformation on the $x_i x_j$ plane) $= \Gamma_{ij}$.

♣ The commutation relations of the translations and the Lorentz transformations:

- $[M_{ij}, M_{kl}] = \eta_{jk} M_{il} + \eta_{il} M_{jk} - \eta_{ik} M_{jl} - \eta_{jl} M_{ik}$.
- $[P_i, M_{jk}] = -\eta_{ik} P_j + \eta_{ij} P_k$.
- $[P_i, P_j] = \frac{1}{R^2} M_{ij} \to 0$. Two translations commute with each other when the radius R is large.

In order to perform the Wigner Inönü contraction, we alter the action as

$$I = \frac{1}{3}Tr(StrM_t^3) - R^2 Tr(StrM_t).$$

The EOM $\frac{\partial I}{\partial M_t} = M_t^2 - R^2 1_{33\times 33} = 0$ possesses a classical solution $\langle M \rangle = \begin{pmatrix} R\Gamma^\sharp \otimes 1_{N\times N} & 0 \\ 0 & R \otimes 1_{N\times N} \end{pmatrix}$.

$$\begin{aligned} M_t &= (\text{classical solution } \langle M \rangle) \quad + (\text{fluctuation } M) \\ &= \begin{pmatrix} R\Gamma^\sharp \otimes 1_{N\times N} & 0 \\ 0 & R \otimes 1_{N\times N} \end{pmatrix} + \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix}. \end{aligned}$$

The action is expressed in terms of the fluctuation as

$$\begin{aligned} I &= R(tr(m_e^2 \Gamma^\sharp) - v^2 - 2i\bar{\phi}_R \psi_L) + (\frac{1}{3}m_e^3 + tr(m_e m_o^2)) \\ &- i(\bar{\phi}_R(m_e + v)\psi_L + \bar{\phi}_L(m_e + v)\psi_R + \bar{\phi}_L m_o \psi_L + \bar{\phi}_R m_o \psi_R) - \frac{1}{3}v^3. \end{aligned}$$

The fluctuation is rescaled as

- $m_t = R\Gamma^\sharp + m = R\Gamma^\sharp + R^{-\frac{1}{2}} m'_e + R^{\frac{1}{4}} m'_o$,
- $v_t = R + v = R + R^{-\frac{1}{2}} v'$,
- $\psi = \psi_L + \psi_R = R^{-\frac{1}{2}} \psi'_L + R^{\frac{1}{4}} \psi'_R$,
- $\bar{\phi} = \bar{\phi}_L + \bar{\phi}_R = R^{\frac{1}{4}} \bar{\phi}'_L + R^{-\frac{1}{2}} \bar{\phi}'_R$.

We obtain the vanishing effective action by integrating out m'_e, ψ'_L, $\bar{\phi}'_R$ and v' .

$$\begin{aligned} &e^{-W} = \int dm'_e d\psi'_L d\bar{\phi}'_R dv e^{-I}, \\ \Rightarrow\ & W = -\frac{1}{4}tr(\Gamma^\sharp \{m_o'^2 + i(\psi'_R \bar{\phi}'_L)\}^2) - \frac{1}{4}(\bar{\phi}'_L \psi_R)^2 + \frac{i}{2}(\bar{\phi}'_L m_o'^2 \psi'_R) = 0. \end{aligned}$$

This gauged model may be related to a topological matrix model.

S. Hirano and M. Kato, *Prog. Theor. Phys.* **98** (1997) 1371, hep-th/9708039

4 Conclusion

- We have investigated the cubic model whose gauge symmetry is the super Lie algebra $OSp(1|32, R) \times U(N)$.
- $osp(1|32, R)$ cubic matrix model possesses a two-fold structure of the $\mathcal{N} = 2$ SUSY of IKKT model.
- We have investigated the $gl(1|32, R) \otimes gl(N)$ gauged model as an extension by means of the Wigner-Inönü contraction.
- The effective action vanishes, and this model is related to a topological matrix model.

(*) In order to grasp the intuitive image of 'Smolin's gauged theory', we consider the following simple example.

$$su(6) = su(3)\check{\otimes}su(2).$$

λ^a: basis of $su(3)$ $(a = 1, 2, \cdots 8)$.
σ^i: basis of $su(2)$ $(i = 1, 2, 3)$.

- $\lambda^a \otimes \sigma^i$ (24 dimensions): The basis of $su(3) \otimes su(2)$, which does not constitute a closed Lie algebra.
- $\lambda^a \otimes 1 + 1 \otimes \sigma^i$ (11 dimensions): The generators of the Lie group $SU(3) \times SU(2)$.
- $su(3)\check{\otimes}su(2) = (su(3)\otimes su(2))\oplus(SU(3)\times SU(2))_{algebra}$ This is a closed 35-dimensional Lie algebra.

$SU(3) \times SU(2)$ is a 11-dimensional Lie group,
while $su(3)\check{\otimes}su(2)$ is a 35-dimensional Lie algebra.