Gauged Linear Sigma Models
for Noncompact Calabi-Yau Varieties

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$\mathcal{N} = (2, 2)$ SUSY gauge theory with matters  (FI : $t \equiv r - i\theta$)

$$\mathcal{L} = \int d^4\theta \left\{ -\frac{1}{e^2} \sum \Sigma + \sum_a \Phi_a e^{2Q_a V} \Phi_a \right\}$$

$$+ \left( \frac{1}{\sqrt{2}} \int d^2\theta \left( -\Sigma t \right) + \text{c.c.} \right) + \left( \int d^2\theta W_{\text{GLSM}}(\Phi_a) + \text{c.c.} \right)$$

$\Phi_a$ : charged chiral superfield, $\bar{D}_\pm \Phi_a = 0$

$\Sigma$ : twisted chiral superfield, $\bar{D}_+ \Sigma = D_- \Sigma = 0$, $\Sigma = \frac{1}{\sqrt{2}} \bar{D}_+ D_- V$

There exist at least two phases:

$\text{FI} \gg 0$ : differential-geometric phase $\rightarrow$ SUSY NLSM

$\text{FI} \ll 0$ : algebro-geometric phase $\rightarrow$ LG, orbifold, SCFT

Calabi-Yau/Landau-Ginzburg correspondence

harmonic forms $\leftrightarrow$ NS-NS chiral primary states

“Mirror geometry” appears in the T-dual theory in terms of twisted chiral superfields $Y_a$

$$Y_a + \bar{Y}_a \equiv 2\bar{\Phi}_a e^{2Q_a V} \Phi_a$$

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**Effective theories**

The potential energy density is given by

\[
U(\phi, \sigma) = \frac{e^2}{2} \mathcal{D}^2 + \sum_a |F_a|^2 + U_\sigma(\phi, \sigma)
\]

\[
\mathcal{D} = \frac{1}{e^2} D = r - \sum_a Q_a |\phi_a|^2, \quad \overline{F}_a = -\frac{\partial}{\partial \phi_a} W_{\text{GLSM}}(\phi), \quad U_\sigma(\phi, \sigma) = 2|\sigma|^2 \sum_a Q_a^2 |\phi_a|^2
\]

The supersymmetric vacuum manifold \( \mathcal{M} \) is defined by

\[
\mathcal{M} = \left\{ (\phi_a, \sigma) \in \mathbb{C}^n \left| \mathcal{D} = F_a = U_\sigma = 0 \right. \right\} / U(1)
\]

In the IR limit \( e \to \infty \), there appears the supersymmetric NLSM on \( \mathcal{M} \) whose coupling is

\[
r = \frac{1}{g^2}
\]

Renormalization of the FI parameter is

\[
r_0 = r_R + s \cdot \log \left( \frac{\Lambda_{\text{UV}}}{\mu} \right), \quad s = \sum_a Q_a
\]

Thus we find that

\[
\begin{align*}
    s > 0 & \quad \rightarrow \quad \text{the theory is asymptotic free} \\
    s = 0 & \quad \rightarrow \quad \text{the theory is conformal} \\
    s < 0 & \quad \rightarrow \quad \text{the theory is infrared free}
\end{align*}
\]
Gauged Linear Sigma Model

for $\mathcal{O}(-N + \ell)$ bundle on $\mathbb{C}P^{N-1}[\ell]$

<table>
<thead>
<tr>
<th>chiral superfield</th>
<th>$S_1$</th>
<th>\cdots</th>
<th>$S_N$</th>
<th>$P_1$</th>
<th>$P_2$</th>
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<tbody>
<tr>
<td>$U(1)$ charge</td>
<td>1</td>
<td>\cdots</td>
<td>1</td>
<td>$-\ell$</td>
<td>$-N + \ell$</td>
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$$W_{\text{GLSM}} = P_1 \cdot G_\ell(S_i)$$

$G_\ell(S_i)$: homogeneous polynomial of degree $\ell$

potential energy density:

$$U = \frac{e^2}{2} \mathcal{D}^2 + |G_\ell(s)|^2 + \sum_{i=1}^{N} |p_1 \partial_i G_\ell(s)|^2 + U_\sigma$$

$$\mathcal{D} = r - \sum_{i=1}^{N} |s_i|^2 + \ell |p_1|^2 + (N - \ell) |p_2|^2$$

$$U_\sigma = +2 |\sigma|^2 \left\{ \sum_{i=1}^{N} |s_i|^2 + \ell^2 |p_1|^2 + (N - \ell)^2 |p_2|^2 \right\}$$

Let us analyze SUSY vacuum manifold $U = 0$ and massless effective theories
The four theories are related to each other via CY/LG correspondence and topology change:

Notice that the above relation is just a conjectured one because we still have no mathematical techniques to check the topological aspects on the noncompact CY.

We also notice that we have obtained various massless effective theories by decomposing all massive modes. Thus they are just approximate descriptions. However the T-dual theory of the GLSM is so powerful to obtain the exact effective theories. Analyzing them exact theories we will re-investigate the massless effective theories in the original GLSM.
\( \mathcal{L} = \int d^4 \theta \left\{ - \frac{1}{e^2} \Sigma \Sigma - \sum_a \left( \frac{1}{2} (Y_a + \bar{Y}_a) \log (Y_a + \bar{Y}_a) \right) \right\} + \left( \frac{1}{\sqrt{2}} \int d^2 \tilde{\theta} \tilde{W} + c.c. \right) \)

\[ \tilde{W} = \Sigma \left( \sum_{i=1}^{N} Y_i - \ell Y_{P_1} - (N - \ell) Y_{P_2} - t \right) + \sum_{i=1}^{N} e^{-Y_i} + e^{-Y_{P_1}} + e^{-Y_{P_2}} \]

Period integral: \( \hat{\Pi} \equiv \int d\Sigma \prod_{i=1}^{N} dY_i dY_{P_1} dY_{P_2} (\ell \Sigma) \exp (- \tilde{W}) \)

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</tr>
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<td>twisted chiral</td>
<td>( Y_1 )</td>
<td>( Y_2 )</td>
<td>( \cdots )</td>
<td>( Y_N )</td>
<td>( Y_{P_1} )</td>
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\( 2\Phi_a e^{2Q_a} V \Phi_a = Y_a + \bar{Y}_a \)

\( U(1) \) phase rotation symmetry on \( \Phi_a \) \( \Rightarrow \) shift symmetry on \( Y_a \): \( Y_a \equiv Y_a + 2\pi i \)

In the IR limit \( e \to \infty \), the gauge field \( \Sigma \) is no longer dynamical and should be integrated out.

In order to obtain LG theory or geometry, we replace \( \Sigma \) to \( \Sigma \to \frac{\partial}{\partial t} \) or \( \Sigma \to \frac{\partial}{\partial Y_P} \)
Twisted Landau-Ginzburg theory:

Solve $Y_{P_1}$ by using the constraint derived from integrating out $\Sigma$:

$$Y_{P_1} = \frac{1}{\ell} \left\{ t - \sum_{i=1}^{N} Y_i + (N - \ell) Y_{P_2} \right\}$$

Field re-definition preserving canonical measure in $\hat{\Pi}$:

$$X_i \equiv e^{-\frac{1}{\ell} Y_i} , \quad X_{P_2} \equiv e^{\frac{N-\ell}{\ell} Y_{P_2}} , \quad X_i \rightarrow \omega_i X_i , \quad X_{P_2} \rightarrow \omega_{P_2} X_{P_2} , \quad (\mathbb{Z}_\ell)^N \text{ symmetry}$$

Thus we obtain the twisted LG superpotential:

$$\left\{ \tilde{W}_\ell = X_1^{\ell} + \cdots + X_N^{\ell} + X_{P_2}^{-\frac{N-\ell}{\ell}} + e^{t/\ell} X_1 \cdots X_N X_{P_2} \right\} / (\mathbb{Z}_\ell)^N$$

The negative power term describes $\mathcal{N} = 2$ Kazama-Suzuki model on $SL(2, \mathbb{R})_k / U(1)$:

$$\frac{\ell}{N - \ell} = k = \frac{2}{Q^2}$$

Thus we argue that

this effective theory is the LG minimal model coupled to the KS model with $(\mathbb{Z}_\ell)^N$ orbifold symmetry
Recall the following two arguments:

- \( \mathcal{N} = 2 \) SCFT on \( SL(2, \mathbb{R})_k/U(1) \) is **equivalent** to \( \mathcal{N} = 2 \) Liouville theory via T-duality.

- If a CFT \( \mathcal{C} \) has an abelian discrete symmetry group \( \Gamma \), the orbifold CFT \( \mathcal{C}' = \mathcal{C}/\Gamma \) has a symmetry group \( \Gamma' \) which is isomorphic to \( \Gamma \). Furthermore a new orbifold CFT \( \mathcal{C}'/\Gamma' \) is **identical** to the original CFT \( \mathcal{C} \).

Thus we insist that

\[
\text{"\{CFT on } \mathbb{C}^1 \otimes \text{LG with } W_{\text{LG}} = \langle p_1 \rangle G_\ell(S) \}/\mathbb{Z}_\ell \text{" in the original GLSM is described by}
\]

\[
\{ \mathcal{N} = 2 \text{ Liouville theory coupled to the LG minimal model with } W_{\text{LG}} \}/\mathbb{Z}_\ell \text{ as an exact effective theory}
\]
Summary

- We found three non-trivial phases and four effective theories in the GLSM
  - two CY sigma models
  - two orbifolded LG theories coupled to 1-dim. SCFT

- We constructed four exact effective theories in the T-dual theory
  - two NLSMs on mirror CY geometries
  - two orbifolded LG theories including a term with negative power $-k$

  This term represents a gauged WZW model on $SL(2, \mathbb{R})_k/U(1)$ at level $k$

- We argue that the LG theories in the original GLSM can be interpreted as $\mathcal{N} = 2$ Liouville theories coupled to LG minimal models
The CY sigma model on noncompact $\mathcal{M}_{CY}$ is related to the $\mathcal{N} = 2$ Liouville \times LG orbifold theory $Z_{\ell}$ through the GLSM (Gepner-Landweber-Smith-Macpherson) model.

$\ell \Sigma \to \frac{\partial}{\partial Y_{P_1}}$ for FI $\gg 0$ and $\ell \Sigma \to \ell \frac{\partial}{\partial t}$ for FI $\ll 0$.

The Hori-Vafa model provides a T-dual connection between the CY sigma model on noncompact $\widetilde{\mathcal{M}}_{\ell}$ and the $SL(2, \mathbb{R})_k/U(1) \times LG (Z_{\ell})^N$ orbifold theory.

These relations are part of a broader framework that includes mirror symmetry and T-duality.
linear dilaton CFT and Liouville theory

\[ \mathbb{R}^{9,1} = \mathbb{R}^{d-1,1} \times X^{2n} \quad \sim \quad \mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times S^1 \times \mathcal{M}/U(1) \]

free SCFT \quad singular CY \quad linear dilaton SCFT \quad \mathcal{N} = 2 \text{ Landau-Ginzburg}

linear dilaton: \( \Phi = -\frac{Q}{2} \phi \)

Landau-Ginzburg: \( W_{\text{LG}} = F(Z_a), \quad F(\lambda^{r_a} Z_a) = \lambda F(Z_a) \)

\[ c_{\text{total}} = c_d + c_{\text{dilaton}} + c_{\text{LG}} \quad \rightarrow \quad 15 = \frac{3}{2}d + \left(\frac{3}{2} + 3Q^2\right) + 3 \sum_{a=1}^{n+1} (1 - 2r_a) \]

\[ \mathcal{N} = 2 \text{ "LG" on } \mathbb{R}_\phi \times S^1 \times \mathcal{M}/U(1): \quad W = -\mu Z_0^{-k} + F(Z_a) \]

\[ k = \frac{1}{r_\Omega} = \frac{2}{Q^2}, \quad r_\Omega \equiv \sum_a r_a - 1 \]

linear dilaton SCFT on \( \mathbb{R}_\phi \times S^1 \) \( \equiv \) "LG" with \( W = -\mu Z_0^{-k} \)

\( \equiv \) Kazama-Suzuki model on \( SL(2, \mathbb{R})_k/U(1) \)

\( \overset{T\text{-dual}}{\equiv} \) Liouville theory of charge \( Q \)

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APPENDIX

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