Monte Carlo studies of the phase transition of finite-temperature large-N gauge theory

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Collaboration with Shingo Takeuchi and Takeshi Morita

Contents

1 Introduction

Finite-temperature Matrix Quantum Mechanics (MQM):

$$Z = \int dX dA e^{-S_{\rm MQM}}, \quad \text{where}$$

$$S_{\rm MQM} = \frac{1}{g^2} \int_0^\beta dt \left\{ \frac{1}{2} {\rm tr} \sum_{I=1}^{D} (D_t X_I(t))^2 - \frac{1}{4} {\rm tr} \sum_{I,J=1}^{D} [X_I(t), X_J(t)]^2 \right\}.$$

- Dimensional reduction of $(1+D)$ Yang-Mills theory ($\beta = \frac{1}{T}$)
- This model is useful in many contexts:
  - * Blackstring/Blackhole phase transition via gauge/gravity correspondence.
  - * Multi-baryon system in the Sakai-Sugimoto model.

    K. Hashimoto and T. Morita, arXiv:1103.5688

Motivation of this work

We would like to compare the two approaches to study the matrix quantum mechanics:

Monte Carlo simulation O. Aharony et. al. hep-th/0406210,0508077, N. Kawahara, J. Nishimura and S. Takeuchi arXiv:0706.3517, 0710.2188

- **Feature : Non-perturbative. Any finite N, D OK.**
- **Demerit: $N \to \infty$ limit is difficult. Numerical errors.**
  Cut off (lattice space) dependence.

$1/D$ expansion G.Mandal, M.Mahato and T.Morita. arXiv:0910.4526

- **Feature: Non-perturbative, $N \gg 1$, $D \gg 1$**
- **Demerit: The $1/D$ expansion is valid in $D \gg N \gg 1$ case.**
  The validity in $N \gg D > 1$ case is subtle.

Large-N phase transition in the MQM

Phase transitions happen in the MQM in the large-N limit.

- **Analogues of the confinement/deconfinement phase transition.**
- **Correspond to a black string/blackhole phase transition via holography.**

This phase transition is know to be resolved at finite N.

Results

- **Is $1/D$ expansion valid at small D?**

  Comparison of Monte Carlo results with the $1/D$ expansion.

  ⇒ Good agreement at low temperature even at small D.

- **Explicit calculation of the finite-N resolution of the phase transition.**

2 Effective action via $1/D$ expansion

Outline of the $1/D$ expansion:

Our goal is to obtain an effective action of A, by integrating out X_I.

G.Mandal, M.Mahato and T.Morita. arXiv:0910.4526

- Rescale the adjoint scalars X_I to gX_I, so that the action $S_{\rm MQM}$ is

$$S_{\rm MQM} = \int_0^\beta dt \left\{ \frac{1}{2}\mathrm{tr} \sum_{I=1}^{D} (D_t X_I(t))^2 - \frac{g^2}{4}\mathrm{tr} \sum_{I,J=1}^{D} [X_I(t), X_J(t)]^2 \right\}.$$

- We define the following matrix

$$M_{ab,cd} = -\frac{1}{4}\{\mathrm{Tr}[\lambda_a, \lambda_c][\lambda_b, \lambda_d] + (a \leftrightarrow b) + (c \leftrightarrow d) + (a \leftrightarrow b)(c \leftrightarrow d)\}$$

λ_a $(a = 1, 2, \cdots, N^2 - 1)$ is the generator of SU(N).

- **The action is rewritten as** $(\frac{1}{g^2} M^{-1}_{ab,cd} B_{cd} = i X_{a,I} X_{b,I},\ X_I = \sum_{a=1}^{N^2-1} \lambda_a X_{a,I})$

$$S = \int_0^\beta dt \left\{ \frac{1}{2} \mathrm{tr}\, (D_t X_{a,I}(t))^2 - \frac{i}{2} B_{ab} X_{a,I}, X_{b,I} + \frac{1}{4g^2} B_{ab} M^{-1}_{ab,cd} B_{cd} \right\}.$$

The degrees of freedom are

$$A \to N^2, \quad B_{ab} \to N^4, \quad X_{a,I} \to DN^2.$$

Limit $D \gg N \gg 1 \Rightarrow X_{a,I}$**'s degree of freedom is dominant.**

Decompose B_{ab} **as** $B_{ab} = i\Delta^2 \delta_{ab} + g b_{ab}(t)$, **so that** $\int dt b_{aa}(t) = 0$.

Δ **becomes nonzero** $\Rightarrow X_{a,I}$ **becomes massive and does not contribute at low energy.**

We take the static and diagonal gauge

$$A = \text{diag}(\alpha_1, \alpha_2, \cdots, \alpha_N)$$

Order parameter for the confinement/deconfinement phase transition

$$u_n = \frac{1}{N}\text{tr}\, U^n = \frac{1}{N}\sum_{a=1}^{N} \exp(in\alpha_a), \quad \text{where}$$

$$U = \mathcal{P}\exp\left(i\int_0^\beta dt A(t)\right) = \text{diag}(e^{i\alpha_1}, \cdots, e^{i\alpha_N}).$$

We take the limit $D \to +\infty$, $N \to +\infty$, $g \to 0$ **with** $D \gg N$ **and fixed** $\tilde{\lambda} = g^2 DN$

Integrate out X_I and b_{ab} $\Rightarrow$ we derive the following effective action for Δ and A:

G.Mandal, M.Mahato and T.Morita. arXiv:0910.4526

$$Z = \int dX d\alpha e^{-S_{\rm MQM}} = \int d\alpha d\Delta e^{-S_{\rm eff}(\Delta,\{u_n\})+O(1/D)},$$

$$S_{\rm eff}(\Delta,\{u_n\})/DN^2 = -\frac{\Delta^4}{8T\tilde{\lambda}^{\frac{1}{3}}} + \frac{\Delta}{2T} + \sum_{n=1}^{+\infty} \frac{1}{n}\left(\frac{1}{D} - \exp\left(-\frac{n\Delta}{T}\right)\right)|u_n|^2.$$

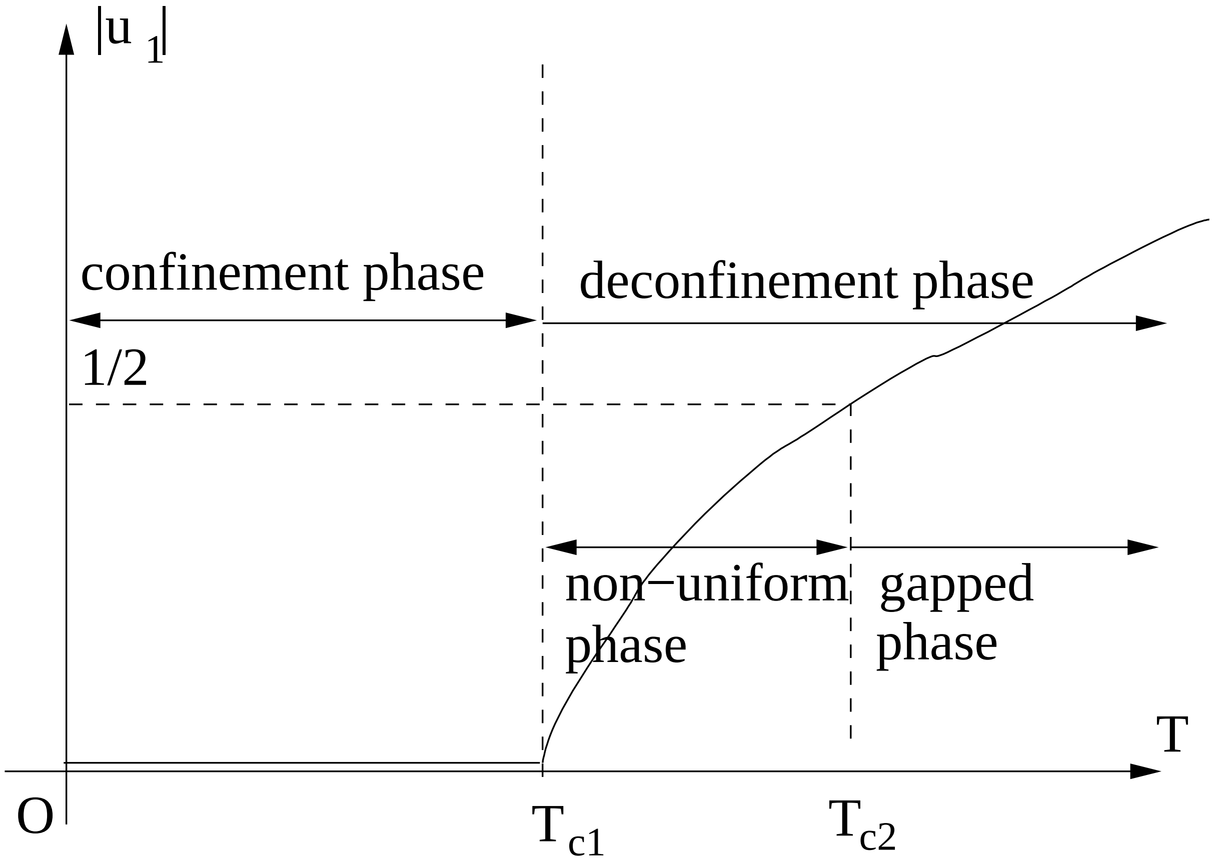
Low temperature (small T) and small u_n $\Rightarrow$ we further integrate out Δ.

We obtain the Landau-Ginzburg (LG) type effective action.

$$S_{\rm eff}(\{u_n\})/DN^2 = \frac{3\tilde{\lambda}^{\frac{1}{3}}}{8T} + b_1|u_1|^4 + \sum_{n=1}^{+\infty} a_n|u_n|^2,$$

$$a_n = \frac{1}{n}\left(\frac{1}{D} - \exp\left(-\frac{n\tilde{\lambda}^{\frac{1}{3}}}{T}\right)\right), \quad b_1 = \frac{\tilde{\lambda}^{\frac{1}{3}}}{3T}\exp\left(-\frac{2\tilde{\lambda}^{\frac{1}{3}}}{T}\right),$$

Phase structure of $S_{\text{eff}}(\{u_n\})$ at large N

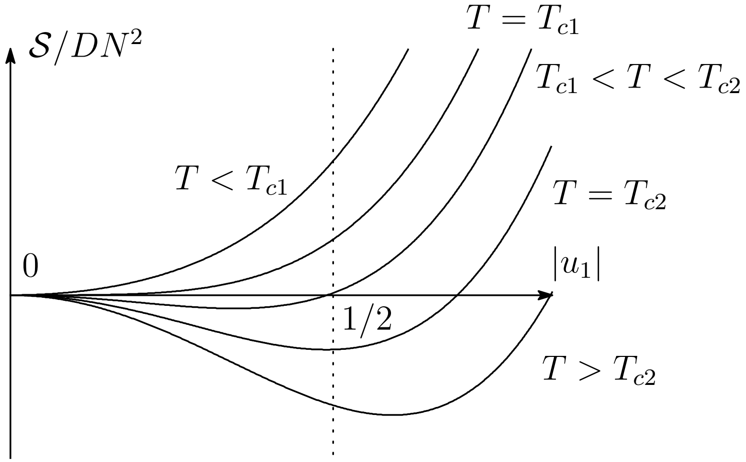


- **Confinement phase** $(T < T_{c1})$: $u_n = 0$ for all n.

- **Deconfinement phase (non-uniform)** $(T_{c1} < T < T_{c2})$:
  $u_1 = \sqrt{-a_1/2b_1} \leq 1/2$, $u_n = 0$ for $n \geq 2$.

- **Deconfinement phase (gapped)** $(T_{c2} < T)$:
  $u_1 \geq 1/2$, $u_n \neq 0$ for $n \geq 2$.

- The transition at $T_{c1} = \left\{ \frac{\log D}{\tilde{\lambda}^{\frac{1}{3}}} \left(1 + \frac{0.523}{D} \right) + O(1/D^2) \right\}^{-1}$:
  $|u_1|$ becomes tachyonic $\Rightarrow$ the phase transition is **second order**.



- The transition at $T_{c2} = \left\{ \frac{1}{T_{c1}} - \frac{1}{\tilde{\lambda}^{\frac{1}{3}}} \times \frac{\log D}{D} \left(\frac{1}{6} + \frac{0.137 \log D + 0.293}{D} \right) + O(1/D^2) \right\}^{-1}$:

  Eigenvalue density of $A = \text{diag}(\alpha_1, \cdots, \alpha_N) \Rightarrow \rho(\alpha) = \frac{1}{N} \sum_{n=1}^{N} \delta(\alpha - \alpha_n)$.

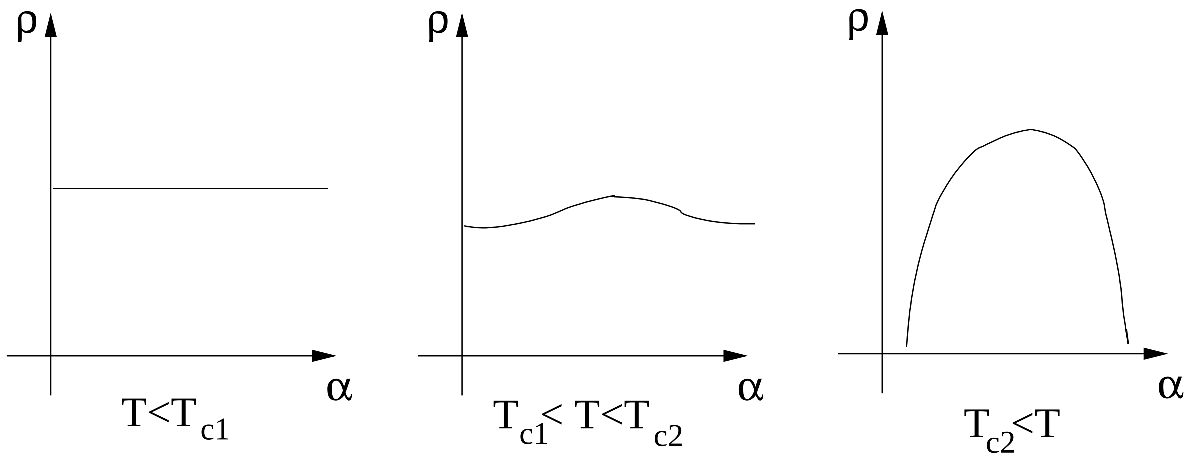
  If $u_n = 0$ (for $n = 2, 3, \cdots$), the density becomes $\rho(\alpha) = \frac{\beta}{2\pi}\{1 + 2|u_1| \cos(\beta\alpha)\}$.

  If $|u_1| = \frac{1}{2}, \Rightarrow \rho\left(\alpha = \frac{\pi}{\beta}\right) = 0$.

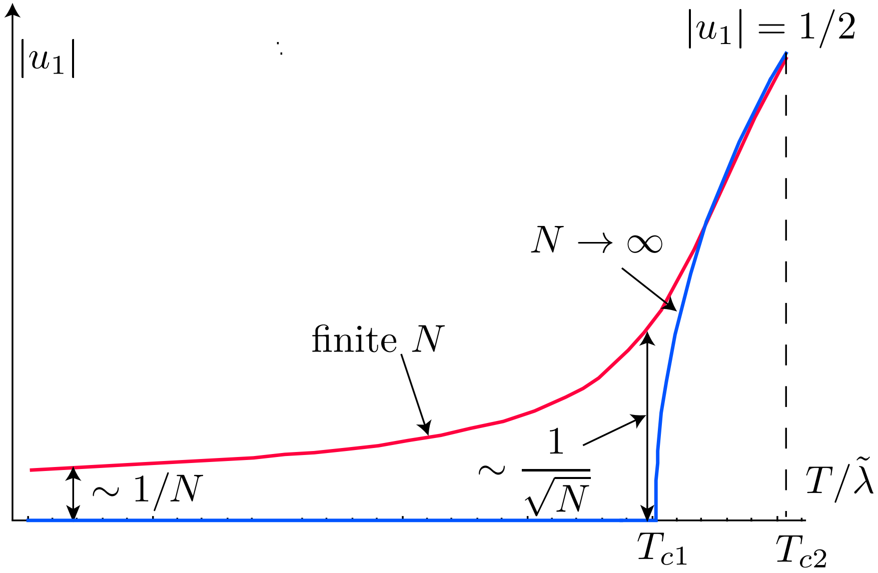
  ρ is positive $\Rightarrow$ a further transition happens there.

  (Gross-Witten-Wadia type phase transition).

  Potential minimum in $S_{\text{eff}}(\{u_n\})$ at $|u_1| = \frac{1}{2} \Rightarrow$ The phase transition is **Gross-Witten-Wadia type third order**.



Resolution of the transitions through $1/N$ effects



We consider the region $|u_1| < \frac{1}{2}$.

⇒ u_n **can be regarded as independent variables:**

$$\langle |u_n| \rangle = \frac{\int du_n du_n^\dagger |u_n| e^{-DN^2 S_n}}{\int du_n du_n^\dagger e^{-DN^2 S_n}}, \text{ where}$$

$$S_1 = a_1 |u_1|^2 + b_1 |u_1|^4, \quad S_n = a_n |u_n|^2 (n \geqq 2)$$

We derive the leading finite-N effects in the path-integral

$$\langle |u_1| \rangle \to \begin{cases} \frac{\sqrt{\pi}}{2N} & (T \to 0) \\ \frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt{N\pi}} \left(\frac{3D}{\log D}\right)^{\frac{1}{4}} & (T = T_{c1}) \end{cases}$$

$$\langle |u_n| \rangle = \frac{1}{2N}\sqrt{\frac{\pi}{Da_n}}, \quad (T \lesssim T_{c2}, n = 2, 3, 4, \cdots)$$

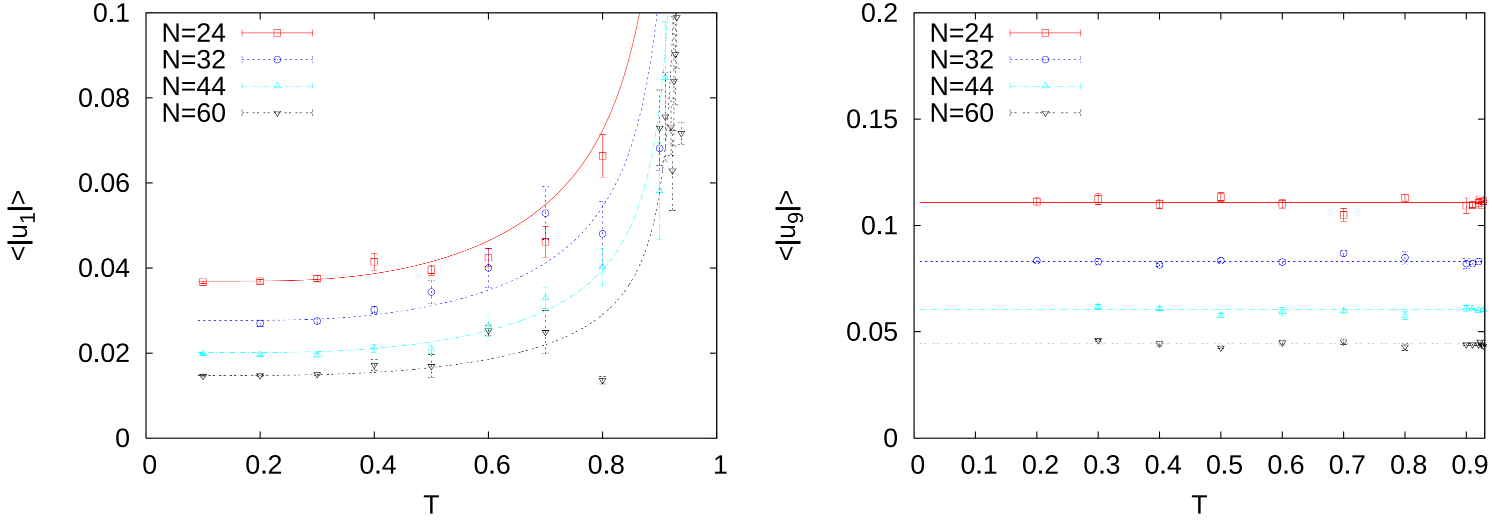
The order parameters u_n are always non-zero. The transitions are resolved to crossovers.

3 $1/D$ expansion v.s. Monte Carlo simulation of MQM

Monte Carlo simulation of the matrix quantum mechanics $S_{\rm MQM}$.

Comparison with the results of the $1/D$ expansion.

Behavior of u_n at low temperatures ($D = 6$)

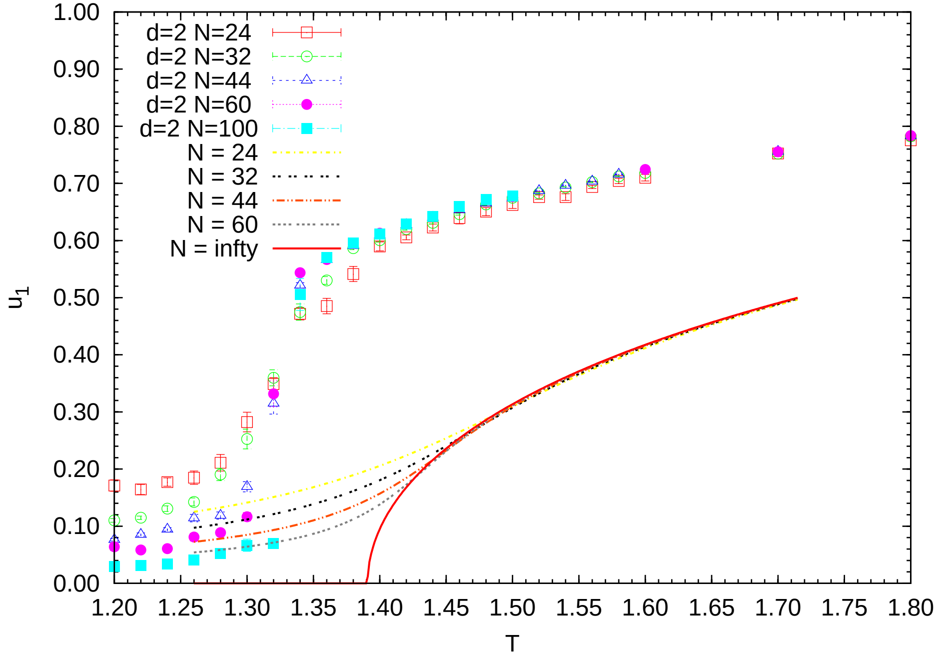


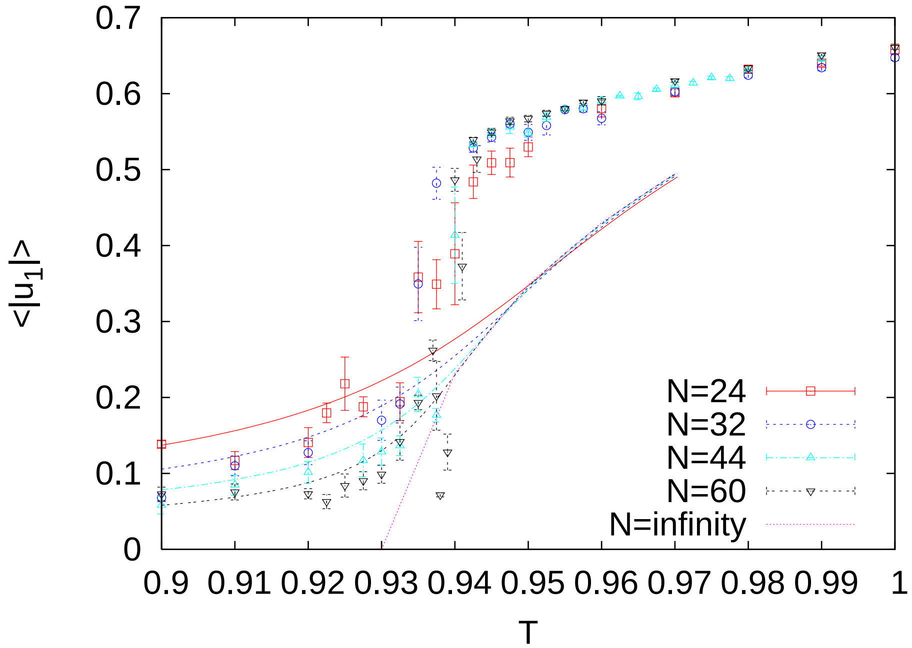
- **In the following, dots are the results from the Monte Carlo simulation of $S_{\rm MQM}$.**
- **Curves in the plots are the results from the $1/D$ expansion up to T_{c2}.**

The Monte Carlo results agree with the $1/D$ expansion even in finite N.

Behavior of u_1 around T_{c1}

(*) left $D = 2$, **right $D = 6$**





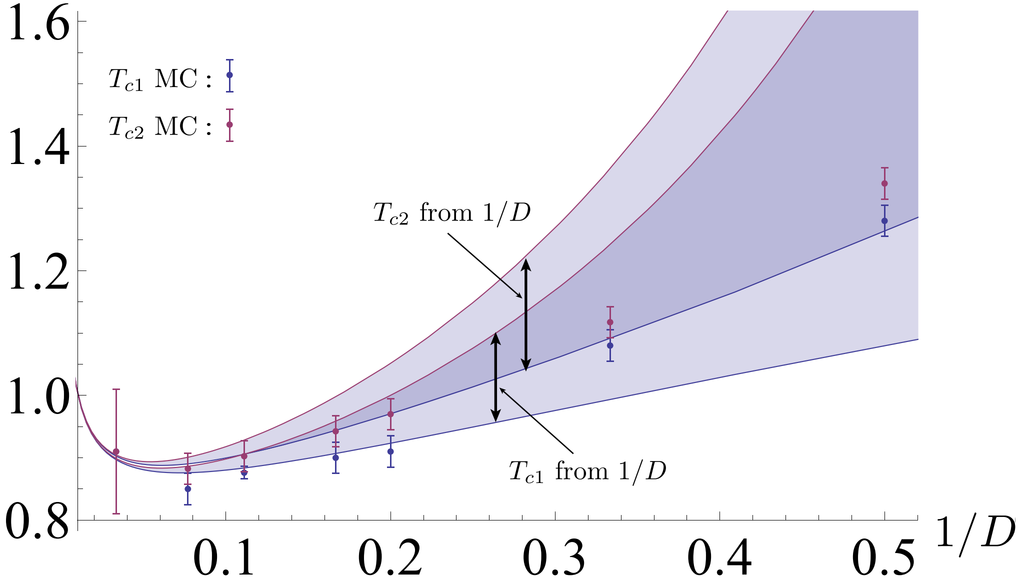
Numerical errors are large near T_{c1} but we can see some similarities.

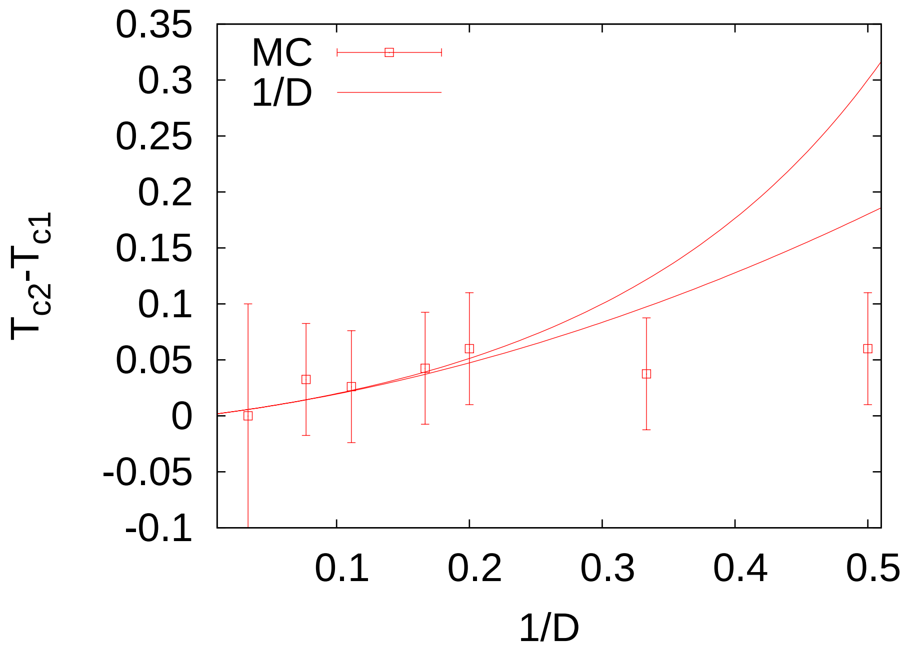
We need a special care to extrapolate the critical temperature at large N from the finite-N Monte Carlo data.

First-order phase transition at $D = 2$?

D dependence of the Critical Temperatures

Preliminary Monte Carlo results of critical temperature $T_{c1,c2}$ versus $1/D$ expansion.





(*) the errorbar of the $1/D$ expansion's result is $T_{c1,c2}(1 \pm 1/D^2)$.

- **The critical temperatures are consistent.**
  The differences are $|(\text{MC data of } S_{\text{MQM}}) - (1/\text{D expansion})| = O(1/D^2)$.
  (the errorbar of the $1/D$ expansion's result is $T_{c1,c2}(1 \pm 1/D^2)$.)

- **There is an ambiguity in the Monte Carlo results of $T_{c1,c2}$, which comes from the extrapolation from finite-N Monte Carlo results.**

- **$T_{c2} - T_{c1}$ for smaller D does not agree well. But the errors in the Monte Carlo are also large and we need to investigate them further.**

Physical quantities in the confinement phase ($T < T_{c1}$)

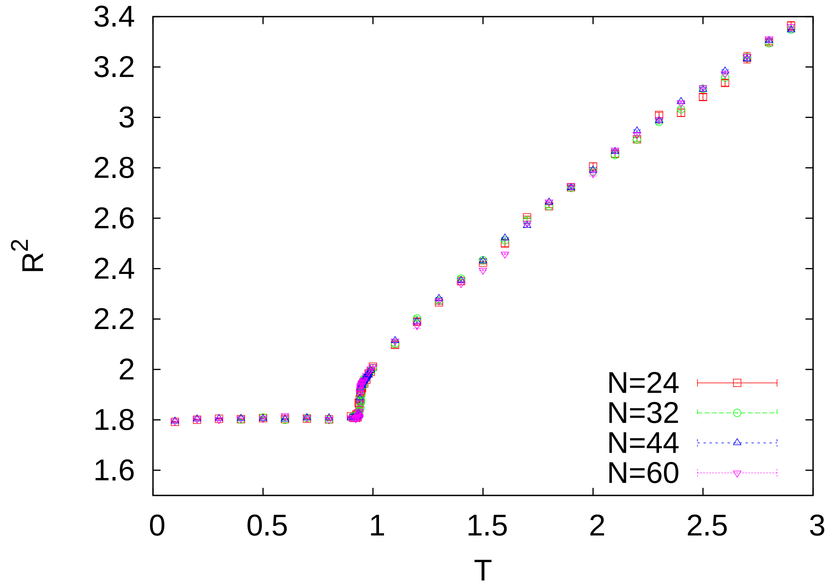
We evaluate the following two quantities:

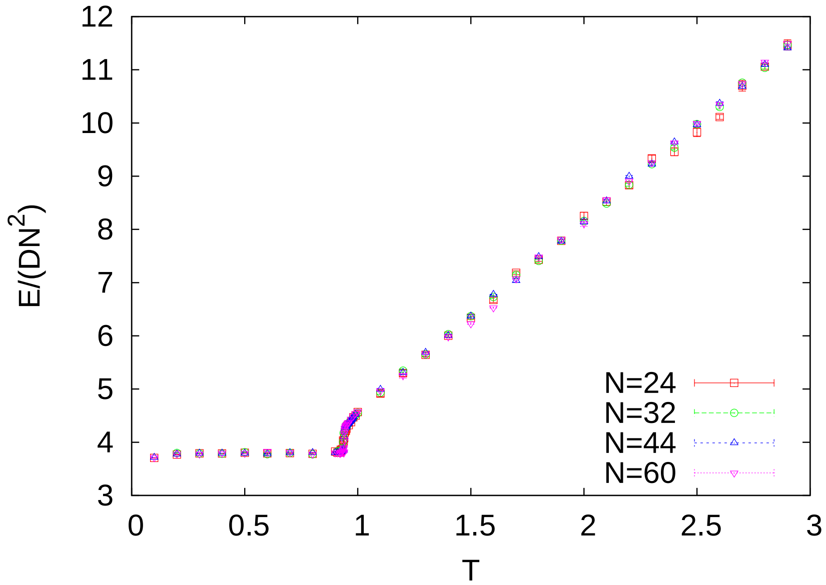
$$R^2 = \frac{T}{g^2 N^2} \int_0^\beta \mathrm{tr}\, X_I^2(t) dt$$

$$\frac{E}{DN^2} = -\frac{3T}{4g^2 N^2 D} \int_0^\beta \mathrm{tr}\, [X_I(t), X_J(t)]^2 dt \qquad \text{(Internal Energy)}$$

Large-N volume independence ⇒ the T dependence is $O(1/N^2)$ at $T < T_{c1}$.

(Monte Carlo results of $S_{\rm MQM}$ for $D = 6$)

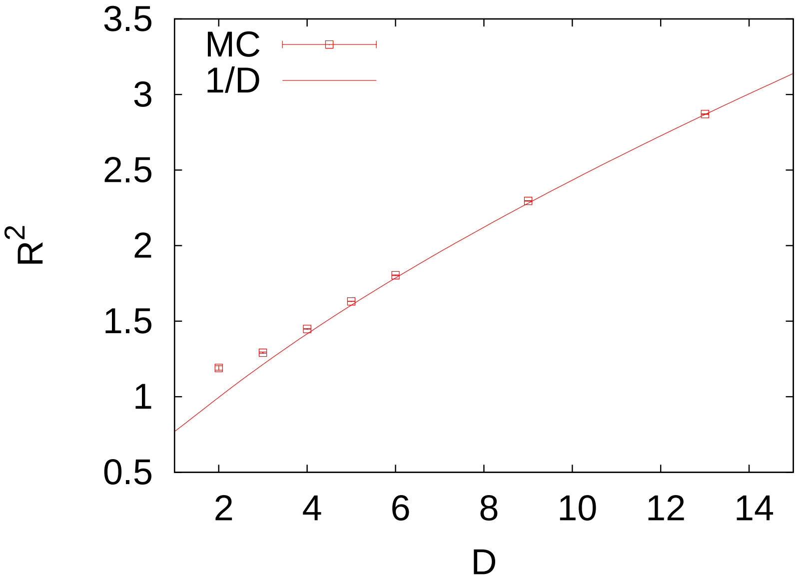


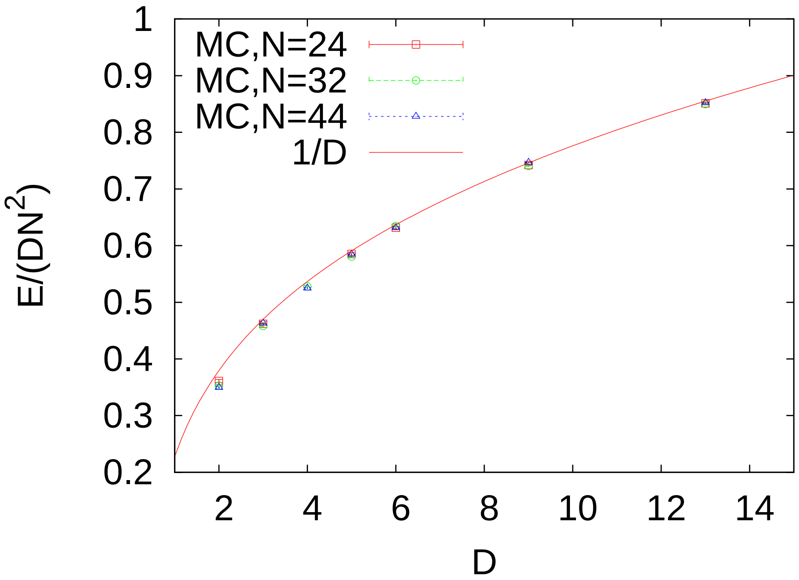


Results from the $1/D$ expansion at $T < T_{c1}$:

$$R^2 = \frac{\tilde{\lambda}^{\frac{1}{3}}}{2}\left(1 + \frac{0.2405}{D}\right) + O(1/N^2, 1/D^2)$$
$$\frac{E}{DN^2} = \tilde{\lambda}^{\frac{1}{3}}\left(\frac{3}{8} - \frac{0.1476}{D}\right) + O(1/N^2, 1/D^2)$$

These quantities also agree very well for various D ($T = 0.5, N = 44$):

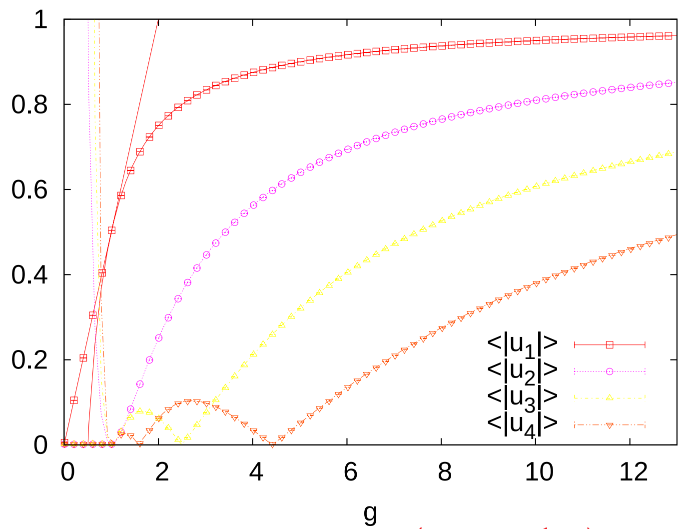




4 Correspondence with GWW model

Comparison of the Monte Carlo result of MQM $S_{\rm MQM}$ with the GWW model

$$Z_{\rm GWW} = \int dU \exp\left(\frac{N}{2} g_{\rm GWW} (\mathrm{tr}\, U + \mathrm{tr}\, U^{\dagger})\right), \text{ where } U = \mathcal{P} \exp\left(i \int_0^{\beta} dt A(t)\right)$$



Third-order GWW phase transition at $g_{\rm GWW} = 1$.

D. J. Gross and E. Witten, Phys. Rev. D 21, 446 (1980).

$$\langle |u_1| \rangle_{\rm GWW} = \begin{cases} \frac{g_{\rm GWW}}{2} & (g_{\rm GWW} \leqq 1) \\ 1 - \frac{1}{2 g_{\rm GWW}} & (g_{\rm GWW} \geqq 1) \end{cases}$$

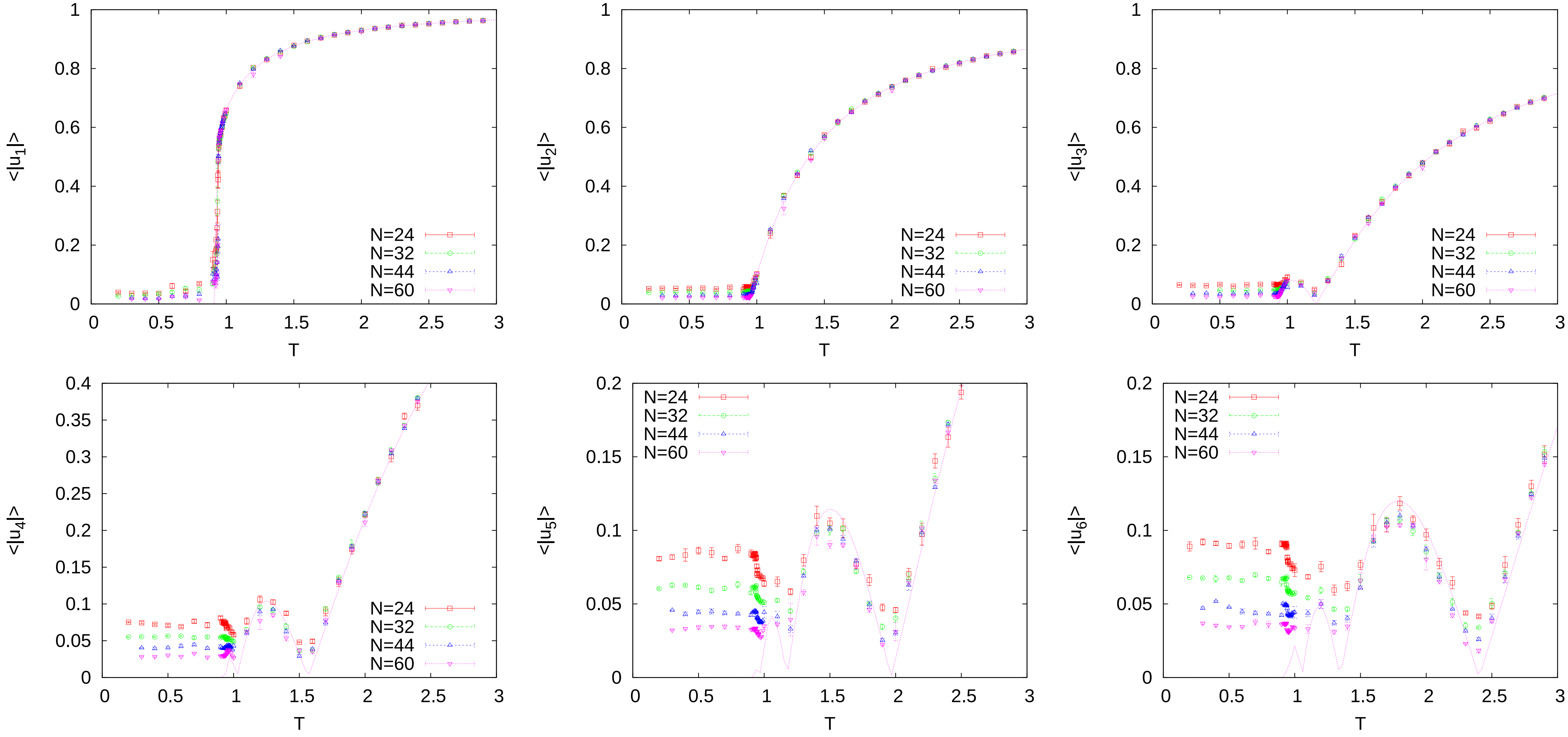
$$\langle |u_n| \rangle_{\rm GWW} = \begin{cases} 0 & (g_{\rm GWW} \leqq 1) \\ \left| \left(1 - \frac{1}{g_{\rm GWW}}\right) \left\{ \frac{1}{n(n+1)} P'_n \left(1 - \frac{2}{g_{\rm GWW}}\right) + \frac{1}{n(n-1)} P'_{n-1} \left(1 - \frac{2}{g_{\rm GWW}}\right) \right\} \right| & (g_{\rm GWW} \geqq 1) \end{cases} \quad (n \geqq 2)$$

where $P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n}$ =(Legendre Polynomial), $\quad P'_n(x) = \frac{dP_n(x)}{dx}$.

Tune the coupling $g_{\rm GWW}$ such that $\langle |u_1(g(T))| \rangle_{\rm GWW} = \langle |u_1(T)| \rangle_{\rm MQM}$ for each temperature.
(where $\langle |u_1(T)| \rangle_{\rm MQM}$ is the result of the MQM $S_{\rm MQM}$)

- **For this coupling $g(T)$, it turns out that $\langle |u_n(g(T))| \rangle_{\rm GWW} \sim \langle |u_n(T)| \rangle_{\rm MQM}$ is satisfied for $n \geqq 2$.**
- **This agreement is trivial at high-temperature.**
  But this agreement holds for any temperature at $T > T_{c2}$, including the region near $T \sim T_{c2}$.

Results of $D = 6$ (for $S_{\rm MQM}$)



5 Conclusion

- **We calculated the finite N effects in the $1/D$ expansion and showed how the $1/N$ effects resolve the transitions.**
- **We compared the predictions from the $1/D$ expansion with Monte Carlo simulation. We found several good agreements at low temperature.**
  $\rightarrow$ $1/D$ works even $D \geq 2$ and finite (but large) N.
- **It seems that the $1/D$ expansion is available without the condition $D \gg N$.**
- **We have compared of the Monte Carlo result of MQM $S_{\rm MQM}$ with the GWW model**
  $\Rightarrow$ Agreement holds for any temperature at $T > T_{c2}$.

Further development

- **Finite N effect vs. finite string coupling effect in holography.**
- **Improvement of the numerical calculation near the critical points.**
- **Determination of the order of phase transition of MQM.**
- **Numerical calculation of $S_{\text{eff}}(\Delta, \{u_n\})$**

  $\rightarrow$ We can evaluate $S_{\text{eff}}(\Delta, \{u_n\})$ for any temperature. (partially done)
- **Effects of matter fields on the confinement/deconfinement phase transition.**

  T. Azuma, T. Morita and S. Takeuchi, in progress

Algorithm for the simulation of finite-temperature matrix quantum mechanics

We adopt the static diagonal gauge

$$A = \frac{1}{\beta}\mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_N),$$

where $\alpha_p \in (-\pi, \pi]$ $(p, q = 1, 2, \cdots, N)$.

We add the corresponding Fadeev-Popov term:

$$S_{\mathrm{f.p.}} = -\sum_{p,q=1,p\neq q}^{N} \log\sin\left|\frac{\alpha_p - \alpha_q}{2}\right|,$$

We discretize the time direction as $t = (\Delta t), 2(\Delta t), \cdots, \underbrace{n_t(\Delta t)}_{=\beta}$.

Finally, we obtain the following discretized action (with $g^2 N = 1$)

(*) In the following, there is no summation unless we have Σ).

$$S_{\mathrm{lat}} = N(\Delta t)\sum_{n=1}^{n_t}\mathrm{tr}\left(\frac{1}{2}\sum_{I=1}^{D}\left\{\frac{1}{(\Delta t)}\mathrm{tr}\left(X_I(n+1) - UX_I(n)U^{\dagger}\right)\right\}^2 - \frac{1}{4}\sum_{I,J=1}^{D}\mathrm{tr}\left[X_I(n), X_J(n)\right]^2\right) + S_{\mathrm{f.p.}},$$

where $U = \exp(i(\Delta t)A) = \mathrm{diag}(e^{i\alpha_1/n_t}, e^{i\alpha_2/n_t}, \cdots, e^{i\alpha_N/n_t})$, $X_I(n) =$ (scalar fields at $t = n(\Delta t)$)

Updating $X_I(n)$ with heat-bath algorithm

We introduce the auxiliary fields $\mathcal{G}_{IJ}(n)$ and rewrite the action (where $G_{IJ}(n) = \{X_I(n), X_J(n)\}$):

$$\tilde{S} = \frac{N(\Delta t)}{2}\sum_{n=1}^{n_t} \mathrm{tr}\left(\sum_{1\leq I<J\leq D} \underbrace{\{\mathcal{G}_{IJ}^2(n) - 2\mathcal{G}_{IJ}(n)G_{IJ}(n) + 4X_I^2(n)X_J^2(n)\}}_{=(\mathcal{G}_{IJ}(n)-G_{IJ}(n))^2-[X_I(n),X_J(n)]^2} + \frac{1}{(\Delta t)^2}\sum_{i=1}^{D}\{X_I^2(n+1) + X_I^2(n) - 2X_I(n+1)UX_I(n)U^{\dagger}\}\right) + S_{\text{f.p.}},$$

Updating the auxiliary fields as

- $(\mathcal{G}_{IJ}(n))_{pp} = \frac{W_p}{\sqrt{N(\Delta t)}} + (G_{IJ}(n))_{pp}$, (diagonal, $p = 1, 2, \cdots, N$)
- $(\mathcal{G}_{IJ}(n))_{pq} = \frac{Y_{pq} + iZ_{pq}}{\sqrt{2N(\Delta t)}} + (G_{IJ}(n))_{pq}$. (non-diagonal, $p \neq q$, $p, q = 1, 2, \cdots, N$)

where W_p, Y_{pq}, Z_{pq} are independent random numbers obeying normal Gaussian distribution

$$P(W_p) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{W_p^2}{2}\right), \quad P(Y_{pq}) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{Y_{pq}^2}{2}\right), \quad P(Z_{pq}) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{Z_{pq}^2}{2}\right)$$

We further rewrite the action as

$$\tilde{S} = -2N\mathrm{tr}\,(T_I(n)X_I(n)) + 4N\mathrm{tr}\,(S_I(n)X_I^2(n)) + S_{\mathrm{f.p.}}, \quad \text{where}$$
$$S_I(n) = \frac{(\Delta t)}{2}\sum_{J\neq I} X_J^2(n),$$
$$T_I(n) = \frac{(\Delta t)}{2}\sum_{J\neq I}(X_J(n)\mathcal{G}_{IJ}(n) + \mathcal{G}_{IJ}(n)X_J(n)) + \frac{1}{2(\Delta t)}(UX_I(n-1)U^\dagger + U^\dagger X_I(n+1)U).$$

Extracting the diagonal part $(X_I(n))_{pp}$ as

$$\tilde{S} = 4N(S_I(n))_{pp}\left\{(X_I(n))_{pp} - \frac{h_p}{(S_I(n))_{pp}}\right\}^2 + \cdots, \quad \text{where}$$
$$h_p = \frac{1}{4}\left\{(T_I(n))_{pp} - 2\sum_{q\neq p}\{(S_I(n))_{qp}(X_I(n))_{pq} + (S_I(n))_{pq}(X_I(n))_{qp}\}\right\}$$

Updating the diagonal part $(X_I(n))_{pp}$ as

$$(X_I(n))_{pp} = \frac{W_p}{\sqrt{8N(S_I(n))_{pp}}} + \frac{h_p}{(S_I(n))_{pp}}, \quad (\text{diagonal}, p = 1, 2, \cdots, N), \quad \text{where}$$
$$h_p = \frac{1}{4}\left\{(T_I(n))_{pp} - 2\sum_{q\neq p}\{(S_I(n))_{qp}(X_I(n))_{pq} + (S_I(n))_{pq}(X_I(n))_{qp}\}\right\}.$$

Extracting the non-diagonal part $(X_I(n))_{pq}$ $(p \neq q)$ as

$$\tilde{S} = 4Nc_{pq}\left|(X_I(n))_{pq} - \frac{h_{pq}}{c_{pq}}\right|^2 + \cdots, \text{ where}$$

$$c_{pq} = (S_I(n))_{pp} + (S_I(n))_{qq}, \quad h_{pq} = \frac{(T_I(n))_{pq}}{2} - \left\{\sum_{r\neq p}(S_I(n))_{pr}(X_I(n))_{rq} + \sum_{r\neq q}(S_I(n))_{rq}(X_I(n))_{pr}\right.$$

Updating the non-diagonal part $(X_I(n))_{pq}$ $(p \neq q)$ as

$$(X_I(n))_{pq} = \frac{X_{pq} + iY_{pq}}{\sqrt{8Nc_{pq}}} + \frac{h_{pq}}{c_{pq}}, \quad (\text{non-diagonal}, p \neq q, \quad p, q = 1, 2, \cdots, N), \text{ where}$$

$$c_{pq} = (S_I(n))_{pp} + (S_I(n))_{qq}, \quad h_{pq} = \frac{(T_I(n))_{pq}}{2} - \left\{\sum_{r\neq p}(S_I(n))_{pr}(X_I(n))_{rq} + \sum_{r\neq q}(S_I(n))_{rq}(X_I(n))_{pr}\right\}.$$

Updating gauge fields A with Metropolis algorithm

Gauge fields' components α_p are updated using accept-reject procedure of Metropolis algorithm.

Consistency check of the code

We use the identity derived from the Schwinger-Dyson equation.

$$0 = \sum_{n=1}^{n_t} \sum_{a=1}^{N^2-1} \sum_{I=1}^{D} \frac{\partial}{\partial X_I^a(n)} \int dM dA \, \mathrm{tr}\,(t^a X_I(n)) e^{-S}.$$

$t^a =$ (basis of the $SU(N)$ Lie algebra)

$$\mathrm{tr}\,(t^a t^b) = \delta^{ab}, \quad \sum_{a=1}^{N^2} (t^a)_{ij} (t^a)_{kl} = \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}.$$

$$\begin{aligned} \sum_{a=1}^{N^2-1} \mathrm{tr}\,(t^a A) \mathrm{tr}\,(t^a B) &= \sum_{a=1}^{N^2-1} A_{ji} B_{lk} (t^a)_{ij} (t^a)_{kl} = A_{ji} B_{lk} (\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}) \\ &= \mathrm{tr}\,(AB) - \frac{1}{N} \mathrm{tr}\, A \,\mathrm{tr}\, B. \end{aligned}$$

The matrices $X_I(n)$ are expanded as $X_I(n) = \sum_{a=1}^{N^2-1} X_I^a(n) t^a$.

We rewrite the Schwinger-Dyson equation as

$$0 = \underbrace{\sum_{n=1}^{n_t} \sum_{a=1}^{N^2-1} \sum_{I=1}^{D} \int dM dA \, \mathrm{tr}\,(t^a t^a) e^{-S}}_{n_t D (N^2-1) e^{-S}} - \sum_{n=1}^{n_t} \sum_{a=1}^{N^2-1} \sum_{I=1}^{D} \int dM dA \, \mathrm{tr}\,(t^a X_I(n)) \frac{\partial S}{\partial X_I^a(n)} e^{-S}.$$

Thus, we obtain (note that $n_t = \frac{\beta}{(\Delta t)}$)

$$n_t D(N^2-1)\langle e^{-S}\rangle = \frac{(N^2-1)D\beta}{(\Delta t)}\langle e^{-S}\rangle = \left\langle \sum_{n=1}^{n_t}\sum_{a=1}^{N^2-1}\sum_{I=1}^{D}\int dM dA \mathrm{tr}\,(t^a X_I(n))\frac{\partial S}{\partial X_I^a(n)}e^{-S}\right\rangle$$

The derivative of the action is obtained as

$$\begin{aligned}\frac{\partial S}{\partial X_I^a(n)} &= N(\Delta t)\mathrm{tr}\,\Big\{t^a\Big(-[X_J(n),[X_I(n),X_J(n)]] + [A,[A,X_I(n)]] \\ &\quad -\frac{i}{(\Delta t)}([A,X_I(n-1)]-[A,X_I(n+1)]) - \frac{1}{(\Delta t)^2}(X_I(n+1)+X_I(n-1)-2X_I(n))\Big)\Big\}.\end{aligned}$$

We obtain the relation (note that $\mathrm{tr}\,X_I = 0$ due to hermiticity)

$$\begin{aligned}\frac{D\beta}{(\Delta t)}\left(1-\frac{1}{N^2}\right) &= (\Delta t)\sum_{n=1}^{n_t}\Big\{-\left\langle\frac{1}{N}\mathrm{tr}\,[X_I(n),X_J(n)]^2\right\rangle \\ &\quad +\Big\langle\frac{1}{N}\mathrm{tr}\,\Big(-\frac{1}{(\Delta t)^2}(X_I(n)X_I(n+1)+X_I(n)X_I(n-1)-2X_I^2(n)) - [A,X_I(n)]^2 \\ &\quad -2i\frac{X_I(n+1)-X_I(n-1)}{2(\Delta t)}[A,X_I(n)]\Big)\Big\rangle\Big\}.\end{aligned}$$

At (Δt) and large N, it is rewritten as

$$\frac{D\beta}{(\Delta t)} = \frac{1}{N}\left\langle\int_0^\beta dt\left(\mathrm{tr}\,(D_tX_I(t))^2 - 2\lambda\mathrm{tr}\,[X_I(t),X_J(t)]^2\right)\right\rangle.$$