Gauged Linear Sigma Models
for Noncompact Calabi-Yau Varieties

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Two-dimensional field theory is a powerful framework

We have studied 2-dim. SUSY nonlinear sigma models...

– 1999: Supersymmetric nonlinear sigma models on hermitian symmetric spaces introducing an auxiliary gauge field
  by Higashijima and Nitta

2000: $1/N$ expansion of SUSY NLSM on $Q^{N-2} = \frac{SO(N)}{SO(N - 2) \times U(1)}$
  two non-trivial vacua, asymptotically free
  by Higashijima, Nitta, Tsuzuki and TK

2001 – 2002: Ricci-flat metrics on noncompact Kähler manifolds (≡ noncompact Calabi-Yau’s)
  by Higashijima, Nitta and TK
<table>
<thead>
<tr>
<th>line bundles</th>
<th>total dim. $D$</th>
<th>dual Coxeter $C$</th>
<th>“orbifolding” $\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{C} \times (\mathbb{C}P^{N-1} = \frac{SU(N)}{SU(N-1) \times U(1)})$</td>
<td>$1 + (N - 1)$</td>
<td>$N$</td>
<td>$N$</td>
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<tr>
<td>$\mathbb{C} \times (Q^{N-2} = \frac{SO(N)}{SO(N-2) \times U(1)})$</td>
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<td>$N - 2$</td>
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</tr>
<tr>
<td>$\mathbb{C} \times E_6/[SO(10) \times U(1)]$</td>
<td>$1 + 16$</td>
<td>$12$</td>
<td>$12$</td>
</tr>
<tr>
<td>$\mathbb{C} \times E_7/[E_6 \times U(1)]$</td>
<td>$1 + 27$</td>
<td>$18$</td>
<td>$18$</td>
</tr>
<tr>
<td>$\mathbb{C} \times (G_{N,M} = \frac{U(N)}{U(N-M) \times U(M)})$</td>
<td>$1 + M(N - M)$</td>
<td>$N$</td>
<td>$MN$</td>
</tr>
<tr>
<td>$\mathbb{C} \times SO(2N)/U(N)$</td>
<td>$1 + \frac{1}{2}N(N - 1)$</td>
<td>$N - 1$</td>
<td>$N(N - 1)$</td>
</tr>
<tr>
<td>$\mathbb{C} \times Sp(N)/U(N)$</td>
<td>$1 + \frac{1}{2}N(N + 1)$</td>
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<td>$N(N + 1)$</td>
</tr>
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\[
\mathcal{K}'_{\text{noncompact}}(\rho, \varphi) = \left( e^{CX} + b \right)^{1/D}
\]

\[
X = \log |\rho^{1/\ell}|^2 + K_{\text{compact}}(\varphi), \quad K_{\mathbb{C}P^{N-1}}(\varphi) = r \log \left( 1 + \sum_{i=1}^{N-1} |\varphi_i|^2 \right)
\]
CFT descriptions (Virasoro- and current-algebras)?

global aspects of noncompact geometries?

and

mirror geometries?

$\mathcal{N} = (2, 2)$ SUSY gauge theory with matters  (FI :  $t \equiv r - i\theta$)

$$\mathcal{L} = \int d^4\theta \left\{ -\frac{1}{e^2} \Sigma \Sigma + \sum_a \Phi_a e^{2Q_a V} \Phi_a \right\} + \left( \frac{1}{\sqrt{2}} \int d^2\tilde{\theta} (\Sigma t) + c.c. \right) + \left( \int d^2\theta W_{\text{GLSM}}(\Phi_a) + c.c. \right)$$

$$\begin{array}{l}
\Phi_a: \text{charged chiral superfield}, \quad \overline{D}_\pm \Phi_a = 0 \\
\Sigma: \text{twisted chiral superfield}, \quad \overline{D}_+ \Sigma = D_- \Sigma = 0, \quad \Sigma = \frac{1}{\sqrt{2}} \overline{D}_+ D_- V \\
\end{array}$$

There exist at least two phases:

- $\text{FI} \gg 0: \text{differential-geometric phase} \rightarrow \text{SUSY NLSM}$
- $\text{FI} \ll 0: \text{algebro-geometric phase} \rightarrow \text{LG, orbifold, SCFT}$

Calabi-Yau/Landau-Ginzburg correspondence

$$\text{harmonic forms} \leftrightarrow \text{NS-NS chiral primary states}$$

“Mirror geometry” appears in the T-dual theory in terms of twisted chiral superfields $Y_a$

$$Y_a + \overline{Y}_a \equiv 2 \overline{\Phi}_a e^{2Q_a V} \Phi_a$$

p. 4
Effective theories

The potential energy density is given by

\[ U(\phi, \sigma) = \frac{e^2}{2}D^2 + \sum_a |F_a|^2 + U_\sigma(\phi, \sigma) \]

\[ D = \frac{1}{e^2} \sqrt{D} = r - \sum_a Q_a |\phi_a|^2, \quad F_a = -\frac{\partial}{\partial \phi_a} W_{\text{GLSM}}(\phi), \quad U_\sigma(\phi, \sigma) = 2|\sigma|^2 \sum_a Q_a^2 |\phi_a|^2 \]

The supersymmetric vacuum manifold \( \mathcal{M} \) is defined by

\[ \mathcal{M} = \left\{ (\phi_a, \sigma) \in \mathbb{C}^n \mid D = F_a = U_\sigma = 0 \right\} / U(1) \]

In the IR limit \( e \to \infty \), there appears the supersymmetric NLSM on \( \mathcal{M} \) whose coupling is

\[ r = \frac{1}{g^2} \]

Renormalization of the FI parameter is

\[ r_0 = r_R + s \cdot \log \left( \frac{\Lambda_{\text{UV}}}{\mu} \right), \quad s = \sum_a Q_a \]

Thus we find that

\[ s > 0 \quad \rightarrow \quad \text{the theory is asymptotic free} \]

\[ s = 0 \quad \rightarrow \quad \text{the theory is conformal} \]

\[ s < 0 \quad \rightarrow \quad \text{the theory is infrared free} \]
**Example**: quintic hypersurface and its mirror

- **CY sigma model on** $\mathbb{CP}^4[5]$
- **LG orbifold theory with** $W_{LG}/\mathbb{Z}_5$
- **GLSM**
- **Hori-Vafa**
- **T-dual**
- **Mirror**
- **$\Sigma \rightarrow \frac{\partial}{\partial Y_P}$**
- **$\Sigma \rightarrow \frac{\partial}{\partial t}$**
- **$\tilde{\text{CY}}$ sigma model on** $\mathbb{CP}^4[5]/(\mathbb{Z}_5)^3$
- **$\tilde{\text{LG}}$ orbifold theory with** $\tilde{W}_{LG}/(\mathbb{Z}_5)^4$

- **FI $\gg 0$**
- **FI $\ll 0$**
Gauged Linear Sigma Model

for $\mathcal{O}(-N + \ell)$ bundle on $\mathbb{C}P^{N-1}[\ell]$

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<th>chiral superfield</th>
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$W_{\text{GLSM}} = P_1 \cdot G_\ell(S_i)$

$G_\ell(S_i)$: homogeneous polynomial of degree $\ell$

potential energy density:

$$
\mathcal{U} = \frac{e^2}{2} \mathcal{D}^2 + |G_\ell(s)|^2 + \sum_{i=1}^{N} |p_1 \partial_i G_\ell(s)|^2 + \mathcal{U}_\sigma
$$

$$
\mathcal{D} = r - \sum_{i=1}^{N} |s_i|^2 + \ell |p_1|^2 + (N - \ell) |p_2|^2
$$

$$
\mathcal{U}_\sigma = +2|\sigma|^2 \left\{ \sum_{i=1}^{N} |s_i|^2 + \ell^2 |p_1|^2 + (N - \ell)^2 |p_2|^2 \right\}
$$

Let us analyze SUSY vacuum manifold $\mathcal{U} = 0$ and massless effective theories
Supersymmetric vacua

- **CY phase on** $\mathcal{M}_{\text{CY}}$
  - conformal sigma model on $\mathcal{M}_{\text{CY}}$

- **orbifold phase on** $\mathcal{M}_{r<0}^1$ (two “LG”s appear)
  - $\{ \text{CFT on } \mathbb{C}^1 \otimes \text{LG with } W_{\text{LG}} = \langle p_1 \rangle G_\ell(S) \}/\mathbb{Z}_\ell$
  - $\{ \text{“LG” with } W_{\text{LG}} = P_1 \cdot G_\ell(S) \}/\mathbb{Z}_{N-\ell}$

- **3rd phase on** $\mathcal{M}_{r<0}^2$ NEW!
  - conformal sigma model on $\mathcal{M}_{r<0}^2$

\[
\mathcal{M}_{\text{CY}} = \left\{ (s_i; p_2) \in \mathbb{C}^{N+2} \ \bigg| \ \mathcal{D} = G_\ell = 0, \ r > 0 \right\}/U(1) \equiv \mathcal{O}(-N + \ell) \text{ bundle on } \mathbb{C}\mathbb{P}^{N-1}[\ell]
\]
\[
\mathcal{M}_{r<0}^1 = \left\{ (p_1, p_2) \in \mathbb{C}^2 \ \bigg| \ \mathcal{D} = 0, \ r < 0 \right\}/U(1) \quad \equiv \ W_{\mathbb{C}\mathbb{P}_1^{\ell,N-\ell}}
\]
\[
\mathcal{M}_{r<0}^2 = \left\{ (s_i; p_2) \in \mathbb{C}^{N+2} \ \bigg| \ \mathcal{D} = G_\ell = 0, \ r < 0 \right\}/U(1)
\]
The four theories are related to each other via CY/LG correspondence and topology change:

We also notice that we have obtained various massless effective theories by decomposing (not by integrating out) all massive modes. Thus they are just approximate descriptions.
**T-dual description** of the GLSM is also powerful to investigate low energy theories.

Analyzing them

we will re-investigate the massless effective theories in the original GLSM.
\[ \mathcal{L} = \int d^4 \theta \left\{ -\frac{1}{e^2} \sum \Sigma - \sum_a \left( \frac{1}{2} (Y_a + \overline{Y}_a) \log (Y_a + \overline{Y}_a) \right) \right\} + \left( \frac{1}{\sqrt{2}} \int d^2 \theta \hat{W} + c.c. \right) \]

\[ \hat{W} = \Sigma \left( \sum_{i=1}^N Y_i - \ell Y_{P_1} - (N - \ell) Y_{P_2} - t \right) + \sum_{i=1}^N e^{-Y_i} + e^{-Y_{P_1}} + e^{-Y_{P_2}} \]

Period integral: \[ \hat{\Pi} \equiv \int d\Sigma \prod_{i=1}^N dY_i dY_{P_1} dY_{P_2} (\ell \Sigma) \exp \left( - \hat{W} \right) \]

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$U(1)$ phase rotation symmetry on $\Phi_a \quad \Rightarrow \quad \text{shift symmetry on } Y_a: \quad Y_a \equiv Y_a + 2\pi i$

In the IR limit $e \to \infty$, the gauge field $\Sigma$ is no longer dynamical and should be integrated out.

\[ 2 \overline{\Phi}_a e^{2Q_a V} \Phi_a = Y_a + \overline{Y}_a \]

In order to obtain \textbf{LG theory} or \textbf{geometry}, we replace $\Sigma$ to $\Sigma \to \frac{\partial}{\partial t}$ or $\Sigma \to \frac{\partial}{\partial Y_P}$
Twisted Landau-Ginzburg theory: There exist consistent solutions

— Solution one: \( \mathbb{Z}_\ell \)-type orbifold symmetry —

Solve \( Y_{P_1} \) by using the constraint derived from integrating out \( \Sigma \):

\[
Y_{P_1} = \frac{1}{\ell} \left\{ t - \sum_{i=1}^{N} Y_i + (N - \ell)Y_{P_2} \right\}
\]

Field re-definition preserving canonical measure in \( \hat{\Pi} \):

\[
X_i \equiv e^{-\frac{1}{\ell}} Y_i, \quad X_{P_2} \equiv e^{\frac{N-\ell}{\ell}} Y_{P_2}, \quad X_i \rightarrow \omega_i X_i, \quad X_{P_2} \rightarrow \omega_{P_2} X_{P_2}, \quad (\mathbb{Z}_\ell)^N \text{ symmetry}
\]

Thus we obtain the twisted LG superpotential:

\[
\left\{ \tilde{W}_\ell = X_1^\ell + \cdots + X_N^\ell + X_{P_2}^{-\frac{\ell}{N-\ell}} + e^{t/\ell} X_1 \cdots X_N X_{P_2} \right\} / (\mathbb{Z}_\ell)^N
\]

The negative power term describes \( \mathcal{N} = 2 \) Kazama-Suzuki model on \( SL(2, \mathbb{R})_k / U(1) \):

\[
\frac{\ell}{N - \ell} = k = \frac{2}{Q^2}
\]

Thus we argue that

this effective theory is the LG minimal model coupled to the KS model with \( (\mathbb{Z}_\ell)^N \) orbifold symmetry
Twisted mirror geometry: There also exist consistent solutions

— Solution one: $\mathbb{Z}_\ell$-type orbifold symmetry —

We replace $\ell \Sigma$ to $\frac{\partial}{\partial Y_{P_1}}$ and obtain

$$\hat{\Pi} = \int \prod_{i=1}^{N} dY_i \left(e^{-Y_{P_1}} dY_{P_1}\right) \delta\left(\sum_i Y_i - \ell Y_{P_1} - (N - \ell) Y_{P_2} - t\right) \exp\left(-\sum_i e^{-Y_i} - e^{-Y_{P_1}} - e^{-Y_{P_2}}\right)$$

Re-defining the variables in order to obtain the canonical measure, we obtain

$$\hat{\mathcal{M}}_\ell = \left\{\mathcal{F}(Z_i) = 0\right\}/\mathbb{C}^* , \mathcal{G}(Z_b; u, v) = 0\right\}/(\mathbb{Z}_\ell)^{N-2}$$

$$\mathcal{F}(Z_i) = Z_1^\ell + \cdots + Z_\ell^\ell + \psi Z_1 \cdots Z_\ell , \quad \psi = e^{t/\ell} Z_{\ell+1} \cdots Z_N$$

$$\mathcal{G}(Z_b; u, v) = Z_{\ell+1}^\ell + \cdots + Z_N^\ell + 1 - uv$$

$Z_a \mapsto \lambda \omega_a Z_a \quad \text{for} \quad a = 1, \cdots, \ell \quad \text{(homogeneous coordinates of $\mathbb{C}P^{\ell-1}[\ell]$)}$

$Z_b \mapsto \omega_b Z_b \quad \text{for} \quad b = \ell + 1, \cdots, N \quad \text{(homogeneous coordinates of $\mathbb{C}^{N-\ell}$)}$

$$\omega_1^\ell = \omega_\ell^\ell = \omega_1 \cdots \omega_N = 1 , \quad \lambda : \mathbb{C}^*-\text{value}$$
Recall the following two arguments:

- \( \mathcal{N} = 2 \) SCFT on \( SL(2, \mathbb{R})_k/\mathbb{U}(1) \) is equivalent to \( \mathcal{N} = 2 \) Liouville theory via T-duality.

- If a CFT \( \mathcal{C} \) has an abelian discrete symmetry group \( \Gamma \), the orbifold CFT \( \mathcal{C}' = \mathcal{C}/\Gamma \) has a symmetry group \( \Gamma' \) which is isomorphic to \( \Gamma \). Furthermore a new orbifold CFT \( \mathcal{C}'/\Gamma' \) is identical to the original CFT \( \mathcal{C} \).

Thus we insist that

\[
\text{"\{CFT on } \mathbb{C}^1 \otimes \text{LG with } W_{\text{LG}} = \langle p_1 \rangle G_\ell(S) \rangle / \mathbb{Z}_\ell \text{" in the original GLSM is described by }
\{\mathcal{N} = 2 \text{ Liouville theory coupled to the LG minimal model with } W_{\text{LG}} \rangle / \mathbb{Z}_\ell \text{" as an exact effective theory."
}
Summary

- We found three non-trivial phases and four effective theories in the GLSM
  - two CY sigma models
  - two orbifolded LG theories coupled to 1-dim. SCFT
- We constructed four exact effective theories in the T-dual theory
  - two NLSMs on mirror CY geometries
  - two orbifolded LG theories including a term with negative power $-k$
    This term represents a gauged WZW model on $SL(2, \mathbb{R})_k/U(1)$ at level $k$
- We argue that the LG theories in the original GLSM can be interpreted as $\mathcal{N} = 2$ Liouville theories coupled to LG minimal models
CY sigma model on noncompact $\mathcal{M}_{CY}$

$\mathcal{N} = 2$ Liouville $\times$ LG
$\mathbb{Z}_\ell$ orbifold theory

GLSM

T-dual equivalent

Mirror

FI $\gg 0$

FI $\ll 0$

$\ell \Sigma \rightarrow \frac{\partial}{\partial Y_{P_1}}$

$\ell \Sigma \rightarrow \ell \frac{\partial}{\partial t}$

CY/LG

Hori-Vafa

$SL(2, \mathbb{R})_k/U(1) \times \tilde{LG}$
$(\mathbb{Z}_\ell)^N$ orbifold theory

CY/LG

T-dual
Hori-Vafa’s T-dual theory is only valid when we consider the GLSM without a superpotential or with a superpotential given simply by a homogeneous polynomial such as $W_{\text{GLSM}} = P \cdot G_{\ell}(S)$. Even though the polynomial $G_{\ell}(S)$ has an additional symmetry, the period integral $\tilde{\Pi}$ cannot recognize the existence of this additional symmetry. Thus the T-dual theory does not map all structures of the CY $\mathcal{M}$ to the mirror geometry completely.
Example: resolved/deformed conifold

\[ S^3 \rightarrow \mathbb{C}P^1 \]

\{-
\begin{align*}
\text{deformed conifold: deformation of complex moduli} \\
\text{resolved conifold: deformation of Kähler moduli}
\end{align*}\}

\begin{align*}
\text{GLSM for resolved conifold} & \quad \text{possible} \\
\text{GLSM for deformed conifold} & \quad \text{now impossible!} \\
\text{Hori-Vafa’s theory for resolved conifold} & \\
\text{Hori-Vafa’s theory for deformed conifold}
\end{align*}
Here we briefly review the T-duality of a generic GLSM without any superpotentials. We start from

\[
\mathcal{L}' = \int d^4 \theta \left\{ -\frac{1}{e^2} \Sigma \Sigma + \sum_a \left( e^{2Q_a V + B_a} - \frac{1}{2} (Y_a + \bar{Y}_a) B_a \right) \right\} + \left( \frac{1}{\sqrt{2}} \int d^2 \tilde{\theta} (-\Sigma t) + (c.c.) \right),
\]

(1)

where \( Y_a \) and \( B_a \) are twisted chiral superfields and a real superfields \( B_a \).

Integrating out twisted chiral superfields \( Y_a \), we obtain \( \overline{D}_+ D_- B_a = D_+ \overline{D}_- B_a = 0 \), whose solutions are written in terms of chiral superfields \( \Psi_a \) and \( \overline{\Psi}_a \) such as \( B_a = \Psi_a + \overline{\Psi}_a \). When we substitute them into (1), \( \mathcal{L}_{\text{GLSM}} \) appears:

\[
\mathcal{L}' \bigg|_{B_a = \Psi_a + \overline{\Psi}_a} = \int d^4 \theta \left\{ -\frac{1}{e^2} \Sigma \Sigma + \sum_a \overline{\Phi}_a e^{2Q_a V} \Phi_a \right\} + \left( \frac{1}{\sqrt{2}} \int d^2 \tilde{\theta} (-\Sigma t) + (c.c.) \right) \equiv \mathcal{L}_{\text{GLSM}},
\]

(2)

where we re-wrote \( \Phi_a = e^\Psi_a \).

On the other hand, when we first integrate out \( B_a \) in \( \mathcal{L}' \), we obtain \( B_a = -2Q_a V + \log \left( \frac{Y_a + \overline{Y}_a}{2} \right) \).

Let us insert these solutions into (1). By using a deformation \( \int d^4 \theta Q_a V Y_a = -\frac{Q_a}{2} \int d^2 \tilde{\theta} \overline{D}_+ D_- V Y_a = -\frac{Q_a}{\sqrt{2}} \int d^2 \tilde{\theta} \Sigma Y_a \), we find that a Lagrangian of twisted chiral superfields appears:

\[
\mathcal{L}_T = \int d^4 \theta \left\{ -\frac{1}{e^2} \Sigma \Sigma - \sum_a \left( \frac{1}{2} (Y_a + \overline{Y}_a) \log (Y_a + \overline{Y}_a) \right) \right\} + \left( \frac{1}{\sqrt{2}} \int d^2 \tilde{\theta} \overline{W} + (c.c.) \right),
\]

\[ \overline{W} = \Sigma \left( \sum_a Q_a Y_a - t \right) + \mu \sum_a e^{-Y_a}. \]

Notice that the twisted superpotential \( \overline{W} \) is corrected by instanton effects where the instantons are the vortices of the gauge theory. In attempt to analyze a model satisfying \( \sum_a Q_a = 0 \), the scale parameter \( \mu \) is omitted by field re-definitions.
Linear dilaton CFT and Liouville theory

\[ \mathbb{R}^{9,1} = \mathbb{R}^{d-1,1} \times X^{2n} \sim \mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times S^1 \times \mathcal{M}/U(1) \]

Free SCFT \quad Singular CY \quad Linear dilaton SCFT \quad \mathcal{N} = 2 \text{ Landau-Ginzburg}

Linear dilaton: \( \Phi = -\frac{Q}{2} \phi \)

Landau-Ginzburg: \( W_{\text{LG}} = F(Z_a), \quad F(\lambda^{r_a}Z_a) = \lambda F(Z_a) \)

\[ c_{\text{total}} = c_d + c_{\text{dilaton}} + c_{\text{LG}} \quad \rightarrow \quad 15 = \frac{3}{2}d + \left( \frac{3}{2} + 3Q^2 \right) + 3 \sum_{a=1}^{n+1} (1 - 2r_a) \]

\[ \mathcal{N} = 2 \text{ "LG" on } \mathbb{R}_\phi \times S^1 \times \mathcal{M}/U(1): \quad W = -\mu Z_0^{-k} + F(Z_a) \]

\[ k = \frac{1}{r_{\Omega}} = \frac{2}{Q^2}, \quad r_{\Omega} \equiv \sum_a r_a - 1 \]

Linear dilaton SCFT on \( \mathbb{R}_\phi \times S^1 \) \equiv "LG" with \( W = -\mu Z_0^{-k} \)

\[ \equiv \text{ Kazama-Suzuki model on } SL(2,\mathbb{R})_k/U(1) \]

\[ \text{T-dual} \quad \equiv \text{ Liouville theory of charge } Q \]

Strictly, we consider the Euclidean black hole: \( SL(2,\mathbb{R})_k/U(1) \rightarrow [SL(2,\mathbb{C})_k/SU(2)]/U(1) \)
Let us study how to obtain the geometry with $\mathbb{Z}_\ell$-type orbifold symmetry. Replacing $\ell \Sigma$ in $\hat{\Pi}$ to

$$\ell \Sigma \to \frac{\partial}{\partial Y_{P_1}},$$

we can perform the integration of $\Sigma$ and obtain

$$\hat{\Pi} = \int \prod_{i=1}^{N} \mathrm{d}Y_i \left( e^{-Y_{P_1}} \mathrm{d}Y_{P_1} \right) \mathrm{d}Y_{P_2} \delta \left( \sum_i Y_i - \ell Y_{P_1} - (N - \ell) Y_{P_2} - t \right) \exp \left( - \sum_i e^{-Y_i} - e^{-Y_{P_1}} - e^{-Y_{P_2}} \right). \quad (3)$$

We perform the re-definitions of the variables $Y_i$, $Y_{P_1}$ and $Y_{P_2}$:

$$e^{-Y_{P_1}} = \tilde{P}_1, \quad e^{-Y_a} = \tilde{P}_1 U_a \quad \text{for} \ a = 1, \ldots, \ell, \quad e^{-Y_{P_2}} = \tilde{P}_2, \quad e^{-Y_b} = \tilde{P}_2 U_b \quad \text{for} \ b = \ell + 1, \ldots, N.$$

Substituting these re-defined variables into (3), we continue the calculation:

$$\hat{\Pi} = \int \prod_{i=1}^{N} \left( \frac{\mathrm{d}U_i}{U_i} \right) \mathrm{d}\tilde{P}_1 \left( \frac{\mathrm{d}\tilde{P}_2}{\tilde{P}_2} \right) \delta \left( \log \left( \prod_i U_i \right) + t \right) \exp \left\{ - \tilde{P}_1 \left( \sum_{a=1}^\ell U_a + 1 \right) - \tilde{P}_2 \left( \sum_{b=\ell+1}^{N} U_b + 1 \right) \right\}
= \int \prod_i \left( \frac{\mathrm{d}U_i}{U_i} \right) \mathrm{d}\tilde{P}_2 \mathrm{d}u \mathrm{d}v \delta \left( \log \left( \prod_i U_i \right) + t \right) \delta \left( \sum_a U_a + 1 \right) \exp \left\{ - \tilde{P}_2 \left( \sum_b U_b + 1 - uv \right) \right\}
= \int \prod_i \left( \frac{\mathrm{d}U_i}{U_i} \right) \mathrm{d}u \mathrm{d}v \delta \left( \log \left( \prod_i U_i \right) + t \right) \delta \left( \sum_a U_a + 1 \right) \delta \left( \sum_b U_b + 1 - uv \right), \quad (4)$$

where we introduced new variables $u$ and $v$ taking values in $\mathbb{C}$ and used a following equation

$$\frac{1}{\tilde{P}_2} = \int \mathrm{d}u \mathrm{d}v \exp \left( \tilde{P}_2 uv \right).$$
It is obvious that (4) still includes a non-canonical integral measure. Thus we perform further redefinitions such as

\[ U_a = e^{-t/\ell} \frac{Z_a^\ell}{Z_1 \cdots Z_N}, \quad U_b = Z_b^\ell. \]

Note that the period integral (4) is invariant under the following transformations acting on the new variables \( Z_i \):

\[ Z_a \mapsto \lambda \omega_a Z_a, \quad Z_b \mapsto \omega_b Z_b, \quad \omega_a^\ell = \omega_b^\ell = \omega_1 \cdots \omega_N = 1, \]

where \( \lambda \) is an arbitrary number taking in \( \mathbb{C}^* \). The \( \omega_i \) come from the shift symmetry of the original variables \( Y_i \equiv Y_i + 2\pi i \).

Combining these transformations we find that \( \hat{\Pi} \) has \( \mathbb{C}^* \times (Z_\ell)^{N-2} \) symmetries. Substituting \( Z_i \) into (4), we obtain

\[ \hat{\Pi} = \int \frac{1}{\text{vol.} (\mathbb{C}^*)} \prod_{i=1}^N dZ_i \, du \, dv \, \delta \left( \sum_{a=1}^\ell Z_a^\ell + e^{t/\ell} Z_1 \cdots Z_N \right) \delta \left( \sum_{b=\ell+1}^N Z_b^\ell + 1 - uv \right), \]

which indicates that the resulting mirror geometry is described by

\[ \hat{\mathcal{M}}_\ell = \left\{(Z_i; u, v) \in \mathbb{C}^{N+2} \mid \left\{ \mathcal{F}(Z_i) = 0 \right\}/\mathbb{C}^*, \quad \mathcal{G}(Z_b; u, v) = 0 \right\}/(Z_\ell)^{N-2}, \]

\[ \mathcal{F}(Z_i) = \sum_{a=1}^\ell Z_a^\ell + \psi Z_1 \cdots Z_\ell, \quad \mathcal{G}(Z_b; u, v) = \sum_{b=\ell+1}^N Z_b^\ell + 1 - uv, \quad \psi = e^{t/\ell} Z_{\ell+1} \cdots Z_N. \]

This is an \((N - 1)\)-dimensional complex manifold.

The equation \( \mathcal{F}(Z_i) = 0 \) denotes that the complex variables \( Z_a \) consist of the degree \( \ell \) hypersurface in the projective space: \( \mathbb{CP}^{\ell-1}[\ell] \). This subspace itself is a compact CY manifold, which is parametrized by a parameter \( \psi \) which is subject to the equation \( \mathcal{G}(Z_b; u, v) = 0 \). Moreover we can also interpret that the total space is a noncompact CY manifold whose compact directions are described by \( Z_i \), while the variables \( u \) and \( v \) run in the noncompact directions under the equations.
\[ \mathcal{N} = (2, 2) \text{ SCFT and (compact) Calabi-Yau geometry} \]

\( (c, c) \) ring element:
- \((p, q)\) charged
- NS-NS chiral primary
- \(p, q \geq 0\)

One-to-one correspondence

Spectral flow
- \(\theta_L = 1, \theta_R = 0\)

harmonic \((d - p, q)\)-form
- on Calabi-Yau \(d\)-fold
- \(d - p, q \geq 0\)
- \(d = c/3\)

\( (a, c) \) ring element:
- \((p - d, q)\) charged
- NS-NS chiral primary
- \(p - d \leq 0\)

Spectral flow
- \(-\theta_L = \theta_R = \frac{1}{2}\)

R-R ground state
- (SUSY ground state)
- \((q_L, q_R) = (p - \frac{d}{2}, q - \frac{d}{2})\)
- \(h_L = h_R = c/24\)

Relation among the NS-NS chiral primary states, R-R ground states and harmonic forms