Matrix model with manifest general coordinate invariance

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Takehiro Azuma
Department of Physics, Kyoto University

University of Tsukuba, Particle Theory Group
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collaborated with H. Kawai

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1 Introduction

Constructive definition of superstring theory

A large $N$ reduced model has been proposed as a nonperturbative formulation of superstring theory.

[IIB matrix model]

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

For a review, hep-th/9908038

\[ S = -\frac{1}{g^2} Tr_{N\times N} \left( \frac{1}{4} \sum_{a,b=0}^{9} [A_a, A_b]^2 - \frac{1}{2} \bar{\psi} \sum_{a=0}^{9} \Gamma^a [A_a, \psi] \right). \]

* $A_a$ and $\psi$ are $N \times N$ Hermitian matrices.
  * $A_a$: 10-dimensional vectors
  * $\psi$: 10-dimensional Majorana-Weyl (i.e. 16-component) spinors

* This model possesses $SU(N)$ gauge symmetry and $SO(9,1)$ Lorentz symmetry.

* Dimensional reduction of $\mathcal{N} = 1$ 10-dimensional SYM to 0 dimension.

* Matrix regularization of the Green-Schwarz action of type IIB superstring theory.
• $\mathcal{N} = 2$ SUSY: This theory must contain spin-2 gravitons if it admits massless particles.

* homogeneous: $\delta^{(1)}_\epsilon A_a = i\epsilon \Gamma_a \psi$, $\delta^{(1)}_\epsilon \psi = \frac{i}{2} \Gamma^{ab}[A_a, A_b] \epsilon$.
* inhomogeneous: $\delta^{(2)}_\xi A_a = 0$, $\delta^{(2)}_\xi \psi = \xi$.
* We obtain the following commutation relations:

(1) $[\delta^{(1)}_\epsilon_1, \delta^{(1)}_\epsilon_2] A_a = 0$, $[\delta^{(1)}_\epsilon_1, \delta^{(1)}_\epsilon_2] \psi = 0$,
(2) $[\delta^{(2)}_\xi_1, \delta^{(2)}_\xi_2] A_a = 0$, $[\delta^{(2)}_\xi_1, \delta^{(2)}_\xi_2] \psi = 0$,
(3) $[\delta^{(1)}_\epsilon, \delta^{(2)}_\xi] A_a = -i\epsilon \Gamma_a \xi$, $[\delta^{(1)}_\epsilon, \delta^{(2)}_\xi] \psi = 0$.

We take the following linear combination

$\tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)}$, $\tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)})$.

This gives a shift of the bosonic variables: ($\alpha, \beta = 1, 2$)

$$
[\tilde{\delta}^{(\alpha)}_\epsilon, \tilde{\delta}^{(\beta)}_\xi] \psi = 0, \\
[\tilde{\delta}^{(\alpha)}_\epsilon, \tilde{\delta}^{(\beta)}_\xi] A_a = -2i\delta^{\alpha\beta}\epsilon \Gamma_a \xi.
$$

$\Rightarrow$ Therefore, the large $N$ matrices $A_a$, per se, represent the spacetime coordinate.
Is it possible to formulate a matrix model which describes the gravitational interaction more manifestly?

Can a matrix model describe the physics in the curved space?

- How is the local Lorentz invariance realized in the matrix model?
- Does a matrix model reduce to the \((\text{type IIB})\) supergravity in the low-energy limit?
2 Matrix as differential operator

We identify infinitely large $N$ matrices with differential operator.

The information of spacetime can be embedded to matrices in various ways.

- Twisted Eguchi-Kawai(TEK) model:
  

  $$A_a \sim \partial_a + a_a.$$ 

  The matrices $A_a$ represent the covariant derivative on the spacetime.

- IIB matrix model:

  $$A_a \sim X_a.$$ 

  $A_a$ itself represent the space-time coordinate.

IIB matrix model with noncommutative background

$$[\hat{p}_a, \hat{p}_b] = iB_{ab}, \quad (B_{ab} = \text{real c-numbers})$$

interpolates these two pictures.

H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, hep-th/9908141

$$Tr_{N \times N} \bar{\psi} \Gamma^a [A_a, \psi]$$ reduces to the fermionic action

$$\int d^d x \bar{\psi}(x) i \Gamma^a (\partial_i \psi(x) + [a_i(x), \psi(x)])$$ in the flat space in the low-energy limit.
- A differential operator acts on a field in the curved space naturally.
- The space of the large $N$ matrices includes the differential operators on an arbitrary spin bundle over an arbitrary manifold simultaneously.
The fermionic action in the curved space:

\[
S_F = \int d^d x e(x) \bar{\psi}(x) i \Gamma^a e^a_i(x) \left( \partial_i \psi(x) + [A_i(x), \psi(x)] + \frac{1}{4} \Gamma^{bc} \omega_{ibc}(x) \psi(x) \right).
\]

- \(a, b, c, \ldots\) : indices of the 10-dimensional Minkowskian spacetime.
- \(i, j, k, \ldots\) : indices of the 10-dimensional curved spacetime.

The correspondence between the matrix model and the continuum limit:

\[
\begin{align*}
Tr_{N \times N} \to & \int d^d x, \\
\psi \to & \Psi(x) = e^{\frac{i}{2}(x)} \psi(x), \\
[A_a, \psi] \to & ie^{\frac{i}{2}(x)} e^a_i(x) \left( \partial_i + [A_i(x), \psi] \right) e^{-\frac{i}{2}(x)}, \\
\{A_{a_1 a_2 a_3}, \psi\} \to & e_{[a_1}^i(x) \omega_{ia_2 a_3]}(x) \psi(x). \\
\end{align*}
\]

The rank-3 matrices correspond to the spin connection!
Commutation relations of (anti)-hermitian operators:

\((1)[h_1, h_2] \in A, \ (2)[h, a] \in H, \ (3)[a_1, a_2] \in A, \ (4)\{h_1, h_2\} \in H, \ (5)\{h, a\} \in A, \ (6)\{a_1, a_2\} \in H.\)

- Hermitian matrices:
  \(H = \{M \in M_{N \times N}(C) | M^\dagger = M\}. \ h, h_1 h_2 \in H.\)

- Anti-hermitian matrices:
  \(A = \{M \in M_{N \times N}(C) | M^\dagger = -M\}. \ a, a_1, a_2 \in A.\)

[Proof of (4)] \(\{h_1, h_2\}^\dagger = (h_1 h_2 + h_2 h_1)^\dagger = h_2^\dagger h_1^\dagger + h_1^\dagger h_2^\dagger = \{h_1, h_2\}.\)

Notation of the gamma matrices:

\(\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}, \) where \(\eta^{ab} = \text{diag}(-1, +1, \cdots, +1),\)

We take the gamma matrices to be real:

\[(\Gamma^a)^\dagger = (^{T}\Gamma^a) = \begin{cases} -\Gamma^a & (a = 0) \\ +\Gamma^a & (a = 1, 2, \cdots, 9) \end{cases}.\]

\(C = \text{(charge conjugation)} = \Gamma^0, \ \Gamma^0 (\Gamma^a)^\dagger \Gamma^0 = \Gamma^a.\)
\[ S_F = \int d^d x \bar{\Psi}(x) e^\frac{i}{2}(x) i \Gamma^a e_i(x) \left\{ \partial_i (e^{-\frac{i}{2}}(x) \Psi(x)) \\
+ [A_i(x), e^{-\frac{i}{2}}(x) \Psi(x)] + \frac{1}{4} \Gamma^{bc} \omega_{ibc}(x) e^{-\frac{i}{2}}(x) \Psi(x) \right\} \]

\[ = \int d^d x \left\{ \bar{\Psi}(x) i \Gamma^a \left[ e_i(x) \partial_i + \frac{1}{2} e_i(x) \omega_{ia}(x) \right] \Psi(x) \\
+ i \bar{\Psi}(x) \Gamma^a e_i(x) [A_i(x), \Psi(x)] \\
+ \frac{i}{4} \bar{\Psi}(x) \Gamma^{a_1a_2a_3} e_{[a_1} e_i(x) \omega_{a_2a_3]}(x) \Psi(x) \right\} \]

\[ = \int d^d x \left\{ \bar{\Psi}(x) i \Gamma^a e_i(x) (\partial_i \Psi(x) + [A_i(x), \Psi(x)]) \\
+ \frac{i}{4} \bar{\Psi}(x) \Gamma^{a_1a_2a_3} e_{[a_1} e_i(x) \omega_{a_2a_3]}(x) \Psi(x) \right\}. \]

In the above, we have utilized the following relationship (when \( \Psi(x) \) is Majorana):

\[ \bar{\Psi}(x) \Gamma^a \Psi(x) = (\bar{\Psi}(x) \Gamma^a \Psi(x))^\dagger = -\Psi^\dagger(x) (\Gamma^a)^\dagger (\Gamma^0)^\dagger \Psi(x) \]

\[ = -\Psi^\dagger(x) \Gamma^0 (\Gamma^0)^\dagger \Psi(x) = -\bar{\Psi}(x) \Gamma^a \Psi(x) = 0. \]

The corresponding matrix model is

\[ S_F \leftrightarrow \frac{1}{2} Tr \bar{\psi} \Gamma^a \{ A_a, \psi \} + \frac{i}{2} \bar{\psi} \Gamma^{abc} \{ A_{abc}, \psi \} \]

\[ = Tr (\bar{\psi} \Gamma^a A_a \psi + i \bar{\psi} \Gamma^{a_1a_2a_3} A_{a_1a_2a_3} \psi). \]

Proof of the equality (only for the boson, when \( \psi \) is Majorana):

\[ \frac{1}{2} Tr (\bar{\psi} \Gamma^a [A_a, \psi]) = \frac{1}{2} \bar{\psi} \Gamma^a A_a \psi^C Tr(t^B t^C) \]

\[ = \frac{1}{2} \bar{\psi} \Gamma^a A_a \psi^C Tr(t^A t^B t^C - t^C t^B t^A) \]

\[ = \frac{1}{2} (\bar{\psi} \Gamma^a A_a \psi^C - \bar{\psi} \Gamma^a A_a \psi^A) Tr(t^A t^B t^C) = Tr (\bar{\psi} \Gamma^a A_a \psi). \]
The symmetry of IIB matrix model: $SO(9, 1)$ and $U(N)$ symmetry is decoupled. The $SO(9, 1) \times U(N)$ symmetry is a tensor product of the group. For $\zeta \in so(9, 1)$ and $u \in u(N)$,

$$\exp(\zeta \otimes 1 + 1 \otimes u) = e^\zeta \otimes e^u.$$ 

The spacetime coordinate is embedded in the eigenvalues of the large $N$ matrices.

$\Rightarrow$ If we are to formulate a matrix model with local Lorentz invariance, the $so(9, 1)$ Lorentz symmetry and the $u(N)$ gauge symmetry must be unified.

(*) $\mathcal{A}, \mathcal{B} = \text{[The Lie algebras whose bases are } \{a_i\} \text{ and } \{b_j\}, \text{ respectively.]}$

- $\mathcal{A} \otimes \mathcal{B}$: The space spanned by the basis $a_i \otimes b_j$. This is not necessarily a closed Lie algebra.

- $\mathcal{A} \hat{\otimes} \mathcal{B}$: The smallest Lie algebra that includes $\mathcal{A} \otimes \mathcal{B}$ as a subset.

The gauge group must close with respect to the commutator

$$[a \otimes A, b \otimes B] = \frac{1}{2} ([a, b] \otimes \{A, B\} + \{a, b\} \otimes [A, B]).$$
(*) In order to grasp the intuitive image of the unified tensor product, we consider the following simple example.

\[ su(6) = su(3) \tilde{\otimes} su(2). \]

\( \lambda^a \): basis of \( su(3) \) \((a = 1, 2, \cdots 8)\).

\( \sigma^i \): basis of \( su(2) \) \((i = 1, 2, 3)\).

- \( \lambda^a \otimes \sigma^i \) (24 dimensions): The basis of \( su(3) \otimes su(2) \), which does not constitute a closed Lie algebra.

- \( \lambda^a \otimes 1 + 1 \otimes \sigma^i \) (11 dimensions): The generators of the Lie group \( SU(3) \times SU(2) \).

- \( su(3) \tilde{\otimes} su(2) = (su(3) \otimes su(2)) \oplus (SU(3) \times SU(2))_{\text{algebra}} \)

This is a closed 35-dimensional Lie algebra.

\( SU(3) \times SU(2) \) is a 11-dimensional Lie group, while \( su(3) \tilde{\otimes} su(2) \) is a 35-dimensional Lie algebra.
Local Lorentz transformation of the matrix model

\[ \delta \psi = \frac{1}{4} \Gamma^{a_1 a_2} e_{a_1 a_2} \psi, \]

instead of \( \delta \psi = \frac{1}{4} \Gamma^{a_1 a_2} \{ e_{a_1 a_2}, \psi \} \) at the cost of the hermiticity of \( \psi \).

At this time, the product \( A_a \psi \) does not directly correspond to the covariant derivative \( (\partial_a \psi(x) + [A_a(x), \psi(x)]) \).

The local Lorentz transformation of the action:

\[ \delta S'_F = \frac{1}{4} Tr \bar{\psi} [\Gamma^a A_a + i \Gamma^{a_1 a_2 a_3} A_{a_1 a_2 a_3}, \Gamma^{b_1 b_2} e_{b_1 b_2}] \psi. \]

However, this action does not close with respect to the local Lorentz transformation:

\[
\begin{align*}
&\frac{i}{2} \{ \Gamma^{a_1 a_2 a_3}, \Gamma^{b_1 b_2} \} \{ A_{a_1 a_2 a_3}, e_{b_1 b_2} \} + \frac{i}{2} \{ \Gamma^{a_1 a_2 a_3}, \Gamma^{b_1 b_2} \} [A_{a_1 a_2 a_3}, e_{b_1 b_2}], \\
&\text{rank 3} \\
&\text{rank 1, 5}
\end{align*}
\]

We need the terms of all odd ranks in order to formulate a local Lorentz invariant matrix model.

The algebra of the local Lorentz transformation must include all the even-rank gamma matrices:

\[
\begin{align*}
&\{ \Gamma^{a_1 a_2} e_{a_1 a_2}, \Gamma^{b_1 b_2} e'_{b_1 b_2} \} \\
&= \frac{1}{2} \{ \Gamma^{a_1 a_2}, \Gamma^{b_1 b_2} \} \{ e_{a_1 a_2}, e'_{b_1 b_2} \} + \frac{1}{2} \{ \Gamma^{a_1 a_2}, \Gamma^{b_1 b_2} \} \{ e_{a_1 a_2}, e'_{b_1 b_2} \}, \\
&\text{rank-2} \\
&\text{rank-0, 4}
\end{align*}
\]

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3 Attempts for a matrix model related to the type IIB supergravity

\[ S = Tr_{N \times N} [tr_{32 \times 32} V(m^2) + \bar{\psi} m \psi] \]

- **Tr(tr):** the trace for the \( N \times N(32 \times 32) \) matrices.
- **m** includes all odd-rank gamma matrices in 10 dimensions:

\[
m = m_a \Gamma^a + \frac{i}{3!} m_{a_1 a_2 a_3} \Gamma^{a_1 a_2 a_3} - \frac{1}{5!} m_{a_1 \ldots a_5} \Gamma^{a_1 \ldots a_5} \]

\[
- \frac{i}{7!} m_{a_1 \ldots a_7} \Gamma^{a_1 \ldots a_7} + \frac{1}{9!} m_{a_1 \ldots a_9} \Gamma^{a_1 \ldots a_9},
\]

where \( m_{a_1 \ldots a_{2n-1}} \) are hermitian matrices:

\[
m_{a_1 \ldots a_{2n-1}} = \frac{i^{n-1}}{32 \times (2n - 1)!} tr(m \Gamma_{a_1 \ldots a_{2n-1}}).
\]

\( m \) satisfies \( \Gamma^0 m^\dagger \Gamma^0 = m \), and the action is hermitian.

We want to identify \( m \) with the Dirac operator.

⇒ We introduce \( D = [(\text{length})^{-1}] \) as an extension of the Dirac operator.

\[
m = \tau_D, \text{ where } \tau = [(\text{length})^2],
\]

\[
D = A_a \Gamma^a + \frac{i}{3!} A_{a_1 a_2 a_3} \Gamma^{a_1 a_2 a_3} - \frac{1}{5!} A_{a_1 \ldots a_5} \Gamma^{a_1 \ldots a_5}
\]

\[
- \frac{i}{7!} A_{a_1 \ldots a_7} \Gamma^{a_1 \ldots a_7} + \frac{1}{9!} A_{a_1 \ldots a_9} \Gamma^{a_1 \ldots a_9}.
\]

\( \tau \) is not an \( N \)-dependent cut-off parameter, but a reference scale (\( \sim l_s^2 \)).
\[ A_{a_1\cdots a_{2n-1}} = \frac{i^{2n-1}}{32(2n-1)!} tr(D\Gamma_{a_1\cdots a_{2n-1}}) \text{ are hermitian differential operators.} \]

⇒ They are expanded by the number of the derivatives:

\[ A_{a_1\cdots a_{2n-1}} = a_{a_1\cdots a_{2n-1}}(x) + \sum_{k=1}^{\infty} \frac{i^k}{2k} \{ \partial_{i_1} \cdots \partial_{i_k}, a^{(i_1\cdots i_k)}_{a_1\cdots a_{2n-1}}(x) \}. \]

\[ (\text{length})^{-1+k} \]

\[ a_a^{(i)}(x) \text{ is identified with the vielbein } e_a^i(x) \text{ in the background metric.} \]

\[ D = e^{\frac{1}{2}x}(x) \left[ ie_a^i(x)\Gamma^a \left( \partial_i + \frac{1}{4} \Gamma^{bc} \omega_{ibc}(x) \right) \right] e^{-\frac{1}{2}x}(x) \]

+ (higher-rank terms) + (higher-derivative terms).

The potential \( V(m^2) \) is generically \( V(m^2) \sim \exp(-(m^2)^\alpha) \).

⇒ The damping factor is naturally included in the bosonic term.

⇒ The trace for the infinitely large \( N \) matrices is finite.

\[ \psi \text{ is a Weyl fermion, but not Majorana.} \]

We need to introduce a damping factor so that the trace should be finite.

\[ \psi = (\chi(x) + \sum_{l=1}^{\infty} \chi^{(i_1\cdots i_l)}(x) \partial_{i_1} \cdots \partial_{i_l}) e^{-(\tau D^2)^\alpha}. \]
Local Lorentz invariance

The action is invariant under the local Lorentz transformation:

\[ \delta m = [m, \varepsilon], \quad \delta \psi = \varepsilon \psi, \quad \delta \bar{\psi} = -\bar{\psi} \varepsilon, \quad \text{where} \]

\[ \varepsilon = -i\varepsilon_0 + \frac{1}{2!} \Gamma^{a_1 a_2} \varepsilon_{a_1 a_2} + \frac{i}{4!} \Gamma^{a_1 \cdots a_4} \varepsilon_{a_1 \cdots a_4} - \frac{1}{6!} \Gamma^{a_1 \cdots a_6} \varepsilon_{a_1 \cdots a_6} \]

\[ -\frac{i}{8!} \Gamma^{a_1 \cdots a_8} \varepsilon_{a_1 \cdots a_8} + \frac{1}{10!} \Gamma^{a_1 \cdots a_{10}} \varepsilon_{a_1 \cdots a_{10}}, \]

- All even-rank gamma matrices are necessary for the local Lorentz transformation algebra to close.
- \( \varepsilon \) satisfies \( \Gamma^0 \varepsilon \Gamma^0 = \varepsilon \), and thus the commutator \( \delta m = [m, \varepsilon] \) actually satisfies \( \Gamma^0 (\delta m)^\dagger \Gamma^0 = \delta m \).

The invariance under the local Lorentz transformation:

\[ \delta S = 2Tr[tr(V^l_S(m^2) m[m, \varepsilon])] + Tr[tr(\bar{\psi}[m, \varepsilon] \psi)] = 0. \]

The cyclic property still holds true of the trace for the large \( N \) matrices, if we assume that the coefficients damp rapidly at infinity:

\[ \lim_{|x| \to \infty} a^{(i_1 \cdots i_k)}_{a_1 \cdots a_{2n-1}}(x) = \lim_{|x| \to \infty} \chi^{(i_1 \cdots i_k)}(x) = 0. \]

[Proof] After integrating in the action, the following commutator vanishes:

\[ Tr([\partial_j, a^{(i_1 \cdots i_k)}_{a_1 \cdots a_{2n-1}}(x)]) \]
\[ = \int d^d x \langle x | (\partial_j a^{(i_1 \cdots i_k)}_{a_1 \cdots a_{2n-1}}(x)) | x \rangle \]
\[ = \int d^d x (\partial_j a^{(i_1 \cdots i_k)}_{a_1 \cdots a_{2n-1}}(x)) \langle x | x \rangle = 0. \]
Heat kernel expansion

The trace of the large $N$ matrices is analyzed through the heat kernel (Seeley de Witt) expansion, which is the expansion around $e^{-\tau \partial_a \partial^a} = e^{-\tau m_0^2}$.

We seek the answers of the following questions:

- Is $m_0 = i\Gamma^a \partial_a$ (the Dirac operator in the flat space) a classical solution? (If so, this model cancels the cosmological constant.)
- Which fields are massive and decoupled in the classical low-energy limit?

If this model is to reduce to the type IIB supergravity, only the following fields must remain massless:

* even-rank antisymmetric tensor $a^{(i)}_{ia_1 \cdots a_{2n}}(x)$
* dilatino $\chi(x)$, and gravitino $\chi^{(i)}(x)$

The computation is performed through the Campbell-Baker-Hausdorff (CBH) formula:

$$Tr(e^{-\tau D^2}) = \int d^d x \langle x | e^{-\tau D^2} | x \rangle$$

$$= Tr \left[ \exp \left( -\tau \partial_a \partial^a \right) \exp \left( -\tau (D^2 - \partial_a \partial^a) \right) e^{-\tau \partial_a \partial^a} \right]$$

$$= Tr \left[ \exp \left( Y + \frac{1}{2} [X, Y] + \frac{1}{12} [X + Y, [X + Y, -X]] + \frac{1}{12} [-X, [-X, X + Y]] + \cdots \right) e^{-X} \right]$$

$$= Tr \left[ \left( 1 + Y + \frac{1}{2} [X, Y] + \frac{1}{6} [X, [X, Y]] + \frac{1}{2} Y^2 + \frac{1}{8} [X, Y]^2 + \frac{1}{3} Y [X, Y] + \frac{1}{6} [X, Y] Y + \cdots \right) e^{-X} \right],$$

$$\langle x | e^{-X} | y \rangle = \frac{1}{(2\pi \tau)^{d/2}} \exp \left( -\frac{1}{4\tau} (x^a - y^a)(x^b - y^b) \eta_{ab} \right).$$
The Laplace transformation of $V(u)$:

$$V(u) = \int_0^\infty ds g(s)e^{-su}.$$  

Then, the bosonic part is expanded as

$$Tr[trV(m^2)] = \int_0^\infty ds g(s)Tr[tre^{-s\tau D^2}]$$

$$= \int \frac{d^d x}{(2\pi \tau)^{d/2}} \left( \sum_{k=-\infty}^{\infty} \left( \int_0^\infty ds g(s)s^{\frac{-d}{2}+k} \right) \tau^k \right) \frac{A_k(x)}{[(\text{length})^{-2k}]}.$$

If $m_0$ is to be a classical solution,

$\Rightarrow$ The linear terms of the fluctuation around $m_0$ should vanish.

- The linear terms of the derivatives vanish after integrating in the action:

$$\int d^d x (\partial_{j_1} \cdots \partial_{j_m} a_a^{(ai_1 \cdots i_{ml})}(x)) = 0.$$

- Only a scalar can constitute a Lorentz invariant linear term.

$\Rightarrow$ We focus on the following terms:

$$a_a^{(ai_1 \cdots i_{ml})}(x) \in \mathcal{A}_{-l}(x).$$

$$[(\text{length})^{2l}]$$

The coefficients $\mathcal{A}_0(x), \mathcal{A}_{-1}(x), \mathcal{A}_{-2}(x) \cdots$ must vanish.

Then, the cosmological constant $\int d^d x \frac{1}{(2\pi \tau)^{d/2}} e(x) \in \mathcal{A}_0(x)$ also vanishes.
Then, the following condition must be satisfied:

\[
\int_0^\infty ds g(s) s^{-\frac{d}{2} - n} = 0, \quad (n = 0, -1, -2, \ldots)
\]

\[
\Leftrightarrow \int_0^\infty du V(u) u^{\frac{d}{2} + n} = 0, \quad (n = -1, 0, 1, 2, \ldots).
\]

\[
\left( \int_0^\infty du V(u) u^{\alpha-1} = \int_0^\infty duds g(s)e^{-su} u^{\alpha-1} = \Gamma(\alpha) \int_0^\infty ds g(s)s^{-\alpha}. \right)
\]

\(V(u)\) is chosen as, for example,

\[
V_0(u) = \frac{\partial^{\frac{d}{2}-1}}{\partial u^{\frac{d}{2}-1}} \left( e^{-u^4} \sin u^4 \right).
\]

The model reduces to the Einstein gravity in the classical low-energy limit.

- The linear term of the vielbein \(a_a^{(a)}(x)\) vanishes.
- The cross terms \(a_a^{(a)}(x)a_b^{(bi_1\cdots i_k)}(x)\) also vanish, due to the general coordinate invariance.

\[
Tr[tr e^{-\tau D^2}] = \int d^d x \frac{32}{(2\pi \tau)^{\frac{d}{2}}} e(x) \frac{R(x)}{6} + \cdots
\]

\[\text{[\text{length}]}^{-2} \in \mathcal{A}_1(x)\]

\(V(u)\) must be chosen so that \(\mathcal{A}_1(x)\) survives in the action.
Which fields are massive or massless?

mass terms: $a_{a_1 \cdots a_{2n-1}(x)}^{(i_1 \cdots i_k)} a_{a_1 \cdots a_{2n-1}(x)}^{(j_1 \cdots j_l)} [\text{length}]^{-2+k+l} A_{1-\frac{k+l}{2}}(x)$

kinetic terms: $\partial_k a_{a_1 \cdots a_{2n-1}(x)}^{(i_1 \cdots i_k)} a_{a_1 \cdots a_{2n-1}(x)}^{(j_1 \cdots j_l)} [\text{length}]^{-4+k+l} \in A_{2-\frac{k+l}{2}}(x)$

- odd-rank antisymmetric tensor $a_{a_1 \cdots a_{2n-1}(x)}$: Mass terms $\in A_1(x)$, Kinetic terms $\in A_2(x)$. These fields are generically massive.

- even-rank anti-symmetric tensor $a^{(i)}_{ia_1 \cdots a_{2n}(x)}$: Mass terms $\in A_0(x)$, Kinetic terms $\in A_1(x)$. They may be massless ??

- Higher-spin fields: $a_{a_1 \cdots a_{2n-1}(x)}^{(i_1 \cdots i_k)} (k = 2, 3, \cdots)$: The mass terms and the kinetic terms are absent. No clue of whether they are massive.
The SUSY transformation of the model:

\[
\delta \psi = 2V'(m^2)\epsilon, \quad \delta \bar{\psi} = 2\bar{\epsilon}V'(m^2), \\
\delta m = \epsilon \bar{\psi} + \psi \bar{\epsilon}.
\]

SUSY invariance of the action

\[
\delta_\epsilon S = Tr \left[ tr \left( (2V'(m^2)m(\epsilon \bar{\psi} + \psi \bar{\epsilon})) + \bar{\psi}(\epsilon \bar{\psi} + \psi \bar{\epsilon})\psi \\
+2\bar{\psi}mV'(m^2)\epsilon + 2\bar{\epsilon}mV'(m^2)\psi \right) \right] = 0.
\]

Commutator of the SUSY transformation on shell:
In the following, we assume that the Taylor expansion of \( V(u) \) around \( u = 0 \) is possible.

\[
[\delta_\epsilon, \delta_\xi]m = 2[\xi \bar{\epsilon} - \epsilon \bar{\xi}, V'(m^2)], \\
[\delta_\epsilon, \delta_\xi]\psi = 2\psi \left( \bar{\epsilon}m \frac{V'(m^2) - V'(0)}{m^2} \xi - \bar{\xi}m \frac{V'(m^2) - V'(0)}{m^2} \epsilon \right).
\]

where we have utilized the equation of motion:

\[
\frac{\partial S}{\partial \bar{\psi}} = 2m\psi = 0, \quad \frac{\partial S}{\partial \psi} = 2\bar{\psi}m = 0.
\]
In order to see the structure of the $\mathcal{N} = 2$ SUSY, we separate the SUSY parameters into the hermitian and the antihermitian parts as

$$\epsilon = \epsilon_1 + i\epsilon_2, \ \xi = \xi_1 + i\xi_2,$$

($\xi_1, \xi_2, \epsilon_1, \epsilon_2$ are Majorana-Weyl fermions.)

The translation of the bosons is attributed to the quartic term in the Taylor expansion of $V(m) = \sum_{k=1}^{\infty} \frac{a_{2k}}{2k} m^{2k}$.

We assume that the SUSY parameters $\epsilon_{1,2}, \xi_{1,2}$ are $c$-numbers (proportional to the unit matrix $1_{N\times N}$).

$$[\delta_\epsilon, \delta_\xi] A_a = \frac{1}{16} tr([\delta_\epsilon, \delta_\xi] m \Gamma_a)$$

$$= \frac{1}{16} \sum_{k=2}^{\infty} a_{2k} tr(\xi \bar{\epsilon} m^{2k-2} \Gamma_a - \epsilon \bar{\xi} m^{2k-2} \Gamma_a$$

$$- m^{2k-2} \xi \bar{\epsilon} \Gamma_a + m^{2k-2} \epsilon \bar{\xi} \Gamma_a)$$

$$= \frac{a_4}{16} \sum_{k=2}^{\infty} a_{2k} (\bar{\xi} [m^{2k-2}, \Gamma_a] \epsilon - \bar{\epsilon} [m^{2k-2}, \Gamma_a] \xi) A_{b_1} A_{b_2} + \cdots$$

$$= \frac{a_4}{16} (\bar{\xi} \Gamma^i \epsilon - \bar{\epsilon} \Gamma^i \xi) [A_i, A_a] + \cdots$$

$$= \frac{a_4}{8} (\bar{\xi}_1 \Gamma^i \epsilon_1 + \bar{\xi}_2 \Gamma^i \epsilon_2) [A_i, A_a] + \cdots,$$

The field $a_a(x)$ receives the translation and the gauge transformation:

$$[A_i, A_a] = [i \partial_i + a_i(x), i \partial_a + a_a(x)] + \cdots$$

$$= \frac{i(\partial_i a_a(x))}{\text{translation}} - i(\partial_a a_i(x)) + [a_i(x), a_a(x)] + \cdots \frac{\text{gauge transformation}}{\text{translation}}.$$
However, the fermions do not receive the translation.

\[ [\delta_\epsilon, \delta_\xi] \psi = - \sum_{k=2}^{n} a_{2k} \psi (\bar{\xi} m^{2k-3} \epsilon - \bar{\epsilon} m^{2k-3} \xi) + \cdots \]
\[ = -a_4 (\bar{\xi} \Gamma^j \epsilon - \bar{\epsilon} \Gamma^j \xi) \psi A_j + \cdots \]
\[ = -2a_4 (\bar{\xi}_1 \Gamma^j \epsilon_1 + \bar{\xi}_2 \Gamma^j \epsilon_2) \psi A_j + \cdots . \]

We explore the term \( \psi A_i \) more carefully:

\[ \psi A_j = i \psi \partial_j + \cdots \]
\[ = \left( \chi(x) \partial_j + \sum_{l=1}^{\infty} \chi^{(i_1 \cdots i_l)}(x) \partial_{i_1} \cdots \partial_{i_l} \partial_j \right) e^{-(\tau D^2)^\alpha} + \cdots . \]

Therefore, each fermionic field is transformed as

\[ [\delta_\epsilon, \delta_\xi] \chi(x) = 0 + \cdots , \]
\[ [\delta_\epsilon, \delta_\xi] \chi^{(i_1 \cdots i_{l+1})}(x) = -2a_4 (\bar{\xi}_1 \Gamma^j \epsilon_1 + \bar{\xi}_2 \Gamma^j \epsilon_2) \chi^{(i_1 \cdots i_{l+1})}(x) \delta^{i_{l+1}j} + \cdots . \]

\((*) \cdots \) denotes the omission of the non-linear terms of the fields.

It is a future problem to surmount this difficulty.
4 Conclusion

- We have pursued the possibility for a matrix model to describe the gravitational interaction in the curved spacetime.

- We have identified the large $N$ matrices with the differential operators.

- In order to describe the local Lorentz invariance in a matrix model, the following two ideas are essential:
  
  * We have identified the higher-rank tensor fields with the spin connection.
  * $so(9, 1)$ Lorentz symmetry and the $u(N)$ gauge symmetry must be coupled.

- We have attempted to build a model which reduces to the type IIB supergravity in the low-energy limit:
  
  * We have elucidated that the bosonic part reduce to the Einstein gravity.
  * There are many problems for the supersymmetric model:
    $\mathcal{N} = 2$ SUSY, the mass of the fields $\cdots$.
**Scalars on \( S_1 \)**

\[
\begin{array}{ccc|c|c}
0 & \frac{1}{N} & \frac{2}{N} & N \times \frac{N}{2} & \frac{N}{2} \\
\hline
\frac{1}{N} & \frac{2}{N} & \frac{1}{N} & \frac{N}{2} & \frac{N}{2} \\
\end{array}
\]

(1) trivial bundle with the periodic condition \( f(1) = f(0) \).

(2) \( Z_2 \)-twisted bundle with the antiperiodic condition \( f(1) = -f(0) \).

**1) Trivial bundle:**

We first consider the trivial bundle with the periodic condition \( f(1) = f(0) \). We discretize the region \( 0 \leq x \leq 1 \) into small slices of spacing \( \epsilon = \frac{1}{N} \).

\[
\partial_x f \left( \frac{k}{N} \right) = \frac{1}{2} \left( \frac{f \left( \frac{k+1}{N} \right) - f \left( \frac{k}{N} \right)}{\epsilon} + \frac{f \left( \frac{k}{N} \right) - f \left( \frac{k-1}{N} \right)}{\epsilon} \right)
\]

\[
= \frac{N}{2} \left( f \left( \frac{k+1}{N} \right) - f \left( \frac{k-1}{N} \right) \right).
\]

\[\partial_x \rightarrow A = \frac{N}{2} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \\ \vdots \\ 1 & -1 & 0 \end{pmatrix}.\]

**2) \( Z_2 \)-twisted bundle**

Now, the periodic condition \( f(1) = -f(0) \) is imposed:

\[
\partial_x \rightarrow A = \frac{N}{2} \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \\ \vdots \\ -1 & -1 & 0 \end{pmatrix}.\]
$i_1, i_2, i_3$ are the neighbours of $i$.

$$\Delta \rightarrow K = \begin{pmatrix}
i_1 & i & i_2 & i_3 \\
\vdots & \vdots & \vdots & \vdots \\
i & \cdots & 1 & -3 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}.$$  

In the space of a large $N$ matrix, the differential operators over various manifolds are embedded.
Hausdorff’s moment problem

[Theorem] (Hausdorff) Let \( f(x) \) be a continuous function. If

\[
\int_0^1 dx f(x)x^n = 0,
\]

for \( n = 0, 1, 2, \cdots \), then \( f(x) = 0 \) for all \( x \in [0, 1] \).

However, this statement does not hold true if we replace \([0, 1]\) with \([0, \infty]\):

[Example] The continuous function

\[
h(x) = \exp(-x^4) \sin(x^4)
\]

satisfy \( \int_0^\infty dx h(x)x^n = 0 \) for all \( n = 0, 1, 2, \cdots \).

[Proof] We note that

\[
\int_0^\infty dy y^m e^{-ay} = m! a^{-m-1}
\]

for \( a = \exp(i\pi/4) = \frac{1+i}{\sqrt{2}} \) and \( m = 0, 1, 2, \cdots \). This is a real number when \( m - 3 \) is a multiple of 4.

Taking the imaginary part of the both hand sides, we obtain

\[
\int_0^\infty dy y^{4n+3} \sin\left(\frac{y}{\sqrt{2}}\right) \exp\left(-\frac{y}{\sqrt{2}}\right) = 0,
\]

for \( n = 0, 1, 2, \cdots \). We make a substitution \( x = \frac{y^4}{4} \) to obtain

\[
\int_0^\infty dx h(x)x^n = 0. \; (Q.E.D.)
\]
1. \([\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}] A_a = 0\), \([\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}] \psi = 0\).

The commutation relation for the bosons is obtained by comparing the following two paths:

\[
A_a \xrightarrow{\delta^{(1)}_{\epsilon_1}} A_a + i\epsilon_2 \Gamma_a \psi \xrightarrow{\delta^{(1)}_{\epsilon_2}} A_a + i(\bar{\epsilon}_1 + \bar{\epsilon}_2) \Gamma_a \psi - \frac{1}{2} \bar{\epsilon}_2 \Gamma_a [A_b, A_c] \Gamma^{bc} \epsilon_1,
\]

\[
A_a \xrightarrow{\delta^{(1)}_{\epsilon_1}} A_a + i\epsilon_1 \Gamma_a \psi \xrightarrow{\delta^{(1)}_{\epsilon_2}} A_a + i(\bar{\epsilon}_1 + \bar{\epsilon}_2) \Gamma_a \psi - \frac{1}{2} \bar{\epsilon}_1 \Gamma_a [A_b, A_c] \Gamma^{bc} \epsilon_2.
\]

Then, the commutator is

\[
[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}] A_a = -\frac{1}{2} \bar{\epsilon}_2 \Gamma_a [A_b, A_c] \Gamma^{bc} \epsilon_1 + \frac{1}{2} \bar{\epsilon}_1 \Gamma_a [A_b, A_c] \Gamma^{bc} \epsilon_2 = [A_a, 2\bar{\epsilon}_1 \Gamma^c \epsilon_2 A_c].
\]

On the other hand, the commutation relation for the fermions is obtained by

\[
\psi \xrightarrow{\delta^{(1)}_{\epsilon_1}} \psi + \frac{a}{2} [A_a, A_b] \Gamma^{ab} \epsilon_2 \xrightarrow{\delta^{(1)}_{\epsilon_2}} \psi + \frac{a}{2} [A_a, A_b] \Gamma^{ab} (\epsilon_1 + \epsilon_2) - [A_a, \bar{\epsilon}_1 \Gamma_b \psi] \Gamma^{ab} \epsilon_2,
\]

\[
\psi \xrightarrow{\delta^{(1)}_{\epsilon_1}} \psi + \frac{a}{2} [A_a, A_b] \Gamma^{ab} \epsilon_1 \xrightarrow{\delta^{(1)}_{\epsilon_2}} \psi + \frac{a}{2} [A_a, A_b] \Gamma^{ab} (\epsilon_1 + \epsilon_2) - [A_a, \bar{\epsilon}_2 \Gamma_b \psi] \Gamma^{ab} \epsilon_1.
\]

By using the formula of Fierz transformation

\[
\bar{\epsilon}_1 \Gamma_b \psi \Gamma^{ab} \epsilon_2 = (\bar{\epsilon}_1 \Gamma^a \epsilon_2) \psi - \frac{7}{16} (\bar{\epsilon}_1 \Gamma^c \epsilon_2) \Gamma_c \Gamma^a \psi
\]

\[
- \frac{1}{16 \times 5!} (\bar{\epsilon}_1 \Gamma^{c_1\cdots c_5} \epsilon_2) \Gamma_{c_1\cdots c_5} \Gamma^a \psi,
\]

and the equation of motion

\[
\frac{dS}{d\psi} = -\frac{1}{g^2} \Gamma^a [A_a, \psi] = 0,
\]

the commutator is computed on shell to be

\[
[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}] \psi = [\psi, 2\bar{\epsilon}_1 \Gamma^c \epsilon_2 A_c].
\]

These commutators are set to be zero by the gauge transformation.
2. \([\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]A_a = 0,\; [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]\psi = 0.\]

This is trivial because the inhomogeneous SUSY transformation is merely a translation of the fermions.

3. \([\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]A_a = -i\bar{\epsilon}\Gamma_a \xi,\; [\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]\psi = 0.\]

This can be proven by taking the difference of these two transformations:

\[
\begin{align*}
A_a &\xrightarrow{\delta^{(2)}_{\xi}} A_a \xrightarrow{\delta^{(1)}_{\epsilon}} A_a + i\bar{\epsilon}\Gamma_a \psi \\
A_a &\xrightarrow{\delta^{(1)}_{\epsilon}} A_a + i\bar{\epsilon}\Gamma_a \psi \xrightarrow{\delta^{(2)}_{\xi}} A_a + i\bar{\epsilon}\Gamma_a (\psi + \xi), \\
\psi &\xrightarrow{\delta^{(2)}_{\xi}} \psi + \xi \xrightarrow{\delta^{(1)}_{\epsilon}} \psi + \xi + \frac{a}{2}\Gamma^{ij}[A_a, A_b] \epsilon \\
\psi &\xrightarrow{\delta^{(1)}_{\epsilon}} \psi + \frac{a}{2}\Gamma^{ij}[A_a, A_b] \epsilon \xrightarrow{\delta^{(2)}_{\xi}} \psi + \xi + \frac{a}{2}\Gamma^{ij}[A_a, A_b] \epsilon.
\end{align*}
\]
Explicit computation of the Seeley de Witt coefficients

We consider the trace of the large $N$ matrices in terms of the heat kernel: The trace of the operators are expressed using the complete system as

$$Trm = \int d^Dx \langle x | m | x \rangle,$$

(1)

where the bracket $|x\rangle$ and $\langle x|$ satisfies $\sum_x |x\rangle \langle x| = 1$. However, it is difficult to consider the trace of a general operator, and we regard the operator as the sum of the Laplacian and the perturbation around it. This is a famous procedure, and the perturbation is expressed in terms of Seeley de Witt coefficient.

It is well known that the Green function is computed to be

$$\langle x | \exp \left( \tau g^{ij}(y) \frac{d}{dx^i} \frac{d}{dx^j} \right) | y \rangle = \frac{e(y)}{(2\pi \tau)^{\frac{d}{2}}} \exp \left( -\frac{(x - y)^i(x - y)^j g_{ij}(y)}{4\tau} \right).$$

(2)

We consider the general elliptic differential operator

$$D^2 = -\left( g^{ij}(x) \frac{d}{dx^i} \frac{d}{dx^j} + A^i(x) \frac{d}{dx^i} + B(x) \right).$$

(3)

And we are now interested in the trace

$$Tr \exp(-\tau D^2) = \int d^d x \langle x | \exp(-\tau D^2) | x \rangle.$$

(4)

To this end, we compute the following quantity utilizing the Campbell-Hausdorff formula:

$$\langle x | \exp(-\tau D^2) | y \rangle = \langle x | \exp(X + Y) | y \rangle,$$

where

$$X = \tau \left( g^{ij}(y) \frac{d}{dx^i} \frac{d}{dx^j} \right),$$

(5)

$$Y = \tau \left( g^{ij}(x) - g^{ij}(y) \right) \frac{d}{dx^i} \frac{d}{dx^j} + A^i(x) \frac{d}{dx^i} + B(x).$$

(6)

The Campbell-Hausdorff formula is

$$e^A e^B = \exp \left( A + B + \frac{1}{2} [A, B] + \frac{1}{12} ([A, [A, B]] + [B, [B, A]]) + \cdots \right).$$

(8)

Since we know that $\langle x | e^X | y \rangle = \frac{e(y)}{(2\pi \tau)^{\frac{d}{2}}} \exp \left( -\frac{1}{4\tau} (x - y)^i(x - y)^j g_{ij}(y) \right)$, the quantity in question is computed as

$$e^{X+Y} e^{-X} = \exp \left( Y + \frac{1}{2} [X, Y] + \frac{1}{12} (2 [X, [X, Y]] - [Y, [Y, X]]) + \cdots \right)$$

$$= \exp \left( Y + \frac{1}{2} [X, Y] + \frac{1}{12} (2 [X, [X, Y]] - [Y, [Y, X]]) + \cdots \right)$$

$$= 1 + Y + \frac{1}{2} [X, Y] + \frac{1}{6} [X, [X, Y]] + \frac{1}{12} [Y, [X, Y]] + \cdots$$

$$+ \frac{1}{2} (Y + \frac{1}{2} [X, Y] + \frac{1}{6} [X, [X, Y]] + \frac{1}{12} [Y, [X, Y]] + \cdots)^2 + \cdots$$

$$= 1 + Y + \frac{1}{2} [X, Y] + \frac{1}{6} [X, [X, Y]] + \frac{1}{2} Y^2 + \frac{1}{8} [X, Y]^2 + \frac{1}{3} Y [X, Y] + \frac{1}{6} [X, Y] Y + \cdots.$$
Before we enter the computation of the quantity \( \langle x | e^{X+Y} | y \rangle \), we summarize the formula of the differentiation of \( e^X \):

\[
\frac{de^X}{dx^i} = -\frac{1}{2\tau} (x - y)^j g_{ij}(y) e^X,
\]

\[
\frac{d^2 e^X}{dx^i dx^j} = \left( -\frac{1}{2\tau} g_{ii}(y) + \frac{1}{4\tau^2} (x - y)^{l_1} (x - y)^{l_2} g_{l_1 l_2}(y) \right) e^X,
\]

\[
\frac{d^3 e^X}{dx^i dx^j dx^k} = \left( \frac{1}{4\tau^2} (x - y)^{l_1} (x - y)^{l_2} g_{l_1 l_2}(y) + g_{ii}(y) g_{ij}(y) + g_{ij}(y) g_{ki}(y) \right) e^X,
\]

\[
= \frac{1}{8\tau^3} (x - y)^{l_1} (x - y)^{l_2} (x - y)^{l_3} g_{l_1 l_2 l_3}(y) e^X.
\]

(10)

**Computation of \( Y e^X \)**

We start with the computation of the easiest case:

\[
Y e^X = \tau \left( (g^{ij}(x) - g^{ij}(y)) \frac{d}{dx} \frac{d}{dx^i} + A^i(x) \frac{d}{dx^i} + B(x) \right) e^X
\]

\[
= \left( \tau B(x) - \frac{1}{2} A^i(x - y)^j g_{ij}(y) + (g^{ij}(x) - g^{ij}(y)) \left( -\frac{1}{2} g_{ij}(y) + \frac{1}{4\tau} (x - y)^{l_1} (x - y)^{l_2} g_{l_1 l_2}(y) \right) \right) e^X.
\]

(12)

Therefore, the trace is obtained by

\[
Tr(Y e^X) = \int d^4x \langle x | Y e^X | x \rangle = \int d^4x \frac{\tau e(x)}{(2\pi\tau)^{\frac{3}{2}}} B(x).
\]

**Computation of \( \frac{1}{2}[X, Y] e^X \)**

We next go on to a bit more complicated case, and we compute the operator \([X, Y]\) itself:

\[
[X, Y] = \tau^2 \left( g^{i_1 i_2}(y) \frac{d}{dx^{i_1}} \frac{d}{dx^{i_2}} + (g^{i_1 j_2}(x) - g^{i_1 j_2}(y)) \frac{d}{dx^{i_1}} A^j(x) \frac{d}{dx^{i_2}} + B(x) \right)
\]

\[
\]

\[
= \tau^2 \left( 2g^{i_1 i_2}(y) \frac{d^3}{dx^{i_1} dx^{i_2} dx^{j_2}} + g^{i_1 i_2}(y) \frac{d^2}{dx^{i_1} dx^{i_2}} \right) e^X.
\]

(13)
Therefore, the trace is computed to be, with the help of the formulae (10),

\[
Tr \left( \frac{1}{2} [X, Y] e^X \right) = \int d^4 x \langle x | \frac{1}{2} [X, Y] e^X | x \rangle = \int d^4 x \frac{e(x)}{(2\pi \tau)^{\frac{3}{2}}} \left\{ \tau \left( \frac{1}{4} g_{i_1 i_2}(x) g_{j_1 j_2}(x) \left( \frac{d^2 g_{i_1 j_2}(x)}{dx_{i_1} dx_{i_2}} - \frac{1}{2} \left( \frac{d^4 A_i(x)}{dx^i} \right) \right) + \frac{\tau^2}{2} g_{i_1 i_2}(x) \left( \frac{d^2 B(x)}{dx_{i_1} dx_{i_2}} \right) \right\}.
\]

(14)

Computations of \([X, [X, Y]] e^X\)

We compute the operator \([X, [X, Y]]\) as

\[
[X, [X, Y]] = \tau^3 \left( 4 g_{i_1 i_2}(y) g^{k_1 k_2}(y) \left( \frac{d^2 g_{i_1 j_2}(x)}{dx_{i_1} dx_{j_2}} \right) \frac{d^4}{dx_{i_1} dx_{j_2} dx_{j_1} dx_{j_2}} + g_{i_1 i_2}(y) g_{k_1 k_2}(y) \left( \frac{d^4 g_{i_1 j_2}(x)}{dx_{i_1} dx_{i_2} dx_{k_1} dx_{k_2}} \right) \frac{d^2}{dx_{i_1} dx_{i_2} dx_{k_1} dx_{k_2}} + 4 g_{i_1 i_2}(y) g^{k_1 k_2}(y) \left( \frac{d^2 A_i(x)}{dx_i} \right) \frac{d}{dx_j} + 4 g_{i_1 i_2}(y) g_{k_1 k_2}(y) \left( \frac{d^2 B(x)}{dx_{i_1} dx_{i_2}} \right) \frac{d}{dx_{j_1} dx_{j_2}} + 4 g_{i_1 i_2}(y) g_{k_1 k_2}(y) \left( \frac{d^2 B(x)}{dx_{i_1} dx_{i_2}} \right) \frac{d}{dx_{j_1} dx_{j_2}} + 4 g_{i_1 i_2}(y) g_{k_1 k_2}(y) \left( \frac{d^2 B(x)}{dx_{i_1} dx_{i_2}} \right) \frac{d}{dx_{j_1} dx_{j_2}} \right).
\]

(15)

Therefore, the trace is computed as

\[
Tr \left( \frac{1}{6} [X, [X, Y]] e^X \right) = \int d^4 x \langle x | \frac{1}{6} [X, [X, Y]] e^X | x \rangle = \int d^4 x \frac{e(x)}{(2\pi \tau)^{\frac{3}{2}}} \left\{ \tau \left( \frac{1}{6} g_{i_1 i_2}(x) g_{j_1 j_2}(x) \left( \frac{d^2 g_{i_1 j_2}(x)}{dx_{i_1} dx_{i_2}} \right) + \frac{1}{3} \left( \frac{d^4 A_i(x)}{dx^i} \right) \right) \right\}

- \tau^2 \left( \frac{1}{12} g_{i_1 i_2}(x) g_{j_1 j_2}(x) g^{k_1 k_2}(x) \left( \frac{d^4 g_{i_1 j_2}(x)}{dx_{i_1} dx_{i_2} dx_{j_1} dx_{j_2}} \right) + \frac{1}{3} g_{i_1 i_2}(x) \left( \frac{d^4 A_i(x)}{dx_{i_1} dx_{i_2}} \right) + \frac{1}{3} g_{i_1 i_2}(x) \left( \frac{d^4 B(x)}{dx_{i_1} dx_{i_2}} \right) \right) + \frac{\tau^3}{6} \left( g_{i_1 i_2}(x) g_{j_1 j_2}(x) \left( \frac{d^4 B(x)}{dx_{i_1} dx_{i_2} dx_{j_1} dx_{j_2}} \right) \right).
\]

(16)

Computations of \(\frac{1}{2} Y^2 e^X\)

The next step is the computation of the term \(\frac{1}{2} Y^2\):

\[
Y^2 = \left( \left( g_{i_1 i_2}(x) - g_{i_1 i_2}(y) \right) \frac{d^2}{dx_{i_1} dx_{i_2}} + A_i(x) \frac{d}{dx^i} + B(x) \right) \left( \left( g_{i_1 j_2}(x) - g_{i_1 j_2}(y) \right) \frac{d^2}{dx_{i_1} dx_{i_2}} + A_j(x) \frac{d}{dx^j} + B(x) \right) + \tau^2 \left( \left( g_{i_1 i_2}(x) - g_{i_1 i_2}(y) \right) \left( g_{i_1 j_2}(x) - g_{i_1 j_2}(y) \right) \frac{d^4}{dx_{i_1} dx_{i_2} dx_{j_1} dx_{j_2}} + \frac{d^3}{dx_{i_1} dx_{i_2} dx_{j_1} dx_{j_2}} + \left( g_{i_1 i_2}(x) - g_{i_1 i_2}(y) \right) \left( \frac{d^2 g_{i_1 j_2}(x)}{dx_{i_1} dx_{i_2}} \right) \frac{d^2}{dx_{j_1} dx_{j_2}} + \frac{d^2}{dx_{i_1} dx_{i_2}} + \frac{d^2}{dx_{j_1} dx_{j_2}} \right) + 2 \left( g_{i_1 i_2}(x) - g_{i_1 i_2}(y) \right) A_j(x) \frac{d^3}{dx_{i_1} dx_{i_2} dx^j} + 2 \left( g_{i_1 i_2}(x) - g_{i_1 i_2}(y) \right) \left( \frac{d^2 A_j(x)}{dx_{i_1} dx_{i_2}} \right) \frac{d^2}{dx_{j_1} dx_{j_2}} + 2 \left( g_{i_1 i_2}(x) - g_{i_1 i_2}(y) \right) \left( \frac{d^2 B(x)}{dx_{i_1} dx_{i_2}} \right) \frac{d^2}{dx_{j_1} dx_{j_2}}
\]

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\[ + (g^{i_{12}}(x) - g^{i_{12}}(y)) \frac{d^2 A_j^i(x)}{dx^i dx^j} + (g^{i_{12}}(x) - g^{i_{12}}(y))B(x) \frac{d^2}{dx^i dx^j} \]
\[ + 2(g^{i_{12}}(x) - g^{i_{12}}(y))(\frac{dB(x)}{dx^{i_{12}}} \frac{d}{dx^j} + (g^{i_{12}}(x) - g^{i_{12}}(y))(\frac{d^2 B(x)}{dx^{i_{12}} dx^j}) \]
\[ + A^i(x)(\frac{dg^{j_{12}}(x)}{dx^i}) \frac{d^2}{dx^j dx^j_2} + A^i(x)A^j(x) \frac{d^2}{dx_i dx^j} + A^i(x)B(x) \frac{d}{dx^i} + A^i(x)(\frac{dB(x)}{dx^i}) \]
\[ (g^{i_{12}}(x) - g^{i_{12}}(y))B(x) \frac{d^2}{dx^j dx^j_2} + B(x)A^i(x) \frac{d}{dx^i} + B(x)B(x) \]

The trace is thus

\[ \text{Tr}(\frac{1}{2} Y^2 e^X) = \int d^d x \langle x | \frac{1}{2} Y^2 e^X | x \rangle \]
\[ = \int d^d x \frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}} \left\{ \tau \left( -\frac{1}{4} A^i(x)g_{j_{12}}(x)(\frac{dg^{j_{12}}(x)}{dx^j}) - \frac{1}{4} A^i(x)A^j(x)g_{ij}(x) \right) \right. \]
\[ + \tau^2 \left( \frac{1}{2} A^i(x)(\frac{dB(x)}{dx^i}) + \frac{1}{2} B(x)B(x) \right) \} \].

Computation of \(\frac{1}{8}[X, Y]^2 e^X\)

We next compute the commutator \([X, Y]^2\), however, from now on, the computation becomes more complicated than before, and we give only the trace:

\[ \text{Tr}(\frac{1}{8}[X, Y]^2 e^X) = \int d^d x \langle x | \frac{1}{8}[X, Y]^2 e^X | x \rangle \]
\[ = \int d^d x \frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}} \left\{ \tau \left( -\frac{1}{16} g^{ik}(x)g_{j_{12}}(x)g_{l_{12}}(x)(\frac{dg^{j_{12}}(x)}{dx^i})(\frac{dg^{l_{12}}(x)}{dx^k}) \right) \right. \]
\[ - \frac{1}{8} g^{ik}(x)g_{j_{12}}(x)g_{l_{12}}(x)(\frac{dg^{j_{12}}(x)}{dx^i})(\frac{dg^{l_{12}}(x)}{dx^k}) - \frac{1}{4} (\frac{dg^{j_{12}}(x)}{dx^i})(\frac{dg^{l_{12}}(x)}{dx^k})g_{l_{12}}(x) \]
\[ - \frac{1}{4} g_{j_{12}}(x)(\frac{dg^{l_{12}}(x)}{dx^i})(\frac{dg^{j_{12}}(x)}{dx^l}) - \frac{1}{4} g_{ij}(x)(\frac{dg^{ip}(x)}{dx^p})(\frac{dg^{j_4}(x)}{dx^q}) \right\} + O(\tau^2) \].

Computation of \(\frac{1}{3} Y[X, Y] e^X\)

\[ \text{Tr}(\frac{1}{3} Y[X, Y] e^X) = \int d^d x \langle x | \frac{1}{3} Y[X, Y] e^X | x \rangle \]
\[ = \int d^d x \frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}} \left\{ \tau \left( \frac{1}{6} A^i(x)(\frac{dg^{j_{12}}(x)}{dx^i})g_{j_{12}}(x) + \frac{1}{3} A^i(x)g_{ij}(x)(\frac{dg^{j_{12}}(x)}{dx^j}) \right) \right. \]
\[ - \tau^2 \left( \frac{1}{6} g^{k_{12}}(x)g_{ij}(x)A^i(x)(\frac{d^2 A^j(x)}{dx^i dx^j}) + \frac{1}{3} A^i(x)(\frac{dB(x)}{dx^i}) \right) \]
\[ + \frac{1}{6} g^{k_{12}}(x)g_{j_{12}}(x)(\frac{d^2 g^{j_{12}}(x)}{dx^i dx^j})B(x) + \frac{1}{6} g^{k_{12}}(x)g_{j_{12}}(x)A^i(x)(\frac{d^3 g^{j_{12}}(x)}{dx^i dx^j dx^k}) \]
\[ + \frac{1}{3} A^i(x)(\frac{d^2 A^j(x)}{dx^i dx^j}) + \frac{1}{3} B(x)(\frac{d^2 A^i(x)}{dx^i}) \]
\[ + \frac{\tau^3}{3} B(x)g^{k_{12}}(x)(\frac{d^2 B(x)}{dx^i dx^j}) \} \].

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Computation of \( \frac{1}{6} [X, Y] Y e^X \)

\[
Tr\left( \frac{1}{6} [X, Y] Y e^X \right) = \int d^d x \langle x | \frac{1}{6} [X, Y] Y e^X | x \rangle \\
= \int d^d x \frac{e(x)}{(2\pi\tau)^{d/2}} \left\{ \tau \left( \frac{1}{12} g^{ki} g_{ki} (x) g_{j_1 j_2} (x) \frac{d g^{i_1 i_2} (x)}{d x^{i_1}} \frac{d g^{j_1 j_2} (x)}{d x^{j_2}} \right) \\
+ \frac{1}{6} g^{ki} g_{ki} (x) g_{j_1 j_2} (x) \frac{d g^{i_1 i_2} (x)}{d x^{i_1}} \frac{d g^{j_1 j_2} (x)}{d x^{j_2}} + \frac{1}{3} g_{j_1 j_2} (x) \frac{d g^{j_1 j_2} (x)}{d x^{j_1}} \frac{d g^{i_1 i_2} (x)}{d x^{i_1}} \\
+ \frac{1}{12} g_{j_1 j_2} (x) \frac{d g^{j_1 j_2} (x)}{d x^{j_1}} A^i (x) + \frac{1}{6} \left( \frac{d g^{j_1 j_2} (x)}{d x^{j_1}} g_{j_1 j_2} (x) A^i (x) \right) + O(\tau^2) \right\}. \quad (21)
\]

Seeley de Witt coefficient of the second lowest order

Now that we have computed all of the contribution of the Seeley de Witt coefficient of the order \( O(\tau^{1-\frac{d}{2}}) \), we sum all the results. Then, the trace is finally rewritten as

\[
Tr(e^{-\tau D^2}) = \int d^d x \langle x | e^{-\tau D^2} | x \rangle = \int d^d x \frac{e(x)}{(2\pi\tau)^{d/2}} (a_0 + \tau a_1 + \cdots). \quad (22)
\]

It goes without stating that the coefficient \( a_0 \) of the lowest order is \( a_0 = 1 \). Then, the subleading effect is

\[
\begin{align*}
A_1 (x) &= B (x) - \frac{1}{2} \frac{d A^i (x)}{d x^i} + \frac{1}{3} \frac{d^2 g^{ij} (x)}{d x^i d x^j} - \frac{1}{12} g^{i_1 i_2} (x) g_{j_1 j_2} (x) \frac{d^2 g^{j_1 j_2} (x)}{d x^{i_1} d x^{i_2}} \\
&+ \frac{1}{12} g_{j_1 j_2} (x) \frac{d g^{j_1 j_2} (x)}{d x^{j_1}} \frac{d g^{i_1 i_2} (x)}{d x^{i_2}} - \frac{1}{4} A^i (x) A^j (x) g_{ij} (x) \\
&+ \frac{1}{2} A^i (x) g_{ij} (x) \frac{d g^{j_1 j_2} (x)}{d x^{j_2}} \\
&+ \frac{1}{48} g^{ki} g_{ki} (x) g_{j_1 j_2} (x) \frac{d g^{i_1 i_2} (x)}{d x^{i_1}} \frac{d g^{j_1 j_2} (x)}{d x^{i_2}} \\
&+ \frac{1}{24} g^{ki} g_{ki} (x) g_{j_1 j_2} (x) \frac{d g^{i_1 i_2} (x)}{d x^{i_1}} \frac{d g^{j_1 j_2} (x)}{d x^{i_2}} \\
&- \frac{1}{12} g^{i_1 i_2} (x) \frac{d g^{i_1 i_2} (x)}{d x^{j_1}} \frac{d g^{j_1 j_2} (x)}{d x^{j_2}} - \frac{1}{4} g_{ij} (x) \frac{d g^{i_1 i_2} (x)}{d x^i} \frac{d g^{j_1 j_2} (x)}{d x^j}. \quad (23)
\end{align*}
\]

Consistency Check with respect to the covariant Laplace Beltrami operator

We now check the consistency of the result (23), by applying the above results to the covariant Laplace Beltrami operator

\[
\Delta (x) = \frac{1}{\sqrt{g (x)}} \left( \frac{d}{d x^i} \sqrt{g (x)} g^{ij} (x) \frac{d}{d x^j} \right) \\
= g^{ij} (x) \frac{d}{d x^i} \frac{d}{d x^j} + \left( \frac{d g^{ij} (x)}{d x^j} - \frac{1}{2} g^{ij} (x) \frac{d g^{kl} (x)}{d x^k} g_{ij} (x) \right) \frac{d}{d x^i}. \quad (24)
\]

where we have utilized the differentiation of the determinant

\[
\delta g (x) = g (x) g^{ij} (x) \delta g_{ij} (x) = -g (x) g_{ij} (x) \delta g^{ij} (x). \quad (25)
\]
Then, the problem corresponds to the case in which

\[
A^i(x) = \left( \frac{dg^{ij}(x)}{dx^j} - \frac{1}{2}g^{ij}(x)\left( \frac{d}{dx^j}g^{kl}(x) \right)g_{kl}(x) \right) B(x) = 0. \tag{26}
\]

In this case, we expect the coefficient \( a_1(x) \) to be

\[
\frac{R(x)}{6} = \frac{1}{6}g^{ij}(x)(-\partial_i\Gamma^k_{kj} + \partial_k\Gamma^i_{kj} - \Gamma^k_{ij}\Gamma^i_{kj} + \Gamma^k_{kj})
\]

\[
= \frac{1}{6}g^{ij}(x)g_{l_1l_2}(x)\left( \frac{d^2g^{l_1l_2}(x)}{dx^ixdx^j} \right) - \frac{1}{6}\left( \frac{d^2g^{l_1l_2}(x)}{dx^il_1dx^l_2} \right) + \frac{1}{6}\left( \frac{dg^{em}(x)}{dx^m} \right)\left( \frac{dg^{l_1l_2}(x)}{dx^e} \right)g_{l_1l_2}(x)
\]

\[
- \frac{5}{24}g^{ij}(x)g_{l_1m_1}(x)g_{l_2m_2}(x)\left( \frac{dg^{l_1l_2}(x)}{dx^i} \right)\left( \frac{dg^{m_1m_2}(x)}{dx^j} \right) + \frac{1}{12}g_{l_1l_2}(x)\left( \frac{dg^{m_2l_1}(x)}{dx^{m_1}} \right)\left( \frac{dg^{m_1l_2}(x)}{dx^{m_2}} \right)
\]

\[
- \frac{1}{24}g^{ij}(x)g_{l_1l_2}(x)g_{m_1m_2}(x)\left( \frac{dg^{l_1l_2}(x)}{dx^i} \right)\left( \frac{dg^{m_1m_2}(x)}{dx^j} \right)
\]  \tag{27}

as investigated in Di Francesco’s textbook.

And when we substitute (26) into the Seeley de Will coefficient \( a_1(x) \), we successfully obtain (27).