

Seiberg-Witten Geometry via Confining Phase Superpotential

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Abstract

We study Seiberg-Witten Geometry to describe the non-perturbative low-energy behavior of $N = 2$ supersymmetric gauge theories in four dimensions. The method of $N = 1$ confining phase superpotential is employed for this purpose. It is shown that the ALE space of type ADE fibered over \mathbf{CP}^1 is natural geometry for the $N = 2$ supersymmetric gauge theories with ADE gauge groups. Furthermore, we obtain in this approach previously unknown Seiberg-Witten geometry for four-dimensional $N = 2$ gauge theory with gauge group E_6 with massive fundamental hypermultiplets. By considering the gauge symmetry breaking in this E_6 gauge theory, we also obtain Seiberg-Witten geometries for $N = 2$ gauge theory with $SO(2N_c)$ ($N_c \leq 5$) with massive spinor and vector hypermultiplets. In a similar way the Seiberg-Witten geometry is determined for $N = 2$ $SU(N_c)$ ($N_c \leq 6$) gauge theory with massive antisymmetric and fundamental hypermultiplets.

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Chapter 1

Introduction

For almost 25 years four-dimensional supersymmetric gauge field theories have been investigated very intensively. One reason for this is that supersymmetric theories have a remarkable property of canceling out the divergence in the self-energies which is desirable to construct a more natural phenomenological model at high energies beyond the standard model. Furthermore a proposal of the unification of the gauge groups of the standard model seems to be more attractive by requiring the theory to have softly broken supersymmetry.

Supersymmetric gauge theories have also been considered as theoretical models to understand the strong coupling effects. These effects such as color confinement and chiral symmetry breaking are difficult to study analytically in the theories without the supersymmetry. On the other hand the action of the supersymmetric field theory is highly constrained by its supersymmetry and in some cases even the exact descriptions of the low-energy theories of these have been obtained on the basis of the idea of duality and holomorphy [1]-[4]. Consequently, the non-perturbative effects in the supersymmetric theory can be evaluated quantitatively.

Another reason for the importance of the study of supersymmetric gauge field theories is their close relation to the superstring theory. Superstring theories receive a lot of current research interest since they are the only known unified models including quantum gravity in a consistent manner and have enough gauge symmetries to contain the standard model. Moreover the superstring theory predicts the spacetime supersymmetry. Therefore supersymmetric gauge field theories naturally appear in the study of the superstring

theory.

In the superstring theory, the supersymmetric gauge field theories appear in two ways. A conventional way is to have a supersymmetric field theory on the lower dimensional spacetime after the ten-dimensional superstring is compactified. The other novel way is that supersymmetric gauge theories in various dimensions are realized on the world volume of D-branes which are higher dimensional objects on which the open strings can end. In the framework of the superstring theory, the gauge field theories with extended supersymmetry * are important because ten-dimensional superstring theories have more supercharges than lower-dimensional $N = 1$ (i.e. minimal) supersymmetric theory and some or all of these supercharges are unbroken if we compactify the superstring theory on a suitably chosen manifold.

In the case of $N = 2$ supersymmetry, a substantial progress was made by Seiberg and Witten [3, 4]. They have shown that the low-energy effective theory of the Coulomb phase of four-dimensional $N = 2$ supersymmetric $SU(2)$ gauge theory can be described by an auxiliary complex curve, called the Seiberg-Witten curve, whose shape depends on the vacuum moduli $u = \text{Tr } \Phi^2$. In this beautiful mathematical description, massless solitons are recognized as vanishing cycles associated with the degeneracy of the curves and their masses are obtained as the integral of certain one-form, which is called the Seiberg-Witten form, over these cycles. Soon after these works, generalizations to the other $N = 2$ supersymmetric gauge theory with the classical gauge groups have been carried out by several groups [5]-[12]. However all these generalizations are based on the assumption that auxiliary complex curves are of hyperelliptic type. Without this assumption, simple extensions of the original work [3, 4] are not promising to determine the curves. Thus it is desired to invent other methods for deriving the curve without the assumption on the types of curves.

To this end, we notice the fact that the singularity of quantum moduli space of the vacua of the theory corresponds to the appearance of massless solitons. Near the singularity, therefore, we observe interesting non-perturbative properties of the theories. Moreover

*Here the extended supersymmetric theory has more supercharges than minimal supersymmetric theory ($N = 1$ supersymmetry). For example, the $N = 2$ supersymmetry is two times as large as the $N = 1$ supersymmetry

the Seiberg-Witten curves are determined almost completely from the information of the locations of singularities on the moduli space. In order to explore physics near $N = 2$ singularities the microscopic superpotential explicitly breaking $N = 2$ to $N = 1$ supersymmetry is often considered [3, 4, 13, 14]. Examining the resulting superpotential for a low-energy effective Abelian theory it is found that the generic $N = 2$ vacuum is lifted and only the singular loci of moduli space remain as the $N = 1$ vacua where monopoles or dyons can condense. The resulting $N = 1$ theory is shown to be in the confining phase in accordance with the old idea of the confinement via the condensation of monopole.

This observation suggests that we may start with a microscopic $N = 1$ theory which we introduce by perturbing an $N = 2$ theory by adding a tree-level superpotential built out of the Casimirs of the adjoint field in the vector multiplet [13, 15, 16] toward the construction of the $N = 2$ curves. Let us concentrate on a phase with a single confined photon in our $N = 1$ theory which corresponds to the classical $SU(2) \times U(1)^{r-1}$ vacua with r being the rank of the gauge group. Then the low-energy effective theory containing non-perturbative effects provides us with the data of the vacua with massless solitons [17, 13]. From this we can identify the singular points in the Coulomb phase of $N = 2$ theories and construct the $N = 2$ Seiberg-Witten curves. This idea, called "confining phase superpotential technique", has been successfully applied to $N = 2$ supersymmetric $SU(N_c)$ pure Yang-Mills theory [15]. We extend their result to the case of $N = 2$ supersymmetric pure Yang-Mills theory with arbitrary classical gauge group [19] as well as $N = 2$ supersymmetric QCD [21] (see also [13]-[20]). The resulting curves are hyperelliptic type and agree with those of [7]-[12].

On the other hand, for exceptional gauge groups there were proposals based on the relation between Seiberg-Witten theory and the integrable systems that Seiberg-Witten curves are not realized by hyperelliptic curves [23, 24, 25]. In [23] it is claimed that the Seiberg-Witten curves for the $N = 2$ supersymmetric pure Yang-Mills theory with arbitrary simple gauge group are given by the spectral curves for the affine Toda lattice which has a form of a foliation over \mathbf{CP}^1 . For G_2 gauge group, this has been confirmed by the confining phase superpotential [26] and the one instanton calculation [27].

The application of the confining phase superpotential technique to the E_6, E_7, E_8 gauge groups seems to be difficult at first sight because of the complicated structure of the

E_n groups. Nonetheless we have shown that this technique can be applied in a unified way in determining the singularity structure of moduli space of the Coulomb phase in supersymmetric pure Yang-Mills theories with ADE gauge groups [28]. Not only the classical case of A_r, D_r groups but the exceptional case of E_6, E_7, E_8 groups can be treated on an equal footing since our discussion is based on the fundamental properties of the root system of the simply-laced Lie algebras. The resulting Riemann surface is described as a foliation over \mathbf{CP}^1 and satisfies the singularity conditions we have obtained from the $N = 1$ confining phase superpotential. This Riemann surface is not of hyperelliptic type for exceptional gauge groups.

In the consideration within the scope of four-dimensional field theory, it was unclear if the Riemann surface in the exact description is an auxiliary object for mathematical setup or a real physical object. It turns out that four-dimensional $N = 2$ gauge theory on \mathbf{R}^4 is realized in the type IIA superstring theory by an Neveu-Schwarz fivebrane on $\mathbf{R}^4 \times \Sigma$ where Σ is the Seiberg-Witten curve [29]. (This fivebrane description of the gauge theory is more transparent in view of 11 dimensional M theory [30].) The T-dual of this curved fivebrane configuration is obtained as type IIB superstring theory compactified on a Calabi-Yau three-fold which is a compact complex Kähler manifold of complex dimension three with vanishing first Chern class. Here we should take this Calabi-Yau three-fold to be a form of $K3$ fibration over \mathbf{CP}^1 with a certain limit which implies the decoupling of gravity. The singularities of $K3$, where some two-cycles get shrinked, are classified by the ADE singularity types and the gauge group of four-dimensional theory corresponds to these ADE singularities of $K3$. From the point of view of four-dimensional theory, this limiting Calabi-Yau three-fold is considered as a higher dimensional generalization of the auxiliary Seiberg-Witten curve and called Seiberg-Witten geometry. For ADE type gauge groups, this Seiberg-Witten geometry may be a more natural object than the curve since the curve depends on the representation of the gauge group, furthermore, there are the Seiberg-Witten geometries which are difficult to be reduced to the curve. Surprisingly it has been shown that this Seiberg-Witten geometry of the form of ADE singularity fibration over \mathbf{CP}^1 naturally appears in the framework of the confining phase superpotential [28, 31, 32] despite that this method has no relation to $K3$ or Calabi-Yau manifold at first sight.

Some extension to include matter hypermultiplets in representations other than the fundamentals can be also considered as the compactification on the Calabi-Yau three-fold [33, 34]. In his approach, however, only massless matters have been treated and the representation of matters are very restricted. On the other hand, the technique of confining phase superpotential can be also applied to supersymmetric theories with matter hypermultiplets and be used to investigate wider class of the theory. Indeed we have succeeded in deriving previously unknown Seiberg-Witten geometries for the $N = 2$ theory with E_6 gauge group with the massive fundamental hypermultiplets [31]. Moreover breaking the E_6 symmetry down to $SO(2N_c)$ ($N_c \leq 5$), we derive the Seiberg-Witten geometry for $N = 2$ $SO(2N_c)$ theory with massive spinor and vector hypermultiplets [32]. In the massless limit, our $SO(10)$ result is in complete agreement with the one obtained in [34]. Breaking of E_6 to $SU(N_c)$ ($N_c \leq 6$) is also considered in [32], and the Seiberg-Witten geometry for the $N = 2$ $SU(N_c)$ theory with antisymmetric matters have been obtained. The singularity structure exhibited by the complex curve obtained by M-theory fivebrane [35, 36] is realized in our result. This is regarded as non trivial evidence for the validity of our results.

As we have described so far the four-dimensional $N = 2$ supersymmetric gauge field theories have very rich physical content and their relation to the superstring theory renders them further interesting subjects to study. In particular the Seiberg-Witten geometry plays a very important role to control the dynamics of $N = 2$ theories. Our aim in this thesis is to understand the Seiberg-Witten geometry for various $N = 2$ supersymmetric theories in the systematic way. In particular, we study the Seiberg-Witten curve and Seiberg-Witten geometry of the $N = 2$ supersymmetric theory using the confining phase superpotential.

The organization of this thesis is as follows. In chapter two, we review the exact description of the low-energy effective theory of the Coulomb phase of four-dimensional $N = 2$ supersymmetric gauge theory in terms of the Seiberg-Witten curve or Seiberg-Witten geometry. In chapter three, we derive the Seiberg-Witten curves of $N = 2$ supersymmetric gauge theories by means of the $N = 1$ confining phase superpotential. In chapter four, we apply the confining phase superpotential method to the $N = 1$ supersymmetric pure Yang-Mills theory with an adjoint matter with classical or ADE gauge

groups. The results can be used to derive the Seiberg-Witten curves for $N = 2$ supersymmetric pure Yang-Mills theory with classical or ADE gauge groups in the form of a foliation over \mathbf{CP}^1 . Transferring the critical points in the $N = 2$ Coulomb phase to the $N = 1$ theories we find non-trivial $N = 1$ SCFT with the adjoint matter field governed by a superpotential. In chapter five, using the confining phase superpotential we determine the curves describing the Coulomb phase of $N = 2$ supersymmetric gauge theories with matter multiplets. For $N = 2$ supersymmetric QCD with classical gauge groups, our results recover the known curves. We also obtain previously unknown Seiberg-Witten geometry for four-dimensional $N = 2$ gauge theory with gauge group E_6 with massive fundamental hypermultiplets. By considering the gauge symmetry breaking in this E_6 gauge theory, we also obtain Seiberg-Witten geometries for $N = 2$ gauge theory with $SO(2N_c)$ ($N_c \leq 5$) with massive spinor and vector hypermultiplets. In a similar way the Seiberg-Witten geometry is determined for $N = 2$ $SU(N_c)$ ($N_c \leq 6$) gauge theory with massive antisymmetric and fundamental hypermultiplets. Finally, chapter six is devoted to our conclusions.

Chapter 2

Seiberg-Witten Geometry

In this chapter we review the exact description of the low-energy effective theory of the Coulomb phase of four-dimensional $N = 2$ supersymmetric gauge theory in terms of the Seiberg-Witten curve or the Seiberg-Witten geometry.

2.1 Seiberg-Witten curve

Let us consider $N = 2$ supersymmetric pure Yang-Mills theory with the gauge group G . This theory contains only an $N = 2$ vectormultiplet in the adjoint representation of G which consists of an $N = 1$ vector multiplet W_α and an $N = 1$ chiral multiplet Φ . The scalar field ϕ belonging to Φ has the potential

$$V(\phi) = \text{Tr}[\phi, \phi^\dagger]^2. \quad (2.1)$$

This is minimized by taking $\phi = \sum \phi_i H^i$, where H^i belongs to the Cartan subalgebra, and thus the classical vacua of this theory are degenerate and parametrized by the Casimirs built out from ϕ_i after being divided by the gauge transformation. The set of Casimirs is a gauge invariant coordinate of the space of inequivalent vacua which is called the moduli space.

The generic classical vacua of the theory have unbroken $U(1)^r$ gauge groups and are called the Coulomb phase where $r = \text{rank } G$. At the singularity of the classical moduli space of vacua, there appears a non-Abelian unbroken gauge group which implies that massless gauge bosons exist there. For the Abelian gauge group case, it is known from supersymmetry that the general low energy effective Lagrangian up to two derivatives is

completely determined by a holomorphic prepotential \mathcal{F} and must be of the form

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[\int d^4\theta K(\Phi, \bar{\Phi}) + \int d^2\theta \left(\frac{1}{2} \sum \tau(\Phi) W^\alpha W_\alpha \right) \right], \quad (2.2)$$

in the $N = 1$ superfield language. Here, $\Phi = \sum_{i=1}^r \Phi_i H^i$, and

$$K(\Phi, \bar{\Phi}) = \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi_i} \bar{\Phi}_i \quad (2.3)$$

is the Kähler potential which prescribes a supersymmetric non-linear σ -model for the field Φ , and

$$\tau(\Phi)_{ij} = \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi_i \partial \Phi_j}. \quad (2.4)$$

This Lagrangian (2.2) contains the terms $\text{Im}(\tau_{ij}) F_i \cdot F_j + \text{Re}(\tau_{ij}) F_i \cdot \tilde{F}_j$, from which we see that

$$\tau(\phi) \equiv \frac{\theta(\phi)}{2\pi} + \frac{4\pi i}{g^2(\phi)} \quad (2.5)$$

represents the complexified effective gauge coupling. Classically, $\mathcal{F}(\Phi) = \frac{1}{2} \tau_0 \text{Tr} \Phi^2$, where τ_0 is the bare coupling constant.

How is this classical moduli space of vacua modified by the quantum effects? Seiberg and Witten have proposed for the $SU(2)$ pure Yang-Mills theory on the basis of holomorphy and duality that the quantum moduli space of vacua is still parametrized by the Casimirs, but all the vacua have only $U(1)$ [3]. Although there are still singularities in the moduli space, the singularities in the quantum moduli space correspond to the appearance of the massless monopoles or dyons, not to the massless gauge bosons. Moreover it has been shown that the prepotential \mathcal{F} , in particular the coupling constant τ , and also the mass of the BPS saturated state are computed from the geometric data of the auxiliary complex curve, called the Seiberg-Witten curve, and a certain meromorphic one-form over it, called the Seiberg-Witten form λ_{SW} . Here the Seiberg-Witten curve is determined as the function over the moduli space of vacua. The Seiberg-Witten type solutions for other $N = 2$ theories with larger gauge groups and matters have been obtained in [4]-[12].

As an illustration of the basic idea of Seiberg-Witten, we briefly review the case of the $N = 2$ supersymmetric $SU(2)$ pure Yang-Mills theory. In this case, there are two singularities corresponding to the appearance of the massless monopole or dyon at $u =$

$\pm 2\Lambda^2$, where $u = \frac{1}{2}\text{Tr}\phi^2$ and Λ is the scale of the theory. Note that the classical singularity at the origin $u = 0$ disappears. The Seiberg-Witten curve is a torus and given by

$$y^2 = (x^2 - u)^2 - 4\Lambda^4 = (x^2 - u + 2\Lambda^2)(x^2 - u - 2\Lambda^2), \quad (2.6)$$

which is degenerate as $y^2 = x^2(x^2 \mp 4\Lambda^2)$ at the singular point $u = \pm 2\Lambda^2$. The Seiberg-Witten one-form takes the form

$$\lambda_{SW} = \frac{1}{\sqrt{2\pi}} x^2 \frac{dx}{y(x, u)}. \quad (2.7)$$

The mass of the BPS state which has electric charge p and magnetic charge q is given in terms of the integral of λ_{SW} over the canonical basis homology cycles of the torus α, β as

$$m = |pa + qa_D|, \quad (2.8)$$

where the period integrals

$$a(u) = \oint_{\alpha} \lambda_{SW}, \quad (2.9)$$

$$a_D(u) = \oint_{\beta} \lambda_{SW} \quad (2.10)$$

are associated with the chiral superfields belonging to the electric $U(1)$ multiplet and its dual magnetic $U(1)$ multiplet respectively. The coupling constant of the low-energy theory is identified with the period matrix of this torus which is written as

$$\tau = \frac{\partial a_D(u)}{\partial a(u)} \quad (2.11)$$

which has the required properties $\text{Im}(\tau) > 0$.

The Seiberg-Witten curves for the other classical gauge groups are also proposed and verified by the one instanton calculation. One for the $N = 2$ $SU(N_c)$ gauge theory is

$$y^2 = P(x)^2 - 4A(x), \quad (2.12)$$

where $P(x) = \langle \det(x - \Phi) \rangle$ is the characteristic equation of Φ which is chosen as the $N_c \times N_c$ matrix of the fundamental representation. For the $N = 2$ $SU(N_c)$ pure Yang-Mills theory, $A(x) \equiv \Lambda^{2N_c}$ [5, 6] and for the $N = 2$ $SU(N_c)$ theory with N_f fundamental flavors (QCD) [9, 10],

$$A(x) \equiv \Lambda^{2N_c - N_f} \det_{N_f}(x + m), \quad (2.13)$$

where m is the $N_f \times N_f$ mass matrix of the fundamental flavors.

The Seiberg-Witten curves for $N = 2$ $SO(2N_c)$ gauge theory read [8]

$$y^2 = P(x)^2 - 4x^2 A(x), \quad (2.14)$$

where $P(x) = \langle \det(x - \Phi) \rangle = P(-x)$ is the characteristic equation of Φ which is chosen as the $2N_c \times 2N_c$ matrix of the fundamental representation. Here for the pure Yang-Mills case $A(x) \equiv \Lambda^{4(N_c-1)}$ and for the QCD case

$$A(x) \equiv \Lambda^{4(N_c-1)-2N_f} \det_{2N_f}(x + m) = A(-x). \quad (2.15)$$

For $SO(2N_c + 1)$ gauge groups, the curves are

$$y^2 = \left(\frac{1}{x}P(x)\right)^2 - 4x^2 A(x), \quad (2.16)$$

with $A(x) \equiv \Lambda^{2(2N_c-1)}$ for the pure Yang-Mills theory [7] and

$$A(x) \equiv \Lambda^{2(2N_c-1)-N_f} \det_{2N_f}(x + m) = A(-x), \quad (2.17)$$

for QCD [11, 12].

The curves for $Sp(2N_c)$ theory are slightly different from the ones for the other gauge groups. They are given by

$$x^2 y^2 = \left(x^2 P(x) + 2B(x)\right)^2 - 4A(x), \quad (2.18)$$

with $B = \Lambda^{2N_c+2}$ and $A(x) \equiv \Lambda^{2(2N_c+2)}$ for the pure Yang-Mills theory [11], whereas

$$B(x) = \Lambda^{2N_c+2-N_f} \text{Pfm} \quad (2.19)$$

and

$$A(x) \equiv \Lambda^{2(2N_c+2)-N_f} \det_{2N_f}(x + m) = A(-x) \quad (2.20)$$

for QCD [11].

There is an interesting connection between the four-dimensional $N = 2$ pure Yang-Mills theory and the integrable systems. The connection is that the Seiberg-Witten curve for the $N = 2$ pure Yang-Mills theory with the gauge group G is identified with the

spectral curve for the periodic Toda theory for the group G [23]. Moreover the Seiberg-Witten form and relevant one-cycles can be also read from the spectral curve. What we want to emphasize here is that this correspondence is true for the arbitrary simple groups, especially for the exceptional groups. However, as we will see just below, for the exceptional gauge group case the Seiberg-Witten curve is not of hyperelliptic type. Introducing the characteristic polynomial in x of order $\dim \mathcal{R}$

$$P_{\mathcal{R}}(x, u_k) = \det(x - \Phi_{\mathcal{R}}), \quad (2.21)$$

where \mathcal{R} is an arbitrary representation of G , the spectral curve is given by

$$\tilde{P}_{\mathcal{R}}(x, z, u_k) \equiv P_{\mathcal{R}}\left(x, u_k + \delta_{k,r}\left(z + \frac{\mu}{z}\right)\right) = 0, \quad (2.22)$$

which has a form of a foliation over \mathbf{CP}^1 . Here $\Phi_{\mathcal{R}}$ is a representation matrix of \mathcal{R} and u_k are Casimirs built out of $\Phi_{\mathcal{R}}$. If we choose \mathcal{R} as a large representation of G , however, the genus of the curve is larger than the rank of G . This means that we should suitably choose $2r$ cycles to define a and a_D since the unbroken gauge group is $U(1)^r$. In particular, for the exceptional gauge groups, the $\dim \mathcal{R}$ is always much larger than r . Although this problem is solved just in terms of the integrable system, it seems somewhat unnatural and we expect that there exists a more transparent formulation. Indeed the generalization of the Seiberg-Witten curve to the complex dimension three manifold, which is called Seiberg-Witten geometry, is motivated by the string theory and is recognized to provide us with a desired formulation. This description is equivalent to the one using the curve for the theory considered in this section and more interestingly available to the $N = 2$ exceptional gauge theory with the matter flavors and $N = 2$ classical gauge theory with matter flavors in the non fundamental representation. They have not been described in term of the curve so far. We will discuss this generalization in the following sections.

2.2 Seiberg-Witten geometry

To see how the Seiberg-Witten geometry arises from the string theory, we first consider the $E_8 \times E_8$ heterotic string theory on $K3 \times T^2$. In the low energy region, this theory becomes effectively four dimensional $N = 2$ supersymmetric theory with possibly non-Abelian gauge bosons and gravitons. To obtain the four dimensional non-Abelian gauge

field theory without gravity, we should take the limit $\alpha' \rightarrow 0$ and simultaneously the weak string coupling limit as

$$\frac{1}{g_{het}^2} = -b \log(\sqrt{\alpha'} \Lambda) \rightarrow \infty, \quad (2.23)$$

where g_{het} is a coupling constant of the heterotic string theory, which is considered as the four dimensional gauge coupling at the plank scale $\alpha'^{-\frac{1}{2}}$ and b is the coefficient of the one loop beta function of the gauge field. The condition (2.23) is required to make the dynamical scale of the non-Abelian gauge theory Λ fixed at a finite value. Although in this setting we can obtain the four dimensional $N = 2$ supersymmetric non-Abelian gauge field theory, it is still difficult to compute the prepotential \mathcal{F} of the Coulomb phase of the theory if the coupling constant g_{het} (more precisely Λ) is not small.

Fortunately there is a duality between the heterotic string theory on $K3 \times T^2$ and the type IIA string theory on a Calabi-Yau three-fold X_3 [37, 38]. What is important is that the type IIA dilaton, whose expectation value is the type IIA string coupling, is in hypermultiplet. Therefore in the type IIA side the exact moduli space of the Coulomb phase can be determined from classical computation. Here we have used the fact that the $N = 2$ supersymmetry prevents couplings between neutral vector and hypermultiplets in the low energy effective action [39]. Note that the heterotic string coupling constant is converted to the geometrical data, Kähler structure moduli. Since the Kähler structure moduli is corrected by the string world sheet instantons, the type IIA description is not sufficiently simple to deal with. Remember here that the mirror symmetry maps the type IIA superstring on X_3 to a type IIB superstring on the mirror Calabi-Yau three-fold \tilde{X}_3 with interchanging the Kähler structure moduli and the complex structure moduli. Thus in this type IIB description classical string sigma model answer for the original vector moduli space is already the full exact result. This implies that the Seiberg-Witten geometry for the gauge field theory is identified with the compactification manifold \tilde{X}_3 . The Calabi-Yau three-fold has the canonical holomorphic three-form Ω and a, a_D are obtained as the integration of Ω over the three-cycles $\Gamma_{\alpha_I}, \Gamma_{\beta^J}, I, J = 1, \dots, h_{11}(\tilde{X}_3) + 1$, which span a integral symplectic basis of H_3 , with the α -type of cycles being dual to the β -type of cycles,

$$a_i = \int_{\Gamma_{\alpha_i}} \Omega, \quad a_{Dj} = \int_{\Gamma_{\beta^j}} \Omega. \quad (2.24)$$

Here i, j runs from one to the rank of the gauge group of the heterotic string theory. The other cycles is not relevant in the field theory limit since its integration diverges in the limit $\alpha' \rightarrow 0$.

To obtain the Seiberg-Witten geometry, we should take the limit $\alpha' \rightarrow 0$ of \tilde{X}_3 . To this end, we introduce the asymptotically local Euclidean space (ALE space) $W_{ADE}(x_i) = 0$ with ADE singularity at the origin. Here the polynomial $W_{ADE}(x_i)$ is given as follows

$$W_{A_r}(x_1, x_2, x_3; v) = x_1^{r+1} + x_2 x_3 + v_2 x_1^{r-1} + v_3 x_1^{r-2} + \cdots + v_r x_1 + v_{r+1}, \quad (2.25)$$

$$W_{D_r}(x_1, x_2, x_3; v) = x_1^{r-1} + x_1 x_2^2 - x_3^2 + v_2 x_1^{r-2} + v_4 x_1^{r-3} + \cdots + v_{2(r-2)} x_1 + v_{2(r-1)} + v_r x_2, \quad (2.26)$$

$$W_{E_6}(x_1, x_2, x_3; w) = x_1^4 + x_2^3 + x_3^2 + w_2 x_1^2 x_2 + w_5 x_1 x_2 + w_6 x_1^2 + w_8 x_2 + w_9 x_1 + w_{12}, \quad (2.27)$$

$$W_{E_7}(x_1, x_2, x_3; w) = x_1^3 + x_1 x_2^3 + x_3^2 - w_2 x_1^2 x_2 - w_6 x_1^2 - w_8 x_1 x_2 - w_{10} x_2^2 - w_{12} x_1 - w_{14} x_2 - w_{18}, \quad (2.28)$$

$$W_{E_8}(x_1, x_2, x_3; w) = x_1^3 + x_2^5 + x_3^2 - w_2 x_1 x_2^3 - w_8 x_1 x_2^2 - w_{12} x_2^3 - w_{14} x_1 x_2 - w_{18} x_2^2 - w_{24} x_2 - w_{30}, \quad (2.29)$$

where v_k and w_k correspond to the degree k Casimirs which resolve the singularity at the origin. Then the mirror Calabi-Yau three-fold \tilde{X}_3 for the $N = 2$ pure Yang-Mills theory is written as

$$W_{\tilde{X}_3}(x_j, z; w_k) = \epsilon \left(z + \frac{\Lambda^{2h}}{z} + W_{ADE}(x_j, w_k) \right) + o(\epsilon^2) = 0, \quad (2.30)$$

where $\epsilon = \alpha'^{\frac{h}{2}}$ and the gauge group is represented by the ADE singularity. Here h is the dual Coxeter number for the ADE Lie algebra. Therefore the Seiberg-Witten geometry for the $N = 2$ pure Yang-Mills theory is obtained as

$$z + \frac{\Lambda^{2h}}{z} + W_{ADE}(x_j, w_k) = 0. \quad (2.31)$$

It is relatively easier for the $SU(N_c)$ gauge group case to see the equivalence of the description using this Seiberg-Witten geometry and the Seiberg-Witten curve [29]. From the fact that the variables x_2, x_3 are both quadratic in $W_{A_{N_c-1}}$, it was shown that these

variables can be "integrated out" from $W_{A_{N_c-1}}$ [29]. Then changing the coordinate $y = -2z + P$, we see that the curve (2.12) is equivalent to the corresponding Seiberg-Witten geometry. For $SO(2N_c)$ case almost the same procedure can be applied, while for E_n case the Seiberg-Witten geometry (2.31) does not resemble to the curve (2.22). This problem is solved by finding a certain transformation of (2.31) to get(2.22) [24, 25].

Chapter 3

Confining Phase Superpotential

In this chapter, we will apply the confining phase superpotential technique to the $N = 2$ supersymmetric pure Yang-Mills theories. A simplest example of the application of this is the $N = 2$ $SU(2)$ gauge theory [13]. We will see that only for this case the confining phase superpotential technique is exact and for other cases this technique is applicable under a mild assumption.

3.1 Simplest example: $SU(2)$ gauge theory

For the $SU(2)$ gauge group, we take a tree-level superpotential $W = mu$, where $u = \frac{1}{2}\text{Tr } \Phi^2$ and m is a mass parameter of the adjoint chiral superfield Φ . If m is very small, we can consider this theory as the $N = 2$ supersymmetric $SU(2)$ pure Yang-Mills theory perturbed by the $N = 1$ small mass term W . The exact low-energy theory of the $N = 2$ theory near the massless monopole singularity has a $U(1)$ vector multiplet and a monopole hypermultiplet with a superpotential determined by the requirement of $N = 2$ supersymmetry

$$W_{N=2}^{eff} = A_D \tilde{M} M, \quad (3.1)$$

where A_D is the dual $U(1)$ vector multiplet and the \tilde{M}, M are monopole hypermultiplet [3]. Note that the bosonic part of A_D is a_D and its VEV determines the mass of the monopole. Thus the equation of motion, which should be satisfied for a supersymmetric ground state, of the theory perturbed by W becomes

$$0 = \frac{\partial W^{eff}}{\partial M} = A_D \tilde{M}, \quad (3.2)$$

$$0 = \frac{\partial W^{eff}}{\partial \tilde{M}} = A_D M, \quad (3.3)$$

$$0 = \frac{\partial W^{eff}}{\partial u} = \frac{\partial A_D}{\partial u} \tilde{M} M + m, \quad (3.4)$$

where $W^{eff} = W_{N=2}^{eff} + W$. The equations (3.2), (3.3) may be reduced to $0 = \langle A \rangle = \langle A_D \rangle$, which means that only the $N = 2$ vacuum where the monopole becomes massless remains as $N = 1$ vacuum. From the equation (3.3), we see that there is a non-zero monopole condensation $\langle \tilde{M} M \rangle = -m / \frac{\partial A_D}{\partial u}$. The non zero monopole condensation is regarded as the source of confinement.

On the other hand, if mass m is very large, then we can integrate out the adjoint chiral superfield Φ and low-energy effective theory becomes the $N = 1$ supersymmetric $SU(2)$ pure Yang-Mills theory which is believed to be in the confining phase. The relation between the high-energy scale Λ and the low-energy scale Λ_L is determined by matching the scale at the adjoint mass m as

$$\Lambda^{2 \cdot 2} = \Lambda_L^{3 \cdot 2} (m)^{-2}. \quad (3.5)$$

Since the gaugino condensation dynamically generates the superpotential in the $N = 1$ $SU(2)$ theory the low-energy effective superpotential takes the form

$$W_L = \pm 2m\Lambda^2. \quad (3.6)$$

Although this effective superpotential is evaluated in the region of large m , it is shown that (3.6) is exact for all values of m [13] by virtue of holomorphy, symmetry and asymptotic dependence on the parameter of the theory [1]. Thus the relation $\langle u \rangle = \partial W_L / \partial m = \pm 2\Lambda^2$ holds exactly. Finally taking the $N = 2$ limit $m \rightarrow 0$, we obtain the correct singularities of the moduli space of the $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory at $u = \pm 2\Lambda^2$.

The $N = 2$ supersymmetric $SU(2)$ QCD, which has fundamental hypermultiplets, has been studied in an analogous way and shown to yields the known singularity structure of the curve [15, 16].

3.2 Outline of confining phase superpotential

In this section, we generalize the above method to the case of other gauge groups. Let us consider the low energy theory for a generic vacuum in the Coulomb phase of $N =$

2 supersymmetric gauge theory with the gauge group G . Generically the low energy behavior of this theory is described by $N = 2$ supersymmetric $U(1)^{\text{rank}G}$ pure Yang-Mills theory. As in the previous section, we add a tree level superpotential $W = \sum_k g_k u_k$, where u_k are the Casimirs built out of the adjoint chiral superfield Φ , to this $N = 2$ theory. According to the technique called the confining phase superpotential [18], we concentrate on investigating the vicinity of a singular point of the $N = 2$ moduli space of vacua where a single monopole or dyon becomes massless. The low energy $N = 1$ theory has a superpotential which is approximately given by

$$W^{eff} = A_D(u_k) \tilde{M} M + \sum_k g_k u_k, \quad (3.7)$$

as in the $SU(2)$ case. The equation of motion of this perturbed theory becomes

$$A_D(u_k) = 0, \quad (3.8)$$

$$\frac{\partial A_D(u_k)}{\partial u_k} \tilde{M} M = -g_k. \quad (3.9)$$

It is important in this equation that only the $N = 2$ vacua with, at least, a single massless monopole or dyon remain as the $N = 1$ vacua. In these $N = 1$ vacua, the monopole or dyon can condense so as to confine a single $U(1)$ photon.

Conversely, we can start with a microscopic $N = 1$ gauge theory which is obtained from an $N = 2$ gauge theory perturbed by W . If we can calculate the low energy effective superpotential as the function of the scale Λ of the original theory and g_i , then by taking the $N = 2$ limit $g_i \rightarrow 0$ we can find the location of the singularity in the moduli space of the $N = 2$ theory.

Let us consider $N = 2$ $SU(3)$ Yang-Mills theory as an illustration of the method. Perturbing by $W = mu + gv$, where $u = \frac{1}{2}\text{Tr}\Phi^2$ and $v = \frac{1}{3}\text{Tr}\Phi^3$, leads to classical vacua with $\Phi = 0$, in which $SU(3)$ is unbroken, and

$$\Phi = \text{diag} \left(\frac{m}{g}, \frac{m}{g}, -2\frac{m}{g} \right), \quad (3.10)$$

in which there is a classically unbroken $SU(2) \times U(1)$. We focus on the vacuum with unbroken $SU(2) \times U(1)$ gauge group. In the semiclassical approximation, the low-energy theory for this vacuum consists of the $N = 1$ $SU(2)$ Yang-Mills theory with a superpotential \tilde{W} and a decoupled $N = 2$ $U(1)$ Yang-Mills theory. (This $U(1)$ theory is free and we

can ignore it in the following consideration.) The scale $\tilde{\Lambda}$ of this $SU(2)$ theory is related to the high-energy $SU(3)$ scale Λ by

$$\tilde{\Lambda}^{2\cdot 2} = \left(\frac{3m}{g}\right)^{-2} \Lambda^{2\cdot 3}, \quad (3.11)$$

which is obtained by matching the $SU(3)$ scale to the $SU(2)$ scale at the scale $(m/g) - (-2m/g) = (3m/g)$ of the W bosons which become massive by the Higgs effect. The superpotential \tilde{W} may be evaluated as

$$\tilde{W} = \frac{1}{2}W''(m/g) \text{Tr}\Phi_{SU(2)}^2 + \frac{1}{3\cdot 2}W'''(m/g) \text{Tr}\Phi_{SU(2)}^3 = \frac{3m}{2}\text{Tr}\Phi_{SU(2)}^2 + \frac{g}{3}\text{Tr}\Phi_{SU(2)}^3, \quad (3.12)$$

where $W(x) = \frac{m}{2}x^2 + \frac{g}{3}x^3$ and $\Phi_{SU(2)}$ is an unbroken $SU(2)$ part of Φ . Note that in \tilde{W} we suppress the terms which are not relevant to the $SU(2)$ theory. Therefore the adjoint chiral superfield $\Phi_{SU(2)}$ has a mass $3m$ and can be integrated out. We are then left with an $N = 1$ $SU(2)$ pure Yang-Mills theory with a scale Λ_L which is related to the scale Λ by

$$\Lambda_L^{3\cdot 2} = (3m)^2 \tilde{\Lambda}^{2\cdot 2} = g^2 \Lambda^6. \quad (3.13)$$

Since the gaugino condensation dynamically generates the superpotential in the $N = 1$ $SU(2)$ pure Yang-Mills theory the low-energy effective superpotential finally takes the form

$$W_L = \frac{m^3}{g^2} \pm 2\Lambda_L^3 = \frac{m^3}{g^2} \pm 2g\Lambda^3, \quad (3.14)$$

where the first term is the tree level term W evaluated for $\Phi = \text{diag}(m/g, m/g, -2m/g)$. We note that to obtain (3.14) we should integrate out all the fields in the original theory then no dynamical fields are remained.

The superpotential (3.14) is certainly correct in the limit $m \gg \Lambda$ and $m/g \gg \Lambda$, where the original theory is broken to our low energy theory at a very high scale. In the case of (3.14), however, we can not directly rule out additive corrections of the form $W_\Delta = \sum_{n=1}^{\infty} a_n (m^3/g^2)(g\Lambda/m)^{6n}$. We will simply assume that (3.14) is exact for all values of the parameters [18]. This assumption is referred to as the assumption of vanishing W_Δ [17]. We will see in the following that this assumption is correct at least for the theory we have investigated. However there is a subtle point concerned with the choice of the basis of the Casimirs of the gauge group. This point is discussed later. It will be seen

also that the statement that $W_\Delta = 0$ seems to reflect the absence of mixing of various classical vacua like θ vacua in QCD.

Once assuming (3.14) is exact, we obtain

$$\langle u \rangle = \frac{\partial W_L}{\partial m} = 3 \left(\frac{m}{g} \right), \quad (3.15)$$

$$\langle v \rangle = \frac{\partial W_L}{\partial g} = -2 \left(\frac{m}{g} \right)^3 \pm 2\Lambda^3. \quad (3.16)$$

In the $N = 2$ limit $m, g \rightarrow 0$, these two vacua of the perturbed theory must lie on the singularities of the moduli space of the Coulomb phase of the $N = 2$ theory since in this limit the vacuum condition (3.9) is valid. Therefore the vacua (3.16) must parameterize the singularities of the Seiberg-Witten curve $y^2 = (x^3 - xu - v)^2 - 4\Lambda^6$ for the $N = 2$ $SU(3)$ pure Yang-Mills theory. The singularities of the curve are indicated by the discriminant locus

$$\Delta_{SU(3)} = 4u^3 - 27v^2 - 108\Lambda^6 \mp 108v\Lambda^3 = 0. \quad (3.17)$$

Indeed, if we eliminate m/g from (3.16) then we obtain $\Delta_{SU(3)} = 0$. We have thus confirmed that the proposed Seiberg-Witten curve for $SU(3)$ pure Yang-Mills theory is correct using the confining phase superpotential. Note that the parameter of the singularities of the $N = 2$ moduli space corresponds to the ratio m/g .

In the following chapters, we will apply this confining phase superpotential technique to various $N = 2$ supersymmetric gauge theories in order to verify the proposed Seiberg-Witten geometries or derive the new Seiberg-Witten geometries if they are unknown.

Chapter 4

$N = 2$ Pure Yang-Mills Theory

4.1 Classical gauge groups

Now we apply the confining phase superpotential method to $N = 2$ supersymmetric pure Yang-Mills theories with classical gauge groups.

First we begin with the $SU(N_c)$ gauge theory [18]. The gauge symmetry breaks down to $U(1)^{N_c-1}$ in the Coulomb phase of $N = 2$ $SU(N_c)$ Yang-Mills theories. Near the singularity of a single massless dyon we have a photon coupled to the light dyon hypermultiplet while the photons for the rest $U(1)^{N_c-2}$ factors remain free. We now perturb the theory by adding a tree-level superpotential

$$W = \sum_{n=1}^{N_c} g_n u_n, \quad u_n = \frac{1}{n} \text{Tr } \Phi^n, \quad (4.1)$$

where Φ is the adjoint $N = 1$ superfield in the $N = 2$ vectormultiplet and g_1 is an auxiliary field implementing $\text{Tr } \Phi = 0$. In view of the macroscopic theory, we see that under the perturbation by (4.1) only the $N = 2$ singular loci survive as the $N = 1$ vacua where a single photon is confined and the $U(1)^{N_c-2}$ factors decouple.

The result should be directly recovered when we start with the microscopic $N = 1$ $SU(N_c)$ gauge theory which is obtained from $N = 2$ $SU(N_c)$ Yang-Mills theory perturbed by (4.1). For this we study the vacuum with unbroken $SU(2) \times U(1)^{N_c-2}$. The classical vacua of the theory are determined by the equation of motion $W'(\Phi) = \sum_{i=1}^{N_c} g_i \Phi^{i-1} = 0$. Then the roots a_i of

$$W'(x) = \sum_{i=1}^{N_c} g_i x^{i-1} = g_{N_c} \prod_{i=1}^{N_c-1} (x - a_i) \quad (4.2)$$

give the eigenvalues of Φ . In particular the unbroken $SU(2) \times U(1)^{N_c-2}$ vacuum is described by

$$\Phi = \text{diag}(a_1, a_1, a_2, a_3, \dots, a_{N_c-1}). \quad (4.3)$$

In the low-energy limit the adjoint superfield for $SU(2)$ becomes massive and will be decoupled. We are then left with an $N = 1$ $SU(2)$ Yang-Mills theory which is in the confining phase and the photon multiplets for $U(1)^{N_c-2}$ are decoupled.

The relation between the high-energy $SU(N_c)$ scale Λ and the low-energy $SU(2)$ scale Λ_L is determined by first matching at the scale of $SU(N_c)/SU(2)$ W bosons and then by matching at the $SU(2)$ adjoint mass M_{ad} . One finds [40], [18]

$$\Lambda^{2N_c} = \Lambda_L^{3 \cdot 2} \left(\prod_{i=2}^{N_c-1} (a_1 - a_i) \right)^2 (M_{\text{ad}})^{-2}. \quad (4.4)$$

To compute M_{ad} we decompose

$$\Phi = \Phi_{cl} + \delta\Phi + \delta\tilde{\Phi}, \quad (4.5)$$

where $\delta\Phi$ denotes the fluctuation along the unbroken $SU(2)$ direction and $\delta\tilde{\Phi}$ along the other directions. Substituting this into W we have

$$\begin{aligned} W &= W_{cl} + \sum_{i=2}^{N_c} g_i \frac{i-1}{2} \text{Tr}(\delta\Phi^2 \Phi_{cl}^{i-2}) + \dots \\ &= W_{cl} + \frac{1}{2} W''(a_1) \text{Tr} \delta\Phi^2 + \dots \\ &= W_{cl} + \frac{1}{2} g_{N_c} \prod_{i=2}^{N_c} (a_1 - a_i) \text{Tr} \delta\Phi^2 + \dots, \end{aligned} \quad (4.6)$$

where $[\delta\Phi, \Phi_{cl}] = 0$ has been used and W_{cl} is the tree-level superpotential evaluated in the classical vacuum. Hence, $M_{\text{ad}} = g_{N_c} \prod_{i=2}^{N_c-1} (a_1 - a_i)$ and the relation (4.4) reduces to

$$\Lambda_L^6 = g_{N_c}^2 \Lambda^{2N_c}. \quad (4.7)$$

Since the gaugino condensation dynamically generates the superpotential in the $N = 1$ $SU(2)$ theory the low-energy effective superpotential finally takes the form [18]

$$W_L = W_{cl} \pm 2\Lambda_L^3 = W_{cl} \pm 2g_{N_c} \Lambda^{N_c}. \quad (4.8)$$

We simply assume here that the superpotential (4.8) is exact for any values of the parameters. (This is equivalent to assume $W_\Delta = 0$ [17], [18].) From (4.8) we obtain

$$\langle u_n \rangle = \frac{\partial W_L}{\partial g_n} = u_n^{cl}(g) \pm 2\Lambda^{N_c} \delta_{n, N_c} \quad (4.9)$$

with u_n^{cl} being a classical value of u_n . As we argued above these vacua should correspond to the singular loci of $N = 2$ massless dyons. This can be easily confirmed by plugging (4.9) in the $N = 2$ $SU(N_c)$ curve [6], [5]

$$y^2 = \langle \det(x - \Phi) \rangle^2 - 4\Lambda^{2N_c} = \left(x^{N_c} - \sum_{i=2}^{N_c} \langle s_i \rangle x^{N_c-i} \right)^2 - 4\Lambda^{2N_c}, \quad (4.10)$$

where

$$ks_k + \sum_{i=1}^k is_{k-i}u_i = 0, \quad k = 1, 2, \dots \quad (4.11)$$

with $s_0 = -1$ and $s_1 = u_1 = 0$. We have

$$\begin{aligned} y^2 &= \left(x^{N_c} - s_2^{cl} x^{N_c-2} - \dots - s_{N_c}^{cl} \right) \left(x^{N_c} - s_2^{cl} x^{N_c-2} - \dots - s_{N_c}^{cl} \pm 4\Lambda^{N_c} \right) \\ &= (x - a_1)^2 (x - a_2) \dots (x - a_{N_c-1}) \left((x - a_1)^2 \dots (x - a_{N_c-1}) \pm 4\Lambda^{N_c} \right). \end{aligned} \quad (4.12)$$

Since the curve exhibits the quadratic degeneracy we are exactly at the singular point of a massless dyon in the $N = 2$ $SU(N_c)$ Yang-Mills vacuum.

Let us now apply our procedure to the $N = 2$ $SO(2N_c)$ Yang-Mills theory. We take a tree-level superpotential to break $N = 2$ to $N = 1$ as

$$W = \sum_{n=1}^{N_c-1} g_{2n} u_{2n} + \lambda v, \quad (4.13)$$

where

$$\begin{aligned} u_{2n} &= \frac{1}{2n} \text{Tr} \Phi^{2n}, \\ v &= \text{Pf} \Phi = \frac{1}{2^{N_c} N_c!} \epsilon_{i_1 i_2 j_1 j_2 \dots} \Phi^{i_1 i_2} \Phi^{j_1 j_2} \dots \end{aligned} \quad (4.14)$$

and the adjoint superfield Φ is an antisymmetric $2N_c \times 2N_c$ tensor. This theory has classical vacua which satisfy the condition

$$W'(\Phi) = \sum_{i=1}^{N_c-1} g_{2i} (\Phi^{2i-1})_{ij} - \frac{\lambda}{2^{N_c} (N_c - 1)!} \epsilon_{i j k_1 k_2 l_1 l_2 \dots} \Phi^{k_1 k_2} \Phi^{l_1 l_2} \dots = 0. \quad (4.15)$$

For the skew-diagonal form of Φ

$$\Phi = \text{diag}(\sigma_2 e_0, \sigma_2 e_1, \sigma_2 e_2, \dots, \sigma_2 e_{N_c-1}), \quad \sigma_2 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (4.16)$$

the vacuum condition (4.15) becomes

$$\sum_{i=1}^{N_c-1} g_{2i} (-1)^{i-1} e_n^{2i-1} + (-i)^{N_c} \frac{\lambda}{2e_n} \prod_{i=0}^{N_c-1} e_i = 0, \quad 0 \leq n \leq N_c - 1. \quad (4.17)$$

Thus we see that $e_n (\neq 0)$ are the roots of $f(x)$ defined by

$$f(x) = \sum_{i=1}^{N_c-1} g_{2i} x^{2i} + d, \quad (4.18)$$

where we put $d = (-i)^{N_c} \frac{1}{2} \lambda \prod_{i=0}^{N_c-1} e_i$.

Since our main concern is the vacuum with a single confined photon we focus on the unbroken $SU(2) \times U(1)^{N_c-1}$ vacuum. Thus writing (4.18) as

$$f(x) = g_{2(N_c-1)} \prod_{i=1}^{N_c-1} (x^2 - a_i^2), \quad (4.19)$$

we take

$$\Phi = \text{diag}(\sigma_2 a_1, \sigma_2 a_1, \sigma_2 a_2, \dots, \sigma_2 a_{N_c-1}) \quad (4.20)$$

with $d = (-i)^{N_c} \frac{1}{2} \lambda a_1^2 \prod_{i=2}^{N_c-1} a_i$. We then make the scale matching between the high-energy $SO(2N_c)$ scale Λ and the low-energy $SU(2)$ scale Λ_L . Following the steps as in the $SU(N_c)$ case yields

$$\Lambda^{2 \cdot 2(N_c-1)} = \Lambda_L^{3 \cdot 2} \left(\prod_{i=2}^{N_c-1} (a_1^2 - a_i^2) \right)^2 (M_{\text{ad}})^{-2}, \quad (4.21)$$

where the factor arising through the Higgs mechanism is easily calculated in an explicit basis of $SO(2N_c)$. In order to evaluate the $SU(2)$ adjoint mass M_{ad} we first substitute the decomposition (4.5) in W and proceed as follows:

$$\begin{aligned} W &= W_{cl} + \sum_{i=1}^{N_c-1} g_i \frac{2i-1}{2} \text{Tr} (\delta\Phi^2 \Phi_{cl}^{2i-2}) + \lambda (\text{Pf}_4 \delta\Phi) (\text{Pf}_{2(N_c-2)} \Phi_{cl}) + \dots \\ &= W_{cl} + \sum_{i=1}^{N_c-1} g_i \frac{2i-1}{2} \text{Tr} (\delta\Phi^2 \Phi_{cl}^{2i-2}) + \lambda \left(\frac{1}{4} \text{Tr} \delta\Phi^2 \right) \left(\prod_{k=2}^{N_c-1} (-ia_k) \right) + \dots \\ &= W_{cl} + \frac{1}{2} \frac{d}{dx} \left(\frac{f(x)}{x} \right) \Big|_{x=a_1} \text{Tr} \delta\Phi^2 + \dots \\ &= W_{cl} + g_{2(N_c-1)} \prod_{i=2}^{N_c-1} (a_1^2 - a_i^2) \text{Tr} \delta\Phi^2 + \dots, \end{aligned} \quad (4.22)$$

where Pf_4 is the Pfaffian of a upper-left 4×4 sub-matrix and $\text{Pf}_{2(N_c-2)}$ is the Pfaffian of a lower-right $2(N_c-2) \times 2(N_c-2)$ sub-matrix. Thus we observe that M_{ad} cancels out the Higgs factor in (4.21), which leads to $\Lambda_L^6 = g_{2(N_c-1)}^2 \Lambda^{4(N_c-1)}$. The low-energy superpotential is now given by

$$W_L = W_{cl} \pm 2\Lambda_L^3 = W_{cl} \pm 2g_{2(N_c-1)}\Lambda^{2(N_c-1)}, \quad (4.23)$$

where the second term is due to the gaugino condensation in the low-energy $SU(2)$ theory.

The vacuum expectation values of gauge invariants are obtained from W_L as

$$\begin{aligned} \langle u_{2n} \rangle &= \frac{\partial W_L}{\partial g_{2n}} = u_{2n}^{cl}(g, \lambda) \pm 2\Lambda^{2(N_c-1)}\delta_{n, N_c-1}, \\ \langle v \rangle &= \frac{\partial W_L}{\partial \lambda} = v_{cl}(g, \lambda). \end{aligned} \quad (4.24)$$

The curve for $N = 2$ $SO(2N_c)$ is known to be [8]

$$\begin{aligned} y^2 &= \langle \det(x - \Phi) \rangle^2 - 4\Lambda^{4(N_c-1)}x^4 \\ &= \left(x^{2N_c} - \sum_{i=1}^{N_c-1} \langle s_{2i} \rangle x^{2(N_c-i)} + \langle v \rangle^2 \right)^2 - 4\Lambda^{4(N_c-1)}x^4, \end{aligned} \quad (4.25)$$

where

$$ks_k + \sum_{i=1}^k i s_{k-i} u_{2i} = 0, \quad k = 1, 2, \dots \quad (4.26)$$

with $s_0 = -1$. At the values (4.24) of the moduli coordinates we see the quadratic degeneracy

$$\begin{aligned} y^2 &= \left(x^{2N_c} - s_2^{cl} x^{2(N_c-1)} - \dots - s_{2(N_c-1)}^{cl} x^2 + v_{cl}^2 \right) \\ &\quad \times \left(x^{2N_c} - s_2^{cl} x^{2(N_c-1)} - \dots - s_{2(N_c-1)}^{cl} x^2 + v_{cl}^2 \pm 4\Lambda^{2(N_c-1)} x^2 \right) \\ &= (x^2 - a_1^2)^2 (x^2 - a_2^2) \dots (x^2 - a_{N_c-1}^2) \\ &\quad \times \left((x^2 - a_1^2)^2 (x^2 - a_2^2) \dots (x^2 - a_{N_c-1}^2) \pm 4\Lambda^{2(N_c-1)} x^2 \right). \end{aligned} \quad (4.27)$$

This is our desired result. Notice that the apparent singularity at $\langle v \rangle = 0$ is not realized in our $N = 1$ theory. Thus the point $\langle v \rangle = 0$ does not correspond to massless solitons in agreement with the result of [8].

Our next task is to study the $SO(2N_c + 1)$ gauge theory. A tree-level superpotential breaking $N = 2$ to $N = 1$ is assumed to be

$$W = \sum_{n=1}^{N_c} g_{2n} u_{2n}, \quad u_{2n} = \frac{1}{2n} \text{Tr} \Phi^{2n}. \quad (4.28)$$

The classical vacua obey $W'(\Phi) = \sum_{i=1}^{N_c} g_{2i} \Phi^{2i-1} = 0$. The eigenvalues of Φ are given by the roots a_i of

$$W'(x) = \sum_{i=1}^{N_c} g_{2i} x^{2i-1} = g_{2N_c} x \prod_{i=1}^{N_c-1} (x^2 - a_i^2). \quad (4.29)$$

As in the previous consideration we take the $SU(2) \times U(1)^{N_c-1}$ vacuum. Notice that there are two ways of breaking $SO(2N_c + 1)$ to $SU(2) \times U(1)^{N_c-1}$. One is to take all the eigenvalues distinct (corresponding to $SO(3) \times U(1)^{N_c-1}$). The other is to choose two eigenvalues coinciding and the rest distinct (corresponding to $SU(2) \times U(1)^{N_c-1}$ with $a_i \neq 0$). Here we examine the latter case

$$\Phi = \text{diag}(\sigma_2 a_1, \sigma_2 a_1, \sigma_2 a_2, \dots, \sigma_2 a_{N_c-1}, 0), \quad \sigma_2 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.30)$$

In this vacuum the high-energy $SO(2N_c + 1)$ scale Λ and the low-energy $SU(2)$ scale Λ_L are related by

$$\Lambda^{2 \cdot (2N_c-1)} = \Lambda_L^{3 \cdot 2} a_1^2 \left(\prod_{i=2}^{N_c-1} (a_1^2 - a_i^2) \right)^2 (M_{\text{ad}})^{-2}, \quad (4.31)$$

where the $SU(2)$ adjoint mass M_{ad} is read off from

$$\begin{aligned} W &= W_{cl} + \sum_{i=1}^{N_c} g_{2i} \frac{2i-1}{2} \text{Tr} (\delta \Phi^2 \Phi_{cl}^{2i-2}) + \dots \\ &= W_{cl} + \frac{1}{2} W''(a_1) \text{Tr} \delta \Phi^2 + \dots \\ &= W_{cl} + g_{2N_c} a_1^2 \prod_{i=2}^{N_c-1} (a_1^2 - a_i^2) \text{Tr} \delta \Phi^2 + \dots \end{aligned} \quad (4.32)$$

So, we obtain $\Lambda_L^6 = g_{2N_c}^2 a_1^2 \Lambda^{2(2N_c-1)}$. The low-energy effective superpotential becomes

$$W_L = W_{cl} \pm 2\Lambda_L^3 = W_{cl} \pm 2g_{2N_c} a_1 \Lambda^{2N_c-1}. \quad (4.33)$$

If we assume $W_\Delta = 0$ the expectation values $\langle u_{2i} \rangle$ are calculated from W_L by expressing a_1 as a function of g_{2i} .

For the sake of illustration let us discuss the $SO(5)$ theory explicitly. From (4.33) we get

$$\begin{aligned}\langle u_2 \rangle &= 2a_1^2 \pm \frac{1}{a_1} \Lambda^3, \\ \langle u_4 \rangle &= a_1^4 \mp a_1 \Lambda^3\end{aligned}\tag{4.34}$$

and $a_1^2 = -g_2/g_4$. We eliminate a_1 from (4.34) to obtain

$$27\Lambda^{12} - \Lambda^6 u_2^3 + 36\Lambda^6 u_2 u_4 - u_2^4 u_4 + 8u_2^2 u_4^2 - 16u_4^3 = 0.\tag{4.35}$$

This should be compared with the $N = 2$ $SO(5)$ discriminant [7]

$$s_2^2(27\Lambda^{12} - \Lambda^6 s_1^3 - 36\Lambda^6 s_1 s_2 + s_1^4 s_2 + 8s_1^2 s_2^2 + 16s_2^3)^2 = 0,\tag{4.36}$$

where $s_1 = u_2$ and $s_2 = u_4 - u_2^2/2$ according to (4.26). Thus we see the discrepancy between (4.35) and (4.36) which implies that our simple assumption of $W_\Delta = 0$ does not work. Inspecting (4.35) and (4.36), however, we notice how to remedy the difficulty. Instead of (4.28) we take a tree-level superpotential

$$W = g_2 s_1 + g_4 s_2 = g_2 u_2 + g_4 \left(u_4 - \frac{1}{2} u_2^2 \right).\tag{4.37}$$

The classical vacuum condition is

$$W'(\Phi) = (g_2 - g_4 u_2) \Phi + g_4 \Phi^3 = 0.\tag{4.38}$$

To proceed, therefore, we can make use of the results obtained in the foregoing analysis just by making the replacement

$$\begin{aligned}g_4 &\rightarrow \tilde{g}_4 = g_4, \\ g_2 &\rightarrow \tilde{g}_2 = g_2 - u_2 g_4.\end{aligned}\tag{4.39}$$

(especially evaluation of M_{ad} is not invalidated because $\text{Tr } \delta\Phi = 0$.) The eigenvalues of Φ are now determined in a self-consistent manner by

$$W'(x) = \tilde{g}_2 x + \tilde{g}_4 x^3 = \tilde{g}_4 x \left(x^2 + \frac{\tilde{g}_2}{\tilde{g}_4} \right) = \tilde{g}_4 x (x^2 - a_1^2) = 0.\tag{4.40}$$

Then we have $u_2^{cl} = 2a_1^2 = -2\tilde{g}_2/\tilde{g}_4$ and $\tilde{g}_2 = -g_2$ from (4.39), which leads to

$$a_1^2 = \frac{g_2}{g_4}. \quad (4.41)$$

Substituting this in (4.33) we calculate $\langle s_i \rangle$ and find the relation of s_i which is precisely the discriminant (4.36) except for the classical singularity at $\langle s_2 \rangle = 0$.

The above $SO(5)$ result indicates that an appropriate mixing term with respect to u_{2i} variables in a microscopic superpotential will be required for $SO(2N_c + 1)$ theories. We are led to assume

$$W = \sum_{i=1}^{N_c-1} g_{2i} u_{2i} + g_{2N_c} s_{N_c} \quad (4.42)$$

for the gauge group $SO(2N_c + 1)$ with $N_c \geq 3$. Then the following analysis is analogous to the $SO(5)$ theory. First of all notice that

$$s_{N_c} = u_{2N_c} - u_{2(N_c-1)}u_2 + (\text{polynomials of } u_{2k}, 1 \leq k < N_c - 1). \quad (4.43)$$

Therefore the eigenvalues of Φ are given by the roots of (4.29) with the replacement

$$\begin{aligned} g_{2N_c} &\rightarrow \tilde{g}_{2N_c} = g_{2N_c}, \\ g_{2(N_c-1)} &\rightarrow \tilde{g}_{2(N_c-1)} = g_{2(N_c-1)} - u_2 g_{2N_c}. \end{aligned} \quad (4.44)$$

Then we have $u_2 = a_1^2 + \sum_{k=1}^{N_c-1} a_k^2 = a_1^2 - \tilde{g}_{2(N_c-1)}/\tilde{g}_{2N_c}$ and find

$$a_1^2 = \frac{g_{2(N_c-1)}}{g_{2N_c}}. \quad (4.45)$$

It follows that the effective superpotential is given by

$$W_L = W_L^{cl} \pm 2\sqrt{g_{2N_c}g_{2(N_c-1)}}\Lambda^{2N_c-1}. \quad (4.46)$$

The vacuum expectation values of gauge invariants are obtained from W_L as

$$\begin{aligned} \langle s_n \rangle &= s_n^{cl}(g), \quad 1 \leq n \leq N_c - 2 \\ \langle s_{N_c-1} \rangle &= s_{N_c-1}^{cl}(g) \pm \frac{1}{a_1} \Lambda^{2N_c-1}, \\ \langle s_{N_c} \rangle &= s_{N_c}^{cl}(g) \pm a_1 \Lambda^{2N_c-1}. \end{aligned} \quad (4.47)$$

For these $\langle s_i \rangle$ the curve describing the $N = 2$ $SO(2N_c + 1)$ theory [7] is shown to be degenerate as follows:

$$\begin{aligned}
y^2 &= \langle \det(x - \Phi) \rangle^2 - 4x^2 \Lambda^{2(2N_c-1)} \\
&= (x^{2N_c} - \langle s_1 \rangle x^{2(N_c-1)} - \dots - \langle s_{N_c-1} \rangle x^2 - \langle s_{N_c} \rangle + 2x \Lambda^{2N_c-1}) \\
&\quad \times (x^{2N_c} - \langle s_1 \rangle x^{2(N_c-1)} - \dots - \langle s_{N_c-1} \rangle x^2 - \langle s_{N_c} \rangle - 2x \Lambda^{2N_c-1}) \\
&= \left\{ (x^2 - a_1^2)^2 (x^2 - a_2^2) \dots (x^2 - a_{N_c-1}^2) \pm \Lambda^{2N_c-1} \left(-\frac{x^2}{a_1} - a_1 + 2x \right) \right\} \\
&\quad \times \left\{ (x^2 - a_1^2)^2 (x^2 - a_2^2) \dots (x^2 - a_{N_c-1}^2) \pm \Lambda^{2N_c-1} \left(-\frac{x^2}{a_1} - a_1 - 2x \right) \right\} \\
&= (x^2 - a_1^2)^2 \left((x + a_1)^2 (x^2 - a_2^2) \dots (x^2 - a_{N_c-1}^2) \mp \frac{\Lambda^{2N_c-1}}{a_1} \right) \\
&\quad \times \left((x - a_1)^2 (x^2 - a_2^2) \dots (x^2 - a_{N_c-1}^2) \mp \frac{\Lambda^{2N_c-1}}{a_1} \right). \tag{4.48}
\end{aligned}$$

Thus we see the theory with the superpotential (4.42) recover the $N = 2$ curve correctly with the assumption $W_\Delta = 0$. As in the $SO(2N_c)$ case, the singularity at $\langle s_{N_c} \rangle = 0$, which corresponds to the classical $SO(3) \times U(1)^{N_c-1}$ vacuum, does not arise in our theory.

We remark that the particular form of superpotential (4.42) is not unique to derive the singularity manifold. In fact we may start with a superpotential

$$W = \sum_{i=1}^{N_c-1} g_{2i} (u_{2i} + h_i(s)) + g_{2N_c} (s_{N_c} + h_{N_c}(s)), \tag{4.49}$$

where $h_i(s)$ are arbitrary polynomials of s_j with $j \geq N_c - 2$, to verify the $N = 2$ curve. However, we are not allowed to take a superpotential such as $W = \sum_{i=1}^{N_c} g_{2i} s_i$, because there are no $SU(2) \times U(1)^{N_c-1}$ vacua (there exist no solutions for $\tilde{g}_{2(N_c-1)}$). Note also that there are no $SO(3) \times U(1)^{N_c-1}$ vacua in the theory with superpotential (4.49).

Finally we discuss the $Sp(2N_c)$ gauge theory. The adjoint superfield Φ is a $2N_c \times 2N_c$ tensor which is subject to

$${}^t \Phi = J \Phi J \iff J \Phi \text{ is symmetric}, \tag{4.50}$$

where $J = \text{diag}(i\sigma_2, \dots, i\sigma_2)$. Let us assume a tree-level superpotential

$$W = \sum_{n=1}^{N_c} g_{2n} u_{2n}, \quad u_{2n} = \frac{1}{2n} \text{Tr} \Phi^{2n}. \tag{4.51}$$

Then our analysis will become quite similar to that for $SO(2N_c + 1)$. The classical vacuum with unbroken $SU(2) \times U(1)^{N_c-1}$ gauge group corresponds to

$$J\Phi = \text{diag}(\sigma_1 a_1, \sigma_1 a_1, \sigma_1 a_2, \dots, \sigma_1 a_{N_c-1}), \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.52)$$

The scale matching relation becomes

$$\Lambda^{2(N_c+1)} = \Lambda_L^{3 \cdot 2 \cdot \frac{1}{2}} a_1^4 \left(\prod_{i=2}^{N_c-1} (a_1^2 - a_i^2) \right)^2 (M_{\text{ad}})^{-1}. \quad (4.53)$$

Since the $SU(2)$ adjoint mass is given by $M_{\text{ad}} = g_{2N_c} a_1^2 \prod_{i=2}^{N_c-1} (a_1^2 - a_i^2)$ we get $\Lambda_L^3 = g_{2N_c} \Lambda^{2(N_c+1)} / a_1^2$. The low-energy effective superpotential thus turns out to be

$$W_L = W_{cl} + 2 \frac{g_{2N_c}}{a_1^2} \Lambda^{2(N_c+1)}. \quad (4.54)$$

Checking the result with $Sp(4)$ we encounter the same problem as in the $SO(5)$ theory. Instead of (4.51), thus, we take a superpotential in the form (4.37), reproducing the $N = 2$ $Sp(4)$ curve [11]. Similarly, for $Sp(2N_c)$ we study a superpotential (4.42). It turns out that $\langle s_i \rangle$ are calculated as

$$\begin{aligned} \langle s_n \rangle &= s_n^{cl}(g), & 1 \leq n \leq N_c - 2, \\ \langle s_{N_c-1} \rangle &= s_{N_c-1}^{cl}(g) - \frac{2}{a_1^4} \Lambda^{2(N_c+1)}, \\ \langle s_{N_c} \rangle &= s_{N_c}^{cl}(g) + \frac{4}{a_1^2} \Lambda^{2(N_c+1)}. \end{aligned} \quad (4.55)$$

These satisfy the $N = 2$ $Sp(2N_c)$ singularity condition [11] since the curve exhibits the quadratic degeneracy

$$\begin{aligned} x^2 y^2 &= \left(x^2 \langle \det(x - \Phi) \rangle + \Lambda^{2(N_c+1)} \right)^2 - \Lambda^{4(N_c+1)} \\ &= \left(x^{2(N_c+1)} - \langle s_1 \rangle x^{2N_c} - \dots - \langle s_{N_c-1} \rangle x^4 - \langle s_{N_c} \rangle x^2 + 2\Lambda^{2(N_c+1)} \right) \\ &\quad \times \left(x^{2(N_c+1)} - \langle s_1 \rangle x^{2N_c} - \dots - \langle s_{N_c-1} \rangle x^4 - \langle s_{N_c} \rangle x^2 \right) \\ &= \left\{ x^2 (x^2 - a_1^2)^2 (x^2 - a_2^2) \dots (x^2 - a_{N_c-1}^2) + 2\Lambda^{2(N_c+1)} \left(\left(\frac{x}{a_1} \right)^4 - 2 \left(\frac{x}{a_1} \right)^2 + 1 \right) \right\} \\ &\quad \times \left(x^2 \det(x - \Phi_{cl}) \right) \\ &= (x^2 - a_1^2)^2 \left(x^2 (x^2 - a_2^2) \dots (x^2 - a_{N_c-1}^2) + \frac{\Lambda^{2(N_c+1)}}{a_1^4} \right) \times \left(x^2 \det(x - \Phi_{cl}) \right). \end{aligned} \quad (4.56)$$

It should be mentioned that our remarks on $SO(2N_c + 1)$ theories also apply here.

4.2 ADE gauge groups

Our purpose in this section is to show that, under an appropriate ansatz, the low-energy effective superpotential for the Coulomb phase is obtained in a unified way for all ADE gauge groups just by using the fundamental properties of the root system Δ . Let us consider the case of the gauge group G is simple and simply-laced, namely, G is of ADE type. Our notation for the root system is as follows. The simple roots of G are denoted as α_i where $1 \leq i \leq r$ with r being the rank of G . Any root is decomposed as $\alpha = \sum_{i=1}^r a^i \alpha_i$. The component indices are lowered by $a_i = \sum_{j=1}^r A_{ij} a^j$ where A_{ij} is the ADE Cartan matrix. The inner product of two roots α, β are then defined by

$$\alpha \cdot \beta = \sum_{i=1}^r a^i b_i = \sum_{i,j=1}^r a^i A_{ij} b^j, \quad (4.57)$$

where $\beta = \sum_{i=1}^r b^i \alpha_i$. For ADE all roots have the equal norm and we normalize $\alpha^2 = 2$.

In our $N = 1$ theory we take a tree-level superpotential

$$W = \sum_{k=1}^r g_k u_k(\Phi), \quad (4.58)$$

where u_k is the k -th Casimir of G constructed from Φ and g_k are coupling constants. The mass dimension of u_k is $e_k + 1$ with e_k being the k -th exponent of G . When $g_k = 0$ Φ is considered as the chiral field in the $N = 2$ vector multiplet and we have $N = 2$ ADE supersymmetric gauge theory.

We first make a classical analysis of the theory with the superpotential (4.58). The classical vacua are determined by the equation of motion $\frac{\partial W}{\partial \Phi} = 0$ and the D -term equation. Due to the D -term equation, we can restrict Φ to take the values in the Cartan subalgebra by the gauge rotation. We denote the vector in the Cartan subalgebra corresponding to the classical value of Φ as $a = \sum_{i=1}^r a^i \alpha_i$. Then the superpotential becomes

$$W(a) = \sum_{k=1}^r g_k u_k(a), \quad (4.59)$$

and the equation of motion reads

$$\frac{\partial W(a)}{\partial a^i} = \sum_{k=1}^r g_k \frac{\partial u_k(a)}{\partial a^i} = 0. \quad (4.60)$$

For $g_k \neq 0$ we must have

$$J(a) \equiv \det \left(\frac{\partial u_j(a)}{\partial a^i} \right) = 0. \quad (4.61)$$

According to [41] it follows that

$$J(a) = c_1 \prod_{\alpha \in \Delta^+} a \cdot \alpha, \quad (4.62)$$

where Δ^+ is a set of positive roots and c_1 is a certain constant.

The condition $J(a) = 0$ means that the vector a hits a wall of the Weyl chamber and there occurs enhanced gauge symmetry. Suppose that the vector a is orthogonal to a root, say, α_1

$$a \cdot \alpha_1 = 0, \quad (4.63)$$

where α_1 may be taken to be a simple root. In this case we have the unbroken gauge group $SU(2) \times U(1)^{r-1}$ where the $SU(2)$ factor is spanned by $\{\alpha_1 \cdot H, E_{\alpha_1}, E_{-\alpha_1}\}$ in the Cartan-Weyl basis. If some other factors of J vanish besides $a \cdot \alpha_1$ the gauge group is further enhanced from $SU(2)$. Since $SU(2) \times U(1)^{r-1}$ is the most generic unbroken gauge group we shall restrict ourselves to this case in what follows.

We remark here that there is the case in which the $SU(2) \times U(1)^{r-1}$ vacuum is not generic. As a simple, but instructive example consider $SU(4)$ theory. Casimirs are taken to be

$$\begin{aligned} u_1 &= \frac{1}{2} \text{Tr } \Phi^2, \\ u_2 &= \frac{1}{3} \text{Tr } \Phi^3, \\ u_3 &= \frac{1}{4} \text{Tr } \Phi^4 - \alpha \left(\frac{1}{2} \text{Tr } \Phi^2 \right)^2, \end{aligned} \quad (4.64)$$

where α is an arbitrary constant. If we set $\alpha = 1/2$ it is observed that the $SU(2) \times U(1)^2$ vacuum exists only for the special values of coupling constants, $(g_2/g_3)^2 = g_1/g_3$. Thus, for $\alpha = 1/2$, the $SU(2) \times U(1)^2$ vacuum is not generic though it does so for $\alpha \neq 1/2$. This points out that we have to choose the appropriate basis for Casimirs when writing down (4.58) to have the $SU(2) \times U(1)^{r-1}$ vacuum generically [19].

Now we assume that there is no mixing between the $SU(2) \times U(1)^{r-1}$ vacuum and other vacua with different unbroken gauge groups. According to the arguments of [49], we

should not consider the broken gauge group instantons. We thus expect that there is only perturbative effect in the energy scale above the scale Λ_{YM} of the low-energy effective $N = 1$ supersymmetric $SU(2)$ Yang-Mills theory.

Our next task is to evaluate the Higgs scale associated with the spontaneous breaking of the gauge group G to $SU(2) \times U(1)^{r-1}$. For this purpose we decompose the adjoint representation of G to irreducible representations of $SU(2)$. We fix the $SU(2)$ direction by taking a simple root α_1 . It is clear that the spin j of every representation obtained in this decomposition satisfies $j \leq 1$ since all roots have the same norm and the $SU(2)$ raising (or lowering) operator shifts a root α to $\alpha + \alpha_1$ (or $\alpha - \alpha_1$). The fact that there is no degeneration of roots indicates that the $j = 1$ multiplet has the roots $(\alpha_1, 0, -\alpha_1)$ corresponding to the unbroken $SU(2)$ generators. The roots orthogonal to α_1 represent the $j = 0$ multiplets. The $j = 1/2$ multiplets have the roots α obeying $\alpha \cdot \alpha_1 = \pm 1$. Let us define a set of these roots by $\Delta_d = \{\alpha | \alpha \in \Delta, \alpha \cdot \alpha_1 = \pm 1\}$. For each root $\alpha \in \Delta_d$ there appears a massive gauge boson. These massive bosons pair up in $SU(2)$ doublets with weights $(\alpha, \alpha \pm \alpha_1)$ which indeed have the same mass $|a \cdot \alpha| = |a \cdot (\alpha \pm \alpha_1)|$ since $a \cdot \alpha_1 = 0$.

We now integrate out the fields that become massive by the Higgs mechanism. The massless $U(1)^{r-1}$ degrees of freedom are decoupled. The resulting theory characterized by the scale Λ_H is $N = 1$ $SU(2)$ theory with an adjoint chiral multiplet. The Higgs scale Λ_H is related to the high-energy scale Λ through the scale matching relation

$$\Lambda^{2h} = \Lambda_H^{2 \cdot 2} \left(\prod_{\beta \in \Delta_d, \beta > 0} a \cdot \beta \right)^\ell, \quad (4.65)$$

where $2h = 4 + \ell n_d/2$, n_d is the number of elements in Δ_d and h stands for the dual Coxeter number of G ; $h = r + 1, 2r - 2, 12, 18, 30$ for $G = A_r, D_r, E_6, E_7, E_8$ respectively. The reason for $\beta > 0$ in (4.65) is that weights $(\beta, \beta \pm \alpha_1)$ of an $SU(2)$ doublet are either both positive or both negative since α_1 is the simple root, and gauge bosons associated with $\beta < 0$ and $\beta > 0$ have the same contribution to the relation (4.65).

To fix ℓ we calculate n_d by evaluating the quadratic Casimir C_2 of the adjoint representation in the following way. Taking hermitian generators we express C_2 in terms of the structure constants f_{abc} through $\sum_{a,b} f_{abc} f_{abc'} = -C_2 \delta_{cc'}$. From the commutation relation

$[\alpha_1 \cdot H, E_\alpha] = (\alpha_1 \cdot \alpha)E_\alpha$ one can check

$$C_2 = \frac{1}{2} \sum_{\alpha \in \Delta} (\alpha_1 \cdot \alpha)^2 = \frac{1}{2} \left(\sum_{\alpha \in \Delta_d} (\alpha_1 \cdot \alpha)^2 + 2(\alpha_1 \cdot \alpha_1)^2 \right) = \frac{1}{2} (n_d + 8). \quad (4.66)$$

On the other hand, the dual Coxeter number h is given by $h = C_2/\theta^2$ with θ being the highest root. We thus find

$$n_d = 4(h - 2) \quad (4.67)$$

and (4.65) becomes

$$\Lambda^{2h} = \Lambda_H^{2 \cdot 2} \prod_{\beta \in \Delta_d, \beta > 0} a \cdot \beta. \quad (4.68)$$

After integrating out the massive fields due to the Higgs mechanism we are left with $N = 1$ $SU(2)$ theory with the massive adjoint. In order to evaluate the mass of the adjoint chiral multiplet Φ we need to clarify some properties of Casimirs. Let σ_β be an element of the Weyl group of G specified by a root $\beta = \sum_{i=1}^r b^i \alpha_i$. The Weyl transformation of a root α is given by

$$\sigma_\beta(\alpha) = \alpha - (\alpha \cdot \beta)\beta. \quad (4.69)$$

When σ_β acts on the Higgs v.e.v. vector $a = \sum_{i=1}^r a^i \alpha_i$ we have

$$a^i = \sum_{j=1}^r S_{\beta^j}^i a^j, \quad S_{\beta^j}^i \equiv \delta^i_j - b^i b_j, \quad (4.70)$$

where $\sigma_\beta(a) = \sum_{i=1}^r a^i \alpha_i$. Since the Casimirs $u_k(a)$ are Weyl invariants it is obvious to see

$$\frac{\partial}{\partial a^i} u_k(a) = \frac{\partial}{\partial a^i} u_k(a') = \sum_{j=1}^r S_{\beta^j}^i \left(\frac{\partial}{\partial a^j} u_k(a) \right) \Big|_{a \rightarrow a'}. \quad (4.71)$$

Let \bar{a} be a particular v.e.v. which is fixed under the action of σ_β , then we find the identity

$$\sum_{j=1}^r (\delta^j_i - S_{\beta^j}^j) \frac{\partial}{\partial a^j} u_k(a) \Big|_{a=\bar{a}} = 0 \quad (4.72)$$

for all i , and thus

$$\sum_{j=1}^r b_j \frac{\partial}{\partial a^j} u_k(a) \Big|_{a=\bar{a}} = 0. \quad (4.73)$$

This implies that for any v.e.v. vector a and root β we can write down

$$\sum_{j=1}^r b^j \frac{\partial}{\partial a^j} u_k(a) = (a \cdot \beta) u_k^\beta(a), \quad (4.74)$$

where $u_k^\beta(a)$ is some polynomial of a^i . If we set $\beta = \alpha_i$, a simple root, we obtain a useful formula

$$\frac{\partial}{\partial a^i} u_k(a) = a_i u_k^{\alpha_i}(a). \quad (4.75)$$

As an immediate application of the above results, for instance, we point out that (4.62) is derived from (4.74) and the fact that the mass dimension of $J(a)$ is given by

$$\sum_{k=1}^r e_k = \frac{1}{2} (\dim G - r), \quad (4.76)$$

where e_k is the k -th exponent of G .

Let us further discuss the properties of $u_k^{\alpha_j}(a)$. Define D_{mn} as

$$D_{mn} \equiv (-1)^{n+m} \det \left(\frac{\partial u_{\tilde{j}}(a)}{\partial a^{\tilde{i}}} \right), \quad 1 \leq m, n \leq r, \quad (4.77)$$

where $1 \leq \tilde{i}, \tilde{j} \leq r$ with $\tilde{i} \neq m, \tilde{j} \neq n$, then D_{1n} is a homogeneous polynomial of a^i with the mass dimension $\sum_{k=1}^r e_k - e_n$. We also denote Δ_e as a set of positive roots where α_1 and $SU(2)$ doublet roots α with $\alpha + \alpha_1 \notin \Delta^+$ are excluded. If we set $a_1 = 0$ and $a \cdot \beta = 0$ where β is any root in Δ_e we see $D_{1n} = 0$ from the identity (4.74). Consequently we can expand

$$D_{1n} = h_n(a) \prod_{\beta \in \Delta_e} (a \cdot \beta) + a_1 f_n(a), \quad (4.78)$$

where $h_n(a), f_n(a)$ are polynomials of a_i . In particular

$$D_{1r} = c_2 \prod_{\beta \in \Delta_e} (a \cdot \beta) + a_1 f_r(a), \quad (4.79)$$

where c_2 is a constant. Notice that the first term on the rhs has the correct mass dimension since the number of roots in Δ_e reads

$$\frac{1}{2} (\dim G - r) - 1 - \frac{n_d}{4} = \sum_{k=1}^r e_k - (h - 1), \quad (4.80)$$

where we have used (4.67) and $e_r = h - 1$.

We are now ready to evaluate the mass of Φ in intermediate $SU(2)$ theory. The fluctuation of $W(a)$ around the classical vacuum yields the adjoint mass. To find the mass relevant for the scale matching we should only consider the components of Φ which are coupled to the unbroken $SU(2)$. The mass M_Φ of these components is then given by

$$2M_\Phi = \left. \frac{\partial^2}{(\partial a^1)^2} W(a) \right| = \left. \frac{\partial}{\partial a^1} (a_1 W_1) \right| = \left. \left(a_1 \frac{\partial}{\partial a^1} W_1 + 2W_1 \right) \right| = 2|W_1|, \quad (4.81)$$

where $W_1 = (\sum_{k=1}^r g_k u_k^{\alpha_1})(a)$ and a^i are understood as solutions of the equation of motion (4.60).

To proceed further it is convenient to rewrite the equation of motion (4.60) and the vacuum condition (4.63) with the simple root α_1 as follows:

$$\begin{aligned} g_1 : g_2 : \cdots : g_r &= D_{11} : D_{12} : \cdots : D_{1r}, \\ a_1 &= 0. \end{aligned} \quad (4.82)$$

The solutions of these equations are expressed as functions of the ratio g_i/g_r . Then we notice that $J(a)$ defined in (4.61) turns out to be

$$J = \sum_{k=1}^r \frac{\partial u_k}{\partial a^1} D_{1k} = \frac{D_{1r}}{g_r} \sum_{k=1}^r g_k \frac{\partial u_k}{\partial a^1} = D_{1r} a_1 \frac{W_1}{g_r}. \quad (4.83)$$

Combining (4.62) and (4.79) we obtain

$$M_{\Phi}^2 = (W_1)^2 = \left(\frac{c_1}{c_2}\right)^2 g_r^2 \prod_{\beta \in \Delta_d, \beta > 0} a \cdot \beta. \quad (4.84)$$

Upon integrating out the massive adjoint we relate the scale Λ_H with the scale Λ_{YM} of the low-energy $N = 1$ $SU(2)$ Yang-Mills theory by

$$\Lambda_H^{2:2} = \Lambda_{YM}^{3:2} / M_{\Phi}^2. \quad (4.85)$$

We finally find from this and (4.65), (4.84) that the scale matching relation becomes

$$\Lambda_{YM}^{3:2} = g_r^2 \Lambda^{2h}, \quad (4.86)$$

where the top Casimir u_r has been rescaled so that we can set $c_1/c_2 = 1$.

Following the previous discussions and the perturbative nonrenormalization theorem for the superpotential, we derive the low-energy effective superpotential

$$W_L = W_{cl}(g) \pm 2\Lambda_{YM}^3 = W_{cl}(g) \pm 2g_r \Lambda^h, \quad (4.87)$$

where the term $\pm 2\Lambda_{YM}^3$ appears as a result of the gaugino condensation in low-energy $SU(2)$ theory and $W_{cl}(g)$ is the tree-level superpotential evaluated at the classical values $a^i(g)$. We will assume that (4.87) is the exact effective superpotential valid for all values of parameters.

The vacuum expectation values of gauge invariants are obtained from W_L

$$\langle u_k \rangle = \frac{\partial W_L}{\partial g_k} = u_k^{cl}(g) \pm 2\Lambda^h \delta_{k,r}. \quad (4.88)$$

We now wish to show that the expectation values (4.88) parametrize the singularities of algebraic curves. For this let us introduce

$$P_{\mathcal{R}}(x, u_k^{cl}) = \det(x - \Phi_{\mathcal{R}}) \quad (4.89)$$

which is the characteristic polynomial in x of order $\dim \mathcal{R}$ where \mathcal{R} is an irreducible representation of G . Here $\Phi_{\mathcal{R}}$ is a representation matrix of \mathcal{R} and u_k^{cl} are Casimirs built out of $\Phi_{\mathcal{R}}$. The eigenvalues of $\Phi_{\mathcal{R}}$ are given in terms of the weights λ_i of the representation \mathcal{R} . Diagonalizing $\Phi_{\mathcal{R}}$ we may express (4.89) as

$$P_{\mathcal{R}}(x, a) = \prod_{i=1}^{\dim \mathcal{R}} (x - a \cdot \lambda_i), \quad (4.90)$$

where a is a Higgs v.e.v. vector, the discriminant of which takes the form

$$\Delta_{\mathcal{R}} = \left(\prod_{i \neq j} a \cdot (\lambda_i - \lambda_j) \right)^2. \quad (4.91)$$

It is seen that, for a which is a solution to (4.60), we have $\Delta_{\mathcal{R}} = 0$, that is

$$P_{\mathcal{R}}(x, u_k^{cl}(a)) = \partial_x P_{\mathcal{R}}(x, u_k^{cl}(a)) = 0 \quad (4.92)$$

for any representation. The solutions of the classical equation of motion thus give rise to the singularities of the level manifold $P_{\mathcal{R}}(x, u_k^{cl}) = 0$.

In order to include the quantum effect what we should do is to modify the top Casimir u_r term so that the gluino condensation in (4.88) is properly taken into account. We are then led to take a curve

$$\tilde{P}_{\mathcal{R}}(x, z, u_k) \equiv P_{\mathcal{R}} \left(x, u_k + \delta_{k,r} \left(z + \frac{\mu}{z} \right) \right) = 0, \quad (4.93)$$

where $\mu = \Lambda^{2h}$ and an additional complex variable z has been introduced. Let us check the degeneracy of the curve at the expectation values (4.88), which means to check if the

following three equations hold

$$\tilde{P}_{\mathcal{R}}(x, z, \langle u_k \rangle) = 0, \quad (4.94)$$

$$\partial_x \tilde{P}_{\mathcal{R}}(x, z, \langle u_k \rangle) = 0, \quad (4.95)$$

$$\partial_z \tilde{P}_{\mathcal{R}}(x, z, \langle u_k \rangle) = \left(1 - \frac{\mu}{z^2}\right) \partial_{u_r} \tilde{P}_{\mathcal{R}}(x, z, \langle u_k \rangle) = 0. \quad (4.96)$$

The last equation (4.96) has an obvious solution $z = \mp\sqrt{\mu}$. Substituting this into the first two equations we see that the singularity conditions reduce to the classical ones (4.92)

$$\tilde{P}_{\mathcal{R}}(x, \mp\sqrt{\mu}, \langle u_k \rangle) = P_{\mathcal{R}}(x, \langle u_k \rangle \mp \delta_{k,r} 2\sqrt{\mu}) = P_{\mathcal{R}}(x, u_k^{cl}) = 0, \quad (4.97)$$

$$\partial_x \tilde{P}_{\mathcal{R}}(x, \mp\sqrt{\mu}, \langle u_k \rangle) = \partial_x P_{\mathcal{R}}(x, \langle u_k \rangle \mp \delta_{k,r} 2\sqrt{\mu}) = \partial_x P_{\mathcal{R}}(x, u_k^{cl}) = 0. \quad (4.98)$$

Thus we have shown that (4.88) parametrize the singularities of the Riemann surface described by (4.93) irrespective of the representation \mathcal{R} .

Let us take the $N = 2$ limit by letting all $g_i \rightarrow 0$ with the ratio g_i/g_r fixed, then (4.93) is the curve describing the Coulomb phase of $N = 2$ supersymmetric Yang-Mills theory with ADE gauge groups. Indeed the curve (4.93) in this particular form of foliation agrees with the one obtained systematically in [23] in view of integrable systems [42],[43],[44]. For E_6 and E_7 see [24],[25].

Finally we remark that there is a possibility of (4.96) having another solutions besides $z = \mp\sqrt{\mu}$. If we take the fundamental representation such solutions are absent for $G = A_r$, and for $G = D_r$ there is a solution with vanishing degree r Casimir (i.e. Pfaffian), but it is known that this is an apparent singularity [8]. For E_r gauge groups there could exist additional solutions. We expect that these singularities are apparent and do not represent physical massless solitons.

4.2.1 $N = 1$ superconformal field theory

We will discuss non-trivial fixed points in our $N = 1$ theory characterized by the microscopic superpotential (4.58). To find critical points we rely on the construction of $N = 2$ superconformal field theories realized at particular points in the moduli space of the Coulomb phase [14],[45],[46],[47]. At these $N = 2$ critical points mutually non-local massless dyons coexist. Thus the critical points lie on the singularities in the moduli

space which are parametrized by the $N = 1$ expectation values (4.88) as was shown in the previous section. This enables us to adjust the microscopic parameters in $N = 1$ theory to the values of $N = 2$ non-trivial fixed points. Doing so in $N = 2$ $SU(3)$ Yang-Mills theory Argyres and Douglas found non-trivial $N = 1$ fixed points [14]. We now show that this class of $N = 1$ fixed points exists in all ADE $N = 1$ theories in general.

Let us start with rederiving $N = 2$ critical behavior based on the curve (4.93). An advantage of using the curve (4.93) is that one can identify higher critical points and determine the critical exponents independently of the details of the curve.

If we set $z = \mp\sqrt{\mu}$ the condition for higher critical points is

$$P_{\mathcal{R}}(x, u_k^c) = \partial_x^n P_{\mathcal{R}}(x, u_k^c) = 0 \quad (4.99)$$

with $n > 2$. Hence there exist higher critical points at $u_k = u_k^{sing} \pm 2\Lambda^h \delta_{k,r}$ where u_k^{sing} are the classical values of u_k for which the gauge group H with rank larger than one is left unbroken. The highest critical point corresponding to the unbroken G is located at $u_k = \pm 2\Lambda^h \delta_{k,r}$.

Near the highest critical point the curve (4.93) behaves as

$$u_r + z + \frac{\mu}{z} = c x^h + \delta u_k x^j, \quad (4.100)$$

where the second term on the rhs with $j = h - (e_k + 1)$ represents a small perturbation around the criticality at $\delta u_k = 0$. A constant c is irrelevant and will be set to $c = 1$. Let $u_r = \pm 2\Lambda^h$, $x = \delta u_k^{1/(h-j)} s$ and $z \pm \Lambda^h = \rho$, then (4.100) becomes

$$\rho \simeq \delta u_k^{\frac{h}{2(h-j)}} (\mp\Lambda^h)^{\frac{1}{2}} (s^h + s^j)^{\frac{1}{2}}. \quad (4.101)$$

We now apply the technique of [46] to verify the scaling behavior of the period integral of the Seiberg-Witten differential λ_{SW} . For the curve (4.93) it is known that $\lambda_{SW} = x dz/z$. Near the critical value $z = \mp\sqrt{\mu}$ we evaluate

$$\begin{aligned} \oint \lambda_{SW} &= \oint x \frac{dz}{z} \simeq \oint x d\rho \\ &\simeq \delta u_k^{\frac{h+2}{2(h-j)}} \oint ds \frac{hs^h + js^j}{(s^h + s^j)^{1/2}}. \end{aligned} \quad (4.102)$$

Since the period has the mass dimension one we read off critical exponents

$$\frac{2(e_k + 1)}{h + 2}, \quad k = 1, 2, \dots, r \quad (4.103)$$

in agreement with the results obtained earlier for $N = 2$ ADE Yang-Mills theories [46],[47].

When our $N = 1$ theory is viewed as $N = 2$ theory perturbed by the tree-level superpotential (4.58) we understand that the mass gap in $N = 1$ theory arises from the dyon condensation [3]. Let us show that the dyon condensate vanishes as we approach the $N = 2$ highest critical point under $N = 1$ perturbation. The $SU(2) \times U(1)^{r-1}$ vacuum in $N = 1$ theory corresponds to the $N = 2$ vacuum where a single monopole or dyon becomes massless. The low-energy effective superpotential takes the form

$$W_m = \sqrt{2}AM\tilde{M} + \sum_{k=1}^r g_k U_k, \quad (4.104)$$

where A is the $N = 1$ chiral superfield in the $N = 2$ $U(1)$ vector multiplet, M, \tilde{M} are the $N = 1$ chiral superfields of an $N = 2$ dyon hypermultiplet and U_k represent the superfields corresponding to Casimirs $u_k(\Phi)$. We will use lower-case letters to denote the lowest components of the corresponding upper-case superfields. Note that $\langle a \rangle = 0$ in the vacuum with a massless soliton.

The equation of motion $dW_m = 0$ is given by

$$-\frac{g_k}{\sqrt{2}} = \frac{\partial A}{\partial U_k} M\tilde{M}, \quad 1 \leq k \leq r \quad (4.105)$$

and $AM = A\tilde{M} = 0$, from which we have

$$\frac{g_k}{g_r} = \frac{\partial a / \partial u_k}{\partial a / \partial u_r}, \quad 1 \leq k \leq r-1, \quad (4.106)$$

when $\langle a \rangle = 0$. The vicinity of $N = 2$ highest criticality may be parametrized by

$$\langle u_k \rangle = \pm 2\Lambda^h \delta_{k,r} + c_k \epsilon^{e_k+1}, \quad c_k = \text{constant}, \quad (4.107)$$

where ϵ is an overall mass scale. From (4.102) one obtains

$$\frac{\partial a}{\partial u_k} \simeq \epsilon^{\frac{h}{2}-e_k}, \quad 1 \leq k \leq r, \quad (4.108)$$

so that

$$\frac{g_k}{g_r} \simeq \epsilon^{h-e_k-1} \longrightarrow 0, \quad 1 \leq k \leq r-1 \quad (4.109)$$

as $\epsilon \rightarrow 0$. The scaling behavior of dyon condensate is easily derived from (4.105)

$$\langle m \rangle = \left(-\frac{g_r}{\sqrt{2}\partial a / \partial u_r} \right)^{1/2} \simeq \sqrt{g_r} \epsilon^{(h-2)/4} \longrightarrow 0. \quad (4.110)$$

Therefore the gap in the $N = 1$ confining phase vanishes. We thus find that $N = 1$ ADE gauge theory with an adjoint matter with a tree-level superpotential

$$W_{\text{crit}} = g_r u_r(\Phi) \tag{4.111}$$

exhibits non-trivial fixed points. The higher-order polynomial $u_r(\Phi)$ is a dangerously irrelevant operator which is irrelevant at the UV gaussian fixed point, but affects the long-distance behavior significantly [40].

Chapter 5

$N = 2$ Gauge Theory with Matter Multiplets

In this chapter, we extend our analysis to describe the Coulomb phase of $N = 1$ supersymmetric gauge theories with N_f flavors of chiral matter multiplets Q^i, \tilde{Q}_j ($1 \leq i, j \leq N_f$) in addition to the adjoint matter Φ . Here Q belongs to an irreducible representation \mathcal{R} of the gauge group G with the dimension $d_{\mathcal{R}}$ and \tilde{Q} belongs to the conjugate representation of \mathcal{R} . A tree-level superpotential consists of the Yukawa-like term $\tilde{Q}\Phi^l Q$ in addition to the Casimir terms built out of Φ , and we shall consider arbitrary classical gauge groups and ADE gauge groups. In the appropriate limit the theory is reduced to $N = 2$ supersymmetric QCD.

5.1 Classical gauge groups and fundamental matters

We start with discussing $N = 1$ $SU(N_c)$ supersymmetric gauge theory with an adjoint matter field Φ , N_f flavors of fundamentals Q and anti-fundamentals \tilde{Q} . We take a tree-level superpotential

$$W = \sum_{n=1}^{N_c} g_n u_n + \sum_{l=0}^r \text{Tr}_{N_f} \lambda_l \tilde{Q} \Phi^l Q, \quad u_n = \frac{1}{n} \text{Tr} \Phi^n, \quad (5.1)$$

where $\text{Tr}_{N_f} \lambda_l \tilde{Q} \Phi^l Q = \sum_{i,j=1}^{N_f} (\lambda_l)_j^i \tilde{Q}_i \Phi^l Q^j$ and $r \leq N_c - 1$. If we set $(\lambda_0)_j^i = m_j^i$ with $[m, m^\dagger] = 0$, $(\lambda_1)_j^i = \delta_j^i$, $(\lambda_l)_j^i = 0$ for $l > 1$ and all $g_i = 0$, eq.(5.1) recovers the superpotential in $N = 2$ $SU(N_c)$ supersymmetric QCD with quark mass m . The second term in (5.1) was considered in a recent work [48].

Let us focus on the classical vacua with $Q = \tilde{Q} = 0$ and an unbroken $SU(2) \times U(1)^{N_c-2}$ symmetry which means $\Phi = \text{diag}(a_1, a_1, a_2, a_3, \dots, a_{N_c-1})$ up to gauge transformations. (Note that the superpotential (5.1) has no classical vacua with unbroken $U(1)^{N_c-1}$.) In this vacuum, we will evaluate semiclassically the low-energy effective superpotential. Our procedure is slightly different from that adopted in [18] upon treating Q and \tilde{Q} . We investigate the tree-level parameter region where the Higgs mechanism occurs at very high energies and the adjoint matter field Φ is quite heavy. Then the massive particles are integrated out and the scale matching relation becomes

$$\Lambda_L^{6-N_f} = g_{N_c}^2 \Lambda^{2N_c-N_f}, \quad (5.2)$$

where Λ is the dynamical scale of high-energy $SU(N_c)$ theory with N_f flavors and Λ_L is the scale of low-energy $SU(2)$ theory with N_f flavors. Eq.(5.2) is derived by following the $SU(N_c)$ Yang-Mills case [18] while taking into account the existence of fundamental flavors at low energies [40].

The semiclassical superpotential in low-energy $SU(2)$ theory with N_f flavors reads

$$W = \sum_{n=1}^{N_c} g_n u_n^{cl} + \sum_{l=0}^r a_1^l \text{Tr}_{N_f} \lambda_l \tilde{Q} Q \quad (5.3)$$

which is obtained by substituting the classical values of Φ and integrating out all the fields except for those coupled to the $SU(2)$ gauge boson. Here, the constraint $\text{Tr} \Phi^{cl} = a_1 + \sum_{i=1}^{N_c-1} a_i = 0$ and the classical equation of motion $\sum_{i=1}^{N_c-1} a_i = -g_{N_c-1}/g_{N_c}$ yield [20]

$$a_1 = \frac{g_{N_c-1}}{g_{N_c}}. \quad (5.4)$$

Below the flavor masses which can be read off from the superpotential (5.3), the low-energy theory becomes $N = 1$ $SU(2)$ Yang-Mills theory with the superpotential

$$W = \sum_{n=1}^{N_c} g_n u_n^{cl}. \quad (5.5)$$

This low-energy theory has the dynamical scale Λ_{YM} which is related to Λ through

$$\Lambda_{YM}^6 = \det \left(\sum_{l=0}^r \lambda_l a_1^l \right) g_{N_c}^2 \Lambda^{2N_c-N_f}. \quad (5.6)$$

As in the previous literatures [18],[19] we simply assume here that the superpotential (5.5) and the scale matching relation (5.6) are exact for any values of the tree-level parameters. Now we add to (5.5) a dynamically generated piece which arises from gaugino condensation in $SU(2)$ Yang-Mills theory. The resulting effective superpotential W_L where all the matter fields have been integrated out is thus given by

$$\begin{aligned} W_L &= \sum_{n=1}^{N_c} g_n u_n^{cl} \pm 2\Lambda_{YM}^3 \\ &= \sum_{n=1}^{N_c} g_n u_n^{cl} \pm 2g_{N_c} \sqrt{A(a_1)} \end{aligned} \quad (5.7)$$

with A being defined as $A(x) \equiv \Lambda^{2N_c - N_f} \det \left(\sum_{l=0}^r \lambda_l x^l \right)$. From $\langle u_n \rangle = \partial W_L / \partial g_n$ we find

$$\langle u_n \rangle = u_n^{cl}(g) \pm \delta_{n, N_c-1} \frac{A'(a_1)}{\sqrt{A(a_1)}} \pm \delta_{n, N_c} \frac{1}{\sqrt{A(a_1)}} (2A(a_1) - a_1 A'(a_1)). \quad (5.8)$$

If we define a hyperelliptic curve

$$y^2 = P(x)^2 - 4A(x), \quad (5.9)$$

where $P(x) = \langle \det(x - \Phi) \rangle$ is the characteristic equation of Φ , the curve is quadratically degenerate at the vacuum expectation values (5.8). This can be seen by plugging (5.8) in $P(x)$

$$P(x) = P_{cl}(x) \mp x \frac{A'(a_1)}{\sqrt{A(a_1)}} \mp \frac{1}{\sqrt{A(a_1)}} (2A(a_1) - a_1 A'(a_1)), \quad (5.10)$$

where $P_{cl}(x) = \det(x - \Phi_{cl})$, and hence

$$P(a_1) = \mp 2\sqrt{A(a_1)}, \quad P'(a_1) = \mp \frac{A'(a_1)}{\sqrt{A(a_1)}}. \quad (5.11)$$

Then the degeneracy of the curve is confirmed by checking $y^2|_{x=a_1} = 0$ and $\frac{\partial}{\partial x} y^2|_{x=a_1} = 0$.

The transition points from the confining to the Coulomb phase are reached by taking the limit $g_i \rightarrow 0$ while keeping the ratio g_i/g_j fixed [18]. These points correspond to the singularities in the moduli space. Therefore the curve (5.9) is regarded as the curve relevant to describe the Coulomb phase of the theory with the tree-level superpotential $W = \sum_{l=0}^r \text{Tr}_{N_f} \lambda_l \tilde{Q} \Phi^l Q$. Indeed, the curve (5.9) agrees with the one obtained in [48].

Especially in the parameter region that has $N = 2$ supersymmetry, we find agreement with the curves for $N = 2$ $SU(N_c)$ QCD with $N_f < 2N_c - 1$ [9],[10],[11].[†]

The procedure discussed above can be also applied to the other classical gauge groups. Let us consider $N = 1$ $SO(2N_c)$ supersymmetric gauge theory with an adjoint matter field Φ which is an antisymmetric $2N_c \times 2N_c$ tensor, and $2N_f$ flavors of fundamentals Q . We assume a tree-level superpotential

$$W = \sum_{n=1}^{N_c-2} g_{2n} u_{2n} + g_{2(N_c-1)} s_{N_c-1} + \lambda v + \frac{1}{2} \sum_{l=0}^r \text{Tr}_{2N_f} \lambda_l Q \Phi^l Q, \quad (5.12)$$

where $r \leq 2N_c - 1$,

$$u_{2n} = \frac{1}{2n} \text{Tr} \Phi^{2n}, \quad v = \text{Pf} \Phi = \frac{1}{2^{N_c} N_c!} \epsilon_{i_1 i_2 j_1 j_2 \dots} \Phi^{i_1 i_2} \Phi^{j_1 j_2} \dots \quad (5.13)$$

and

$$k s_k + \sum_{i=1}^k i s_{k-i} u_{2i} = 0, \quad s_0 = -1, \quad k = 1, 2, \dots \quad (5.14)$$

Here, ${}^t \lambda_l = (-1)^l \lambda_l$ and the $N = 2$ supersymmetry is present when we set $(\lambda_0)_j^i = m_j^i$ where $[m, m^\dagger] = 0$, $(\lambda_1)_j^i = \text{diag}(i\sigma_2, i\sigma_2, \dots)$ with $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $(\lambda_l)_j^i = 0$ for $l > 1$ and all $g_i = 0$.

As in the case of $SU(N_c)$, we concentrate on the unbroken $SU(2) \times U(1)^{N_c-1}$ vacua with $\Phi = \text{diag}(a_1 \sigma_2, a_1 \sigma_2, a_2 \sigma_2, a_3 \sigma_2, \dots, a_{N_c-1} \sigma_2)$ and $Q = 0$. By virtue of using s_{N_c} instead of u_{2N_c} in (5.12) the degenerate eigenvalue of Φ_{cl} is expressed by g_i

$$a_1^2 = \frac{g_{2(N_c-2)}}{g_{2(N_c-1)}} \quad (5.15)$$

as found for the $SO(2N_c+1)$ case [19]. Note that the superpotential (5.12) has no classical vacua with unbroken $SO(4) \times U(1)^{N_c-1}$ when $g_{2(N_c-2)} \neq 0$. We also note that the fundamental representation of $SO(2N_c)$ is decomposed into two fundamental representations of $SU(2)$ under the above embedding. It is then observed that the scale matching relation between the high-energy $SO(2N_c)$ scale Λ and the scale Λ_L of low-energy $SU(2)$ theory with $2N_f$ fundamental flavors is given by

$$\Lambda_L^{6-2N_f} = g_{2(N_c-1)}^2 \Lambda^{4(N_c-1)-2N_f}. \quad (5.16)$$

[†]For $N_f = 2N_c - 1$ an instanton may generate a mass term and shift the bare quark mass in $A(x)$. If we include this effect the curve (5.9) completely agrees with the result in [11].

The superpotential for low-energy $N = 1$ $SU(2)$ QCD with $2N_f$ flavors can be obtained in a similar way to the $SU(N_c)$ case, but the duplication of the fundamental flavors are taken into consideration. After some manipulations it turns out that the superpotential for low-energy $N = 1$ $SU(2)$ QCD with $2N_f$ flavors is written as

$$W = \sum_{n=1}^{N_c-2} g_{2n} u_{2n}^{cl} + g_{2(N_c-1)} s_{N_c-1}^{cl} + \lambda v^{cl} + \sum_{l=0}^r a_1^l \text{Tr}_{2N_f} \lambda_l \widetilde{\mathbf{Q}} \mathbf{Q}, \quad (5.17)$$

where

$$\mathbf{Q}^j = \frac{1}{\sqrt{2}} \begin{pmatrix} Q_1^j - iQ_2^j \\ Q_3^j - iQ_4^j \end{pmatrix}, \quad \widetilde{\mathbf{Q}}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} Q_1^j + iQ_2^j \\ Q_3^j + iQ_4^j \end{pmatrix}. \quad (5.18)$$

Upon integrating out the $SU(2)$ flavors we have the scale matching between Λ and Λ_{YM} for $N = 1$ $SU(2)$ Yang-Mills theory

$$\Lambda_{YM}^6 = \det \left(\sum_{l=0}^r \lambda_l a_1^l \right) g_{2(N_c-1)}^2 \Lambda^{4(N_c-1)-2N_f}, \quad (5.19)$$

and we get the effective superpotential

$$\begin{aligned} W_L &= \sum_{n=1}^{N_c-2} g_n u_n^{cl} + g_{2(N_c-1)} s_{N_c-1}^{cl} + \lambda v^{cl} \pm 2\Lambda_{YM}^3 \\ &= \sum_{n=1}^{N_c-2} g_n u_n^{cl} + g_{2(N_c-1)} s_{N_c-1}^{cl} + \lambda v^{cl} \pm 2g_{2(N_c-1)} \sqrt{A(a_1)}, \end{aligned} \quad (5.20)$$

where A is defined by $A(x) \equiv \Lambda^{4(N_c-1)-2N_f} \det \left(\sum_{l=0}^r \lambda_l x^l \right) = A(-x)$.

The vacuum expectation values of gauge invariants are obtained from W_L as

$$\begin{aligned} \langle s_n \rangle &= s_n^{cl}(g) \pm \delta_{n, N_c-2} \frac{A'(a_1)}{\sqrt{A(a_1)}} \pm \delta_{n, N_c-1} \frac{1}{\sqrt{A(a_1)}} \left(2A(a_1) - a_1^2 A'(a_1) \right), \\ \langle v \rangle &= v^{cl}(g), \end{aligned} \quad (5.21)$$

where $A'(x) = \frac{\partial}{\partial x^2} A(x)$. It is now easy to see that a curve

$$y^2 = P(x)^2 - 4x^4 A(x) \quad (5.22)$$

with $P(x) = \langle \det(x - \Phi) \rangle$ is degenerate at these values of $\langle s_n \rangle$, $\langle v \rangle$, and reproduces the known $N = 2$ curve [12], [11].

The only difference between $SO(2N_c)$ and $SO(2N_c + 1)$ is that the gauge invariant $\text{Pf } \Phi$ vanishes for $SO(2N_c + 1)$. Thus, taking a tree-level superpotential

$$W = \sum_{n=1}^{N_c-1} g_{2n} u_{2n} + g_{2N_c} s_{N_c} + \frac{1}{2} \sum_{l=0}^r \text{Tr}_{2N_f} \lambda_l Q \Phi^l Q, \quad r \leq 2N_c, \quad (5.23)$$

we focus on the unbroken $SU(2) \times U(1)^{N_c-1}$ vacuum which has the classical expectation values $\Phi = \text{diag}(a_1 \sigma_2, a_1 \sigma_2, a_2 \sigma_2, \dots, a_{N_c-1} \sigma_2, 0)$ and $Q = 0$ [19]. As in the $SO(2N_c)$ case we make use of the scale matching relation between the high-energy scale Λ and the low-energy $N = 1$ $SU(2)$ Yang-Mills scale Λ_{YM}

$$\Lambda_{YM}^6 = \det \left(\sum_{l=0}^r \lambda_l a_1^l \right) g_{2N_c} g_{2(N_c-1)} \Lambda^{2(2N_c-1-N_f)}. \quad (5.24)$$

As a result we find the effective superpotential

$$\begin{aligned} W_L &= \sum_{n=1}^{N_c-1} g_{2n} u_n^{cl} + g_{2N_c} s_{N_c}^{cl} \pm 2\Lambda_{YM}^3 \\ &= \sum_{n=1}^{N_c-1} g_{2n} u_n^{cl} + g_{2N_c} s_{N_c}^{cl} \pm 2\sqrt{g_{2N_c} g_{2(N_c-1)} A(a_1)}, \end{aligned} \quad (5.25)$$

where A is defined as $A(x) \equiv \Lambda^{2(2N_c-1-N_f)} \det \left(\sum_{l=0}^r \lambda_l x^l \right)$.

Noting the relation $a_1^2 = g_{2(N_c-1)}/g_{2N_c}$ [19] we calculate the vacuum expectation values of gauge invariants

$$\begin{aligned} \langle s_n \rangle = s_n^{cl}(g) &\pm \delta_{n, N_c-1} \frac{1}{\sqrt{A(a_1)}} \left(\frac{A(a_1)}{a_1} + a_1 A'(a_1) \right) \\ &\pm \delta_{n, N_c} \frac{1}{\sqrt{A(a_1)}} \left(a_1 A(a_1) - a_1^3 A'(a_1) \right). \end{aligned} \quad (5.26)$$

For these $\langle s_n \rangle$ we observe the quadratic degeneracy of the curve

$$y^2 = \left(\frac{1}{x} P(x) \right)^2 - 4x^2 A(x), \quad (5.27)$$

where $P(x) = \langle \det(x - \Phi) \rangle$. In the $N = 2$ limit we see agreement with the curve constructed in [12],[11]. The confining phase superpotential for the $SO(5)$ gauge group was obtained also in [26].

Let us now turn to $Sp(2N_c)$ gauge theory. We take for matter content an adjoint field Φ and $2N_f$ fundamental fields Q . The $2N_c \times 2N_c$ tensor Φ is subject to ${}^t\Phi = J\Phi J$ with $J = \text{diag}(i\sigma_2, \dots, i\sigma_2)$. Our tree-level superpotential reads

$$W = \sum_{n=1}^{N_c-1} g_{2n} u_{2n} + g_{2N_c} s_{N_c} + \frac{1}{2} \sum_{l=0}^r \text{Tr}_{2N_f} \lambda_l Q J \Phi^l Q, \quad (5.28)$$

where ${}^t\lambda_l = (-1)^{l+1} \lambda_l$ and $r \leq 2N_c - 1$. The classical vacuum with the unbroken $SU(2) \times U(1)^{N_c-1}$ gauge group corresponds to

$$J\Phi = \text{diag}(\sigma_1 a_1, \sigma_1 a_1, \sigma_1 a_2, \dots, \sigma_1 a_{N_c-1}), \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (5.29)$$

where $a_1^2 = g_{2(N_c-1)}/g_{2N_c}$. The scale Λ_L for low-energy $SU(2)$ theory with $2N_f$ flavors is expressed as [19]

$$\Lambda_L^{6-2N_f} = \left(\frac{g_{2N_c}^2}{g_{2(N_c-1)}} \right)^2 \Lambda^{2(2N_c+2-N_f)}. \quad (5.30)$$

There exists a subtle point in the analysis of $Sp(2N_c)$ theory. When $Sp(2N_c)$ is broken to $SU(2) \times U(1)^{N_c-1}$ the instantons in the broken part of the gauge group play a role since the index of the embedding of the unbroken $SU(2)$ in $Sp(2N_c)$ is larger than one (see eq.(5.30)) [49],[50]. The possible instanton contribution to W_L will be of the same order in Λ as low-energy $SU(2)$ gaugino condensation. Therefore even in the lowest quantum corrections the instanton term must be added to W_L .

For clarity we begin with discussing $Sp(4)$ Yang-Mills theory. In this theory by the symmetry and holomorphy the effective superpotential is determined to take the form $W_L = f\left(\frac{g_4}{g_2} \Lambda^2\right) \frac{g_4^2}{g_2} \Lambda^6$ with f being certain holomorphic function. If we assume that there is only one-instanton effect, the precise form of W_L including the low-energy gaugino condensation effect may be given by

$$W_L = 2 \frac{g_4^2}{g_2} \Lambda^6 \pm 2 \frac{g_4^2}{g_2} \Lambda^6, \quad (5.31)$$

as in the case of $SO(4) \simeq SU(2) \times SU(2)$ breaking to the diagonal $SU(2)$. This is due to the fact $Sp(4) \simeq SO(5)$ and the natural embedding of $SO(4)$ in $SO(5)$. Our low-energy $SU(2)$ gauge group is identified with the one diagonally embedded in $SO(4) \simeq SU(2) \times SU(2)$ [49],[51]. Accordingly, in $Sp(2N_c)$ Yang-Mills theory, we first make the

matching at the scale of $Sp(2N_c)/Sp(4)$ W bosons by taking all the $a_1 - a_i$ large. Then the low-energy superpotential is found to be

$$W_L = W_{cl} + 2\frac{g_{2N_c}}{a_1^2}\Lambda^{2(N_c+1)} \pm 2\frac{g_{2N_c}}{a_1^2}\Lambda^{2(N_c+1)}. \quad (5.32)$$

Let us turn on the coupling to fundamental flavors Q and evaluate the instanton contribution. When flavor masses vanish there is a global $O(2N_f) \simeq SO(2N_f) \times \mathbf{Z}_2$ symmetry. The couplings λ_l and instantons break a ‘‘parity’’ symmetry \mathbf{Z}_2 . We treat this \mathbf{Z}_2 as unbroken by assigning odd parity to the instanton factor $\Lambda^{2N_c+2-N_f}$ and $O(2N_f)$ charges to λ_l . Symmetry consideration then leads to the one-instanton factor proportional to $B(a_1)$ where

$$B(x) = \Lambda^{2N_c+2-N_f} \text{Pf} \left(\sum_{l \text{ even}} \lambda_l x^l \right). \quad (5.33)$$

Note that $B(x)$ is parity invariant since Pfaffian has odd parity. Thus the superpotential for low-energy $N = 1$ $SU(2)$ QCD with $2N_f$ flavors including the instanton effect turns out to be

$$W = \sum_{n=1}^{N_c-1} g_{2n} u_{2n}^{cl} + g_{2N_c} s_{N_c}^{cl} + \sum_{l=0}^r a_1^l \text{Tr}_{2N_f} \lambda_l \widetilde{\mathbf{Q}} \mathbf{Q} + 2\frac{g_{2N_c}^2}{g_{2(N_c-1)}} B(a_1), \quad (5.34)$$

where

$$\mathbf{Q}^j = \begin{pmatrix} Q_1^j \\ Q_3^j \end{pmatrix}, \quad \widetilde{\mathbf{Q}}_j = \begin{pmatrix} Q_2^j \\ Q_4^j \end{pmatrix}. \quad (5.35)$$

When integrating out the $SU(2)$ flavors, the scale matching relation between Λ and the scale Λ_{YM} of $N = 1$ $SU(2)$ Yang-Mills theory becomes

$$\Lambda_{YM}^6 = \det \left(\sum_{l=0}^r \lambda_l a_1^l \right) \left(\frac{g_{2N_c}^2}{g_{2(N_c-1)}} \right)^2 \Lambda^{2(2N_c+2-N_f)}, \quad (5.36)$$

and we finally obtain the effective superpotential

$$\begin{aligned} W_L &= \sum_{n=1}^{N_c-1} g_n u_n^{cl} + g_{2N_c} s_{N_c}^{cl} \pm 2\Lambda_{YM}^3 + 2\frac{g_{2N_c}^2}{g_{2(N_c-1)}} B(a_1) \\ &= \sum_{n=1}^{N_c-1} g_n u_n^{cl} + g_{2N_c} s_{N_c}^{cl} + 2\frac{g_{2N_c}^2}{g_{2(N_c-1)}} \left(B(a_1) \pm \sqrt{A(a_1)} \right), \end{aligned} \quad (5.37)$$

where $A(x) \equiv \Lambda^{2(2N_c+2-N_f)} \det \left(\sum_{l=0}^r \lambda_l x^l \right)$.

The gauge invariant expectation values $\langle s_n \rangle$ are

$$\begin{aligned} \langle s_n \rangle = s_n^{cl}(g) &+ \delta_{n, N_c-1} \frac{1}{a_1^4} \left(-2B(a_1) + 2a_1^2 B'(a_1) \pm \frac{1}{\sqrt{A(a_1)}} \left(-2A(a_1) + a_1^2 A'(a_1) \right) \right) \\ &+ \delta_{n, N_c} \frac{1}{a_1^2} \left(4B(a_1) - 2a_1^2 B'(a_1) \pm \frac{1}{\sqrt{A(a_1)}} \left(4A(a_1) - a_1^2 A'(a_1) \right) \right). \end{aligned} \quad (5.38)$$

Substituting these into a curve

$$x^2 y^2 = \left(x^2 P(x) + 2B(x) \right)^2 - 4A(x), \quad (5.39)$$

we see that the curve is degenerate at (5.38). In this case too our result (5.39) agrees with the $N = 2$ curve obtained in [11].

Before concluding this section, we should note that the effective superpotentials W_L obtained in this section are also confirmed in the approach based on the brane dynamics [53, 54].

5.2 ADE gauge groups and various matters

Let us consider $N = 1$ gauge theory with the ADE gauge group and N_f flavors of chiral matter multiplets Q^i, \tilde{Q}_j in addition to the adjoint matter Φ . We take a tree-level superpotential

$$W = \sum_{k=1}^r g_k u_k(\Phi) + \sum_{l=0}^q \text{Tr}_{N_f} \gamma_l \tilde{Q} \Phi_R^l Q, \quad (5.40)$$

where Φ_R is a $d_R \times d_R$ matrix representation of Φ in \mathcal{R} and $(\gamma_l)_{ij}$, $1 \leq i, j \leq N_f$, are the coupling constants and q should be restricted so that $\tilde{Q} \Phi_R^l Q$ is irreducible in the sense that it cannot be factored into gauge invariants. If we set $(\gamma_0)_j^i = m_j^i$ with $[m, m^\dagger] = 0$, $(\gamma_1)_j^i = \sqrt{2} \delta_j^i$, $(\gamma_l)_j^i = 0$ for $l > 1$ and all $g_i = 0$, (5.40) reduces to the superpotential in $N = 2$ supersymmetric Yang-Mills theory with massive N_f hypermultiplets.

Let us focus on the classical vacua of the Coulomb phase with $Q = \tilde{Q} = 0$ and an unbroken $SU(2) \times U(1)^{r-1}$ gauge group symmetry. The vacuum condition for Φ is given by (4.82) and the classical vacuum takes the form as in the Yang-Mills case

$$\Phi_R = \text{diag}(a \cdot \lambda_1, a \cdot \lambda_2, \dots, a \cdot \lambda_{d_R}), \quad (5.41)$$

where λ_i are the weights of the representation \mathcal{R} . In this vacuum, we will evaluate semiclassically the low-energy effective superpotential in the tree-level parameter region where the Higgs mechanism occurs at very high energies and the adjoint matter field Φ is quite heavy. Then the massive particles are integrated out and we get low-energy $SU(2)$ theory with flavors.

This integrating-out process results in the scale matching relation which is essentially the same as the Yang-Mills case (4.86) except that we here have to take into account flavor loops. The one instanton factor in high-energy theory is given by $\Lambda^{2h-l(\mathcal{R})N_f}$. Here the index $l(\mathcal{R})$ of the representation \mathcal{R} is defined by $l(\mathcal{R})\delta_{ab} = \text{Tr}(T_a T_b)$ where T_a is the representation matrix of \mathcal{R} with root vectors normalized as $\alpha^2 = 2$. The index is always an integer [52]. The scale matching relation becomes

$$\Lambda_L^{3\cdot 2-l(\mathcal{R})N_f} = g_r^2 \Lambda^{2h-l(\mathcal{R})N_f}, \quad (5.42)$$

where Λ_L is the scale of low-energy $SU(2)$ theory with massive flavors.

To consider the superpotential for low-energy $SU(2)$ theory with N_f flavors we decompose the matter representation \mathcal{R} of G in terms of the $SU(2)$ subgroup. We have

$$\mathcal{R} = \bigoplus_{s=1}^{n_{\mathcal{R}}} \mathcal{R}_{SU(2)}^s \oplus \text{singlets}, \quad (5.43)$$

where $\mathcal{R}_{SU(2)}^s$ stands for a non-singlet $SU(2)$ representation. Accordingly Q^i is decomposed into $SU(2)$ singlets and \mathbf{Q}_s^i ($1 \leq i \leq N_f$, $1 \leq s \leq n_{\mathcal{R}}$) in an $SU(2)$ representation $\mathcal{R}_{SU(2)}^s$. \tilde{Q}_i is decomposed in a similar manner. The singlet components are decoupled in low-energy $SU(2)$ theory.

The semiclassical superpotential for $SU(2)$ theory with N_f flavors is now given by

$$W = \sum_{k=1}^r g_k u_k^{cl} + \sum_{s=1}^{n_{\mathcal{R}}} \sum_{l=0}^q (a \cdot \lambda_{\mathcal{R}_s})^l \text{Tr}_{N_f} \gamma_l \tilde{\mathbf{Q}}_s \mathbf{Q}_s, \quad (5.44)$$

where $\lambda_{\mathcal{R}_s}$ is a weight of \mathcal{R} which branches to the weights in $\mathcal{R}_{SU(2)}^s$. Here we assume that \mathcal{R} is a representation which does not break up into integer spin representations of $SU(2)$; otherwise we would be in trouble when $\gamma_0 = 0$. The fundamental representations of ADE groups except for E_8 are in accord with this assumption.

We now integrate out massive flavors to obtain low-energy $N = 1$ $SU(2)$ Yang-Mills theory with the dynamical scale Λ_{YM} . Reading off the flavor masses from (5.44) we get

the scale matching

$$\begin{aligned}\Lambda_{YM}^{3.2} &= g_r^2 A(a), \\ A(a) &\equiv \Lambda^{2h-l(\mathcal{R})N_f} \prod_{s=1}^{n_{\mathcal{R}}} \left\{ \det \left(\sum_{l=0}^q \gamma_l(a \cdot \lambda_{\mathcal{R}_s})^l \right)^{l(\mathcal{R}_{SU(2)}^s)} \right\},\end{aligned}\quad (5.45)$$

where $l(\mathcal{R}_{SU(2)}^s)$ is the index of $\mathcal{R}_{SU(2)}^s$ which is related to $l(\mathcal{R})$ through

$$l(\mathcal{R}) = \sum_{s=1}^{n_{\mathcal{R}}} l(\mathcal{R}_{SU(2)}^s). \quad (5.46)$$

The index of the spin $m/2$ representation of $SU(2)$ is given by $m(m+1)(m+2)/6$.

Including the effect of $SU(2)$ gaugino condensation we finally arrive at the effective superpotential for low-energy $SU(2)$ theory

$$W_L = W_{cl}(g) \pm 2\Lambda_{YM}^3 = W_{cl}(g) \pm 2g_r \sqrt{A(a)}, \quad (5.47)$$

The expectation values $\langle u_k \rangle = \partial W_L / \partial g_k$ are found to be

$$\begin{aligned}\langle u_j \rangle &= u_j^{cl} \pm 2 \frac{\partial \sqrt{A}}{\partial g_j'}, \quad 1 \leq j \leq r-1, \\ \langle u_r \rangle &= u_r^{cl} \pm 2 \left(\sqrt{A} + g_r \sum_{k=1}^{r-1} \frac{\partial g_k'}{\partial g_r} \frac{\partial \sqrt{A}}{\partial g_k'} \right) \\ &= u_r^{cl} \pm 2 \left(\sqrt{A} - \sum_{k=1}^{r-1} g_k' \frac{\partial \sqrt{A}}{\partial g_k'} \right),\end{aligned}\quad (5.48)$$

where we have set $g_k' = g_k/g_r$ and used the fact that u_k^{cl} and A are functions of g_k' since a_i in (5.47) are solutions of (4.60) (see also (4.82)).

Let us show that the vacuum expectation values (5.48) obey the singularity condition for the family of $(r-1)$ -dimensional complex manifolds defined by $\mathcal{W} = 0$ with coordinates z, x_1, \dots, x_{r-1} where

$$\mathcal{W} = z + \frac{A(x_n)}{z} - \sum_{i=1}^r x_i (u_i - u_i^{cl}(x_n)). \quad (5.49)$$

Here we have introduced the variables x_i ($1 \leq i \leq r-1$) instead of g_i' to express $A(g_n')$ and $u_i^{cl}(g_n')$, $x_r = 1$ and u_i are moduli parameters. The manifold $\mathcal{W} = 0$ is singular when

$$\frac{\partial \mathcal{W}}{\partial z} = 0, \quad \frac{\partial \mathcal{W}}{\partial x_i} = 0. \quad (5.50)$$

Then, if we set $z = \pm\sqrt{A(x_k)}$, $x_k = g'_k$ and $u_j = \langle u_j \rangle$ it is easy to show that the singularity conditions are satisfied

$$\begin{aligned}
\mathcal{W}| &= \pm 2\sqrt{A(g'_k)} - \sum_{i=1}^r g'_i (\langle u_i \rangle - u_i^{cl}(g'_k)) = 0, \\
\left. \frac{\partial \mathcal{W}}{\partial z} \right| &= 0, \\
\left. \frac{\partial \mathcal{W}}{\partial x_j} \right| &= \pm \frac{1}{\sqrt{A(g'_k)}} \frac{\partial A(g'_k)}{\partial g'_j} - \langle u_j \rangle + \frac{\partial}{\partial g'_j} \left(\sum_{i=1}^r g'_i u_i^{cl}(g'_k) \right) \\
&= -u_j^{cl}(g'_k) + g_r \frac{\partial}{\partial g_j} \left(\frac{W_{cl}(g)}{g_r} \right) = 0, \quad 1 \leq j \leq r-1. \tag{5.51}
\end{aligned}$$

Thus the singularities of the manifold defined by $\mathcal{W} = 0$ are parametrized by the expectation values $\langle u_k \rangle$.

Let us explain how the known curves for $SU(N_c)$ and $SO(2N_c)$ supersymmetric QCD are reproduced from (5.49). First we consider $SU(N_c)$ theory with N_f fundamental flavors. Here we denote the degree i Casimir by u_i and correspondingly change the notations for x_j and g'_j . It is shown in [20],[21] that

$$A = \Lambda^{2N_c - N_f} \det_{N_f} \left(\sum_{l=0}^q (a^1)^l \gamma_l \right), \quad a^1 = g'_{N_c-1}, \tag{5.52}$$

and hence (5.49) becomes

$$\mathcal{W} = z + \frac{A(x_{N_c-1})}{z} - \sum_{i=2}^{N_c} x_i (u_i - u_i^{cl}(x_n)). \tag{5.53}$$

Since A depends only on x_{N_c-1} one can eliminate other variables x_1, \dots, x_{N_c-2} by imposing $\partial \mathcal{W} / \partial x_j = 0$ to get the relation

$$u_j^{cl}(x_n) = u_j \tag{5.54}$$

for $2 \leq j \leq N_c - 2$, and then

$$\mathcal{W} = z + \frac{A(x_{N_c-1})}{z} - (u_{N_c} - u_{N_c}^{cl}(x_n)) - x_{N_c-1} (u_{N_c-1} - u_{N_c-1}^{cl}(x_n)). \tag{5.55}$$

Remember that

$$0 = \det(a^1 - \Phi_{cl}) = (a^1)^{N_c} - s_2^{cl} (a^1)^{N_c-1} - \dots - s_{N_c}^{cl}, \tag{5.56}$$

where

$$ks_k + \sum_{i=1}^k i s_{k-1} u_i = 0, \quad u_n = \frac{1}{n} \text{Tr} \Phi^n, \quad k = 1, 2, \dots \quad (5.57)$$

with $s_0 = -1$ and $s_1 = u_1 = 0$. We see with the aid of (5.56) that

$$\begin{aligned} u_{N_c}^{cl} + x_{N_c-1} u_{N_c-1}^{cl} &= (u_{N_c}^{cl} - s_{N_c}^{cl}) + x_{N_c-1} (u_{N_c-1}^{cl} - s_{N_c-1}^{cl}) + (s_{N_c}^{cl} + x_{N_c-1} s_{N_c-1}^{cl}) \\ &= (u_{N_c} - s_{N_c}) + x_{N_c-1} (u_{N_c-1} - s_{N_c-1}) \\ &\quad + \left((x_{N_c-1})^{N_c} - s_2 (x_{N_c-1})^{N_c-1} - \dots - s_{N_c-2} \right), \end{aligned} \quad (5.58)$$

where (5.54) and the fact that $s_{N_c} = u_{N_c} + (\text{polynomial of } u_k, 2 \leq k \leq N_c - 2)$ have been utilized. We now rewrite (5.55) as

$$\begin{aligned} \mathcal{W} &= z + \frac{A(x)}{z} - (u_{N_c} + x u_{N_c-1}) + (u_{N_c}^{cl} + x u_{N_c-1}^{cl}) \\ &= z + \frac{A(x)}{z} + x^{N_c} - s_2 x^{N_c-1} - \dots - s_{N_c}, \end{aligned} \quad (5.59)$$

where x_{N_c-1} was replaced by x for notational simplicity. This reproduces the hyperelliptic curve derived in [48],[21] after making a change of variable $y = z - A(x)/z$ and agrees with the $N = 2$ curve obtained in [9],[10],[11] in the $N = 2$ limit .

Next we consider $SO(2N_c)$ theory with $2N_f$ fundamental flavors Q . Following [21] we take a tree-level superpotential

$$W = \sum_{n=k}^{N_c-2} g_{2k} u_{2k} + g_{2(N_c-1)} s_{N_c-1} + \lambda v + \frac{1}{2} \sum_{l=0}^q \text{Tr}_{2N_f} \gamma_l Q \Phi^l Q, \quad (5.60)$$

where

$$\begin{aligned} u_{2k} &= \frac{1}{2k} \text{Tr} \Phi^{2k}, \quad 1 \leq k \leq N_c - 1, \\ v &= \text{Pf} \Phi = \frac{1}{2^{N_c} N_c!} \epsilon_{i_1 i_2 j_1 j_2 \dots} \Phi^{i_1 i_2} \Phi^{j_1 j_2} \dots \end{aligned} \quad (5.61)$$

and

$$ks_k + \sum_{i=1}^k i s_{k-i} u_{2i} = 0, \quad s_0 = -1, \quad k = 1, 2, \dots \quad (5.62)$$

According to [19] we have

$$(a^1)^2 = g'_{2(N_c-2)}, \quad \lambda' = 2 \prod_{j=2}^{N_c-1} (-i a^j), \quad v^{cl} = -g'_{2(N_c-2)} \lambda' / 2 \quad (5.63)$$

and [21]

$$A = \Lambda^{4(N_c-1)-2N_f} \det_{2N_f} \left(\sum_{l=0}^q (a^1)^l \gamma_l \right), \quad (5.64)$$

and thus

$$\mathcal{W} = z + \frac{A(x_{N_c-2})}{z} - \sum_{i=1}^{N_c-1} x_i (u_{2i} - u_{2i}^{cl}(x_n)) - x(v - v^{cl}(x_n)), \quad (5.65)$$

where $\lambda' = \lambda/g_{2(N_c-1)}$ was replaced by x and $g_{2i}/g_{2(N_c-1)}$ by x_i .

In view of (5.64) we again notice that there are redundant variables which can be eliminated by imposing the condition $\partial\mathcal{W}/\partial x_j = 0$ so as to obtain

$$u_{2j}^{cl}(x_n) = u_{2j} \quad (5.66)$$

for $1 \leq j \leq N_c - 3$. We then find

$$\begin{aligned} \mathcal{W} = z + \frac{A(x_{N_c-2})}{z} & - (u_{2(N_c-1)} - u_{2(N_c-1)}^{cl}(x_n)) - x_{N_c-2}(u_{2(N_c-2)} - u_{2(N_c-2)}^{cl}(x_n)) \\ & - x(v - v^{cl}(x_n)). \end{aligned} \quad (5.67)$$

Using $\det(a^1 - \Phi_{cl}) = 0$ we proceed further as in the $SU(N_c)$ case. The final result reads

$$\begin{aligned} \mathcal{W} & = z + \frac{A(y)}{z} + \frac{1}{y} (y^{N_c} - s_1 y^{N_c-1} - \dots - s_{N_c-1} y + v^{cl}(x_n)^2) \\ & \quad - x(v - v^{cl}(x_n)) \\ & = z + \frac{A(y)}{z} - \frac{1}{4} x^2 y + y^{N_c-1} - s_1 y^{N_c-2} - \dots - s_{N_c-1} - vx, \end{aligned} \quad (5.68)$$

where we have set $y = x_{N_c-2}$ and used (5.63). It is now easy to check that imposing $\partial\mathcal{W}/\partial x = 0$ to eliminate x yields the known curve in [21] which has the correct $N = 2$ limit [12],[11].

It should be noted here that adding gaussian variables in (5.59) and (5.68) we have

$$\begin{aligned} \mathcal{W}_{A_{n-1}} & = z + \frac{A(y_1)}{z} + y_1^n - s_2 y_1^{n-1} - \dots - s_n + y_2^2 + y_3^2, \\ \mathcal{W}_{D_n} & = z + \frac{A(y_1)}{z} - \frac{1}{4} y_2^2 y_1 + y_1^{n-1} - s_1 y_1^{n-2} - \dots - s_{n-1} - v y_2 + y_3^2. \end{aligned} \quad (5.69)$$

These are equations describing ALE spaces of AD type fibered over \mathbf{CP}^1 . Inclusion of matter hypermultiplets makes fibrations more complicated than those for pure Yang-Mills theory. For A_n the result is rather obvious, but for D_n it may be interesting to follow how

two variables y_1, y_2 come out naturally from (5.49). These variables are traced back to coupling constants $g_{2(n-2)}/g_{2(n-1)}, \lambda/g_{2(n-1)}$, respectively, and their degrees indeed agree $[y_1] = [g_{2(n-2)}/g_{2(n-1)}] = 2, [y_2] = [\lambda/g_{2(n-1)}] = n - 2$.

This observation suggests a possibility that even in the E_n case we may eliminate redundant variables and derive the desired ALE form of Seiberg-Witten geometry directly from (5.49). This issue is considered in the next subsection.

5.2.1 E_6 theory with fundamental matters

In this subsection we will show that an extension of [28] enables us to obtain exceptional Seiberg-Witten geometry with fundamental hypermultiplets. The resulting manifold takes the form of a fibration of the ALE space of type E_6 .

Let us consider $N = 1$ E_6 gauge theory with N_f fundamental matters Q^i, \tilde{Q}_j ($1 \leq i, j \leq N_f$) and an adjoint matter Φ . Q^i, \tilde{Q}_j are in **27** and $\overline{\mathbf{27}}$, and Φ in **78** of E_6 . The coefficient of the one-loop beta function is given by $b = 24 - 6N_f$, and hence the theory is asymptotically free for $N_f = 0, 1, 2, 3$ and finite for $N_f = 4$. We take a tree-level superpotential

$$W = \sum_{k \in \mathcal{S}} g_k s_k(\Phi) + \text{Tr}_{N_f} \gamma_0 \tilde{Q}Q + \text{Tr}_{N_f} \gamma_1 \tilde{Q}\Phi Q, \quad (5.70)$$

where $\mathcal{S} = \{2, 5, 6, 8, 9, 12\}$ denotes the set of degrees of E_6 Casimirs $s_k(\Phi)$ and $g_k, (\gamma_a)_i^j$ ($1 \leq i, j \leq N_f$) are coupling constants. A basis for the E_6 Casimirs will be specified momentarily. When we put $(\gamma_0)_j^i = \sqrt{2}m_j^i$ with $[m, m^\dagger] = 0, (\gamma_1)_i^j = \sqrt{2}\delta_i^j$ and all $g_k = 0$, (5.70) is reduced to the superpotential in $N = 2$ supersymmetric Yang-Mills theory with massive N_f hypermultiplets.

We now look at the Coulomb phase with $Q = \tilde{Q} = 0$. Since Φ is restricted to take the values in the Cartan subalgebra we express the classical value of Φ in terms of a vector *

$$a = \sum_{i=1}^6 a_i \alpha_i \quad (5.71)$$

with α_i being the simple roots of E_6 . Then the classical vacuum is parametrized by

$$\Phi^{cl} = \text{diag}(a \cdot \lambda_1, a \cdot \lambda_2, \dots, a \cdot \lambda_{27}), \quad (5.72)$$

*Our notation is slightly different from [28]. Here we use a_i with lower index instead of a^i in [28].

where λ_i are the weights for **27** of E_6 . For the notation of roots and weights we follow [52]. We define a basis for the E_6 Casimirs $u_k(\Phi)$ by

$$\begin{aligned}
u_2 &= -\frac{1}{12} \chi_2, & u_5 &= -\frac{1}{60} \chi_5, & u_6 &= -\frac{1}{6} \chi_6 + \frac{1}{6 \cdot 12^2} \chi_2^3, \\
u_8 &= -\frac{1}{40} \chi_8 + \frac{1}{180} \chi_2 \chi_6 - \frac{1}{2 \cdot 12^4} \chi_2^4, & u_9 &= -\frac{1}{7 \cdot 6^2} \chi_9 + \frac{1}{20 \cdot 6^3} \chi_2^2 \chi_5, \\
u_{12} &= -\frac{1}{60} \chi_{12} + \frac{1}{5 \cdot 6^3} \chi_6^2 + \frac{13}{5 \cdot 12^3} \chi_2 \chi_5^2 \\
&\quad + \frac{5}{2 \cdot 12^3} \chi_2^2 \chi_8 - \frac{1}{3 \cdot 6^4} \chi_2^3 \chi_6 + \frac{29}{10 \cdot 12^6} \chi_2^6,
\end{aligned} \tag{5.73}$$

where $\chi_n = \text{Tr } \Phi^n$. The standard basis $w_k(\Phi)$ are written in terms of u_k as follows

$$\begin{aligned}
w_2 &= \frac{1}{2} u_2, & w_5 &= -\frac{1}{4} u_5, & w_6 &= \frac{1}{96} (u_6 - u_2^3), \\
w_8 &= \frac{1}{96} \left(u_8 + \frac{1}{4} u_2 u_6 - \frac{1}{8} u_2^4 \right), & w_9 &= -\frac{1}{48} \left(u_9 - \frac{1}{4} u_2^2 u_5 \right), \\
w_{12} &= \frac{1}{3456} \left(u_{12} + \frac{3}{32} u_6^2 - \frac{3}{4} u_2^2 u_8 - \frac{3}{16} u_2^3 u_6 + \frac{1}{16} u_2^6 \right).
\end{aligned} \tag{5.74}$$

The basis $\{u_k\}$ and (5.74) were first introduced in [24].[†] In our superpotential (5.70) we then set

$$s_2 = w_2, \quad s_5 = w_5, \quad s_6 = w_6, \quad s_8 = w_8, \quad s_9 = w_9, \quad s_{12} = w_{12} - \frac{1}{4} w_6^2. \tag{5.75}$$

We will discuss later why this particular form is assumed.

The equations of motion are given by

$$\frac{\partial W(a)}{\partial a_i} = \sum_{k \in \mathcal{S}} g_k \frac{\partial s_k(a)}{\partial a_i} = 0. \tag{5.76}$$

Let us focus on the classical vacua with an unbroken $SU(2) \times U(1)^5$ gauge symmetry. Fix the $SU(2)$ direction by choosing the simple root α_1 , then we have the vacuum condition

$$a \cdot \alpha_1 = 2a_1 - a_2 = 0. \tag{5.77}$$

It follows from (5.76), (5.77) that

$$\frac{g_9}{g_{12}} = \frac{D_{1,9}}{D_{1,12}}$$

[†]The Casimirs $u_1, u_2, u_3, u_4, u_5, u_6$ in [24] are denoted here as $u_2, u_5, u_6, u_8, u_9, u_{12}$, respectively.

$$\begin{aligned}
&= -\frac{1}{8} \left(2a_1a_5a_4 - a_4a_3^2 + a_5^2a_4 + a_4^2a_3 - a_3a_6^2 + a_3^2a_6 \right. \\
&\quad \left. - 2a_1a_5^2 + 2a_1a_6^2 - 2a_4^2a_1 - a_5a_4^2 - 2a_1a_3a_6 + 2a_4a_3a_1 \right), \\
\frac{g_8}{g_{12}} &= \frac{D_{1,8}}{D_{1,12}} \\
&= -\frac{1}{48} \left(12a_1a_5^2a_4 - 6a_1^2a_5^2 - 6a_1^2a_6^2 - 4a_1^3a_3 + 4a_3^3a_1 + 2a_3^3a_4 + 2a_3^3a_6 - a_4^4 \right. \\
&\quad \left. - 3a_3^2a_6^2 - 3a_3^2a_4^2 - a_3^4 - a_6^4 - 12a_1a_5a_4^2 - 2a_5a_4a_6^2 + 8a_1a_3a_6^2 + 3a_1^4 \right. \\
&\quad \left. + 6a_1^2a_4a_3 - 8a_4a_3^2a_1 - 2a_5a_4a_3^2 - 2a_4^2a_3a_6 + 6a_1^2a_5a_4 - 2a_4a_3a_6^2 \right. \\
&\quad \left. + 2a_5a_4^2a_3 - 2a_5^2a_3a_6 + 2a_4a_3^2a_6 - a_5^4 - 2a_5^2a_4a_3 - 2a_1^2a_3^2 - 6a_1^2a_4^2 \right. \\
&\quad \left. + 2a_5^2a_6^2 + 2a_4^2a_6^2 + 2a_5^2a_3^2 + 6a_1^2a_3a_6 - 8a_1a_3^2a_6 + 8a_4^2a_3a_1 - 4a_5^2a_1a_3 \right. \\
&\quad \left. + 2a_5a_4a_3a_6 + 4a_5a_4a_1a_3 + 2a_3a_4^3 - 3a_4^2a_5^2 + 2a_4a_5^3 + 2a_4^3a_5 + 2a_6^3a_3 \right), \\
\frac{g_6}{g_{12}} &= \frac{D_{1,6}}{D_{1,12}} \\
&= \frac{1}{192} \left(4a_3^3a_1a_5^2 - 18a_4^3a_1^2a_5 + 13a_3^4a_1^2 - a_3^4a_5^2 - 7a_3^3a_6^3 + 9a_1^2a_6^4 + \dots \right), \quad (5.78)
\end{aligned}$$

where $D_{1,k}$ is the cofactor for a $(1, k)$ element of the 6×6 matrix $[\partial s_i(a)/\partial a_j]$, $i \in \mathcal{S}$ and $j = 1, \dots, 6$ [28]. In (5.78) the explicit expression for g_6/g_{12} is too long to be presented here, and hence suppressed. Denoting $y_1 = g_9/g_{12}$, $y_2 = g_8/g_{12}$, $y_3 = g_6/g_{12}$, we find that the others are expressed in terms of y_1, y_2

$$\frac{g_2}{g_{12}} = \frac{D_{2,2}}{D_{2,12}} = y_1^2 y_2, \quad \frac{g_5}{g_{12}} = \frac{D_{2,5}}{D_{2,12}} = y_1 y_2. \quad (5.79)$$

This means that our superpotential specified with Casimirs (5.75) realizes the $SU(2) \times U(1)^5$ vacua only when the coupling constants are subject to the relation (5.79).

Notice that reading off degrees of y_1, y_2, y_3 from (5.78) gives $[y_1] = 3$, $[y_2] = 4$, $[y_3] = 6$. Thus, if we regard y_1, y_2, y_3 as variables to describe the E_6 singularity, (5.78) and (5.79) may be identified as relevant monomials in versal deformations of the E_6 singularity. In fact we now point out an intimate relationship between classical solutions corresponding to the symmetry breaking $E_6 \supset SU(2) \times U(1)^5$ and the E_6 singularity. For this we examine the superpotential (5.40) at classical solutions

$$\begin{aligned}
W_{cl} &= g_{12} \sum_{k \in \mathcal{S}} \left(\frac{g_k}{g_{12}} \right) s_k^{cl}(a) \\
&= g_{12} \left(s_2^{cl} y_1^2 y_2 + s_5^{cl} y_1 y_2 + s_6^{cl} y_3 + s_8^{cl} y_2 + s_9^{cl} y_1 + s_{12}^{cl} \right). \quad (5.80)
\end{aligned}$$

Evaluating the RHS with the use of (5.77)-(5.79) leads to

$$W_{cl} = -g_{12} \left(2y_1^2 y_3 + y_2^3 - y_3^2 \right). \quad (5.81)$$

It is also checked explicitly that

$$\begin{aligned} -4y_1 y_3 &= 2s_2^{cl} y_1 y_2 + s_5^{cl} y_2 + s_9^{cl}, \\ -3y_2^2 &= s_2^{cl} y_1^2 + s_5^{cl} y_1 + s_8^{cl}, \\ -2y_1^2 + 2y_3 &= s_6^{cl}. \end{aligned} \quad (5.82)$$

To illustrate the meaning of (5.80)-(5.82) let us recall the standard form of versal deformations of the E_6 singularity

$$W_{E_6}(x_1, x_2, x_3; w) = x_1^4 + x_2^3 + x_3^2 + w_2 x_1^2 x_2 + w_5 x_1 x_2 + w_6 x_1^2 + w_8 x_2 + w_9 x_1 + w_{12}, \quad (5.83)$$

where the deformation parameters w_k are related to the E_6 Casimirs via (5.74) [24]. Then what we have observed in (5.80)-(5.82) is that when we express w_k in terms of a_i as $w_k = w_k^{cl}(a)$ the equations

$$W_{E_6} = \frac{\partial W_{E_6}}{\partial x_1} = \frac{\partial W_{E_6}}{\partial x_2} = \frac{\partial W_{E_6}}{\partial x_3} = 0 \quad (5.84)$$

can be solved by [‡]

$$x_1 = y_1(a), \quad x_2 = y_2(a), \quad x_3 = i \left(y_3(a) - y_1(a)^2 - \frac{s_6^{cl}(a)}{2} \right) \quad (5.85)$$

under the condition (5.77). This observation plays a crucial role in our analysis.

When applying the technique of confining phase superpotentials we usually take all coupling constants g_k as independent moduli parameters. To deal with $N = 1$ E_6 theory with fundamental matters, however, we find it appropriate to proceed as follows. First of all, motivated by the above observations for classical solutions, we keep three coupling constants $g'_6 = g_6/g_{12}$, $g'_8 = g_8/g_{12}$ and $g'_9 = g_9/g_{12}$ adjustable while the rest is fixed as $g'_2 = g'_8 g_9^2$, $g'_5 = g'_8 g'_9$ with $g'_k = g_k/g_{12}$. Taking this parametrization it is seen that the equations of motion are satisfied by virtue of (5.79) in the $SU(2) \times U(1)^5$ vacua (5.77).

[‡]We have observed a similar relation between the symmetry breaking solutions $SU(r+1)$ (or $SO(2r)$) $\supset SU(2) \times U(1)^{r-1}$ and the A_r (or D_r) singularity.

Note here that originally there exist six classical moduli a_i among which one is fixed by (5.77) and three are converted to $g'_9 = y_1(a)$, $g'_8 = y_2(a)$ and $g'_6 = y_3(a)$, and hence we are left with two classical moduli which will be denoted as ξ_i . Without loss of generality one may choose $\xi_2 = s_2^{cl}(a)$ and $\xi_5 = s_5^{cl}(a)$.

We now evaluate the low-energy effective superpotential in the $SU(2) \times U(1)^5$ vacua. $U(1)$ photons decouple in the integrating-out process. The standard procedure yields the effective superpotential for low-energy $SU(2)$ theory [18],[28]

$$W_L = -g_{12} \left(2y_1^2 y_3 + y_2^3 - y_3^2 \right) \pm 2\Lambda_{YM}^3, \quad (5.86)$$

where the second term takes account of $SU(2)$ gaugino condensation with Λ_{YM} being the dynamical scale for low-energy $SU(2)$ Yang-Mills theory. The low-energy scale Λ_{YM} is related to the high-energy scale Λ through the scale matching [28]

$$\begin{aligned} \Lambda_{YM}^6 &= g_{12}^2 A(a), \\ A(a) &\equiv \Lambda^{24-6N_f} \prod_{s=1}^6 \det_{N_f} (\gamma_0 + \gamma_1(a \cdot \lambda_s)), \end{aligned} \quad (5.87)$$

where λ_s are weights of **27** which branch to six $SU(2)$ doublets respectively under $E_6 \supset SU(2) \times U(1)^5$. Explicitly they are given in the Dynkin basis as

$$\begin{aligned} \lambda_1 &= (1, 0, 0, 0, 0, 0), & \lambda_2 &= (1, -1, 0, 0, 1, 0), \\ \lambda_3 &= (1, -1, 0, 1, -1, 0), & \lambda_4 &= (1, -1, 1, -1, 0, 0), \\ \lambda_5 &= (1, 0, -1, 0, 0, 1), & \lambda_6 &= (1, 0, 0, 0, 0, -1). \end{aligned} \quad (5.88)$$

Notice that $\sum_{s=1}^6 \lambda_s = 3\alpha_1$.

Let us first discuss the $N_f = 0$ case, i.e. E_6 pure Yang-Mills theory, for which $A(a)$ in (5.87) simply equals Λ^{24} . The vacuum expectation values are calculated from (5.86)

$$\begin{aligned} \frac{\partial W_L}{\partial g_{12}} &= \langle \widetilde{W}(y_1, y_2, y_3; s) \rangle = - \left(2y_1^2 y_3 + y_2^3 - y_3^2 \right) \pm 2\Lambda^{12}, \\ \frac{1}{g_{12}} \frac{\partial W_L}{\partial y_1} &= \left\langle \frac{\partial \widetilde{W}(y_1, y_2, y_3; s)}{\partial y_1} \right\rangle = -4y_1 y_3, \\ \frac{1}{g_{12}} \frac{\partial W_L}{\partial y_2} &= \left\langle \frac{\partial \widetilde{W}(y_1, y_2, y_3; s)}{\partial y_2} \right\rangle = -3y_2^2, \\ \frac{1}{g_{12}} \frac{\partial W_L}{\partial y_3} &= \left\langle \frac{\partial \widetilde{W}(y_1, y_2, y_3; s)}{\partial y_3} \right\rangle = -2y_1^2 + 2y_3, \end{aligned} \quad (5.89)$$

where y_1, y_2, y_3 and g_{12} have been treated as independent parameters as discussed before and

$$\widetilde{W}(y_1, y_2, y_3; s) = s_2 y_1^2 y_2 + s_5 y_1 y_2 + s_6 y_3 + s_8 y_2 + s_9 y_1 + s_{12}. \quad (5.90)$$

Define a manifold by $\mathcal{W}_0 = 0$ with four coordinate variables $z, y_1, y_2, y_3 \in \mathbf{C}$ and

$$\mathcal{W}_0 \equiv z + \frac{\Lambda^{24}}{z} - \left(2y_1^2 y_3 + y_2^3 - y_3^2 + \widetilde{W}(y_1, y_2, y_3; s) \right) = 0. \quad (5.91)$$

It is easy to show that the expectation values (5.89) parametrize the singularities of the manifold where

$$\frac{\partial \mathcal{W}_0}{\partial z} = \frac{\partial \mathcal{W}_0}{\partial y_1} = \frac{\partial \mathcal{W}_0}{\partial y_2} = \frac{\partial \mathcal{W}_0}{\partial y_3} = 0. \quad (5.92)$$

Making a change of variables $y_1 = x_1, y_2 = x_2, y_3 = -ix_3 + x_1^2 + s_6/2$ in (5.91) we have

$$z + \frac{\Lambda^{24}}{z} - W_{E_6}(x_1, x_2, x_3; w) = 0. \quad (5.93)$$

Thus the ALE space description of $N = 2$ E_6 Yang-Mills theory [29],[24] is obtained from the $N = 1$ confining phase superpotential.

We next turn to considering the fundamental matters. In the $N = 2$ limit we have $A(a) = \Lambda^{24-6N_f} \cdot 8^{N_f} \prod_{i=1}^{N_f} f(a, m_i)$ with $f(a, m) = \prod_{s=1}^6 (m + a \cdot \lambda_s)$. After some algebra we find

$$f(a, m) = m^6 + 2\xi_2 m^4 - 8m^3 y_1 + (\xi_2^2 - 12y_2) m^2 + 4\xi_5 m - 4y_2 \xi_2 - 8y_3, \quad (5.94)$$

where we have used (5.75)-(5.78). Let us recall that, in viewing (5.86), we think of $(y_1, y_2, y_3, \xi_2, \xi_5, g_{12})$ as six independent parameters. Then the quantum expectation values are given by

$$\begin{aligned} \frac{\partial W_L}{\partial g_{12}} &= \langle \widetilde{W}(y_1, y_2, y_3; s) \rangle = - \left(2y_1^2 y_3 + y_2^3 - y_3^2 \right) \pm 2\sqrt{A(y_1, y_2, y_3; \xi, m)}, \\ \frac{1}{g_{12}} \frac{\partial W_L}{\partial y_1} &= \left\langle \frac{\partial \widetilde{W}(y_1, y_2, y_3; s)}{\partial y_1} \right\rangle = -4y_1 y_3 \pm 2 \frac{\partial}{\partial y_1} \sqrt{A(y_1, y_2, y_3; \xi, m)}, \\ \frac{1}{g_{12}} \frac{\partial W_L}{\partial y_2} &= \left\langle \frac{\partial \widetilde{W}(y_1, y_2, y_3; s)}{\partial y_2} \right\rangle = -3y_2^2 \pm 2 \frac{\partial}{\partial y_2} \sqrt{A(y_1, y_2, y_3; \xi, m)}, \\ \frac{1}{g_{12}} \frac{\partial W_L}{\partial y_3} &= \left\langle \frac{\partial \widetilde{W}(y_1, y_2, y_3; s)}{\partial y_3} \right\rangle = -2y_1^2 + 2y_3 \pm 2 \frac{\partial}{\partial y_3} \sqrt{A(y_1, y_2, y_3; \xi, m)}. \end{aligned} \quad (5.95)$$

Similarly to the $N_f = 0$ case one can check that these expectation values satisfy the singularity condition for a manifold defined by

$$z + \frac{1}{z}A(y_1, y_2, y_3; \xi, m) - \left(2y_1^2y_3 + y_2^3 - y_3^2 + \widetilde{W}(y_1, y_2, y_3; s)\right) = 0. \quad (5.96)$$

Note that s_k in \widetilde{W} are quantum moduli parameters. What about ξ_2, ξ_5 in the one-instanton factor A ? Classically we have $\xi_i = s_i^{cl}$ as was seen before. The issue is thus whether the classical relations $\xi_i = s_i^{cl}$ receive any quantum corrections at the singularities. If there appear no quantum corrections, ξ_i in A can be replaced by quantum moduli parameters s_i . Let us simply assume here that $\xi_i = s_i^{cl} = \langle s_i \rangle$ for $i = 2, 5$ in the $N = 1$ $SU(2) \times U(1)^5$ vacua. This assumption seems quite plausible as long as we have inspected possible forms of quantum corrections due to gaugino condensates.

Now we find that Seiberg-Witten geometry of $N = 2$ supersymmetric QCD with gauge group E_6 is described by

$$z + \frac{1}{z}A(x_1, x_2, x_3; w, m) - W_{E_6}(x_1, x_2, x_3; w) = 0, \quad (5.97)$$

where a change of variables from y_i to x_i as in (5.93) has been made in (5.96) and

$$\begin{aligned} & A(x_1, x_2, x_3; w, m) \\ = & \Lambda^{24-6N_f} \cdot 8^{N_f} \prod_{i=1}^{N_f} \left(m_i^6 + 2w_2m_i^4 - 8m_i^3x_1 + (w_2^2 - 12x_2)m_i^2 \right. \\ & \left. + 4w_5m_i - 4w_2x_2 - 8(x_1^2 - ix_3 + w_6/2) \right). \end{aligned} \quad (5.98)$$

The manifold takes the form of ALE space of type E_6 fibered over the base \mathbf{CP}^1 . Note an intricate dependence of the fibering data over \mathbf{CP}^1 on the hypermultiplet masses. This is in contrast with the ALE space description of $N = 2$ $SU(N_c)$ and $SO(2N_c)$ gauge theories with fundamental matters. In (5.97), letting $m_i \rightarrow \infty$ while keeping $\Lambda^{24-6N_f} \prod_{i=1}^{N_f} m_i^6 \equiv \Lambda_0^{24}$ finite we recover the pure Yang-Mills result (5.93).

As a non-trivial check of our proposal (5.97) let us examine the semi-classical singularities. In the semi-classical limit $\Lambda \rightarrow 0$ the discriminant Δ for (5.97) is expected to take the form $\Delta \propto \Delta_G \Delta_M$ where Δ_G is a piece arising from the classical singularities associated with the gauge symmetry enhancement and Δ_M represents the semi-classical

singularities at which squarks become massless. When the N_f matter hypermultiplets belong to the representation \mathcal{R} of the gauge group G we have

$$\Delta_M = \prod_{i=1}^{N_f} \det_{d \times d}(m_i \mathbf{1} - \Phi^{cl}) = \prod_{i=1}^{N_f} P_G^{\mathcal{R}}(m_i; u), \quad (5.99)$$

where $d = \dim \mathcal{R}$, m_i are the masses, Φ^{cl} denotes the classical Higgs expectation values and $P_G^{\mathcal{R}}(x; u)$ is the characteristic polynomial for \mathcal{R} with u_i being Casimirs constructed from Φ^{cl} .

For simplicity, let us consider the case in which all the quarks have equal bare masses. Then we can change a variable x_3 to \tilde{x}_3 so that $A = A(\tilde{x}_3; w, m)$ is independent of x_1 and x_2 . Eliminating x_1 and x_2 from (5.97) by the use of

$$\frac{\partial W_{E_6}}{\partial x_1} = \frac{\partial W_{E_6}}{\partial x_2} = 0, \quad (5.100)$$

we obtain a curve which is singular at the discriminant locus of (5.97). The curve is implicitly defined through

$$\overline{W}_{E_6} \left(\tilde{x}_3; w_i - \delta_{i,12} \left(z + \frac{A(\tilde{x}_3; w, m)}{z} \right) \right) = 0, \quad (5.101)$$

where $\overline{W}_{E_6}(\tilde{x}_3; w_i) = W_{E_6}(x_1(\tilde{x}_3, w_i), x_2(\tilde{x}_3, w_i), \tilde{x}_3; w_i)$ and $x_1(\tilde{x}_3, w_i), x_2(\tilde{x}_3, w_i)$ are solutions of (5.100). Now the values of \tilde{x}_3 and z at singularities of this curve can be expanded in powers of $\Lambda^{\frac{24-6N_f}{2}}$. Then it is more or less clear that the classical singularities corresponding to massless gauge bosons are produced. Furthermore, if we denote as $R(\overline{W}, A)$ the resultant of $\overline{W}_{E_6}(\tilde{x}_3; w_i)$ and $A(\tilde{x}_3; w, m)$, then $R(\overline{W}, A) = 0$ yields another singularity condition of the curve in the limit $\Lambda \rightarrow 0$. We expect that $R(\overline{W}, A) = 0$ corresponds to the semi-classical massless squark singularities as is observed in the case of $N = 2$ $SU(N_c)$ QCD [9],[22]. Indeed, we have checked this by explicitly computing $R(\overline{W}, A)$ at sufficiently many points in the moduli space. For instance, taking $w_2 = 2, w_5 = 5, w_6 = 7, w_8 = 9, w_9 = 11$ and $w_{12} = 13$ in the $N_f = 1$ case, we get

$$\begin{aligned} & R(\overline{W}, A) \\ &= m^2 \left(3m^{10} + 12m^8 + \dots \right) \left(26973m^{27} + 258552m^{25} + \dots \right)^3 \\ & \quad \left(m^{27} + 24m^{25} + 240m^{23} + 240m^{22} + 2016m^{21} + 3360m^{20} + 16416m^{19} \right) \end{aligned}$$

$$\begin{aligned}
& +34944 m^{18} + 88080 m^{17} + 216576 m^{16} + 448864 m^{15} + 607488 m^{14} \\
& +2198272 m^{13} - 296000 m^{12} + 4177792 m^{11} - 3407104 m^{10} + 7796224 m^9 \\
& +10664448 m^8 - 31708160 m^7 + 41183232 m^6 - 21889792 m^5 + 15575040 m^4 \\
& -17125120 m^3 - 38456320 m^2 - 3461120 m + 9798656), \tag{5.102}
\end{aligned}$$

while the E_6 characteristic polynomial for **27** is given by

$$\begin{aligned}
& P_{E_6}^{\mathbf{27}}(x; u) \\
& = x^{27} + 12w_2x^{25} + 60w_2^2x^{23} + 48w_5x^{22} + (96w_6 + 168w_2^3)x^{21} + 336w_2w_5x^{20} \\
& \quad + (528w_2w_6 + 294w_2^4 + 480w_8)x^{19} + (1344w_9 + 1008w_2^2w_5)x^{18} + \dots \tag{5.103}
\end{aligned}$$

We now find a remarkable result that the last factor of (5.102) precisely coincides with $P_{E_6}^{\mathbf{27}}(m; u)$! Hence the manifold described by (5.97) correctly produces all the semi-classical singularities in the moduli space of $N = 2$ supersymmetric E_6 QCD.

If we choose another form of the superpotential (5.70), say, the superpotential with $s_i = w_i$ for $i \in \mathcal{S}$ instead of (5.75) we are unable to obtain Δ_M in (5.99). As long as we have checked the choice made in (5.75) is judicious in order to pass the semi-classical test. At present, we have no definite recipe to fix the tree-level superpotential which produces the correct semi-classical singularities, though it is possible to proceed by trial and error. In fact we can find Seiberg-Witten geometry for $N = 2$ $SO(2N_c)$ gauge theory with spinor matters and $N = 2$ $SU(N_c)$ gauge theory with antisymmetric matters [31].

In our result (5.97) it may be worth mentioning that the gaussian variable x_3 of the E_6 singularity appears in the fibering term.

5.2.2 Gauge symmetry breaking in Seiberg-Witten geometry

Starting with the $N = 2$ Seiberg-Witten geometry with E_6 gauge group with massive fundamental matters, we construct the Seiberg-Witten geometry with $SU(N_c)$ and $SO(2N_c)$ gauge groups with various matter contents, in the rest of this section. All these geometries we will obtain take the form of a fibration of the ALE spaces over a sphere.

To this end, we first discuss how to implement the gauge symmetry breaking in the general Seiberg-Witten geometry by giving appropriate VEV to the adjoint scalar field in the $N = 2$ vector multiplet.

Classically the VEV of the adjoint Higgs Φ is chosen to take the values in the Cartan subalgebra. The classical moduli space is then parametrized by a Higgs VEV vector $a = \sum_{i=1}^r a^i \alpha_i$. At the generic points in the classical moduli space, the gauge group G is completely broken to $U(1)^r$. However there are singular points where G is broken only partially to $\prod_i G'_i \times U(1)^l$ with G'_i being a simple subgroup of G . If we fix the gauge symmetry breaking scale to be large, the theory becomes $N = 2$ supersymmetric gauge theory with the gauge group $\prod_i G'_i \times U(1)^l$ and the initial Seiberg-Witten geometry reduces to the one describing the gauge group G'_i after taking an appropriate scaling limit.

We begin with the case of $N = 2$ supersymmetric $SU(r + 1)$ gauge theory with fundamental flavors. The Seiberg-Witten curve for this theory is given by [5, 9]

$$y^2 = \det_{r+1}(x - \Phi_{\mathcal{R}})^2 - \Lambda^{2(r+1)-N_f} \prod_{i=1}^{N_f} (m_i - x). \quad (5.104)$$

Choosing the classical value $\langle \Phi_{\mathcal{R}} \rangle_{cl}$ as

$$\begin{aligned} \langle \Phi_{\mathcal{R}} \rangle_{cl} &= \text{diag}(\langle a^1 \rangle, \langle a^2 \rangle - \langle a^1 \rangle, \langle a^3 \rangle - \langle a^2 \rangle, \dots, \langle a^r \rangle - \langle a^{r-1} \rangle, -\langle a^r \rangle) \\ &= \text{diag}(M, M, M, \dots, M, -rM), \end{aligned} \quad (5.105)$$

where M is a constant, we break the gauge group $SU(r + 1)$ down to $SU(r) \times U(1)$. Note that this parametrization is equivalent to $\langle a^j \rangle = jM$ which means $\langle a_j \rangle = \delta_{j,r}(r + 1)M$. Setting $a_i = \delta_{j,r}(r + 1)M + \delta a_i$ and $m_i = M + m'_i$, we take the scaling limit $M \rightarrow \infty$ with $\Lambda^{2r-N_f} = \frac{\Lambda^{2(r+1)-N_f}}{(r+1)M^2}$ held fixed. Then we are left with the Seiberg-Witten curve corresponding to the gauge group $SU(r)$

$$(y')^2 = \left\{ (x' - \delta a^1) (x' - (\delta a^2 - \delta a^1)) \cdots (x' - (-\delta a^{r-1})) \right\}^2 - \Lambda'^{2r-N_f} \prod_{i=1}^{N_f} (m'_i - x'), \quad (5.106)$$

where $y' = \frac{y}{\sqrt{r+1}M}$ and $x' = x - M$. Notice that we must shift the masses m_i to obtain the finite masses of hypermultiplets in the $SU(r)$ theory with N_f flavors.

Now we consider the case of $N = 2$ theory with a simple gauge group G . When we assume the nonzero VEV of the adjoint scalar, the largest non-Abelian gauge symmetry which is left unbroken has rank $r - 1$. As we will see shortly, this largest unbroken gauge symmetry is realized by choosing

$$\langle a_i \rangle = M \delta_{i,i_0}, \quad 1 \leq i \leq r, \quad (5.107)$$

where M is an arbitrary constant and i_0 is some fixed value. Under this symmetry breaking (5.107), a gauge boson which corresponds to a generator E_b , where the subscript $b = \sum_i b^i \alpha_i$ indicates a corresponding root, has a mass proportional to $\langle a \rangle \cdot b = M b^{i_0}$. This is seen from $[\langle a \rangle \cdot H, E_b] = (\langle a \rangle \cdot b) E_b$ where H_i are the generators of the Cartan subalgebra. Thus the massless gauge bosons correspond to the roots which satisfy $b^{i_0} = 0$ and the unbroken gauge group becomes $G'_i \times U(1)$ where the Dynkin diagram of G' is obtained by removing a node corresponding to the i_0 -th simple root in the Dynkin diagram of G . The Cartan subalgebra of G is decomposed into the Cartan subalgebra of G' and the additional $U(1)$ factor. The former is generated by $E_{\alpha_k} \in G$ obeying $[E_{\alpha_k}, E_{\alpha_{-k}}] \simeq \alpha_k \cdot H$ with $k \neq i_0$, while the latter is generated by $\alpha_{i_0} \cdot H$. Therefore, we set

$$a^i = (A^{-1})^{i i_0} M + \delta a^i, \quad (5.108)$$

where scalars corresponding to G' have been denoted as δa with $\delta a^{i_0} = 0$. Note that the $U(1)$ sector decouples completely from the G' sector and the Weyl group of G' naturally acts on δa out of which the Casimirs of G' are constructed.

When the gauge symmetry is broken as above, we have to decompose the matter representation \mathcal{R} of G in terms of the subgroup G' as well. We have

$$\mathcal{R} = \bigoplus_{s=1}^{n_{\mathcal{R}}} \mathcal{R}_s, \quad (5.109)$$

where \mathcal{R}_s stands for an irreducible representation of G' . Accordingly Q^i is decomposed into \mathbf{Q}_s^i ($1 \leq i \leq N_f$, $1 \leq s \leq n_{\mathcal{R}}$) in a G' representation \mathcal{R}_s . \tilde{Q}_i is decomposed in a similar manner. After the massive components in Φ are integrated out, the low-energy theory becomes $N = 2$ $G' \times U(1)$ gauge theory. The $U(1)$ sector decouples from the G' sector and we consider the G' sector only. The semiclassical superpotential for this theory can be read off from (5.40). We have

$$W = \sum_{i=1}^{N_f} \left(\sqrt{2} \sum_{s=1}^{n_{\mathcal{R}}} (\langle a \rangle \cdot \lambda_{\mathcal{R}_s} + m_i) \tilde{\mathbf{Q}}_{is} \mathbf{Q}_s^i + \sqrt{2} \sum_{s=1}^{n_{\mathcal{R}}} \tilde{\mathbf{Q}}_{is} \Phi_{\mathcal{R}_s} \mathbf{Q}_s^i \right), \quad (5.110)$$

where $\lambda_{\mathcal{R}_s}$ is a weight of \mathcal{R} which branches to the weights in \mathcal{R}_s . This implies that we should shift the mass m_i as

$$m_i = -\langle a \rangle \cdot \lambda_{\mathcal{R}_{s_i}} + m'_i = -M (\lambda_{\mathcal{R}_{s_i}})^{i_0} + m'_i \quad (5.111)$$

to obtain the G' theory with appropriate matter hypermultiplets. Note that we can choose \mathcal{R}_{s_i} for each hypermultiplet separately. This enables us to obtain the N_f matters in different representations of G' from the N_f matters in a single representation of G . In the limit $M \rightarrow \infty$, some hypermultiplets have infinite masses and decouple from the theory. Then the superpotential (5.110) becomes

$$W = \sqrt{2} \sum_{i=1}^{N_f} m'_i \tilde{\mathbf{Q}}_{i s_i} \mathbf{Q}_{s_i}^i + \sqrt{2} \sum_{i=1}^{N_f} \tilde{\mathbf{Q}}_{i s_i} \Phi_{\mathcal{R}_{s_i}} \mathbf{Q}_{s_i}^i, \quad (5.112)$$

and the resulting theory becomes $N = 2$ theory with gauge group G' with hypermultiplets belonging to the representation \mathcal{R}_{s_i} . Note that $\langle a \rangle \cdot \lambda_{\mathcal{R}_s}$ is proportional to its additional $U(1)$ charge.

In the known cases, the low-energy effective theory in the Coulomb phase is described by the Seiberg-Witten geometry which is described by a three-dimensional complex manifold in the form of the ALE space of ADE type fibered over \mathbf{CP}^1

$$z + \frac{1}{z} \Lambda^{2h-l(\mathcal{R})N_f} \prod_{i=1}^{N_f} X_G^{\mathcal{R}}(x_1, x_2, x_3; a, m_i) - W_G(x_1, x_2, x_3; a) = 0, \quad (5.113)$$

where z parametrizes \mathbf{CP}^1 , h is the dual Coxeter number of G and $l(\mathcal{R})$ is the index of the representation \mathcal{R} of the matter. Here $W_G(x_1, x_2, x_3; a) = 0$ is a simple singularity of type G and $X_G^{\mathcal{R}}(x_1, x_2, x_3; a, m_i)$ is some polynomial of the indicated variables. Note that the simple singularity W_G depends only on the gauge group G , but the $X_G^{\mathcal{R}}(x_1, x_2, x_3; a, m_i)$ depends on the matter content of the theory.

Starting with (5.113) let us consider the symmetry breaking in the Seiberg-Witten geometry. In the limit $M \rightarrow \infty$, the gauge symmetry G is reduced to the smaller one G' . The Seiberg-Witten geometry is also reduced to the one with gauge symmetry G' in this limit. We can see this by substituting $a = \langle a \rangle + \delta a$ into (5.113) and keeping the leading order in M . To leave the j -th flavor of hypermultiplets in the G' theory, its mass m_j is also shifted as in (5.111). After taking the appropriate coordinate (x'_1, x'_2, x'_3) we should have

$$\begin{aligned} W_G(x_1, x_2, x_3; a) &= M^{h-h'} W_{G'}(x'_1, x'_2, x'_3; \delta a) + o(M^{h-h'}), \\ X_G^{\mathcal{R}}(x_1, x_2, x_3; a, m_j) &= M^{l(\mathcal{R})-l(\mathcal{R}_{s_j})} X_{G'}^{\mathcal{R}_{s_j}}(x'_1, x'_2, x'_3; \delta a, m'_j) + o(M^{l(\mathcal{R})-l(\mathcal{R}_{s_j})}), \end{aligned} \quad (5.114)$$

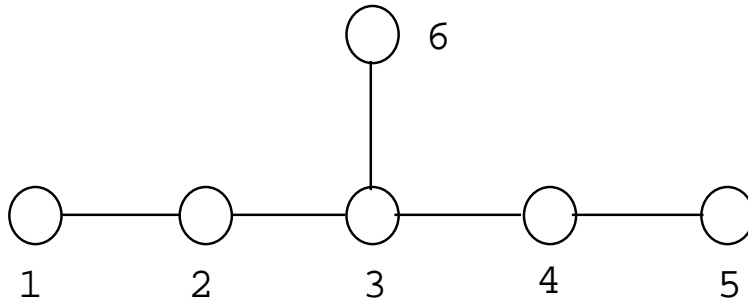


Figure 5.1: E_6 Dynkin diagram

where $W_{G'}$ is a simple singularity of type G' , $X_{G'}^{\mathcal{R}_{s_j}}$ is some polynomial of the indicated variables, h' is the dual Coxeter number of G' and $l(\mathcal{R}_{s_j})$ is the index of the representation \mathcal{R}_{s_j} of G' . The dependence on M can be understood from the scale matching relation between theories with gauge group G and G'

$$\Lambda'^{2h' - \sum_{j=1}^{N_f} l(\mathcal{R}_{s_j})} = \frac{\Lambda^{2h - l(\mathcal{R})N_f}}{M^{2(h-h') - (l(\mathcal{R})N_f - \sum_j l(\mathcal{R}_{s_j}))}}, \quad (5.115)$$

where Λ' is the scale of the G' theory. Thus, in the limit $M \rightarrow \infty$, the Seiberg-Witten geometry becomes

$$z' + \frac{1}{z'} \Lambda'^{2h' - \sum_{j=1}^{N_f} l(\mathcal{R}_{s_j})} \prod_{j=1}^{N_f} X_{G'}^{\mathcal{R}_{s_j}}(x'_1, x'_2, x'_3; \delta a, m'_j) - W_{G'}(x'_1, x'_2, x'_3; \delta a) = 0, \quad (5.116)$$

where $z' = z/M^{h-h'}$.

Next, we will apply this reduction procedure explicitly to the $N = 2$ gauge theory with gauge group E_6 with N_f fundamental hypermultiplets.

5.2.3 Breaking E_6 gauge group to $SO(10)$

There are two ways of removing a node from the Dynkin diagram of E_6 to obtain a simple group G' (see fig.5.1). When a node corresponding to α_5 (or α_6) is removed, we have $G' = SO(10)$ (or $SU(6)$). The former corresponds to the case of $G' = SO(10)$ and the latter to $G' = SU(6)$. First we consider the breaking of E_6 gauge group down to $SO(10)$ by tuning VEV of Φ as $\langle a_i \rangle = M\delta_{i,5}$. Using the inverse of the Cartan matrix we get $\langle a^i \rangle = (\frac{2}{3}M, \frac{4}{3}M, \frac{6}{3}M, \frac{5}{3}M, \frac{4}{3}M, M)$.

The Seiberg-Witten geometry for $N = 2$ gauge theory with gauge group E_6 with N_f fundamental matters is proposed in [31]

$$z + \frac{1}{z} \Lambda^{24-6N_f} \prod_{i=1}^{N_f} X_{E_6}^{27}(x_1, x_2, x_3; w, m_i) - W_{E_6}(x_1, x_2, x_3; w) = 0, \quad (5.117)$$

where

$$W_{E_6}(x_1, x_2, x_3; w) = x_1^4 + x_2^3 + x_3^2 + w_2 x_1^2 x_2 + w_5 x_1 x_2 + w_6 x_1^2 + w_8 x_2 + w_9 x_1 + w_{12}, \quad (5.118)$$

and

$$\begin{aligned} & X_{E_6}^{27}(x_1, x_2, x_3; w, m_i) \\ &= 8 \left(m_i^6 + 2w_2 m_i^4 - 8m_i^3 x_1 + (w_2^2 - 12x_2) m_i^2 \right. \\ & \quad \left. + 4w_5 m_i - 4w_2 x_2 - 8(x_1^2 - i x_3 + w_6/2) \right). \end{aligned} \quad (5.119)$$

Here $w_k = w_k(a)$ is the degree k Casimir of E_6 made out of a_j and the degrees of x_1, x_2 and x_3 are 3, 4 and 6 respectively. Now, substituting $a_i = M\delta_{i,5} + \delta a_i$ into $w_k(a)$ and setting $\delta a^5 = 0$, we expand W_{E_6} and $X_{E_6}^{27}$ in M . As discussed in the previous section, there should be coordinates (x'_1, x'_2, x'_3) which can eliminate the terms depending upon M^l ($5 \leq l \leq 12$) in W_{E_6} . Indeed, we can find such coordinates as,

$$\begin{aligned} x_1 &= -\frac{2}{27}M^3 - \frac{1}{4}Mx'_1 - \frac{1}{6}Mw_2, \\ x_2 &= \frac{1}{54}M^4 + \frac{1}{12}M^2x'_1 + \frac{1}{9}M^2w_2 + \frac{1}{8}x'_2 + \frac{1}{6}w_2^2, \\ x_3 &= -i\frac{1}{16}M^2x'_3. \end{aligned} \quad (5.120)$$

Then the E_6 singularity W_{E_6} is written as

$$W_{E_6}(x_1, x_2, x_3; w) = \left(\frac{1}{4}M\right)^4 W_{D_5}(x'_1, x'_2, x'_3; v) + O(M^3), \quad (5.121)$$

where

$$W_{D_5}(x_1, x_2, x_3; v) = x_1^4 + x_1 x_2^2 - x_3^2 + v_2 x_1^3 + v_4 x_1^2 + v_6 x_1 + v_8 + v_5 x_2, \quad (5.122)$$

and $v_k = v_k(\delta a)$ is the degree k Casimir of $SO(10)$ constructed from δa_i . If we represent Φ as a 10×10 matrix of the fundamental representation of $SO(10)$, we have $v_{2l} = \frac{1}{2l} \text{Tr} \Phi^{2l}$

and $v_5 = 2i\text{Pf}\Phi$. Thus we see in the $M \rightarrow \infty$ limit that the Seiberg-Witten geometry for $N = 2$ pure Yang-Mills theory with gauge group E_6 becomes that with gauge group $SO(10)$.

Next we consider the effect of symmetry breaking in the matter sector. The fundamental representation **27** of E_6 is decomposed into the representations of $SO(10) \times U(1)$ as

$$\mathbf{27} = \mathbf{16}_{-\frac{1}{3}} \oplus \mathbf{10}_{\frac{2}{3}} \oplus \mathbf{1}_{-\frac{4}{3}}, \quad (5.123)$$

where the subscript denotes the $U(1)$ charge $\alpha_5 \cdot \lambda_i$ ($1 \leq i \leq 27$). The indices of the spinor representation **16** and the vector representation **10** are four and two, respectively. Let us first take the scaling limit in such a way that the spinor matters of $SO(10)$ survive. Then the terms with M^l ($l \geq 3$) in $X_{E_6}^{27}$ must be absent after a change of variables (5.120) and the mass shift $m_i = \frac{1}{3}M + m_{s_i}$ (see (5.111)). In fact we find that

$$X_{E_6}^{27}(x_1, x_2, x_3; w, m_i) = M^2 X_{D_5}^{16}(x'_1, x'_2, x'_3; v, m_{s_i}) + O(M), \quad (5.124)$$

where

$$X_{D_5}^{16}(x_1, x_2, x_3; v, m) = m^4 + \left(x_1 + \frac{1}{2}v_2\right)m^2 - mx_2 + \frac{1}{2}x_3 - \frac{1}{4}\left(v_4 - \frac{1}{4}v_2^2\right) - \frac{1}{4}v_2x_1 - \frac{1}{2}x_1^2. \quad (5.125)$$

In order to make the vector matter of $SO(10)$ survive, we shift masses as $m_i = -\frac{2}{3}M + m_{v_i}$. The result reads

$$X_{E_6}^{27}(x_1, x_2, x_3; v, m_i) = M^4 X_{D_5}^{10}(x'_1, x'_2, x'_3; v, m_{v_i}) + O(M^3), \quad (5.126)$$

where

$$X_{D_5}^{10}(x_1, x_2, x_3; v, m) = m^2 - x_1. \quad (5.127)$$

Assembling (5.121), (5.124), (5.126) and taking the limit $M \rightarrow \infty$ with

$$\Lambda_{SO(10)N_s N_v}^{16-4N_s-2N_v} = 2^{16+3N_s+3N_v} M^{-(8-2N_s-4N_v)} \Lambda^{24-6N_f} \quad (5.128)$$

kept fixed, we now obtain the Seiberg-Witten geometry for $N = 2$ $SO(10)$ gauge theory with N_s spinor and N_v vector hypermultiplets

$$z + \frac{1}{z} \Lambda_{SO(10)N_s N_v}^{16-4N_s-2N_v} \prod_{i=1}^{N_s} X_{D_5}^{16}(x_1, x_2, x_3; v, m_{s_i}) \prod_{j=1}^{N_v} X_{D_5}^{10}(x_1, x_2, x_3; v, m_{v_j}) - W_{D_5}(x_1, x_2, x_3; v) = 0, \quad (5.129)$$

where $N_f = N_s + N_v$. In the massless case $m_{s_i} = m_{v_j} = 0$, our result agrees with that obtained from the analysis of the compactification of Type IIB string theory on the suitably chosen Calabi-Yau threefold [34]. This is non-trivial evidence in support of (5.117). Moreover the Seiberg-Witten geometry derived in [34] is only for the massless matters with $N_s - N_v = -2$. Here our expression is valid for massive matters of arbitrary number of flavors.

Next we examine the gauge symmetry breaking in the $N = 2$ $SO(10)$ gauge theory with spinor matters. When Φ acquires the VEV $\langle a_i \rangle = M\delta_{i,1}$, namely $\langle a^i \rangle = \left(M, M, M, \frac{M}{2}, \frac{M}{2}\right)$, the gauge group $SO(10)$ breaks to $SO(8)$. (we rename δa_i to a_i henceforth.) Note that the spinor representation of $SO(10)$ reduces to the spinor $\mathbf{8s}$ and its conjugate $\mathbf{8c}$ of $SO(8)$. Upon taking the limit $M \rightarrow \infty$ with $a_i = \langle a_i \rangle + \delta a_i$, we make a change of variables in (5.122)

$$\begin{aligned}x_1 &= x'_1, \\x_2 &= iMx'_2, \\x_3 &= Mx'_3.\end{aligned}\tag{5.130}$$

In terms of these variables, the D_5 singularity is shown to be

$$W_{D_5}(x_1, x_2, x_3; v) = (-M^2) W_{D_4}(x'_1, x'_2, x'_3; u) + O(M),\tag{5.131}$$

where

$$W_{D_4}(x_1, x_2, x_3; u) = x_1^3 + x_1x_2^2 + x_3^2 + u_2x_1^2 + v_4x_1 + u_6 + 2i\tilde{v}_4x_2,\tag{5.132}$$

u_k is the degree k Casimir of $SO(8)$ constructed from δa_i and $\tilde{v}_4 = \text{Pfaffian}$. The contribution (5.125) coming from the matters becomes

$$X_{D_5}^{\mathbf{16}}(x_1, x_2, x_3; v, m_{s_i}) = M^2 X_{D_4}^{\mathbf{8s}}(x'_1, x'_2, x'_3; u, m'_{s_i}) + O(M^3),\tag{5.133}$$

where

$$X_{D_4}^{\mathbf{8s}}(x_1, x_2, x_3; u, m) = m^2 + \frac{1}{2}x_1 - i\frac{1}{2}x_2 + \frac{1}{4}u_2.\tag{5.134}$$

In the above limit, we have taken $m_{s_i} = \frac{1}{2}M + m'_{s_i}$ which corresponds to the spinor representation of $SO(8)$. If we instead take $m_{s_i} = -\frac{1}{2}M + m'_{s_i}$, which corresponds to the conjugate spinor representation, then x_2 is replaced with $-x_2$ in $X_{D_4}^{\mathbf{8s}}$.

If we consider the vector matters of $SO(10)$, we see that a change of variables (5.130) without the shift of mass does not affect $m_{v_i} - x_1$. Therefore, in taking the limit $M \rightarrow \infty$ with

$$\Lambda_{SO(8)N_s N_v}^{12-2N_s-2N_v} = M^{-(4-2N_s)} \Lambda_{SO(10)N_s N_v}^{16-4N_s-2N_v} \quad (5.135)$$

being fixed, we conclude that the Seiberg-Witten geometry for $N = 2$ $SO(8)$ gauge theory with N_s spinor and N_v vector flavors is

$$z + \frac{1}{z} \Lambda_{SO(8)N_s N_v}^{12-2N_s-2N_v} \prod_{i=1}^{N_s} X_{D_4}^{\mathbf{8s}}(x_1, x_2, x_3; u, m'_{si}) \prod_{j=1}^{N_v} X_{D_4}^{\mathbf{8v}}(x_1, x_2, x_3; u, m_{vj}) - W_{D_4}(x_1, x_2, x_3; u) = 0, \quad (5.136)$$

where $X_{D_4}^{\mathbf{8v}}(x_1, x_2, x_3; u, m) = m^2 - x_1$.

There is a \mathbf{Z}_2 action in the triality of $SO(8)$ which exchanges the vector representation and the spinor representation. Accordingly the $SO(8)$ Casimirs are exchanged as

$$\begin{aligned} v_2 &\leftrightarrow v_2, \\ v_4 &\leftrightarrow -\frac{1}{2}v_4 + 3\text{Pf} + \frac{3}{8}v_2^2, \\ \text{Pf} &\leftrightarrow \frac{1}{2}\text{Pf} + \frac{1}{4}v_4 - \frac{1}{16}v_2^2, \\ v_6 &\leftrightarrow v_6 + \frac{1}{16}v_2^3 - \frac{1}{4}v_4 v_2 + \frac{1}{2}\text{Pf} v_2. \end{aligned} \quad (5.137)$$

Thus the \mathbf{Z}_2 action is expected to exchange $X_{D_4}^{\mathbf{8s}}$ and $X_{D_4}^{\mathbf{8v}}$ in (5.136) after an appropriate change of coordinates x_i . Actually, using the new coordinates (x'_1, x'_2) introduced by

$$\begin{aligned} x_1 &= -\frac{1}{2}x'_1 + i\frac{1}{2}x'_2 - \frac{1}{4}v_2, \\ x_2 &= -i\frac{3}{2}x'_1 + \frac{1}{2}x'_2 - i\frac{1}{4}v_2, \end{aligned} \quad (5.138)$$

we see that the D_4 singularity (5.132) remains intact except for (5.137) and $X_{D_4}^{\mathbf{8s}} \leftrightarrow X_{D_4}^{\mathbf{8v}}$.

One may further break the gauge group $SO(8)$ to $SO(6)$ following the breaking pattern $SO(10)$ to $SO(8)$. Suitable coordinates are found to be $x_1 = x'_1$, $x_2 = iMx'_2$ and $x_3 = Mx'_3$. The resulting Seiberg-Witten geometry for $N = 2$ $SO(6)$ gauge theory with N_s spinor flavors and N_v vector flavors is

$$z + \frac{1}{z} \Lambda_{SO(6)N_s N_v}^{8-N_s-2N_v} \prod_{i=1}^{N_s} \left(\frac{1}{2}x_2 \pm m_{si}\right) \prod_{j=1}^{N_v} (m_{vj}^2 - x_1) - W_{D_3}(x_1, x_2, x_3; u) = 0, \quad (5.139)$$

where $W_{D_3}(x_1, x_2, x_3; u) = x_1^2 + x_1x_2^2 + x_3^2 + u_2x_1 + u_4 + 2i\text{Pf}\Phi x_2$. The sign ambiguity in (5.139) arises from the two possible choices of the shift of masses in $SO(8)$ theory.

When $N_s = 0$, it is seen that the present $SO(2N_c)$ results yield the well-known curves for $SO(2N_c)$ theory with vector matters [11, 12].

5.2.4 Breaking E_6 gauge group to $SU(6)$

Now we wish to break the E_6 gauge group down to $SU(6)$ by giving the VEV $\langle a_i \rangle = M\delta_{i,6}$ to Φ , that is, $\langle a^i \rangle = (M, 2M, 3M, 2M, M, 2M)$. As in the previous section, we first substitute $a_i = M\delta_{i,6} + \delta a_i$ into $w_k(a)$ in (5.117) and set $\delta a^6 = 0$. Then we expand W_{E_6} and $X_{E_6}^{27}$ in M , and look for the coordinates (x'_1, x'_2, x'_3) which eliminate the terms depending on M^l ($7 \leq l \leq 12$) in (5.117). We can find such coordinates as

$$\begin{aligned} x_1 &= -\frac{5}{8}M^2 x'_1 - \frac{3}{4}x'_1 w_2, \\ x_2 &= \frac{1}{16}M^4 + \left(\frac{1}{4}x'_2 + \frac{1}{4}x'^2_1 + \frac{1}{12}w_2\right)M^2, \\ x_3 &= \frac{1}{160}M^6 + \left(-\frac{1}{8}x'_2 + \frac{3}{160}w_2\right)M^4 \\ &\quad + \frac{1}{8}(x'_3 - x'^2_2 - 3x'_2 x'^2_1 - x'_2 w_2 + \frac{2}{15}w_2^2 - 3x'^4_1)M^2 + \frac{1}{2}w_5 x'_1 - \frac{1}{10}w_6, \end{aligned} \quad (5.140)$$

in terms of which the E_6 singularity W_{E_6} is represented as

$$W_{E_6}(x_1, x_2, x_3; w) = \left(\frac{1}{2}M\right)^6 W_{A_5}(x'_1, x'_2, x'_3; v) + O(M^5), \quad (5.141)$$

where

$$W_{A_r}(x_1, x_2, x_3; v) = x_1^r + x_2x_3 + v_2x_1^{r-1} + v_3x_1^{r-2} + \cdots + v_r x_1 + v_{r+1}, \quad (5.142)$$

and $v_k = v_k(\delta a)$ is the degree k Casimir of $SU(6)$ build out of δa_i . Hence it is seen in the $M \rightarrow \infty$ limit that the Seiberg-Witten geometry for $N = 2$ pure Yang-Mills theory with gauge group E_6 becomes that with gauge group $SU(6)$.

The fundamental representation **27** of E_6 is decomposed into the representations of $SU(6) \times U(1)$ as

$$\mathbf{27} = \mathbf{15}_0 \oplus \mathbf{6}_1 \oplus \bar{\mathbf{6}}_{-1}, \quad (5.143)$$

where the subscript denotes the $U(1)$ charge $\alpha_6 \cdot \lambda_i$ ($1 \leq i \leq 27$). The indices of the antisymmetric representation **15** and the fundamental representation **6** are four and one,

respectively. Thus the terms with M^l ($l \geq 3$) in $X_{E_6}^{27}$ must be absent after taking the coordinates (x'_1, x'_2, x'_3) defined in (5.140). Note that there is no need to shift the mass to make the antisymmetric matter survive. We indeed obtain a desired expression

$$X_{E_6}^{27}(x_1, x_2, x_3; w, m_i) = -M^2 X_{A_5}^{15}(x'_1, x'_2, x'_3; v, m_i) + O(M), \quad (5.144)$$

where

$$X_{A_5}^{15}(x_1, x_2, x_3; v, m) = m^4 - 2m^3 x_1 + 3 \left(\frac{1}{3} v_2 + x_1^2 + x_2 \right) m^2 + m v_3 - x_3 + x_1^4 + 2v_2 x_1^2 + 3x_2 x_1^2 + v_3 x_1 + x_2^2 + v_2 x_2 + v_4. \quad (5.145)$$

If we shift the mass as $m_i = M + m_{f_i}$ in order to make the vector matter survive, we find that

$$X_{E_6}^{27}(x_1, x_2, x_3; v, m_i) = 2M^5 X_{A_5}^6(x'_1, x'_2, x'_3; v, m_{f_i}) + O(M^4), \quad (5.146)$$

where $X_{A_5}^6(x_1, x_2, x_3; v, m) = m + x_1$. The shift of masses $m_i = -M + m_{f_i}$ also corresponds to making the vector matter survive, but the factor (-1) is needed in the RHS of (5.146).

From these observations we can obtain the Seiberg-Witten geometry for $N = 2$ $SU(6)$ gauge theory with N_a antisymmetric and N'_f fundamental matters by taking the limit $M \rightarrow \infty$ while

$$\Lambda_{SU(6)N_a N'_f}^{12-4N_a-N'_f} = (-1)^{N_a} 2^{12+2N'_f} M^{-(12-2N_a-5N'_f)} \Lambda^{24-6N_f} \quad (5.147)$$

held fixed. Our result reads

$$z + \frac{1}{z} \Lambda_{SU(6)N_a N'_f}^{12-4N_a-N'_f} \prod_{i=1}^{N_a} X_{A_5}^{15}(x_1, x_2, x_3; v, m_{a_i}) \prod_{j=1}^{N'_f} X_{A_5}^6(x_1, x_2, x_3; v, m_{f_j}) - W_{A_5}(x_1, x_2, x_3; v) = 0, \quad (5.148)$$

where $N_f = N_a + N'_f$.

We are now able to break $SU(r+1)$ gauge group to $SU(r)$ successively by putting $\langle a_i \rangle = M \delta_{i,r}$. In sect.2 we have seen that the proper coordinates are chosen to be $x_1 = x'_1 + M/(r+1)$, $x_2 = x'_2$ and $x_3 = M x'_3$ in terms of which $W_{A_r}(x_1, x_2, x_3; v) = M W_{A_{r-1}}(x'_1, x'_2, x'_3; v') + O(M^0)$. Note that the degrees of x_1, x_2 and x_3 are 1, 2 and

$r - 1$, respectively. The antisymmetric representation of $SU(r + 1)$ is decomposed into the antisymmetric and fundamental representations of $SU(r) \times U(1)$ as follows

$$\frac{\mathbf{r}(\mathbf{r} + \mathbf{1})}{\mathbf{2}} = \frac{(\mathbf{r} - \mathbf{1})\mathbf{r}}{\mathbf{2}} \oplus_{\frac{2}{r+1}} \mathbf{r}_{-\frac{r-1}{r+1}}, \quad (5.149)$$

where the subscript denotes the $U(1)$ charge. After some computations we can see that the Seiberg-Witten geometry for $N = 2$ $SU(r+1)$ ($r \leq 5$) gauge theory with N_a antisymmetric and N'_f fundamental hypermultiplets turns out to be

$$z + \frac{1}{z} \Lambda_{SU(r+1)N_a N'_f}^{2(r+1)-(r-1)N_a - N'_f} \prod_{i=1}^{N_a} X_{A_r}^{\frac{\mathbf{r}(\mathbf{r}+1)}{\mathbf{2}}}(x_1, x_2, x_3; v, m_{ai}) \prod_{j=1}^{N'_f} (x_1 - m_{f_j}) - W_{A_r}(x_1, x_2, x_3; v) = 0, \quad (5.150)$$

where $X_{A_r}^{\frac{\mathbf{r}(\mathbf{r}+1)}{\mathbf{2}}}$ is defined as

$$X_{A_r}^{\frac{\mathbf{r}(\mathbf{r}+1)}{\mathbf{2}}}(x_j; v, m_{ai} = \frac{2M}{r+1} + m'_{ai}) = M X_{A_{r-1}}^{\frac{(\mathbf{r}-1)\mathbf{r}}{\mathbf{2}}}(x'_j; v', m'_{ai}) + O(M^0), \quad (5.151)$$

and $\Lambda_{SU(r+1)N_a N'_f}^{2(r+1)-(r-1)N_a - N'_f} = M^{2-N_a} \Lambda_{SU(r)N_a N'_f}^{2r-(r-2)N_a - N'_f}$. Explicit calculations yield

$$\begin{aligned} X_{A_4}^{\mathbf{10}}(x_j; v, m_{ai}) &= m^3 - m^2 x_1 + (2x_2 + 2x_1^2 + v_2)m + 2x_1^2 - x_3 + x_2 x_1 + v_2 x_1 + v_3, \\ X_{A_3}^{\mathbf{6}}(x_j; v, m_{ai}) &= m^2 + x_2 - x_3 + 2x_1^2 + v_2, \\ X_{A_2}^{\mathbf{3}}(x_j; v, m_{ai}) &= m + x_1 - x_3. \end{aligned} \quad (5.152)$$

We also see that

$$\begin{aligned} X_{A_5}^{\mathbf{15}}(x_j; v, m_{ai} = -\frac{2}{3}M + m'_{f_i}) &= M^3(x'_1 - m'_{f_i}) + O(M^2), \\ X_{A_4}^{\mathbf{10}}(x_j; v, m_{ai} = -\frac{3}{5}M + m'_{f_i}) &= -M^2(x'_1 - m'_{f_i}) + O(M^1), \\ X_{A_3}^{\mathbf{6}}(x_j; v, m_{ai} = -\frac{1}{2}M + m'_{f_i}) &= M(x'_1 - m'_{f_i} - x'_3) + O(M^0) \end{aligned} \quad (5.153)$$

by shifting masses in such a way that the fundamental matters remain.

We now check our $SU(N_c)$ results. First of all, for $SU(3)$ gauge group, the antisymmetric representation is identical to the fundamental representation. Thus (5.150) should be equivalent to the well-known $SU(3)$ curve. In fact, if we integrate out variables x_2 and x_3 , the Seiberg-Witten geometry (5.150) yields the $SU(3)$ curve with $N_a + N'_f$ fundamental flavors.

Let us next turn to the case of $SU(4)$ gauge group. Since the Lie algebra of $SU(4)$ is the same as that of $SO(6)$, the antisymmetric and fundamental representations of $SU(4)$ correspond to the vector and spinor representations of $SO(6)$ respectively. This relation is realized in (5.150) and (5.139) as follows. If we set $x_1 = \frac{1}{2}x'_2$, $x_2 = ix'_3 - \frac{1}{2}x'_1 - \frac{1}{4}x'_2{}^2 - \frac{1}{2}v_2$ and $x_3 = ix'_3 + \frac{1}{2}x'_1 + \frac{1}{4}x'_2{}^2 + \frac{1}{2}v_2$, we find

$$W_{A_3}(x_i; v) = -\frac{1}{4}W_{D_3}(x'_i; u), \quad (5.154)$$

where u is related to v through $u_2 = 2v_2$, $u_4 = -4v_4 + v_2^2$ and $\text{Pf} = iv_3$. Moreover we obtain $X_{A_3}^6(x_j; v, m_{a_i}) = m_{a_i}^2 - x'_1$ and $x_1 - m_{f_j} = \frac{1}{2}x'_2 - m_{f_j}$. Thus our $SU(4)$ result is in accordance with what we have anticipated. This observation provides a consistency check of our procedure since both $SO(6)$ and $SU(4)$ results are deduced from the E_6 theory via two independent routes associated with different symmetry breaking patterns.

Checking the $SU(5)$ gauge theory result is most intricate. Complex curves describing $N = 2$ $SU(N_c)$ gauge theory with matters in one antisymmetric representation and fundamental representations are obtained in [35, 36] using brane configurations. Let us concentrate on $SU(5)$ theory with one massless antisymmetric matter and no fundamental matters in order to compare with our result (5.139). The relevant curve is given by [35]

$$y^3 + xy^2(x^5 + v_2x^3 - v_3x^2 + v_4x - v_5) - y\Lambda^7(3x^5 + 3v_2x^3 - v_3x^2 + 3v_4x - v_5) + 2\Lambda^{14}(x^4 + v_2x^2 + v_4) = 0. \quad (5.155)$$

The discriminant of (5.155) has the form

$$\Delta_{Brane} = F_0(v)\Lambda^{105}(27\Lambda^7v_4^2 + v_5^3)(H_{50}(v, L))^2(H_{35}(v, L))^6, \quad (5.156)$$

where F_0 is some polynomial in v , H_n is a degree n polynomial in v and $L = -\Lambda^7/4$. If we set $v_2 = v_3 = 0$ for simplicity, then

$$\begin{aligned} H_{50}(v, L) &= 65536 v_4^{10} v_5^2 + 1048576 v_4^9 L^2 - 33587200 v_4^7 v_5^3 L + 1600000 v_4^5 v_5^6 \\ &\quad - 539492352 v_4^6 v_5 L^3 + 3261440000 v_4^4 v_5^4 L^2 + 390000000 v_4^2 v_5^7 L \\ &\quad + 9765625 v_5^{10} + 143947517952 v_4^3 v_5^2 L^4 + 5378240000 v_4 v_5^5 L^3 \\ &\quad + 1457236279296 v_4^2 L^6 + 53971714048 v_5^3 L^5, \\ H_{35}(v, L) &= 32v_5^7 + 432Lv_4^2v_5^2 + 17496L^3v_4v_5^2 + 177147L^5. \end{aligned} \quad (5.157)$$

We have also calculated the discriminant Δ_{ALE} of our expression (5.150) with $r = 4$ and found it in the factorized form. Evaluating Δ_{Brane} and Δ_{ALE} at sufficiently many points in the moduli space, we observe that Δ_{ALE} also contains a factor $H_{50}(v, L)$ with $\Lambda_{SU(4)1,0}^7 = L$. This fact may be regarded as a non-trivial check for the compatibility of the M-theory/brane dynamics result and our ALE space description. It is thus inferred that only the zeroes of a common factor $H_{50}(v, L)$ in the discriminants represent the physical singularities in the moduli space.[§]

Moreover it is shown that the Seiberg-Witten geometries obtained in this section by breaking the E_6 Seiberg-Witten geometry can be rederived using the method of $N = 1$ confining phase superpotentials [32].

[§]A similar phenomenon is observed in $SU(4)$ gauge theory. We have checked that the discriminant of the curve for $SU(4)$ theory with one massive antisymmetric hypermultiplet proposed in [35] and that of our ALE formula (5.150) with $r = 3$ carry a common factor. See also [55]

Chapter 6

Conclusions

In this thesis, we have studied the $N = 2$ Seiberg-Witten Geometry via the Confining Phase superpotential technique. In particular, we have shown that the ALE spaces of type ADE fibered over \mathbf{CP}^1 is natural geometry for the $N = 2$ supersymmetric gauge theories with ADE gauge groups.

In chapter two, we have reviewed the exact description of the low-energy effective theory of the Coulomb phase of four-dimensional $N = 2$ supersymmetric gauge theory in terms of the Seiberg-Witten curve or Seiberg-Witten geometry. The Seiberg-Witten geometry has been derived from the superstring theory compactified on the suitably chosen Calabi-Yau three-fold.

In chapter three, we have shown how to derive the Seiberg-Witten curves for the Coulomb phase of $N = 2$ supersymmetric gauge theories by means of the $N = 1$ confining phase superpotential. To put it concretely, we have obtained a low-energy effective superpotential for a phase with a single confined photon in $N = 1$ gauge theory. The expectation values of gauge invariants built out of the adjoint field parametrize the singularities of moduli space of the $N = 2$ Coulomb phase. According to this derivation it is clearly observed that the quantum effect in the Seiberg-Witten curve has its origin in the $SU(2)$ gluino condensation in view of $N = 1$ gauge theory dynamics.

In chapter four, we have applied the confining phase superpotential to the $N = 1$ supersymmetric pure Yang-Mills theory with an adjoint matter with classical or ADE gauge groups. The results can be used to derive the Seiberg-Witten curves for $N = 2$ supersymmetric pure Yang-Mills theory with classical or ADE gauge groups in the form

of a foliation over \mathbf{CP}^1 which is identical to the spectral curves for the periodic Toda lattice. Transferring the critical points in the $N = 2$ Coulomb phase to the $N = 1$ theories we have found non-trivial $N = 1$ SCFT with the adjoint matter field governed by a superpotential.

In chapter five, using the technique of confining phase superpotential we have determined the curves describing the Coulomb phase of $N = 1$ supersymmetric gauge theories with adjoint and fundamental matters with classical gauge groups. In the $N = 2$ limit our results recover the curves for the Coulomb phase in $N = 2$ QCD. For the gauge group $Sp(2N_c)$, in particular, we have observed that taking into account the instanton effect in addition to $SU(2)$ gaugino condensation is crucial to obtain the effective superpotential for the phase with a confined photon. This explains in terms of $N = 1$ theory a peculiar feature of the $N = 2$ $Sp(2N_c)$ curve when compared to the $SU(N_c)$ and $SO(N_c)$ cases.

Next we have proposed Seiberg-Witten geometry for $N = 2$ supersymmetric gauge theory with gauge group E_6 with massive N_f fundamental hypermultiplets employing the confining phase superpotentials method. The resulting manifold takes the form of a fibration of the ALE space of type E_6 .

Starting with the Seiberg-Witten geometry for $N = 2$ supersymmetric gauge theory with gauge group E_6 with massive fundamental hypermultiplets, we have obtained the Seiberg-Witten geometry for $SO(2N_c)$ ($N_c \leq 5$) theory with massive spinor and vector hypermultiplets by implementing the gauge symmetry breaking in the E_6 theory. The other symmetry breaking pattern has been used to derive the Seiberg-Witten geometry for $N = 2$ $SU(N_c)$ ($N_c \leq 6$) theory with massive antisymmetric and fundamental hypermultiplets. All the Seiberg-Witten geometries we have obtained are of the form of ALE fibrations over a sphere. Whenever possible our results have been compared with those obtained in the approaches based on the geometric engineering and the brane dynamics. It is impressive to find an agreement in spite of the fact that the methods are fairly different.

Thus our study of the confining phase superpotentials supports that Seiberg-Witten geometry of the form of ALE fibrations over \mathbf{CP}^1 is a canonical description for wide classes of the four-dimensional $N = 2$ supersymmetric gauge field theories. It is highly desirable to develop such a scheme explicitly for non-simply-laced gauge groups.

Although we have not discussed in this thesis, in order to analyze the mass of the BPS states and other interesting properties of the theory, one has to know the Seiberg-Witten three-form and appropriate cycles in the ALE fibration space. For $N = 2$ $SO(10)$ theory with massless spinor and vector hypermultiplets, these objects may be obtained in principle from the Calabi-Yau threefold on which the string theory is compactified [34]. It is important to find the Seiberg-Witten three-form and appropriate cycles for the Seiberg-Witten geometry when the massive hypermultiplets exist.

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