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Generalized/doubled/nongeometric string backgrounds

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(Lower dimensional) supergravity related to this topic

J. Maharana, J.H. Schwarz hep-th/9207016

L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré, T. Magri hep-th/9605032 P. Fré hep-th/9512043

N. Kaloper, R.C. Myers hep-th/9901045

E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, A. Van Proeyen hep-th/0103233

M.B. Schulz hep-th/0406001 S. Gurrieri hep-th/0408044 T.W. Grimm hep-th/0507153

B. de Wit, H. Samtleben, M. Trigiante hep-th/0507289

EOM, SUSY, and Bianchi identities on a geometry with $SU(3) \times SU(3)$ structures

M. Graña, J. Louis, D. Waldram hep-th/0505264 hep-th/0612237

D. Cassani, A. Bilal arXiv:0707.3125 D. Cassani arXiv:0804.0595

A.K. Kashani-Poor, R. Minasian hep-th/0611106 A. Tomasiello arXiv:0704.2613 B.y. Hou, S. Hu, Y.h. Yang arXiv:0806.3393

M. Graña, R. Minasian, M. Petrini, D. Waldram arXiv:0807.4527

AdS$_4$ SUSY vacua

D. Lüst, D. Tsimpis hep-th/0412250

C. Kounnas, D. Lüst, P.M. Petropoulos, D. Tsimpis arXiv:0707.4270 P. Koerber, D. Lüst, D. Tsimpis arXiv:0804.0614

C. Caviezel, P. Koerber, S. Kors, D. Lüst, D. Tsimpis, M. Zagermann arXiv:0806.3458

D-branes, orientifold projection, calibration, and smeared sources

B.S. Acharya, F. Benini, R. Valandro hep-th/0607223

M. Graña, R. Minasian, M. Petrini, A. Tomasiello hep-th/0609124

L. Martucci, P. Smyth hep-th/0507099 P. Koerber, D. Tsimpis arXiv:0706.1244 P. Koerber, L. Martucci arXiv:0707.1038

M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell, A. Westerberg hep-th/9611159

E. Bergshoeff, P.K. Townsend hep-th/9611173

Doubled formalism

C.M. Hull hep-th/0406102 hep-th/0605149 hep-th/0701203 C.M. Hull, R.A. Reid-Edwards hep-th/0503114 arXiv:0711.4818

J. Shelton, W. Taylor and B. Wecht hep-th/0508133 A. Dabholkar, C.M. Hull hep-th/0512005

A. Lawrence, M.B. Schulz, B. Wecht hep-th/0602025

G. Dall'Agata, S. Ferrara hep-th/0502066

G. Dall'Agata, M. Prezas, H. Samtleben, M. Trigiante arXiv:0712.1026 G. Dall'Agata, N. Prezas arXiv:0806.2003

C. Albertsson, TK, R.A. Reid-Edwards arXiv:0806.1783

and more...

Introduction

- $\mathcal{N} = 2$ supergravity

  highly symmetric (controllable), dynamical (non-trivial), connectable to Seiberg-Witten, etc..

  governed by holomorphic functionals (prepotentials)

- $\mathcal{N} = 1$ supergravity

  highly dynamical, less symmetric, connectable to (SUSY) GUTs, etc..

  governed by Kähler potential and superpotential

many ways to derive them from ten-dimensional type II and heterotic string theories

10-dim. equations of motion (with sources)

$\Updownarrow$

SUSY variations	+	EOM for form fields (with sources)	+	Bianchi identities (with sources)

SUSY variation : compactified geometry

EOM for form fields : SUSY solutions

Bianchi identities : no-go theorem (sources as D-branes, orientifold planes)

Moduli in $\mathcal{N} = 2$ supergravity: Appendix

	vector multiplet	hypermultiplet
generic	coord. of Hodge-Kähler	coord. of quaternionic
IIA on Calabi-Yau	Kähler moduli	complex moduli + RR
IIB on Calabi-Yau	complex moduli	Kähler moduli + RR

Duality relations in $\mathcal{N} = 2$ theories:

type IIA	$\longleftrightarrow$	type IIB	T-duality, mirror symmetry
type $\mathrm{II}/\mathrm{CY}_3$	$\longleftrightarrow$	heterotic$/[K3 \times T^2]$	S-duality

Reduction to $\mathcal{N} = 1$ supergravity is given in terms of orientifold planes

$$K^{\mathsf{KS}} = -\log\left(\frac{4}{3}\int_{\mathrm{CY}_3} J \wedge J \wedge J\right)$$

$$K^{\mathsf{CS}} = -\log\left(i\int_{\mathrm{CY}_3} \Omega \wedge \overline{\Omega}\right)$$

$$W_{\mathrm{IIA},\mathsf{RR}} = i\,\mathrm{e}^{\phi}\int_{\mathrm{CY}_3} G_{\mathrm{A}} \wedge \mathrm{e}^{-B-iJ}$$

$$W_{\mathrm{IIB},\mathsf{RR}} = i\,\mathrm{e}^{\phi}\int_{\mathrm{CY}_3} G_{\mathrm{B}} \wedge \Omega$$

$$W_{H\text{-}\mathsf{flux}} = \int_{\mathrm{CY}_3} H_3 \wedge \Omega$$

$$F_n \;=\; \mathrm{d}C_{n-1} - H_3 \wedge C_{n-3} \;\equiv\; \mathrm{e}^{B} G$$

$$G_{\mathrm{A}} \;=\; G_0 + G_2 + G_4 + G_6 \qquad\qquad G_{\mathrm{B}} \;=\; G_3$$

$$J \wedge J \wedge J \;=\; \frac{3i}{4}\Omega \wedge \overline{\Omega} \qquad\qquad J \wedge \Omega \;=\; 0 \;=\; B \wedge \Omega$$

Question 1: Generic supersymmetric effective theory beyond Calabi-Yau geometry?

- ☹ condition on geometry from supersymmetry? --→ $SU(3)$-structure manifold Appendix
- ☹ identify "light" modes?
- ☹ generic form of Kähler potentials and superpotentials?

$$\begin{aligned} \mathrm{d}s^2_{1,9} &= \mathrm{e}^{2A} g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + g_{ij}\, \mathrm{d}y^i \otimes \mathrm{d}y^j \\ \delta\psi_i &= \Big(\partial_i + \frac{1}{4}\omega_{iab}\,\gamma^{ab}\Big)\eta - \frac{1}{4} H_{ijk}\gamma^{jk}\eta + \ldots \ \equiv\ 0 \\ \delta\lambda &= -\frac{1}{4}\Big(\gamma^i\partial_i\phi - \frac{1}{6} H_{ijk}\gamma^{ijk}\Big)\eta + \ldots \ \equiv\ 0 \end{aligned}$$

$$(\mathrm{d} - H_3\wedge)(\mathrm{e}^{4A} *_6 F) \ =\ 0 \qquad\qquad (\mathrm{d} - H_3\wedge)F \ =\ \delta(\text{source})$$

$$\mathrm{d}(\mathrm{e}^{4A-2\phi} *_6 H_3) \ =\ \mp \mathrm{e}^{4A} F_n \wedge *_6 F_{n+2}$$

$$\mathrm{d}H_3 \ =\ 0$$

Question 2: Modification of dualities among string theories by fluxes?

- ☹ T-duality (mirror symmetry) from (non-)Calabi-Yau to what?
- ☹ S-duality and U-duality symmetries?
- ☹ Find more non-trivial relations?

$$\begin{array}{ccc} \wedge^{\text{even}} T^* \mathcal{M}_6 & & \wedge^{\text{odd}} T^* \mathcal{M}_6 \\ \mathrm{e}^{-B-iJ} & \longleftrightarrow & \Omega \\ G_{\mathrm{A}} = G_0 + G_2 + G_4 + G_6 & & G_{\mathrm{B}} = G_3 \end{array}$$

Generically, a Calabi-Yau with non-trivial fluxes does not yield a supersymmetric solution...

How should we derive modified Kähler/superpotentials?

How are string dualities realized?

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Extend geometrical information of compactified space

N.J. Hitchin

Generalized geometry

J on $T\mathcal{M}_d$, ω on $T^*\mathcal{M}_d \dashrightarrow \mathcal{J}_\pm$ on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$

"Cliff(6) pure spinor $\eta_\pm$" on $T\mathcal{M}_6$

$\dashrightarrow$ "Cliff(6,6) pure spinor $\Phi_\pm$" on $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$

Evaluate spaces of $\Phi_\pm$ to provide Kähler/superpotentials in supergravity

C.M. Hull

Doubled formalism

T^d with B-field $\dashrightarrow T^d \times T^d$ (with B-field)

Regard T-duality transformation as a part of transition function

Go beyond (non)-abelian gauged supergravity with B-field

and its duality transformation

- Generalized geomety provides...

  Kähler potentials and superpotentials in the most generic description

  signals of nongeometric fluxes from genuinely stringy effects

- Doubled formalism presents...

  extension of Lie algebra of gauge symmetry in four-dimensional physics

  concrete expressions of stringy (or nongeometric) backgrounds

Generalized geometry

Decomposition of vector bundle on ten-dimensional spacetime:

$$T\mathcal{M}_{1,9} = T_{1,3} \oplus F$$

$$\begin{cases} T_{1,3}: & \text{a real } SO(1,3) \text{ vector bundle} \\ F: & \text{an } SO(6) \text{ vector bundle which admits a pair of } SU(3) \text{ structures} \end{cases}$$

10-dimensional spacetime itself is not decomposed yet, i.e., do not yet consider truncation of modes.

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Decomposition of Lorentz symmetry:

$$Spin(1,9) \to Spin(1,3) \times Spin(6) = SL(2,\mathbb{C}) \times SU(4)$$

$$\mathbf{16}_1 = (\mathbf{2},\mathbf{4})_1 \oplus (\overline{\mathbf{2}},\overline{\mathbf{4}})_1 \qquad \mathbf{16}_2 = (\mathbf{2},\overline{\mathbf{4}})_2 \oplus (\overline{\mathbf{2}},\mathbf{4})_2$$

Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\begin{cases} \epsilon^1_{\text{IIA}} = \xi^1_+ \otimes (a\eta^1_+) + \xi^1_- \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\text{IIA}} = \xi^2_+ \otimes (\overline{b}\eta^2_-) + \xi^2_- \otimes (b\eta^2_+) \end{cases} \qquad \begin{cases} \epsilon^1_{\text{IIB}} = \xi^1_+ \otimes (a\eta^1_+) + \xi^1_- \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\text{IIB}} = \xi^2_+ \otimes (b\eta^2_+) + \xi^2_- \otimes (\overline{b}\eta^2_-) \end{cases}$$

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Set $SU(3)$ invariant spinor η^A_+ s.t. $D^{(T)}\eta^A_+ = 0$ $(A = 1,2)$: Appendix

a pair of $SU(3)$ on F (η^1_+, η^2_+) $\longleftrightarrow$ a single $SU(3)$ on F $(\eta^1_+ = \eta^2_+ = \eta_+)$

Requirement that we have a pair of $SU(3)$ structures means there is a sub-supermanifold

$$\mathcal{N}^{1,9|4+4} \subset \mathcal{M}^{1,9|16+16}$$

$$\left(\begin{array}{ll} \text{(1,9):} & \text{bosonic degrees} \\ \text{4+4:} & \text{eight Grassmann variables as spinors of } Spin(1,3) \text{ and singlet of } SU(3)\text{s} \end{array} \right)$$

Equivalence such as

eight SUSY theory reformulation of type II supergravity
$\Updownarrow$
a pair of $SU(3)$ structures on vector bundle F
$\Updownarrow$
an $SU(3) \times SU(3)$ structure on extended $F \oplus F^*$

$\uparrow$

Entrance Gate to generalized geometry

Introduce a generalized almost complex structure $\mathcal{J}$ on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$ s.t.

$$\mathcal{J} : \; T\mathcal{M}_d \oplus T^*\mathcal{M}_d \;\longrightarrow\; T\mathcal{M}_d \oplus T^*\mathcal{M}_d$$

$$\mathcal{J}^2 \;=\; -\mathbb{1}_{2d}$$

$$^{\exists}\; O(d,d) \text{ invariant metric } L, \text{ s.t. } \; \mathcal{J}^T L \mathcal{J} \;=\; L$$

Structure group on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$:

$^{\exists}L$	$GL(2d)$	$\dashrightarrow$	$O(d,d)$
$\mathcal{J}^2 = -\mathbb{1}_{2d}$	$O(d,d)$	$\dashrightarrow$	$U(d/2,d/2)$
$\mathcal{J}_1, \mathcal{J}_2$	$U_1(d/2,d/2) \cap U_2(d/2,d/2)$	$\dashrightarrow$	$U(d/2) \times U(d/2)$
integrable $\mathcal{J}_{1,2}$	$U(d/2) \times U(d/2)$	$\dashrightarrow$	$SU(d/2) \times SU(d/2)$

- Integrability is discussed by "(0,1)" part of the complexified $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$:

$$\Pi \equiv \frac{1}{2}(\mathbb{1}_{2d} - i\mathcal{J})$$

$$\Pi A = A \quad \text{where } A = v + \zeta \text{ is a section of } T\mathcal{M}_d \oplus T^*\mathcal{M}_d$$

We call this A i-eigenbundle $L_{\mathcal{J}}$, whose dimension is $\dim L_{\mathcal{J}} = d$.

Integrability condition of $\mathcal{J}$ is

$$\overline{\Pi}\big[\Pi(v+\zeta), \Pi(w+\eta)\big]_{\mathrm{C}} = 0 \qquad v, w \in T\mathcal{M}_d \qquad \zeta, \eta \in T^*\mathcal{M}_d$$

$$[v+\zeta, w+\eta]_{\mathrm{C}} = [v,w] + \mathcal{L}_v \eta - \mathcal{L}_w \zeta - \frac{1}{2}\mathrm{d}(\iota_v \eta - \iota_w \zeta) : \text{ Courant bracket}$$

- Two typical examples of generalized almost complex structures:

$$\mathcal{J}_1 = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I^T \end{pmatrix} \qquad \text{w/ } I^2 = -\mathbb{1}_d\text{: almost complex structure}$$

$$\mathcal{J}_2 = \begin{pmatrix} \mathbf{0} & -J^{-1} \\ J & \mathbf{0} \end{pmatrix} \qquad \text{w/ } J\text{: almost symplectic form}$$

integrable $\mathcal{J}_1 \quad \leftrightarrow \quad$ integrable I

integrable $\mathcal{J}_2 \quad \leftrightarrow \quad$ integrable J

On a usual geometry, J_{ij} can be given by an $SU(3)$ invariant (pure) spinor η_+ as

$$J_{ij} = -2i\eta_+^\dagger \gamma_{ij} \eta_+ \qquad \gamma^m \eta_+ = 0 \qquad \gamma^{\overline{n}} \eta_+ \neq 0$$

In a similar analogy, we want to find Cliff$(6,6)$ pure spinor(s) Φ.

∵) Compared to almost complex structures, (pure) spinors can be easily utilized in supergravity framework.

On $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$, we can define Cliff$(6,6)$ algebra and $Spin(6,6)$ spinor Φ:

$$\{\Gamma^i, \Gamma^j\} = 0 \qquad \{\Gamma^i, \Gamma_j\} = \delta^i_j \qquad \{\Gamma_i, \Gamma_j\} = 0$$

Irreducible repr. of $Spin(6,6)$ spinor is a Majorana-Weyl

$\rightarrow$ a generic $Spin(6,6)$ spinor bundle S splits to $S^{\pm}$ (Weyl)

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Weyl spinor bundles $S^\pm$ are isomorphic to bundles of forms on $T^*\mathcal{M}_6$:

$$S^+ \text{ on } T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \quad \sim \quad \wedge^{\text{even}} T^*\mathcal{M}_6$$

$$S^- \text{ on } T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \quad \sim \quad \wedge^{\text{odd}} T^*\mathcal{M}_6$$

Thus we often regard a Cliff$(6,6)$ spinor as a form on $\wedge^{\text{even/odd}} T^*\mathcal{M}_6$

A form-valued representation of the algebra

$$\Gamma^i = \mathrm{d}x^i \wedge \qquad \Gamma_j = \iota_j$$

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IF Φ is annihilated by half numbers of the Cliff$(6,6)$ generators:

$\rightarrow$ Φ is called a pure spinor

cf.) $SU(3)$ invariant spinor η_+ is a Cliff(6) pure spinor: $\gamma^m \eta_+ = 0$

An equivalent definition of a Cliff$(6,6)$ pure spinor is given by "Clifford action":

$$(v+\zeta)\cdot\Phi \;=\; v^i \iota_{\partial_i}\Phi + \zeta_i \, \mathrm{d}x^i \wedge \Phi \qquad \text{w/ } v\text{: vector} \quad \zeta\text{: one-form}$$

Define the annihilator of a spinor as

$$L_\Phi \;\equiv\; \Big\{ v+\zeta \in T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \,\Big|\, (v+\zeta)\cdot\Phi = 0 \Big\}$$
$$\dim L_\Phi \;\leq\; d$$

If $\dim L_\Phi = 6$ (maximally isotropic) $\rightarrow$ Φ is a pure spinor

Correspondence between pure spinors and generalized almost complex structures:

$$\mathcal{J} \leftrightarrow \Phi \qquad \text{if} \;\; L_{\mathcal{J}} = L_{\Phi} \quad \text{with} \;\; \dim L_{\Phi} = 6$$

More precisely: $\mathcal{J} \leftrightarrow$ a line bundle of pure spinor Φ

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Then, we can rewrite the generalized almost complex structure as

$$\boxed{\mathcal{J}_{\pm \Pi\Sigma} \ = \ \big\langle \mathrm{Re}\Phi_{\pm}, \Gamma_{\Pi\Sigma}\, \mathrm{Re}\Phi_{\pm} \big\rangle}$$

w/ Mukai pairing:

even forms: $\big\langle \Psi_+, \Phi_+ \big\rangle = \Psi_6 \wedge \Phi_0 - \Psi_4 \wedge \Phi_2 + \Psi_2 \wedge \Phi_4 - \Psi_0 \wedge \Phi_6$

odd forms: $\big\langle \Psi_-, \Phi_- \big\rangle = \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Phi_3 + \Psi_1 \wedge \Phi_5$

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$\mathcal{J}$ is integrable $\longleftrightarrow$ $^{\exists}$ vector v and one-form ζ s.t. $\mathrm{d}\Phi = (v \llcorner + \zeta \wedge)\Phi$

generalized CY $\longleftrightarrow$ $^{\exists}\Phi$ is pure s.t. $\mathrm{d}\Phi = 0$

"twisted" GCY $\longleftrightarrow$ $^{\exists}\Phi$ is pure, and H is closed s.t. $(\mathrm{d} - H\wedge)\Phi = 0$

A Cliff$(6,6)$ spinor can also be mapped to a bispinor:

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \cdots i_k} \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \cdots i_k} \, \gamma^{i_1 \cdots i_k}_{\alpha\beta}$$

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On a geometry of a single $SU(3)$-structure, the following two $SU(3,3)$ spinors:

$$\Phi_{0+} = \eta_+ \otimes \eta_+^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_+^\dagger \gamma_{i_k \cdots i_1} \eta_+ \, \gamma^{i_1 \cdots i_k} = \frac{1}{8} \mathrm{e}^{-iJ}$$

$$\Phi_{0-} = \eta_+ \otimes \eta_-^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_-^\dagger \gamma_{i_k \cdots i_1} \eta_+ \, \gamma^{i_1 \cdots i_k} = -\frac{i}{8} \Omega$$

Check purity: $(\delta + iI)_i{}^j \, \gamma_j \, \eta_+ \otimes \eta_\pm^\dagger = 0 = \eta_+ \otimes \eta_\pm^\dagger \, \gamma_j (\delta \mp iI)^j{}_i$

One-to-one correspondence: $\Phi_{0-} \leftrightarrow \mathcal{J}_1, \quad \Phi_{0+} \leftrightarrow \mathcal{J}_2$

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On a generic geometry of a pair of $SU(3)$-structure defined by (η_+^1, η_+^2): Appendix

$$\Phi_{0+} = \eta_+^1 \otimes \eta_+^{2\dagger} = \frac{1}{8} \big(\bar{c}_\parallel \mathrm{e}^{-ij} - i \bar{c}_\perp w \big) \wedge \mathrm{e}^{-iv \wedge v'}$$

$$\Phi_{0-} = \eta_+^1 \otimes \eta_-^{2\dagger} = -\frac{1}{8} \big(c_\perp \mathrm{e}^{-ij} + i c_\parallel w \big) \wedge (v + iv')$$

$$|c_\parallel|^2 + |c_\perp|^2 = 1$$

$$\Phi_\pm = \mathrm{e}^{-\mathcal{B}} \Phi_{0\pm}$$

Each $\Phi_{\pm}$ defines an $SU(3,3)$ structure on E. Common structure is $SU(3) \times SU(3)$.

(F is extended to E by including $\mathrm{e}^{-\mathcal{B}}$)

Compatibility requires

$$\left\langle \Phi_+, V \cdot \Phi_- \right\rangle = \left\langle \overline{\Phi}_+, V \cdot \Phi_- \right\rangle = 0 \qquad \text{for } \forall V = x + \xi$$

$$\left\langle \Phi_+, \overline{\Phi}_+ \right\rangle = \left\langle \Phi_-, \overline{\Phi}_- \right\rangle$$

Start with a real form $\chi_f \in \wedge^{\text{even/odd}} F^*$ (associated with a real $Spin(6,6)$ spinor χ_s)

Regard χ_f as a stable form satisfying

$$\begin{aligned} q(\chi_f) &= -\frac{1}{4}\langle \chi_f, \Gamma_{\Pi\Sigma}\chi_f\rangle\langle \chi_f, \Gamma^{\Pi\Sigma}\chi_f\rangle \ \in \ \wedge^6 F^* \otimes \wedge^6 F^* \\ U &= \{\chi_f \in \wedge^{\text{even/odd}} F^* \ : \ q(\chi_f) < 0\} \end{aligned}$$

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Define a Hitchin function

$$H(\chi_f) \ \equiv \ \sqrt{-\frac{1}{3}q(\chi_f)} \ \in \ \wedge^6 F^*$$

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which gives an integrable complex structure on U

Then we can get another real form $\hat{\chi}_f$ and a complex form Φ_f by Mukai pairing

$$\langle \hat{\chi}_f, \chi_f\rangle \;=\; -\mathrm{d}H(\chi_f) \qquad \text{i.e.,} \quad \hat{\chi}_f \;=\; -\frac{\partial H(\chi_f)}{\partial \chi_f}$$

$$\dashrightarrow \qquad \Phi_f \;\equiv\; \frac{1}{2}(\chi_f + i\hat{\chi}_f) \qquad H(\Phi_f) \;=\; i\langle \Phi_f, \overline{\Phi}_f\rangle$$

Hitchin showed: Φ_f is a (form corresponding to) pure spinor!

N.J. Hitchin math/0010054 math/0107101 math/0209099

Consider the space of pure spinors Φ ...

Mukai pairing $\langle *, * \rangle$	$\longrightarrow$	symplectic structure
Hitchin function $H(*)$	$\longrightarrow$	complex structure

$$\Downarrow$$

The space of pure spinor is Kähler (or, rather rigid special Kähler)!

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$$\Downarrow$$

The space of pure spinor is Kähler (or, rather rigid special Kähler)!

Quotienting this space by the $\mathbb{C}^*$ action $\Phi \to \lambda\Phi$ for $\lambda\mathbb{C}^*$

$\dashrightarrow$ The space becomes a local special Kähler geometry with Kähler potential K:

$$\mathrm{e}^{-K} \ = \ H(\Phi) \ = \ i\big\langle \Phi, \overline{\Phi} \big\rangle \ = \ i\big(\overline{\mathcal{Z}}^I \mathcal{F}_I - \mathcal{Z}^I \overline{\mathcal{F}}_I \big) \ \in \ \wedge^6 F^*$$

$\mathcal{Z}^I$: holomorphic homogeneous coordinates

$\mathcal{F}_I$: derivative of prepotential $\mathcal{F}$, i.e., $\mathcal{F}_I = \partial \mathcal{F} / \partial \mathcal{Z}^I$

These are nothing but objects which we want to introduce in $\mathcal{N} = 2$ supergravity!

Space of pure spinors $\Phi_{\pm}$ on $F \oplus F^*$ with $SU(3) \times SU(3)$ structure

$$\|$$

special Kähler geometry of local type $=$ Hodge-Kähler geometry

$$\mathrm{e}^{-K_{\pm}} \; = \; H(\Phi_{\pm}) \; = \; i\big\langle \Phi_{\pm}, \overline{\Phi}_{\pm} \big\rangle \; = \; i\big(\overline{\mathcal{Z}}{}^I_{\pm} \mathcal{F}_{\pm I} - \mathcal{Z}^I_{\pm} \overline{\mathcal{F}}_{\pm I} \big) \in \; \wedge^6 F^*$$

For a single $SU(3)$-structure case:

$$\Phi_+ \; = \; -\frac{1}{8} \mathrm{e}^{-\mathcal{B} - iJ} \qquad K_+ \; = \; -\log \left(\frac{1}{48} J \wedge J \wedge J \right)$$

$$\Phi_- \; = \; -\frac{i}{8} \mathrm{e}^{-\mathcal{B}} \Omega \qquad K_- \; = \; -\log \left(\frac{i}{64} \Omega \wedge \overline{\Omega} \right)$$

Structure of forms is exactly same as the one in the case of Calabi-Yau compactification!

We should truncate Kaluza-Klein massive modes from these forms to obtain 4-dimensional supergravity.

As introduced, we want to obtain four-dimensional $\mathcal{N} = 1, 2$ supergravity theories

Type $\mathrm{IIA}/\mathrm{IIB}$ supergravity theories have 32 supercharges with field multiplets

1	gravity multiplet
6	gravitino multiplets
15	vector multiplets
9	hypermultiplets
1	tensor multiplet

in the language of "$\mathcal{N} = 2$" multiplets

As introduced, we want to obtain four-dimensional $\mathcal{N} = 1, 2$ supergravity theories

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in the language of "$\mathcal{N} = 2$" multiplets

Consider truncation of 6 gravitino multiplets in terms of group theoretical descriptions

Let us discuss group-theoretical properties of massless fields

on a generalized tangent bundle $T_{3,1} \oplus F \oplus F^*$ with $SU(3) \times SU(3)$ structure

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First, consider decomposition of $\mathbf{8}_{\mathrm{S}}$, $\mathbf{8}_{\mathrm{C}}$, $\mathbf{8}_{\mathrm{V}}$ of $SO(8)$ (i.e., light-cone gauge)

$SO(8)$	$\to$	$SO(2) \times SO(6)$	$\to$	$SO(2) \times SU(3)$
$\mathbf{8}_{\mathrm{S}}$	$\to$	$\mathbf{4}_{\frac{1}{2}} \oplus \overline{\mathbf{4}}_{-\frac{1}{2}}$	$\to$	$\mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{\frac{1}{2}} \oplus \overline{\mathbf{3}}_{-\frac{1}{2}}$
$\mathbf{8}_{\mathrm{C}}$	$\to$	$\mathbf{4}_{-\frac{1}{2}} \oplus \overline{\mathbf{4}}_{\frac{1}{2}}$	$\to$	$\mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{-\frac{1}{2}} \oplus \overline{\mathbf{3}}_{\frac{1}{2}}$
$\mathbf{8}_{\mathrm{V}}$	$\to$	$\mathbf{1}_{1} \oplus \mathbf{1}_{-1} \oplus \mathbf{6}_{0}$	$\to$	$\mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{0} \oplus \overline{\mathbf{3}}_{0}$

Using this, consider the decompositions of (NS,R), (R,NS), (NS,NS) and (R,R) sectors...

$\mathbf{a}_{\mathrm{b}}$ denotes a field in the $SU(3)$ repr. $\mathbf{a}$ and 4-dimensional helicity $\mathbf{b}$. $\mathbf{T}$ denotes an antisymmetric tensor.

- Fermions: (R,NS) and (NS,R) sectors:

	$SO(8)_{\rm L} \times SO(8)_{\rm R}$	$\to$	$SO(2) \times SU(3)_{\rm L} \times SU(3)_{\rm R}$
IIA/IIB	$(\mathbf{8}_{\rm S}, \mathbf{8}_{\rm V})$	$\to$	$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\frac{3}{2},-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-\frac{3}{2},\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\pm\frac{1}{2}}$ $\oplus (\mathbf{3},\mathbf{3})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-\frac{1}{2}}$
IIB	$(\mathbf{8}_{\rm V}, \mathbf{8}_{\rm S})$	$\to$	$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{\frac{3}{2},-\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{-\frac{3}{2},\frac{1}{2}}$ $\oplus (\mathbf{3},\mathbf{3})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-\frac{1}{2}}$
IIA	$(\mathbf{8}_{\rm V}, \mathbf{8}_{\rm C})$	$\to$	$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{-\frac{3}{2},\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\frac{3}{2},-\frac{1}{2}}$ $\oplus (\mathbf{3},\mathbf{3})_{-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{\frac{1}{2}}$

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	$SO(8)_{\rm L} \times SO(8)_{\rm R}$	$\to$	$SO(2) \times SU(3)_{\rm L} \times SU(3)_{\rm R}$
IIA/IIB	$(\mathbf{8}_{\rm S}, \mathbf{8}_{\rm V})$	$\to$	$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\frac{3}{2},-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-\frac{3}{2},\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\pm\frac{1}{2}}$ $\oplus (\mathbf{3},\mathbf{3})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-\frac{1}{2}}$
IIB	$(\mathbf{8}_{\rm V}, \mathbf{8}_{\rm S})$	$\to$	$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{\frac{3}{2},-\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{-\frac{3}{2},\frac{1}{2}}$ $\oplus (\mathbf{3},\mathbf{3})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-\frac{1}{2}}$
IIA	$(\mathbf{8}_{\rm V}, \mathbf{8}_{\rm C})$	$\to$	$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2},\pm\frac{1}{2}} \oplus (\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})_{-\frac{3}{2},\frac{1}{2}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\frac{3}{2},-\frac{1}{2}}$ $\oplus (\mathbf{3},\mathbf{3})_{-\frac{1}{2}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\frac{1}{2}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{\frac{1}{2}}$

$(\mathbf{1},\mathbf{1})_{\pm\frac{3}{2}}$: 2 gravitinos in gravity multiplet

$(\mathbf{3},\mathbf{1})_{\pm\frac{3}{2}}$ etc.: 6 gravitinos in gravitino multiplets

$(\mathbf{3},\mathbf{1})_{\pm\frac{1}{2}}$ etc.: fermions in gravitino multiplets

(should not be included in $\mathcal{N}=2$ theory)

- Bosons: (NS,NS) sector:

$$\mathbf{8_V} \times \mathbf{8_V} \;=\; \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \;=\; (\phi, \mathcal{B}_{MN}, \mathcal{G}_{MN})$$

$SO(8)_{\mathrm{L}} \times SO(8)_{\mathrm{R}}$	$\rightarrow$	$SO(2) \times SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}}$	
		$\mathcal{E}_{\mu\nu}$	$(\mathbf{1},\mathbf{1})_{\pm\mathbf{2}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{T}}$
		$\mathcal{E}_{\mu i}$	$(\mathbf{1},\mathbf{3})_{\pm\mathbf{1}} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\pm\mathbf{1}}$
$\mathcal{E}_{MN} = \mathcal{G}_{MN} + \mathcal{B}_{MN}$	$\rightarrow$	$\mathcal{E}_{i\nu}$	$(\mathbf{3},\mathbf{1})_{\pm\mathbf{1}} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm\mathbf{1}}$
		$\mathcal{E}_{ij}$	$(\mathbf{3},\mathbf{3})_{\mathbf{0}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\mathbf{0}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{\mathbf{0}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{\mathbf{0}}$

- Bosons: (NS,NS) sector:

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$SO(8)_{\rm L} \times SO(8)_{\rm R}$	$\to$	$SO(2) \times SU(3)_{\rm L} \times SU(3)_{\rm R}$	
		$\mathcal{E}_{\mu\nu}$	$(\mathbf{1},\mathbf{1})_{\pm\mathbf{2}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{T}}$
		$\mathcal{E}_{\mu i}$	$(\mathbf{1},\mathbf{3})_{\pm 1} \oplus (\mathbf{1},\overline{\mathbf{3}})_{\pm 1}$
$\mathcal{E}_{MN} = \mathcal{G}_{MN} + \mathcal{B}_{MN}$	$\to$	$\mathcal{E}_{i\nu}$	$(\mathbf{3},\mathbf{1})_{\pm 1} \oplus (\overline{\mathbf{3}},\mathbf{1})_{\pm 1}$
		$\mathcal{E}_{ij}$	$(\mathbf{3},\mathbf{3})_{\mathbf{0}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\mathbf{0}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{\mathbf{0}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{\mathbf{0}}$

- Bosons: (R,R) sector:

	$SO(8)_{\rm L} \times SO(8)_{\rm R}$	$\to$	$SO(2) \times SU(3)_{\rm L} \times SU(3)_{\rm R}$
IIA	$(\mathbf{8}_{\rm S}, \mathbf{8}_{\rm C})$	$\to$	$(\mathbf{1},\mathbf{1})_{\pm 1, 0} \oplus (\mathbf{3},\mathbf{3})_{\mathbf{0}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{\mathbf{0}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\mathbf{1}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-\mathbf{1}}$
IIB	$(\mathbf{8}_{\rm S}, \mathbf{8}_{\rm S})$	$\to$	$(\mathbf{1},\mathbf{1})_{\pm 1, 0} \oplus (\mathbf{3},\mathbf{3})_{\mathbf{1}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-\mathbf{1}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\mathbf{0}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{\mathbf{0}}$

Field expressions:

IIA	$\mathcal{A}_0^- = \mathcal{A}_{(0,1)} + \mathcal{A}_{(0,3)} + \mathcal{A}_{(0,5)}$	$\simeq (\mathbf{1},\mathbf{1})_{\mathbf{0}} \oplus (\mathbf{3},\mathbf{3})_{\mathbf{0}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{\mathbf{0}}$
	$\mathcal{A}_1^+ = \mathcal{A}_{(1,0)} + \mathcal{A}_{(1,2)} + \mathcal{A}_{(1,4)} + \mathcal{A}_{(1,6)}$	$\simeq (\mathbf{1},\mathbf{1})_{\pm\mathbf{1}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\mathbf{1}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{-\mathbf{1}}$
IIB	$\mathcal{A}_0^+ = \mathcal{A}_{(0,0)} + \mathcal{A}_{(0,2)} + \mathcal{A}_{(0,4)} + \mathcal{A}_{(0,6)}$	$\simeq (\mathbf{1},\mathbf{1})_{\mathbf{0}} \oplus (\mathbf{3},\overline{\mathbf{3}})_{\mathbf{0}} \oplus (\overline{\mathbf{3}},\mathbf{3})_{\mathbf{0}}$
	$\mathcal{A}_1^- = \mathcal{A}_{(1,1)} + \mathcal{A}_{(1,3)} + \mathcal{A}_{(1,5)}$	$\simeq (\mathbf{1},\mathbf{1})_{\mathbf{1}} \oplus (\mathbf{3},\mathbf{3})_{\mathbf{1}} \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}})_{-\mathbf{1}}$

where $\mathcal{A}_{(p,q)}$ is a "4-dimensional" p-form and a "6-dimensional" q-form

RR field strength is $\mathcal{G}^{\pm} = {\rm d}\mathcal{A}_0^{\mp}$, whose gauge potential is $\mathcal{C} = {\rm e}^{\mathcal{B}} \mathcal{A}$ w/ $\mathcal{F} = {\rm d}\mathcal{C} - \mathcal{H}_3 \wedge \mathcal{C} = {\rm e}^{\mathcal{B}} \mathcal{G}$

- Reduction: effective theory with two gravitinos

$\rightarrow$ all repr. of the form $(\mathbf{3},\mathbf{1}),(\overline{\mathbf{3}},\mathbf{1}),(\mathbf{1},\mathbf{3}),(\mathbf{1},\overline{\mathbf{3}})$ (6 gravitino multiplets) are projected out!

type IIA multiplet	$SU(3)\times SU(3)$ repr.	bosonic field content
gravity multiplet	$(\mathbf{1},\mathbf{1})$	$g_{\mu\nu}$ $\mathcal{A}_1^+$
tensor multiplet	$(\mathbf{1},\mathbf{1})$	$\mathcal{B}_{\mu\nu}$ ϕ $\mathcal{A}_0^-$
vector multiplet	$(\mathbf{3},\overline{\mathbf{3}})$	$\mathcal{A}_1^+$ $\delta\Phi^+$
hypermultiplet	$(\mathbf{3},\mathbf{3})$	$\delta\Phi^-$ $\mathcal{A}_0^-$

type IIB multiplet	$SU(3)\times SU(3)$ repr.	bosonic field content
gravity multiplet	$(\mathbf{1},\mathbf{1})$	$g_{\mu\nu}$ $\mathcal{A}_1^-$
tensor multiplet	$(\mathbf{1},\mathbf{1})$	$\mathcal{B}_{\mu\nu}$ ϕ $\mathcal{A}_0^+$
vector multiplet	$(\mathbf{3},\mathbf{3})$	$\mathcal{A}_1^-$ $\delta\Phi^-$
hypermultiplet	$(\mathbf{3},\overline{\mathbf{3}})$	$\delta\Phi^+$ $\mathcal{A}_0^+$

Notice that all fields are still living on 10-dimensional space, i.e., all KK modes are included.

In case of a tangent bundle $T_{3,1} \oplus F \oplus F^*$ w/ a single $SU(3)$-structure (i.e., $\eta^1_+ = \eta^2_+$):

Ten-dimensional fields are decomposed as

$\mathcal{G}_{MN}$	$g_{\mu\nu}$	$\mathbf{1}_{\pm 2}$
	$\mathcal{G}_{\mu i}$	$(\mathbf{3} + \overline{\mathbf{3}})_{\pm 1}$
	$\mathcal{G}_{ij}$	$\mathbf{1_0} + (\mathbf{6} + \overline{\mathbf{6}})_{\mathbf{0}} + \mathbf{8_0}$
$\mathcal{B}_{MN}$	$\mathcal{B}_{\mu\nu}$	$\mathbf{1_T}$
	$\mathcal{B}_{\mu i}$	$(\mathbf{3} + \overline{\mathbf{3}})_{\pm 1}$
	$\mathcal{B}_{ij}$	$\mathbf{1_0} + (\mathbf{3} + \overline{\mathbf{3}})_{\mathbf{0}} + \mathbf{8_0}$
ϕ	ϕ	$\mathbf{1_0}$

Ψ_M	Ψ_μ	$\mathbf{1}_{\pm\frac{3}{2}} + \mathbf{3}_{\pm\frac{3}{2}}$
	Ψ_i	$\mathbf{1}_{\pm\frac{1}{2}} + \mathbf{3}_{\pm\frac{1}{2}} + 2 \times \overline{\mathbf{3}}_{\pm\frac{1}{2}} + \mathbf{6}_{\pm\frac{1}{2}} + \mathbf{8}_{\pm\frac{1}{2}}$
λ	λ	$\mathbf{1}_{\pm\frac{1}{2}} + \mathbf{3}_{\pm\frac{1}{2}}$

$\mathcal{C}_M$	$\mathcal{C}_\mu$	$\mathbf{1}_{\pm 1}$
	$\mathcal{C}_i$	$(\mathbf{3} + \overline{\mathbf{3}})_{\mathbf{0}}$
$\mathcal{C}_{MNP}$	$\mathcal{C}_{\mu\nu k}$	$(\mathbf{3} + \overline{\mathbf{3}})_{\mathbf{T}}$
	$\mathcal{C}_{\mu jk}$	$\mathbf{1_T} + (\mathbf{3} + \overline{\mathbf{3}})_{\pm 1} + \mathbf{8}_{\pm 1}$
	$\mathcal{C}_{ijk}$	$(\mathbf{1} + \mathbf{1})_{\mathbf{0}} + (\mathbf{3} + \overline{\mathbf{3}})_{\pm 1} + (\mathbf{6} + \overline{\mathbf{6}})_{\mathbf{0}}$

$\mathcal{C}_0$	$\mathcal{C}_0$	$\mathbf{1_0}$
$\mathcal{C}_{MN}$	$\mathcal{C}_{\mu\nu}$	$\mathbf{1_T}$
	$\mathcal{C}_{\mu i}$	$(\mathbf{3} + \overline{\mathbf{3}})_{\pm 1}$
	$\mathcal{C}_{ij}$	$\mathbf{1_0} + (\mathbf{3} + \overline{\mathbf{3}})_{\mathbf{0}} + \mathbf{8_0}$
$\mathcal{C}_{MNPQ}$	$\mathcal{C}_{\mu jkl}$	$\frac{1}{2}[(\mathbf{1} + \mathbf{1})_{\pm 1} + (\mathbf{3} + \overline{\mathbf{3}})_{\pm 1} + (\mathbf{6} + \overline{\mathbf{6}})_{\pm 1}]$
	$\mathcal{C}_{ijkl}/\mathcal{C}_{\mu\nu kl}$	$\mathbf{1_0} + (\mathbf{3} + \overline{\mathbf{3}})_{\mathbf{0}} + \mathbf{8_0}$

Standard four-dimensional $\mathcal{N} = 2$ supergravity $=$ "absence of 6 gravitino multiplets"

IIA multiplets	$SU(3)$ repr.	field contents
gravity multiplet	$\mathbf{1}$	$g_{\mu\nu}$ $\mathcal{C}_{\mu}$ Ψ_{μ}
tensor multiplet	$\mathbf{1}$	$\mathcal{B}_{\mu\nu}$ ϕ $\mathcal{C}_{ijk}$ λ
vector multiplet	$\mathbf{8}+\mathbf{1}$	$\mathcal{C}_{\mu jk}$ $\mathcal{G}_{ij}$ $\mathcal{B}_{ij}$ Ψ_{i}
hypermultiplet	$\mathbf{6}$	$\mathcal{G}_{ij}$ $\mathcal{C}_{ijk}$ Ψ_{i}

IIB multiplets	$SU(3)$ repr.	field contents
gravity multiplet	$\mathbf{1}$	$g_{\mu\nu}$ $\mathcal{C}_{\mu jkl}$ Ψ_{μ}
tensor multiplet	$\mathbf{1}$	$\mathcal{B}_{\mu\nu}$ $\mathcal{C}_{\mu\nu}$ ϕ $\mathcal{C}_{0}$ λ
vector multiplet	$\mathbf{6}$	$\mathcal{C}_{\mu jkl}$ $\mathcal{G}_{ij}$ Ψ_{i}
hypermultiplet	$\mathbf{8}+\mathbf{1}$	$\mathcal{G}_{ij}$ $\mathcal{B}_{ij}$ $\mathcal{C}_{ij}$ $\mathcal{C}_{ijkl}$ Ψ_{i}

Notice that all fields are still living on 10-dimensional space, i.e., all KK modes are included.

Analyze potential (interaction) terms:

given in the supersymmetry transformation of 4-dimensional $\mathcal{N}=2$ gravitinos ψ_μ^A

$$\hat{\Psi}_\mu^A \equiv \Psi_\mu^A + \frac{1}{2}\gamma_\mu{}^i \Psi_i^A = \psi_{A\mu+} \otimes \eta_\pm^A + \psi_{A\mu-} \otimes \eta_\mp^A + \ldots$$

$$\delta\psi_{A\mu} = D_\mu \xi_A + i\gamma_\mu S_{AB}\,\xi^B \qquad A = 1,2$$

$$S_{AB} = \frac{i}{2}e^{\frac{1}{2}K_V}\,\sigma_{AB}^x\,\mathcal{P}^x \qquad \sigma_{AB}^x = \begin{pmatrix} \delta^{x1} - i\delta^{x2} & -\delta^{x3} \\ -\delta^{x3} & -\delta^{x1} - i\delta^{x2} \end{pmatrix} \qquad x = 1,2,3$$

$\mathcal{P}^x$: $\mathcal{N}=2$ Killing prepotentials, which yield $\mathcal{N}=1$ superpotentials

M. Graña, J. Louis, D. Waldram hep-th/0612237

To get S_{AB}, project the SUSY transformation $\delta\hat{\Psi}_\mu$ onto $SU(3)$-singlet parts from

$$\begin{aligned} \delta\Psi_M &= D_M \epsilon - \frac{1}{96} \mathrm{e}^{-\phi} \Big(\gamma_M{}^{PQR} \mathcal{H}_{PQR} - 9 \gamma^{PQ} \mathcal{H}_{MPQ} \Big) \mathcal{P} \, \epsilon \\ & \quad - \sum_n \frac{1}{64 n!} \mathrm{e}^{\frac{5-n}{4}\phi} \Big[(n-1) \gamma_M{}^{N_1 \cdots N_n} - n(9-n) \delta_M{}^{N_1} \gamma^{N_2 \cdots N_n} \Big] \mathcal{F}_{N_1 \cdots N_n} \mathcal{P}_n \, \epsilon \end{aligned}$$

To get S_{AB}, project the SUSY transformation $\delta\hat{\Psi}_\mu$ onto $SU(3)$-singlet parts from

$$\begin{aligned}\delta\Psi_M &= D_M\epsilon - \frac{1}{96}\mathrm{e}^{-\phi}\Big(\gamma_M{}^{PQR}\mathcal{H}_{PQR} - 9\gamma^{PQ}\mathcal{H}_{MPQ}\Big)\mathcal{P}\,\epsilon \\ &\qquad - \sum_n \frac{1}{64n!}\mathrm{e}^{\frac{5-n}{4}\phi}\Big[(n-1)\gamma_M{}^{N_1\cdots N_n} - n(9-n)\delta_M{}^{N_1}\gamma^{N_2\cdots N_n}\Big]\mathcal{F}_{N_1\cdots N_n}\mathcal{P}_n\,\epsilon\end{aligned}$$

In type IIB case (w/ $\mathcal{F}^- = \mathcal{F}_1 + \mathcal{F}_3 + \mathcal{F}_5$, $\sigma(\mathcal{F}^-) = -\mathcal{F}_1 + \mathcal{F}_3 - \mathcal{F}_5$):

$$\begin{aligned}\begin{pmatrix} \delta\psi^1_{\mu+} \\ \delta\psi^2_{\mu+} \end{pmatrix} &= \begin{pmatrix} D_\mu\xi^1_+ \\ D_\mu\xi^2_+ \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \gamma_\mu\xi^1_-\ \overline{\eta}^1_-\gamma^i D_i\eta^1_+ \\ \gamma_\mu\xi^2_-\ \overline{\eta}^2_-\gamma^i D_i\eta^2_+ \end{pmatrix} + \frac{1}{48}\begin{pmatrix} \gamma_\mu\xi^1_-\ \mathcal{H}_{ijk}\,\overline{\eta}^1_-\gamma^{ijk}\eta^1_+ \\ -\gamma_\mu\xi^2_-\ \mathcal{H}_{ijk}\,\overline{\eta}^2_-\gamma^{ijk}\eta^2_+ \end{pmatrix} \\ &\qquad - \frac{1}{8}\begin{pmatrix} -\gamma_\mu\xi^2_-\ \mathrm{e}^\phi\frac{1}{n!}\mathcal{F}^-_{i_1\cdots i_n}\,\overline{\eta}^1_-\gamma^{i_1\cdots i_n}\eta^2_+ \\ \gamma_\mu\xi^1_-\ \mathrm{e}^\phi\frac{1}{n!}\sigma(\mathcal{F}^-)_{i_1\cdots i_n}\,\overline{\eta}^2_-\gamma^{i_1\cdots i_n}\eta^1_+ \end{pmatrix}\end{aligned}$$

Then we obtain

$$\begin{aligned}
S_{11} &= \frac{i}{2}\overline{\eta}{}^1_- \gamma^i D_i \eta^1_+ - \frac{i}{48}\mathcal{H}_{ijk}\,\overline{\eta}{}^1_- \gamma^{ijk}\eta^1_+ = -\frac{1}{8}\big\langle \Phi_-, \mathrm{d}\Phi_+ \big\rangle \\
S_{22} &= \frac{i}{2}\overline{\eta}{}^2_- \gamma^i D_i \eta^2_+ + \frac{i}{48}\mathcal{H}_{ijk}\,\overline{\eta}{}^2_- \gamma^{ijk}\eta^2_+ = \frac{1}{8}\big\langle \Phi_-, \mathrm{d}\overline{\Phi}_+ \big\rangle \\
S_{12} &= \frac{i}{8n!}\mathrm{e}^{\phi}\mathcal{F}^-_{i_1\cdots i_n}\,\overline{\eta}{}^1_- \gamma^{i_1\cdots i_n}\eta^2_+ = \frac{1}{8}\big\langle \Phi_-, \mathcal{G}^- \big\rangle \\
S_{21} &= \frac{i}{8n!}\mathrm{e}^{\phi}\sigma(\mathcal{F})^-_{i_1\cdots i_n}\overline{\eta}{}^2_- \gamma^{i_1\cdots i_n}\eta^1_+ = \frac{1}{8}\big\langle \Phi_-, \mathcal{G}^- \big\rangle \\
\mathcal{F} &= \mathrm{d}\mathcal{C} - \mathcal{H}_3 \wedge \mathcal{C} = \mathrm{e}^{\mathcal{B}}\mathcal{G} \qquad \mathcal{C} = \mathrm{e}^{\mathcal{B}}\mathcal{A} \qquad \mathcal{G}^{\pm} = \mathrm{d}\mathcal{A}^{\mp}_0
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\end{aligned}
$$

Summarizing information, we obtain (also for type IIA)

$$
\begin{aligned}
S^{(4)}_{AB}(\mathrm{IIB}) &= \frac{1}{8}\mathrm{e}^{\frac{1}{2}K_-}\begin{pmatrix} -\mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}}\big\langle \Phi_-, \mathrm{d}\Phi_+ \big\rangle & -\mathrm{e}^{2\phi^{(4)}}\big\langle \Phi_-, \mathcal{G}^- \big\rangle \\ -\mathrm{e}^{2\phi^{(4)}}\big\langle \Phi_-, \mathcal{G}^- \big\rangle & \mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}}\big\langle \Phi_-, \mathrm{d}\overline{\Phi}_+ \big\rangle \end{pmatrix} \\
S^{(4)}_{AB}(\mathrm{IIA}) &= \frac{1}{8}\mathrm{e}^{\frac{1}{2}K_+}\begin{pmatrix} \mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}}\big\langle \Phi_+, \mathrm{d}\Phi_- \big\rangle & \mathrm{e}^{2\phi^{(4)}}\big\langle \Phi_+, \mathcal{G}^+ \big\rangle \\ \mathrm{e}^{2\phi^{(4)}}\big\langle \Phi_+, \mathcal{G}^+ \big\rangle & -\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}}\big\langle \Phi_+, \mathrm{d}\overline{\Phi}_- \big\rangle \end{pmatrix} \\
g^{(4)}_{\mu\nu} &= \mathrm{e}^{-2\phi^{(4)}}g_{\mu\nu} \qquad \phi^{(4)} \;=\; \phi - \frac{1}{4}\log\det\mathcal{G}_{ij}
\end{aligned}
$$

$\mathcal{N} = 1$ superpotentials and Kähler potentials can be read as

$$\delta\psi_\mu = D_\mu \xi + i\mathrm{e}^{K/2} W \gamma_\mu \xi^c \qquad K = K_+ + K_- + 2\phi^{(4)}$$

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Most generic form of $\mathcal{N}=1$ superpotentials on $SU(3)\times SU(3)$ structure:

$$W_{\mathrm{IIA}} \;=\; \cos^2\alpha\, \mathrm{e}^{i\beta}\big\langle \Phi_+, \mathrm{d}\Phi_-\big\rangle - \sin^2\alpha\, \mathrm{e}^{-i\beta}\big\langle \Phi_+, \mathrm{d}\overline{\Phi}_-\big\rangle + \sin 2\alpha\, \mathrm{e}^{\phi}\big\langle \Phi_+, \mathcal{G}^+\big\rangle$$

$$W_{\mathrm{IIB}} \;=\; -\cos^2\alpha\, \mathrm{e}^{i\beta}\big\langle \Phi_-, \mathrm{d}\Phi_+\big\rangle + \sin^2\alpha\, \mathrm{e}^{-i\beta}\big\langle \Phi_-, \mathrm{d}\overline{\Phi}_+\big\rangle - \sin 2\alpha\, \mathrm{e}^{\phi}\big\langle \Phi_-, \mathcal{G}^-\big\rangle$$

$$\mathcal{G}^+ \;=\; \mathcal{G}_0 + \mathcal{G}_2 + \mathcal{G}_4 + \mathcal{G}_6 \qquad\qquad \mathcal{G}^- \;=\; \mathcal{G}_1 + \mathcal{G}_3 + \mathcal{G}_5$$

$$\mathcal{G}^\pm \;=\; \mathrm{d}\mathcal{A}_0^\mp \qquad\qquad \mathcal{C} = \mathrm{e}^{\mathcal{B}}\mathcal{A} \qquad\qquad \mathcal{F} = \mathrm{d}\mathcal{C} - \mathcal{H}_3 \wedge \mathcal{C} \;=\; \mathrm{e}^{\mathcal{B}}\mathcal{G}$$

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$$W_{\mathrm{IIB}} \;=\; -\cos^2\alpha\,\mathrm{e}^{i\beta}\big\langle\Phi_-, \mathrm{d}\Phi_+\big\rangle + \sin^2\alpha\,\mathrm{e}^{-i\beta}\big\langle\Phi_-, \mathrm{d}\overline{\Phi}_+\big\rangle - \sin 2\alpha\,\mathrm{e}^{\phi}\big\langle\Phi_-, \mathcal{G}^-\big\rangle$$

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Reducing to single $SU(3)$-structure by $\eta_+^1 = \eta_+^2 \equiv \eta_+$, we obtain well-known forms:

$2\alpha = -\beta = \dfrac{\pi}{2}$ in W_{IIB}	$W_{\text{GVW}} \;=\; -i\mathrm{e}^{\phi}\big\langle\mathcal{F}_3 - \tau\mathcal{H}_3, \Omega\big\rangle$
$\alpha = \dfrac{\pi}{4}$, $\mathrm{d}\Phi_- = 0$ in W_{IIA}	$W_{\mathrm{IIA,RR}} \;=\; \mathrm{e}^{\phi}\big\langle\mathrm{e}^{-\mathcal{B}-iJ}, \mathcal{G}^+\big\rangle$
$\beta = \dfrac{\pi}{2}$, $\mathcal{G}^+ = 0$ in W_{IIA}	$W_{\text{half-flat}} \;=\; i\big\langle\mathrm{e}^{-\mathcal{B}-iJ}, \mathrm{d}(\mathrm{Re}\Omega)\big\rangle$

$a = \cos\alpha\mathrm{e}^{-i\beta/2}$, $b = \sin\alpha\,\mathrm{e}^{i\beta/2}$, $\tau = \mathcal{C}_0 + i\mathrm{e}^{-\phi}$

We have obtained Kähler potentials and superpotentials

which should appear in four-dimensional $\mathcal{N} = 1, 2$ supergravity theories

in the language of ten-dimensional fields:

$$\begin{aligned} e^{-K_\pm} &= i\big\langle \Phi_\pm, \overline{\Phi}_\pm \big\rangle = i\big(\overline{\mathcal{Z}}{}^I_\pm \mathcal{F}_{\pm I} - \mathcal{Z}^I_\pm \overline{\mathcal{F}}_{\pm I} \big) \\ W_{\mathrm{IIA/IIB}} &= \pm \cos^2\alpha\, e^{i\beta} \big\langle \Phi_\pm, d\Phi_\mp \big\rangle \mp \sin^2\alpha\, e^{-i\beta} \big\langle \Phi_\pm, d\overline{\Phi}_\mp \big\rangle \pm \sin 2\alpha\, e^{\phi} \big\langle \Phi_\pm, \mathcal{G}^\pm \big\rangle \end{aligned}$$

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Next task is to find a suitable truncation of massive modes

by decomposition $\mathcal{M}_{1,9} = \mathcal{M}_{1,3} \times_{\mathrm{W}} \mathcal{M}_6$ with $T_{1,3} \equiv T\mathcal{M}_{1,3}$ and $F \equiv T\mathcal{M}_6$

We truncate ten-dimensional theory

with keeping only a finite number of light modes in the spectrum.

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✓ If $\mathcal{M}_6$ is a generic geometry (w/ torsion)

Existence of finite number of harmonic forms are not guaranteed..

Instead, we assume existence of a certain finite-dimensional subspace of $\wedge^* T^* \mathcal{M}_6$

If there exists harmonic forms on $\mathcal{M}_6$, we can evaluate the dimensions of the forms via Index theorem: T. Kimura arXiv:0704.2111

Assumption the existence of finite-dimensional subset of p-forms:

$$\wedge^p_{\text{finite}} \subset \wedge^p T^* \mathcal{M}_6 \qquad\qquad U^{\text{finite}} = U \cap \wedge^*_{\text{finite}}$$

Note: the truncation should not break supersymmetry

$\dashrightarrow$ special Kähler geometry on U should give special Kähler geometry on U^{finite}

i.e., we require $\begin{cases} \text{Mukai pairing } \langle *, * \rangle \text{ is non-degenerate on } \wedge^p_{\text{finite}} \\ \text{if } \chi \in U^{\text{finite}}, \text{ then } \hat{\chi} \in U^{\text{finite}} \end{cases}$

First we introduce a set of basis forms (w/ Mukai pairing as symplectic structure):

$$\text{even forms :} \quad \Sigma_+ = \{\omega_A, \widetilde{\omega}^B\}, \quad \int_{\mathcal{M}_6} \langle \omega_A, \widetilde{\omega}^B \rangle = \delta_A{}^B, \quad A, B = 0, \ldots, b^+$$

$$\text{odd forms :} \quad \Sigma_- = \{\alpha_K, \beta^L\}, \quad \int_{\mathcal{M}_6} \langle \alpha_K, \beta^L \rangle = \delta_K{}^L, \quad K, L = 0, \ldots, b^-$$

Using this, the pure spinors $\Phi_\pm$ are expanded

$$\Phi_+ = \mathrm{e}^{-\mathcal{B}} \Phi_{0+} = \mathcal{X}^A \omega_A - \mathcal{G}_A \tilde{\omega}^A$$

$$\Phi_- = \mathrm{e}^{-\mathcal{B}} \Phi_{0-} = \mathcal{Z}^K \alpha_K - \mathcal{F}_K \beta^K$$

The compatibility is read as (w/ using $\forall V = x + \xi \in E$)

$$\langle \omega_A, V \cdot \alpha_K \rangle = \langle \omega_A, V \cdot \beta^K \rangle = \langle \widetilde{\omega}^A, V \cdot \alpha_K \rangle = \langle \widetilde{\omega}^A, V \cdot \beta^K \rangle = 0$$

The truncated Kähler potentials by $\int_{\mathcal{M}_6} \langle \omega_A, \widetilde{\omega}^B \rangle = \delta_A{}^B$ and $\int_{\mathcal{M}_6} \langle \alpha_K, \beta^L \rangle = \delta_K{}^L$ are

$$\mathrm{e}^{-K_+} \ = \ i \int_{\mathcal{M}_6} \big\langle \Phi_+, \overline{\Phi}_+ \big\rangle \ = \ i \Big(\overline{\mathcal{X}}^A \mathcal{G}_A - \mathcal{X}^A \overline{\mathcal{G}}_A \Big)$$

$$\mathrm{e}^{-K_-} \ = \ i \int_{\mathcal{M}_6} \big\langle \Phi_-, \overline{\Phi}_- \big\rangle \ = \ i \Big(\overline{\mathcal{Z}}^K \mathcal{F}_K - \mathcal{Z}^K \overline{\mathcal{F}}_K \Big)$$

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RR fields are also expanded as

type IIA:
$$\begin{cases} \mathcal{A}_0^- \;=\; \xi^K \alpha_K + \widetilde{\xi}_L \beta^L \\ \mathcal{A}_1^+ \;=\; A_1^A \omega_A + \widetilde{A}_{1B} \widetilde{\omega}^B \end{cases}$$
w/ $\xi^K, \widetilde{\xi}_L$: scalars; $A_1^A, \widetilde{A}_{1B}$: vectors

type IIB:
$$\begin{cases} \mathcal{A}_0^+ \;=\; \xi^A \omega_A + \widetilde{\xi}_B \widetilde{\omega}^B \\ \mathcal{A}_1^- \;=\; A_1^K \alpha_K + \widetilde{A}_{1L} \beta^L \end{cases}$$
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$$\mathrm{e}^{-K_+} \;=\; i \int_{\mathcal{M}_6} \big\langle \Phi_+, \overline{\Phi}_+ \big\rangle \;=\; i\Big(\overline{\mathcal{X}}^A \mathcal{G}_A - \mathcal{X}^A \overline{\mathcal{G}}_A \Big)$$

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type IIB: $\begin{cases} \mathcal{A}_0^+ = \xi^A \omega_A + \widetilde{\xi}_B \widetilde{\omega}^B \\ \mathcal{A}_1^- = A_1^K \alpha_K + \widetilde{A}_{1L} \beta^L \end{cases}$ w/ $\xi^A, \widetilde{\xi}_B$: scalars; $A_1^K, \widetilde{A}_{1L}$: vectors

Convenient to define dual antisymmetric tensor fields of $\mathcal{A}_0^-$ and $\mathcal{A}_0^+$:

$$\mathcal{A}_0^- \;\leftrightarrow\; \mathcal{A}_2^- \;\equiv\; \widetilde{C}_2^K \alpha_K + C_{2L} \beta^L \qquad\qquad \mathcal{A}_0^+ \;\leftrightarrow\; \mathcal{A}_2^+ \;\equiv\; \widetilde{C}_2^A \omega_A + C_{2B} \widetilde{\omega}^B$$

$$\xi^K \;\leftrightarrow\; C_{2K} \qquad \widetilde{\xi}_K \;\leftrightarrow\; \widetilde{C}_2^K \qquad \xi^A \;\leftrightarrow\; C_{2A} \qquad \widetilde{\xi}_A \;\leftrightarrow\; \widetilde{C}_2^A$$

The most general differential conditions which can be imposed on basis forms are

$$\mathrm{d}\alpha_K \sim p_K{}^A \omega_A + e_{KA} \widetilde{\omega}^A \qquad \mathrm{d}\beta^K \sim q^{KA} \omega_A + m^K{}_A \widetilde{\omega}^A$$

$$\mathrm{d}\omega_A \sim m^K{}_A \alpha_K - e_{KA} \beta^K \qquad \mathrm{d}\widetilde{\omega}^A \sim -q^{KA} \alpha_K + p_K{}^A \beta^K$$

$p_K{}^A$, q^{KA}, e_{KA} and $m^K{}_A$ are $(b^+ + 1) \times (b^- + 1)$-dimensional constant matrices

Not necessary to be closed as in Calabi-Yau

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$p_K{}^A$, q^{KA}, e_{KA} and $m^K{}_A$ are $(b^+ + 1) \times (b^- + 1)$-dimensional constant matrices

Not necessary to be closed as in Calabi-Yau

Introduce a notation $\Sigma_+ = \begin{pmatrix} \omega_A \\ \widetilde{\omega}^B \end{pmatrix}$, $\Sigma_- = \begin{pmatrix} \alpha_K \\ \beta^L \end{pmatrix}$ and $\mathcal{Q} = \begin{pmatrix} p_K{}^A & e_{KB} \\ q^{LA} & m^L{}_B \end{pmatrix}$.

In terms of them the above differential condition is

$$\mathrm{d}\Sigma_- \sim \mathcal{Q}\Sigma_+ \qquad \mathrm{d}\Sigma_+ \sim \mathcal{S}_+ \mathcal{Q}^T (\mathcal{S}_-)^{-1} \Sigma_-$$

$\mathcal{S}_\pm$: the symplectic structures on $U^\pm$

Imposing $\mathrm{d}^2 = 0$ on the charged matrix as $\mathcal{Q}\mathcal{S}_+\mathcal{Q}^T = 0 = \mathcal{Q}^T(\mathcal{S}_-)^{-1}\mathcal{Q}$, we obtain

$$q^{KA} m_A{}^L - m^K{}_A q^{AL} = 0 \qquad p_K{}^A e_{AL} - e_{KA} p^A{}_L = 0 \qquad p_K{}^A m_A{}^L - e_{KA} q^{AL} = 0$$

$$q^{AK} p_K{}^B - p^A{}_K q^{KB} = 0 \qquad m_A{}^K e_{KB} - e_{AK} m^K{}_B = 0 \qquad m_A{}^K p_K{}^B - e_{AK} q^{KB} = 0$$

Kinetic terms $|\mathcal{G}_n|^2$ generate mass terms via truncation of fields:

- Type IIA:

$$\begin{aligned} \mathcal{G}_{2p} &= \mathrm{d}\mathcal{A}_{2p-1} \sim \mathrm{d}_6\mathcal{A}_2^- + \mathrm{d}_4\mathcal{A}_1^+ \equiv D_2^A\,\omega_A + \widetilde{D}_{2A}\,\widetilde{\omega}^A \\ D_2^A &= \mathrm{d}_4 A_1^A + \widetilde{C}_2^K\,p_K{}^A + C_{2K}\,q^{AK} \\ \widetilde{D}_{2A} &= \mathrm{d}_4 \widetilde{A}_1^A + \widetilde{C}_2^K\,e_{AK} + C_{2K}\,m^K{}_A \end{aligned}$$

- Type IIB:

$$\begin{aligned} \mathcal{G}_{2p+1} &= \mathrm{d}\mathcal{A}_{2p} \sim \mathrm{d}_6\mathcal{A}_2^+ + \mathrm{d}_4\mathcal{A}_1^- \equiv D_2^K\,\alpha_K + \widetilde{D}_{2K}\,\beta^K \\ D_2^K &= \mathrm{d}_4 A_1^K - \widetilde{C}_2^A\,m^K{}_A + C_{2A}\,q^{AK} \\ \widetilde{D}_{2K} &= \mathrm{d}_4 \widetilde{A}_1^K + \widetilde{C}_2^A\,e_{AK} - C_{2A}\,p_K{}^A \end{aligned}$$

Then charge matrices give massive modes of RR fields:

	e_{AK}	$m^K{}_A$	$p_K{}^A$	q^{KA}
IIA	massive A_μ^A	massive A_μ^A	massive $\widetilde{C}_2^K$	massive C_{2K}
IIB	massive A_μ^K	massive $\widetilde{C}_2^A$	massive A_μ^K	massive C_{2A}

M. Graña, J. Louis, D. Waldram hep-th/0612237

Recall that $\Phi_\pm$ *are expanded in terms of truncation bases* Σ_+ *and* Σ_-.

Whenever $c_\| \neq 0$, *the structure* Φ_+ *contains a scalar. This implies that at least one of the forms in the basis* Σ_+ *contains a scalar. Let us call this element* Σ_+^1, *and take the simple case where the only non-zero elements of* $\mathcal{Q}$ *are those of the form* ${\mathcal{Q}_{\hat{I}}}^1$ *(where* $\hat{I} = 1, \ldots, 2b^- + 2$*).*

Thus $\mathrm{d}(\Sigma_-)_{\hat{I}} = {\mathcal{Q}_{\hat{I}}}^1 \Sigma_+^1$ *and so if* ${\mathcal{Q}_{\hat{I}}}^1 \neq 0$ *then* $(\mathrm{d}\Sigma_-)_{\hat{I}}$ *contains a scalar.*

But this is not possible if d *is an honest exterior derivative, acting as* $\mathrm{d} : \Lambda^p \longrightarrow \Lambda^{p+1}$.

The same is true if $c_\|$ *is zero. In this case, there may be no scalars in any of the even forms* Σ_-, *and for an "honest"* d *operator, there should be then no one-forms in* $\mathrm{d}\Sigma_-$. *But we again see from that* Φ_- *contains a one-form, and as a consequence so do some of the elements in* Σ_-.

One way to generate a completely general charge matrix $\mathcal{Q}$ in this picture is to consider a modified operator d which is now a generic map $\mathrm{d} : U^{+} \longrightarrow U^{-}$ which satisfies $\mathrm{d}^2 = 0$ but does not transform the degree of a form properly.

In particular, the operator d can map a p-form to a $(p-1)$-form.
Of course, this d does not act this way in conventional geometrical compactifications.

One is thus led to conjecture that to obtain a generic $\mathcal{Q}$ we must consider non-geometrical compactifications. One can still use the structures

$$\mathrm{d}\Sigma_{-} \ \sim \ \mathcal{Q}\Sigma_{+} \,, \qquad \mathrm{d}\Sigma_{+} \ \sim \ \mathcal{S}_{+}\mathcal{Q}^{T}(\mathcal{S}_{-})^{-1}\Sigma_{-}$$

to derive sensible effective actions, expanding in bases Σ_{+} and Σ_{-} with a generalised d operator, but there is of course now no interpretation in terms of differential forms and the exterior derivative.

$\dashrightarrow$ introduce generalized fluxes
(not only geometrical H- and f-fluxes, but also Q- and R-fluxes)

For a geometrical background it is natural to consider forms of the type

$$\omega \;=\; \mathrm{e}^{-B}\omega_{m_1\ldots m_p}\, e^{m_1}\wedge\cdots\wedge e^{m_p} \qquad \text{w/ } \omega_{m_1\ldots m_p} \text{ constant}$$

Acting with d on ω we find

$$\mathrm{d}\omega \;=\; -H\wedge\omega + f\cdot\omega\,, \qquad (f\cdot\omega)_{m_1\ldots m_{p+1}} \;=\; f^a{}_{[m_1 m_2|}\omega_{a|m_3\ldots m_{p+1}]}$$

The natural nongeometrical extension is then to an operator $\mathcal{D}$ such that

$$\mathcal{D}\omega \;:=\; -H\wedge\omega + f\cdot\omega + Q\cdot\omega + R\llcorner\omega,$$

$$(Q\cdot\omega)_{m_1\ldots m_{p-1}} \;=\; Q^{ab}{}_{[m_1}\omega_{|ab|m_2\ldots m_{p-1}]}\,, \qquad (R\llcorner\omega)_{m_1\ldots m_{p-3}} \;=\; R^{abc}\omega_{abcm_1\ldots m_{p-3}}$$

Requiring $\mathcal{D}^2 = 0$ implies that same conditions on fluxes as arose from the Jacobi identities for the extended Lie algebra

$$\begin{aligned} [Z_a, Z_b] &= f_{ab}{}^c Z_c + H_{abc} X^c \\ [X^a, X^b] &= Q^{ab}{}_c X^c + R^{abc} Z_c \\ [X^a, Z_b] &= f^a{}_{bc} X^c - Q^{ac}{}_b Z_c \end{aligned}$$

We can see nongeometrical information in $\mathcal{Q}$ as contribution from Q and R.

- Type IIA Killing prepotentials $\mathcal{P}^x$ in S_{AB} w/ $\mathcal{G}^+ = \mathrm{d}\mathcal{A}_0^- + G^A_{(\mathsf{RR})}\omega_A + \widetilde{G}_{(\mathsf{RR})A}\widetilde{\omega}^A$:

$$\begin{aligned}
\mathcal{P}^1 + i\mathcal{P}^2 &= -2\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_+, \mathrm{d}\Phi_- \rangle \\
&= 2\mathrm{e}^{\frac{1}{2}K_- + \phi^{(4)}} \Big(-\mathcal{X}^A e_{AK} \mathcal{Z}^K + \mathcal{X}^A m_A{}^K \mathcal{F}_K - \mathcal{G}_A p^A{}_K \mathcal{Z}^K + \mathcal{G}_A q^{AK} \mathcal{F}_K \Big) \\
\mathcal{P}^3 &= \mathrm{e}^{2\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_+, \mathcal{G}^+ \rangle \\
&= \mathrm{e}^{2\phi^{(4)}} \Big[\mathcal{X}^A \big(\widetilde{G}_{(\mathsf{RR})A} + e_{AK}\xi^K + m_A{}^K \widetilde{\xi}_K \big) + \mathcal{G}_A \big(G^A_{(\mathsf{RR})} + p^A{}_K \xi^K + q^{AK} \widetilde{\xi}_K \big) \Big]
\end{aligned}$$

$\mathcal{N} = 1$ superpotential W_{IIA} is given by

$$W_{\mathrm{IIA}} = \cos^2\alpha\, \mathrm{e}^{i\beta} \int_{\mathcal{M}_6} \langle \Phi_+, \mathrm{d}\Phi_- \rangle - \sin^2\alpha\, \mathrm{e}^{-i\beta} \int_{\mathcal{M}_6} \langle \Phi_+, \mathrm{d}\overline{\Phi}_- \rangle + \sin 2\alpha\, \mathrm{e}^{\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_+, \mathcal{G}^+ \rangle$$

- Type IIB Killing prepotentials $\mathcal{P}^x$ in S_{AB} w/ $\mathcal{G}^- = \mathrm{d}\mathcal{A}_0^+ + G_{\mathsf{(RR)}}^K \alpha_K + \widetilde{G}_{\mathsf{(RR)}L} \beta^L$:

$$\begin{aligned}\mathcal{P}^1 - i\mathcal{P}^2 &= -2\mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_-, \mathrm{d}\Phi_+ \rangle \\ &= 2\mathrm{e}^{\frac{1}{2}K_+ + \phi^{(4)}} \Big(- \mathcal{Z}^K e_{KA} \mathcal{X}^A - \mathcal{Z}^K p_K{}^A \mathcal{G}_A + \mathcal{F}_K m^K{}_A \mathcal{X}^A + \mathcal{F}_K q^{KA} \mathcal{G}_A \Big) \\ \mathcal{P}^3 &= -\mathrm{e}^{2\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_-, \mathcal{G}^- \rangle \\ &= -\mathrm{e}^{2\phi^{(4)}} \Big[\mathcal{Z}^K \big(\widetilde{G}_{\mathsf{(RR)}K} - e_{KA} \xi^A + p_K{}^A \widetilde{\xi}_A \big) + \mathcal{F}_K \big(G_{\mathsf{(RR)}}^K + m^K{}_A \xi^A - q^{KA} \widetilde{\xi}_A \big) \Big]\end{aligned}$$

$\mathcal{N} = 1$ superpotential W_{IIB} is given by

$$W_{\mathrm{IIB}} = -\cos^2\alpha\, \mathrm{e}^{i\beta} \int_{\mathcal{M}_6} \langle \Phi_-, \mathrm{d}\Phi_+ \rangle + \sin^2\alpha\, \mathrm{e}^{-i\beta} \int_{\mathcal{M}_6} \langle \Phi_-, \mathrm{d}\overline{\Phi}_+ \rangle - \sin 2\alpha\, \mathrm{e}^{\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_-, \mathcal{G}^- \rangle$$

Generically, scalar potential V in four-dimensional theory is

$$V = \mathrm{e}^K \Big(g^{a\bar{b}} D_a W \, \overline{D_b W} - 3|W|^2 \Big)$$

$$g_{a\bar{b}} = \partial_a \overline{\partial}_{\bar{b}} \big(K_+ + K_- + 2\phi^{(4)} \big) \qquad D_a W = \big(\partial_a + \partial_a K \big) W$$

Expanded the scalar potential V by "scalar fields" $\{\mathcal{X}^A, \xi^A, \widetilde{\xi}_A, \mathcal{Z}^K, \xi^K, \widetilde{\xi}_K\}$,

we would obtain non-trivial mass terms in $\mathcal{N} = 1$ theory

--→ so-called moduli stabilization

S.B. Giddings, S. Kachru, J. Polchinski hep-th/0105097 S. Kachru, M.B. Schulz, S. Trivedi hep-th/0201028
R. Kallosh hep-th/0510024 S. Bellucci, S. Ferrara, R. Kallosh, A. Marrani arXiv:0711.4547 L. Anguelova arXiv:0806.3820
and references therein (more than hundreds papers!)

- Introduce a pair of $SU(3)$ structures on F ~ $SU(3) \times SU(3)$ structure on $F \oplus F^*$
- Define generalized complex structures $\mathcal{J}_i$
- Construct $Spin(6,6)$ pure spinors $\Phi_\pm$
- Evaluate the space of pure spinors, and define Hitchin functional $H(\Phi_\pm)$
- Derive Kähler potentials $K_\pm$ and superpotentials $W_{\mathrm{IIA/IIB}}$
- Truncation of ten-dimensional fields

Remaining problem of flux compactification in type $\mathrm{IIA/IIB}$ is...

to find concrete dimensions $b^\pm$ of (non-)harmonic forms on compactified geometry $\mathcal{M}_6$!

$\longrightarrow$ a (mathematical) future problem

Doubled formalism

Start from low energy effective field theory for ten-dimensional string theory including

$$S = \int \mathrm{d}^{10}x\sqrt{-\mathcal{G}}\,\mathrm{e}^{-2\Phi}\Big\{\mathcal{R} + 4(\nabla\Phi)^2 - \frac{1}{12}\mathcal{H}_{MNP}\mathcal{H}^{MNP}\Big\}$$

$$\mathcal{H} = \mathrm{d}\mathcal{B}$$

Consider the field theory compactified on (twisted) torus in the presence of B-field.

motivation

duality relations among flux vacua

N. Kaloper, R.C. Myers hep-th/9901045

Decomposition of fields by Kaluza-Klein compactification on a flat d-torus

$$\mathrm{d}s^2 = \mathcal{G}_{\mu\nu}(x,y)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \mathcal{G}_{ij}(x,y)\big(\mathrm{d}y^i + \mathcal{V}^i{}_\mu(x,y)\mathrm{d}x^\mu\big) \otimes \big(\mathrm{d}y^j + \mathcal{V}^j{}_\nu(x,y)\mathrm{d}x^\nu\big)$$

$$\mathcal{B} = \frac{1}{2}\mathcal{B}_{\mu\nu}(x,y)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \mathcal{B}_{\mu i}(x,y)\mathrm{d}x^\mu \wedge \mathrm{d}y^i + \frac{1}{2}\mathcal{B}_{ij}(x,y)\mathrm{d}y^i \wedge \mathrm{d}y^j$$

with Ansatz (truncation of massive Kaluza-Klein modes)

$$\mathcal{G}_{\mu\nu}(x,y) = g_{\mu\nu}(x)\,, \qquad \mathcal{G}_{ij}(x,y) = g_{ij}(x)\,, \qquad \mathcal{V}^i{}_\mu(x,y) = V^i{}_\mu(x)$$

$$\mathcal{B}_{\mu\nu}(x,y) = \mathcal{B}_{\mu\nu}(x)\,, \qquad \mathcal{B}_{\mu i}(x,y) = \mathcal{B}_{\mu i}(x)\,, \qquad \mathcal{B}_{ij}(x,y) = \mathcal{B}_{ij}(x)$$

$$\Phi(x,y) = \phi(x) + \frac{1}{4}\log\big|\det g_{ij}(x)\big|$$

Reduced degrees of freedom to demonstrate manifest gauge invariance:

$$B_{ij} = \mathcal{B}_{ij}\,, \qquad B_{\mu i} = \mathcal{B}_{\mu i} + B_{ij}V^j{}_\mu$$

$$B_{\mu\nu} = \mathcal{B}_{\mu\nu} + V^i{}_{[\mu}B_{\nu]i} - B_{ij}V^i{}_\mu V^j{}_\nu$$

Reduced D-dim. action compactified on a flat d-torus $(D = 10 - d)$:

$$S = \int \mathrm{d}^D x \sqrt{-g}\, \mathrm{e}^{-2\phi} \Big\{ R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{8} L_{IJ} \nabla_\mu \mathcal{M}^{JK} L_{KL} \nabla^\mu \mathcal{M}^{LI} - \frac{1}{4} F^I_{\mu\nu} L_{IJ} \mathcal{M}^{JK} L_{KL} F^{L\mu\nu} \Big\}$$

This theory has $U(1)^{2d}$ gauge symmetry and a manifest global $O(d,d)$ symmetry with

$$\mathcal{M}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik}\, g^{kl}\, B_{lj} & B_{ik}\, g^{kj} \\ -g^{ik}\, B_{kj} & g^{ij} \end{pmatrix}$$: moduli, taking values in $\dfrac{O(d,d)}{O(d) \times O(d)}$

$$F^I = \mathrm{d}A^I\,, \qquad A^I_\mu = \begin{pmatrix} V^i{}_\mu \\ B_{\mu i} \end{pmatrix}, \qquad H_{\mu\nu\rho} = 3\,\partial_{[\mu} B_{\nu\rho]} - \frac{3}{2} A^I_{[\mu}\, L_{|IJ|}\, F^J_{\nu\rho]}$$

$$L^{IJ} \equiv \begin{pmatrix} \mathbf{0}_d & \mathbb{1}_d \\ \mathbb{1}_d & \mathbf{0}_d \end{pmatrix}$$: $O(d,d)$ invariant metric s.t. $\forall M \in O(d,d),\ M L M^T = L$

Non-abelian gauge symmetry from a $2d$-dimensional subgroup G of $O(d,d)$:

Fundamental repr. of $O(d,d)$ becomes adjoint repr. of G under embedding

$$[T_I, T_J] \;=\; t_{IJ}{}^K \, T_K \,, \qquad T_I \;=\; \frac{1}{2} \Theta_I{}^{JK} \, \mathfrak{m}_{JK}$$

$$\begin{cases} T_I : & \text{generators of } G \text{ with structure constant } t_{IJ}{}^K \\ \mathfrak{m}_{JK} : & \text{generators of } O(d,d) \\ \Theta_I{}^{JK} : & \text{embedding tensor} \end{cases}$$

Then, D-dimensional theory with gauge symmetry G is

$$S = \int \mathrm{d}^D x \sqrt{-g}\, \mathrm{e}^{-2\phi} \Big\{ R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{8} L_{IJ} \mathcal{D}_\mu \mathcal{M}^{JK} L_{KL} \mathcal{D}^\mu \mathcal{M}^{LI} - \frac{1}{4} F^I_{\mu\nu} \mathcal{M}^{JK} L_{KL} F^{L\mu\nu} - g^2 W(\mathcal{M}) \Big\}$$

with covariantized form (via Scherk-Schwarz reduction with $t_{IJK} = t_{IJ}{}^L L_{KL}$)

$$\mathcal{D}_\mu \mathcal{M}^{IJ} = \partial_\mu \mathcal{M}^{IJ} - g t_{KL}{}^I A^K_\mu \mathcal{M}^{LJ} - g t_{KL}{}^J A^K_\mu \mathcal{M}^{IL}$$

$$F = \mathrm{d}A + gA \wedge A$$

$$H = \mathrm{d}B - \frac{1}{2} \mathrm{tr} \Big(A \wedge F + \frac{2g}{3} A \wedge A \wedge A \Big)$$

$$W(\mathcal{M}) = \frac{1}{12} \mathcal{M}^{II'} \mathcal{M}^{JJ'} \mathcal{M}^{KK'} t_{IJK}\, t_{I'J'K'} - \frac{1}{4} \mathcal{M}^{II'} L^{JJ'} L^{KK'} t_{IJK}\, t_{I'J'K'}$$

A. Dabholkar, C.M. Hull hep-th/0512005

T_I are (non-)abelian generators for gauge fields $A^I_\mu = \big(V^i{}_\mu, B_{\mu i} \big)^T$:

$$T_I \ni \begin{cases} Z_i : & \text{generators for } V^i{}_\mu \\ X^i : & \text{generators for } B_{\mu i} \end{cases} \dashrightarrow \quad \begin{aligned} [Z_i, Z_j] &= f_{ij}{}^k \, Z_k + h_{ijk} \, X^k \\ [X^i, X^j] &= 0 \\ [X^i, Z_j] &= f^i{}_{jk} \, X^k \end{aligned}$$

$f_{ij}{}^k$:	structure constant of twisted torus
h_{ijk} :	(minus) VEV of three-form H_{ijk}

$$f^l{}_{i'[i} f_{jk]}{}^{i'} = 0 \quad \text{Jacobi id.}$$
$$h_{i'[ij} f_{kl]}{}^{i'} = 0 \quad \mathrm{d}H_3 = 0$$
$$f^i{}_{ij} = 0 \quad \text{invariance of } \sqrt{-g}$$

- Twisted torus is introduced by vielbein $\mathrm{d}y^i \to e^a \;=\; e^a{}_i(y)\,\mathrm{d}y^i$:

$$g_{ij}(x) \;\to\; G_{ij}(x,y) \;=\; g_{ab}(x)\, e_i{}^a(y)\, e^b{}_j(y)$$

$$g_{ij}(x)\big(\mathrm{d}y^i + V^i{}_\mu \mathrm{d}x^\mu\big)\big(\mathrm{d}y^j + V^j{}_\nu \mathrm{d}x^\nu\big) \;\to\; g_{ab}(x)\big(e^a(y) + V^a{}_\mu \mathrm{d}x^\mu\big)\big(e^b(y) + V^b{}_\nu \mathrm{d}x^\nu\big)$$

We often switch off 4-dim. fluctuations: $g_{ab}(x) \to \delta_{ab}$, $B_{ab}(x) \to 0$, $G_{ij}(x,y) \to G_{ij}(y)$.

$$
\begin{aligned}
[Z_a, Z_b] &= f_{ab}{}^c Z_c + h_{abc} X^c \\
[X^a, X^b] &= 0 \\
[X^a, Z_b] &= f^a{}_{bc} X^c
\end{aligned}
$$

$$\begin{aligned} [Z_a, Z_b] &= f_{ab}{}^c Z_c + h_{abc} X^c \\ [X^a, X^b] &= 0 \\ [X^a, Z_b] &= f^a{}_{bc} X^c \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} [Z_a, Z_b] &= f_{ab}{}^c Z_c + h_{abc} X^c \\ [X^a, X^b] &= Q^{ab}{}_c X^c + R^{abc} Z_c \\ [X^a, Z_b] &= f^a{}_{bc} X^c - Q^{ac}{}_b Z_c \end{aligned}$$

Why should we study additional structure constants $Q^{ab}{}_c$ and R^{abc}?

$$\begin{aligned} [Z_a, Z_b] &= f_{ab}{}^c Z_c + h_{abc} X^c \\ [X^a, X^b] &= 0 \\ [X^a, Z_b] &= f^a{}_{bc} X^c \end{aligned}$$

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Why should we study additional structure constants $Q^{ab}{}_c$ and R^{abc}?

$$\downarrow$$

Because they are related via T-duality transformations

They also appear in generalized geometry

J. Shelton, W. Taylor, B. Wecht hep-th/0508133 A. Dabholkar, C.M. Hull hep-th/0512005

$$h_{abc}$$

$$\downarrow$$ T-duality transformation

$$f^a{}_{bc}$$

$$\downarrow$$ T-duality transformation

$$Q^{ab}{}_c$$

$$\downarrow$$ T-duality transformation

$$R^{abc}$$

h_{abc}	Flat torus with three-form flux
$\downarrow$	T-duality transformation
${f^a}_{bc}$	
$\downarrow$	T-duality transformation
${Q^{ab}}_c$	
$\downarrow$	T-duality transformation
R^{abc}	

h_{abc}	Flat torus with three-form flux
$\downarrow$	T-duality transformation
$f^a{}_{bc}$	Twisted torus with non-trivial isometry group
$\downarrow$	T-duality transformation
$Q^{ab}{}_c$	
$\downarrow$	T-duality transformation
R^{abc}	

h_{abc}	Flat torus with three-form flux
$\downarrow$	T-duality transformation
$f^a{}_{bc}$	Twisted torus with non-trivial isometry group
$\downarrow$	T-duality transformation
$Q^{ab}{}_c$	T-fold globally nongeometric, locally geometric: *stringy*
$\downarrow$	T-duality transformation
R^{abc}	

h_{abc}	Flat torus with three-form flux
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$\downarrow$	T-duality transformation
R^{abc}	Nongeometric background even locally: *stringy*

h_{abc}	Flat torus with three-form flux
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$Q^{ab}{}_c$	T-fold globally nongeometric, locally geometric: *stringy*
$\downarrow$	T-duality transformation
R^{abc}	Nongeometric background even locally: *stringy*

In order to include the above information,

we double the *compactified geometry from* $\mathcal{M}_d$ *to* $\mathcal{M}_{2d} = \mathcal{M}_d \times \widetilde{\mathcal{M}}_d$

and study an extended sigma model on it. $\dashrightarrow$ Doubled Formalism

C.M. Hull hep-th/0406102 hep-th/0605149 C.M. Hull, R.A. Reid-Edwards hep-th/0503114 arXiv:0711.4818

Glue two local patches of a conventional string background with transition function by

diffeomorphism

and

duality transformations

Let Y^i be fields in sigma model corresponding to coordinates y^i on a space $\mathcal{M}_d$.

In formulating CFT on $\mathcal{M}_d$,

extra d coordinates $\widetilde{Y}_i$ on a dual space $\widetilde{\mathcal{M}}_d$ are needed

Start with a sigma model on a space $\mathcal{M}_d$ with metric $G_{ij}(Y)$ and B-field $B_{ij}(Y)$:

$$S_c = \frac{1}{2} \int_{\Sigma} \Big(G_{ij} \, \mathrm{d}Y^i \wedge * \mathrm{d}Y^j + B_{ij} \, \mathrm{d}Y^i \wedge \mathrm{d}Y^j \Big)$$

Start with a sigma model on a space $\mathcal{M}_d$ with metric $G_{ij}(Y)$ and B-field $B_{ij}(Y)$:

$$S_c = \frac{1}{2} \int_{\Sigma} \Big(G_{ij}\, \mathrm{d}Y^i \wedge * \mathrm{d}Y^j + B_{ij}\, \mathrm{d}Y^i \wedge \mathrm{d}Y^j \Big)$$

Extend to the action with the Wess-Zumino term on a doubled space $\mathcal{M}_{2d}\,(= G/\Gamma)$

$$S = \frac{1}{4} \int_{\Sigma} \mathcal{M}_{AB}\, \mathcal{P}^A \wedge * \mathcal{P}^B + \frac{1}{12} \int_V t_{ABC}\, \mathcal{P}^A \wedge \mathcal{P}^B \wedge \mathcal{P}^C$$

- Σ: string worldsheet (without boundary)
- V: an extension of Σ s.t. $\partial V = \Sigma$
- G: $2d$-dim. (non-)compact Lie group with $[T_A, T_B] = t_{AB}{}^C T_C$
- Γ: a discrete subgroup of G chosen s.t. $\mathcal{M}_{2d}$ is compact

Constituents of the action $S = \frac{1}{4}\int_{\Sigma} \mathcal{M}_{AB}\,\mathcal{P}^A \wedge *\mathcal{P}^B + \frac{1}{12}\int_V t_{ABC}\,\mathcal{P}^A \wedge \mathcal{P}^B \wedge \mathcal{P}^C$

Constituents of the action $S = \frac{1}{4}\int_{\Sigma} \mathcal{M}_{AB}\,\mathcal{P}^A \wedge *\mathcal{P}^B + \frac{1}{12}\int_V t_{ABC}\,\mathcal{P}^A \wedge \mathcal{P}^B \wedge \mathcal{P}^C$

✓ Scalar fields of doubled coordinates and doubled vielbeins:

$$\mathbb{Y}^I \,; \quad \mathcal{P} = \mathfrak{g}^{-1}\mathrm{d}\mathfrak{g} = \mathcal{P}^A{}_I \big(r\,T_A\big)\,\mathrm{d}\mathbb{Y}^I \quad \text{with} \quad \mathfrak{g} \in G \subset O(d,d)$$

Constituents of the action $S = \frac{1}{4}\int_{\Sigma} \mathcal{M}_{AB}\, \mathcal{P}^A \wedge *\mathcal{P}^B + \frac{1}{12}\int_V t_{ABC}\, \mathcal{P}^A \wedge \mathcal{P}^B \wedge \mathcal{P}^C$

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✓ Bianchi identity (Maurer-Cartan eq.):

$$\mathrm{d}\mathcal{P}^A = -\frac{r}{2}\, t_{BC}{}^A\, \mathcal{P}^B \wedge \mathcal{P}^C$$

Constituents of the action $S = \frac{1}{4}\int_\Sigma \mathcal{M}_{AB}\,\mathcal{P}^A \wedge *\mathcal{P}^B + \frac{1}{12}\int_V t_{ABC}\,\mathcal{P}^A \wedge \mathcal{P}^B \wedge \mathcal{P}^C$

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✓ Bianchi identity (Maurer-Cartan eq.):

$$\mathrm{d}\mathcal{P}^A = -\frac{r}{2}\,t_{BC}{}^A\,\mathcal{P}^B \wedge \mathcal{P}^C$$

✓ Doubled metric from doubled vielbeins:

$$\mathcal{M}_{AB} = \mathcal{P}_A{}^I\,\mathcal{M}_{IJ}\,\mathcal{P}^J{}_B, \quad \mathcal{M}_{IJ} \text{ takes values in a coset } \frac{O(d,d)}{O(d) \times O(d)}$$

Constituents of the action $S = \frac{1}{4}\int_{\Sigma} \mathcal{M}_{AB}\, \mathcal{P}^A \wedge *\mathcal{P}^B + \frac{1}{12}\int_V t_{ABC}\, \mathcal{P}^A \wedge \mathcal{P}^B \wedge \mathcal{P}^C$

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✓ Self-duality constraint (to go back to conventional system):

$$\mathcal{P}^A = L^{AB} \mathcal{M}_{BC} * \mathcal{P}^C$$

Generators of the Lie algebra T_A given by $2d \times 2d$ matrix projectors $\Pi^A{}_B$, $\widetilde{\Pi}^A{}_B$:

$$\Pi^A{}_B \Pi^B{}_C \;=\; \Pi^A{}_C \,, \quad \Pi^A{}_B \widetilde{\Pi}^B{}_C \;=\; 0 \,, \quad \Pi^A{}_B + \widetilde{\Pi}^A{}_B \;=\; \delta^A{}_B$$

$$\Pi^A{}_B \;\equiv\; \begin{pmatrix} \Pi^a{}_B \\ 0 \end{pmatrix} , \quad \widetilde{\Pi}^A{}_B \;\equiv\; \begin{pmatrix} 0 \\ \widetilde{\Pi}_{aB} \end{pmatrix}$$

$$X^a \;=\; \Pi^a{}_B \, L^{AB} T_B \,, \quad Z_a \;=\; \widetilde{\Pi}_{aB} \, L^{AB} T_B$$

Then the doubled coordinates, metric and vielbeins are polarized as

$$Y^I \;\equiv\; \Pi^I{}_J \mathbb{Y}^J \;=\; \begin{pmatrix} Y^i \\ 0 \end{pmatrix} , \quad \widetilde{Y}^I \;\equiv\; \widetilde{\Pi}^I{}_J \mathbb{Y}^J \;=\; \begin{pmatrix} 0 \\ \widetilde{Y}_i \end{pmatrix}$$

$$\mathcal{M}_{IJ} \;=\; \begin{pmatrix} G_{ij} - B_{ik} \, G^{kl} \, B_{lj} & B_{ik} \, G^{kj} \\ -G^{ik} \, B_{kj} & G^{ij} \end{pmatrix}$$

$$\mathcal{P}^A{}_I \;=\; \begin{pmatrix} e^a{}_i & 0 \\ -e_a{}^j B_{ji} & e_a{}^i \end{pmatrix}$$

$$\mathcal{M}_{IJ} \text{ takes values in a coset } \frac{O(d,d)}{O(d) \times O(d)}$$

This sigma model on the doubled space $\mathcal{M}_{2d}$ has

- $O(d,d)$ global symmetry by $\rho \in O(d,d)$ with $\rho^A{}_C L^{CD} \rho_D{}^B = L^{AB}$:

$$\begin{aligned} \mathbb{Y}^I \ &\rightarrow \ \mathbb{Y}'^I = \rho^I{}_J \mathbb{Y}^J \\ \mathcal{P}^A{}_I(\mathbb{Y}) \ &\rightarrow \ \mathcal{P}'^A{}_I(\mathbb{Y}') = \rho^A{}_B \, \mathcal{P}^B{}_J(\mathbb{Y}') \, \rho^J{}_I \\ \mathcal{M}_{IJ}(\mathbb{Y}) \ &\rightarrow \ \mathcal{M}'_{IJ}(\mathbb{Y}') = \rho_I{}^K \, \mathcal{M}_{KL}(\mathbb{Y}') \rho^L{}_J \end{aligned}$$

Basis vector is kept invariant under the transformation: so-called "active transformation"

- $O(d) \times O(d)$ local symmetry: $\mathcal{P}^A{}_I(\mathbb{Y}) \rightarrow \mathcal{P}'^A{}_I(\mathbb{Y}) = h^A{}_B(\mathbb{Y}) \, \mathcal{P}^B{}_I(\mathbb{Y})$

$$\rho \in O(d,d); \quad \mathbb{Y}^I \;\to\; \mathbb{Y}'^I = \rho^I{}_J \mathbb{Y}^J, \quad \mathcal{P}^A{}_I(\mathbb{Y}) \;\to\; \mathcal{P}'^A{}_I(\mathbb{Y}') = \rho^A{}_B \, \mathcal{P}^B{}_J(\mathbb{Y}') \, \rho^J{}_I$$

$$\rho \in O(d,d); \quad \mathbb{Y}^I \; \to \; \mathbb{Y}'^I = \rho^I{}_J \mathbb{Y}^J, \quad \mathcal{P}^A{}_I(\mathbb{Y}) \; \to \; \mathcal{P}'^A{}_I(\mathbb{Y}') = \rho^A{}_B \, \mathcal{P}^B{}_J(\mathbb{Y}') \, \rho^J{}_I$$

A realization of fractional transformation of $M_{ij} = G_{ij} + B_{ij}$

$$\rho = \begin{pmatrix} A & \beta \\ \Theta & D \end{pmatrix} : \qquad M \to \big(DM + \Theta\big)\big(\beta M + A\big)^{-1}$$

- Θ : gauge transformation of B-field $B \to B + \Theta$
- D, A : diffeomorphism
- β : duality transformation with mixing Y^i and $\widetilde{Y}_i$

$$\rho \in O(d,d); \quad \mathbb{Y}^I \;\to\; \mathbb{Y}'^I = \rho^I{}_J \mathbb{Y}^J, \quad \mathcal{P}^A{}_I(\mathbb{Y}) \;\to\; \mathcal{P}'^A{}_I(\mathbb{Y}') = \rho^A{}_B \, \mathcal{P}^B{}_J(\mathbb{Y}') \, \rho^J{}_I$$

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T-duality transformation (ex. $d = 3$ case):

$$\rho_i = \begin{pmatrix} \mathbb{1}_3 - \mathrm{T}_i & \mathrm{T}_i \\ \mathrm{T}_i & \mathbb{1}_3 - \mathrm{T}_i \end{pmatrix} \in O(3,3;\mathbb{Z})$$

$$\mathrm{T}_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad \mathrm{T}_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad \mathrm{T}_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

This action exchanges physical coordinates Y^i with dual coordinates $\widetilde{Y}_i$

Reduction of (co)tangent bundle of doubled space $\mathcal{M}_{2d}$

$$L^{AB} = \langle \mathcal{P}^A, \mathcal{P}^B \rangle = \begin{pmatrix} \mathbf{0}_d & \mathbb{1}_d \\ \mathbb{1}_d & \mathbf{0}_d \end{pmatrix}, \quad L^{IJ} \equiv \langle \mathrm{d}\mathbb{Y}^I, \mathrm{d}\mathbb{Y}^J \rangle = \mathcal{P}^I{}_A \, L^{AB} \, \mathcal{P}_B{}^J$$

This implies $T^* \mathcal{M}_{2d} = T\mathcal{M}_d \oplus T^* \mathcal{M}_d$ s.t.

$$\langle \mathrm{d}Y^i, \mathrm{d}\widetilde{Y}_j \rangle = \delta^i_j \quad \to \quad \mathrm{d}\widetilde{Y}_i = \frac{\partial}{\partial Y^i}$$

$$\mathcal{P}^A = \mathcal{P}^A{}_I \, \mathrm{d}\mathbb{Y}^I = \begin{pmatrix} e^a{}_i \, \mathrm{d}Y^i \\ e_a{}^i (\mathrm{d}\widetilde{Y}_i - B_{ij} \, \mathrm{d}Y^j) \end{pmatrix} = \begin{pmatrix} e^a{}_i \, \mathrm{d}Y^i \\ e_a{}^i (\frac{\partial}{\partial Y^i} - B_{ij} \, \mathrm{d}Y^j) \end{pmatrix}$$

a connection to Generalized Geometry

Using the worldsheet coordinates σ^α, we see the self-duality constraint as

$$\mathcal{P}^A = L^{AB}\mathcal{M}_{BC} * \mathcal{P}^C \qquad \longleftrightarrow \qquad \mathrm{d}\mathbb{Y}^I = L^{IJ}\mathcal{M}_{JK} * \mathrm{d}\mathbb{Y}^K$$

$$\therefore \quad \Big(\partial_\alpha \mathbb{Y}^I - \sqrt{-\eta}\,\varepsilon_\alpha{}^\beta\, L^{IJ}\mathcal{M}_{JK}\,\partial_\beta \mathbb{Y}^K\Big)\,\mathrm{d}\sigma^\alpha = 0 \qquad \text{w/} \quad \begin{cases} \eta_{\alpha\beta} = \mathrm{diag.}(+,-) \\ \varepsilon_{01} = 1 = \varepsilon^{10} \end{cases}$$

Taking the polarization, we obtain a set of non-trivial equations:

$$\boxed{(\partial_0 \pm \partial_1)\widetilde{Y}_i = \Big(B_{ij}(Y,\widetilde{Y}) \mp G_{ij}(Y,\widetilde{Y})\Big)\,(\partial_0 \pm \partial_1) Y^j}$$

Then the dual coordinates $\widetilde{Y}_i$ are related to the physical coordinates Y^i.

Start from a flat three-torus T^3 with a three-form flux H given by the following forms:

$$\mathrm{d}s^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \,, \qquad H = \mathrm{d}B = m\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

with a symmetric gauge $B = k\Big(x\, \mathrm{d}y \wedge \mathrm{d}z + y\, \mathrm{d}z \wedge \mathrm{d}x + z\, \mathrm{d}x \wedge \mathrm{d}y \Big) \,, \qquad k = \frac{m}{3}$

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Doubled vielbein $\mathcal{P}^A{}_I$ and doubled metric $\mathcal{M}_{IJ} = \mathcal{P}_I{}^A \delta_{AB} \mathcal{P}^B{}_J$ are given as

$$\mathcal{P}^A{}_I = \begin{pmatrix} e^a{}_i & \mathbf{0} \\ -e_a{}^j B_{ji} & e_a{}^i \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -kz & ky & 1 & 0 & 0 \\ kz & 0 & -kx & 0 & 1 & 0 \\ -ky & kx & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\mathcal{M}_{IJ} = \left(\begin{array}{ccc|ccc} 1 + k^2 y^2 + k^2 z^2 & -k^2 xy & -k^2 zx & 0 & kz & -ky \\ -k^2 xy & 1 + k^2 x^2 + k^2 z^2 & -k^2 yz & -kz & 0 & kx \\ -k^2 zx & -k^2 yz & 1 + k^2 x^2 + k^2 y^2 & ky & -kx & 0 \\ \hline 0 & -kz & ky & 1 & 0 & 0 \\ kz & 0 & -kx & 0 & 1 & 0 \\ -ky & kx & 0 & 0 & 0 & 1 \end{array}\right)$$

- Bianchi identity of doubled vielbein $\mathcal{P}^A = \begin{pmatrix} \mathcal{P}^a \\ \widetilde{\mathcal{P}}_a \end{pmatrix}$ gives a structure constant $t_{AB}{}^C$:

$$\begin{aligned} \mathrm{d}\mathcal{P}^1 &= 0\,, & \mathrm{d}\mathcal{P}^2 &= 0\,, & \mathrm{d}\mathcal{P}^3 &= 0 \\ \mathrm{d}\widetilde{\mathcal{P}}_1 &= \frac{2m}{3}\,\mathcal{P}^2 \wedge \mathcal{P}^3\,, & \mathrm{d}\widetilde{\mathcal{P}}_2 &= \frac{2m}{3}\,\mathcal{P}^3 \wedge \mathcal{P}^1\,, & \mathrm{d}\widetilde{\mathcal{P}}_3 &= \frac{2m}{3}\,\mathcal{P}^2 \wedge \mathcal{P}^3 \end{aligned}$$

$$\therefore \qquad \mathrm{d}\mathcal{P}^a = 0\,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_a = -\frac{r}{2}\,t_{abc}\,\mathcal{P}^b \wedge \mathcal{P}^c$$

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$$\therefore \quad \mathrm{d}\mathcal{P}^a = 0\,, \quad \mathrm{d}\widetilde{\mathcal{P}}_a = -\frac{r}{2} t_{abc}\,\mathcal{P}^b \wedge \mathcal{P}^c$$

Then we can read the structure constant $t_{abc} \equiv h_{abc}$ of the Lie algebra as

$$[Z_a, Z_b] = h_{abc}\,X^c\,, \qquad h_{123} = -\frac{2m}{3r} \equiv -H_{123}$$

We can also fix the scaling factor in the Bianchi identity:

$$r = \frac{2}{3}\,, \qquad \mathrm{d}\mathcal{P}^A = -\frac{1}{3} t_{BC}{}^A\,\mathcal{P}^B \wedge \mathcal{P}^C$$

- Periodicity of physical coordinates Y^i and dual coordinates $\widetilde{Y}_i$:

$$(x, \widetilde{y}, \widetilde{z}) \sim (x+1, \widetilde{y} + kz, \widetilde{z} - ky) \qquad \widetilde{x} \sim \widetilde{x} + 1$$

$$(y, \widetilde{z}, \widetilde{x}) \sim (y+1, \widetilde{z} + kx, \widetilde{x} - kz) \qquad \widetilde{y} \sim \widetilde{y} + 1$$

$$(z, \widetilde{x}, \widetilde{y}) \sim (z+1, \widetilde{x} + ky, \widetilde{y} - kx) \qquad \widetilde{z} \sim \widetilde{z} + 1$$

This does not change the metric G_{ij} and the B-field B_{ij}.

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$$(y, \widetilde{z}, \widetilde{x}) \sim (y+1, \widetilde{z} + kx, \widetilde{x} - kz) \qquad \widetilde{y} \sim \widetilde{y} + 1$$

$$(z, \widetilde{x}, \widetilde{y}) \sim (z+1, \widetilde{x} + ky, \widetilde{y} - kx) \qquad \widetilde{z} \sim \widetilde{z} + 1$$

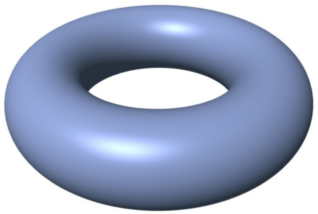
This does not change the metric G_{ij} and the B-field B_{ij}.

- Self-duality constraint: $Y^i = (x, y, z)$, $\widetilde{Y}_i = (\widetilde{x}, \widetilde{y}, \widetilde{z})$ and $\sigma^{\pm} \equiv \sigma^0 \pm \sigma^1$

$$\partial_{\pm} \widetilde{Y}_i = \big(B_{ij}(Y) \mp \delta_{ij} \big) \partial_{\pm} Y^j$$

We can completely take the projection onto the physical space

--→ geometric background



- Doubled vielbein by T-duality along z-direction:

$$(\mathcal{P}_f)^A{}_I = (\rho_z)^A{}_B \mathcal{P}^B{}_J (\rho_z)^J{}_I = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -ky & kx & 1 & 0 & 0 & 0 \\ \hline 0 & -k\widetilde{z} & 0 & 1 & 0 & ky \\ k\widetilde{z} & 0 & 0 & 0 & 1 & -kx \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(\mathcal{M}_f)_{IJ} = (\rho_z)_I{}^K \mathcal{M}_{KL} (\rho_z)^L{}_J \equiv \begin{pmatrix} G_f - B_f G_f^{-1} B_f & B_f G_f^{-1} \\ -G_f^{-1} B_f & G_f^{-1} \end{pmatrix}$$

"Metric" G_f and "B-field" B_f can be read from the doubled metric as

$$(G_f)_{ij} = \begin{pmatrix} 1 + k^2 y^2 & -k^2 xy & -ky \\ -k^2 xy & 1 + k^2 x^2 & kx \\ -ky & kx & 1 \end{pmatrix}, \quad (B_f)_{ij} = \begin{pmatrix} 0 & k\,\widetilde{z} & 0 \\ -k\,\widetilde{z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Bianchi identity of doubled vielbein $\mathcal{P}^A = \begin{pmatrix} \mathcal{P}^a \\ \widetilde{\mathcal{P}}_a \end{pmatrix}$ gives a structure constant $t_{AB}{}^C$:

$$\mathrm{d}\mathcal{P}^1 = 0\,, \qquad \mathrm{d}\mathcal{P}^2 = 0\,, \qquad \mathrm{d}\mathcal{P}^3 = \frac{2m}{3}\,\mathcal{P}^1 \wedge \mathcal{P}^2$$

$$\mathrm{d}\widetilde{\mathcal{P}}_1 = \frac{2m}{3}\,\mathcal{P}^2 \wedge \widetilde{\mathcal{P}}_3\,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_2 = \frac{2m}{3}\,\widetilde{\mathcal{P}}_3 \wedge \mathcal{P}^1\,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_3 = 0$$

$$\therefore \quad \mathrm{d}\mathcal{P}^a = -\frac{1}{3}\,t^a{}_{bc}\,\mathcal{P}^b \wedge \mathcal{P}^c\,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_a = -\frac{1}{3}\,t_{ab}{}^c\,\mathcal{P}^b \wedge \widetilde{\mathcal{P}}_c$$

Then we can read the structure constant $t_{ab}{}^c \equiv f_{ab}{}^c$ as

$$[Z_a, Z_b] = f_{ab}{}^c\,Z_c\,, \qquad [X^a, Z_b] = f^a{}_{bc}\,X^c\,, \qquad f^1{}_{23} = -m$$

- Periodicity of physical coordinates Y^i and dual coordinates $\widetilde{Y}_i$:

$$
\begin{aligned}
(x, \widetilde{y}, z) &\sim (x+1, \widetilde{y} + kz, z - ky) & \widetilde{x} &\sim \widetilde{x} + 1 \\
(y, z, \widetilde{x}) &\sim (y+1, z + kx, \widetilde{x} - kz) & \widetilde{y} &\sim \widetilde{y} + 1 \\
z &\sim z + 1 & (\widetilde{z}, \widetilde{x}, \widetilde{y}) &\sim (\widetilde{z} + 1, \widetilde{x} + k\widetilde{y}, \widetilde{y} - k\widetilde{x})
\end{aligned}
$$

$$
\mathrm{d}s^2 = \big(\mathrm{d}x\big)^2 + \big(\mathrm{d}y\big)^2 + \big(\mathrm{d}z - ky\,\mathrm{d}x + kx\,\mathrm{d}y\big)^2, \qquad B = k\widetilde{z}\,\mathrm{d}x \wedge \mathrm{d}y
$$

The metric is invariant and the B-field is shifted via this periodic shift.

- Periodicity of physical coordinates Y^i and dual coordinates $\widetilde{Y}_i$:

$$(x, \widetilde{y}, z) \sim (x+1, \widetilde{y}+kz, z-ky) \qquad \widetilde{x} \sim \widetilde{x}+1$$

$$(y, z, \widetilde{x}) \sim (y+1, z+kx, \widetilde{x}-kz) \qquad \widetilde{y} \sim \widetilde{y}+1$$

$$z \sim z+1 \qquad (\widetilde{z}, \widetilde{x}, \widetilde{y}) \sim (\widetilde{z}+1, \widetilde{x}+k\widetilde{y}, \widetilde{y}-k\widetilde{x})$$

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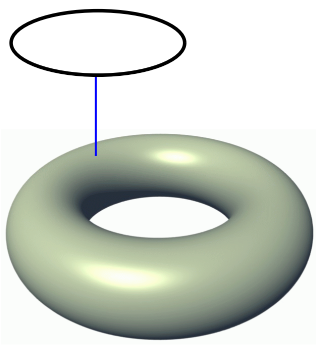
The metric is invariant and the B-field is shifted via this periodic shift.

- Self-duality constraint is $\partial_\pm \widetilde{Y}_i = \big(B_{ij}(\widetilde{Y}) \mp G_{ij}(Y)\big)\,\partial_\pm Y^j$

We can completely take the projection on the physical space

$\dashrightarrow$ geometric background



- Doubled vielbein by T-duality along (y,z)-directions:

$$(\mathcal{P}_Q)^A{}_I = (\rho_y\rho_z)^A{}_B\,\mathcal{P}^B{}_J(\rho_z\rho_y)^J{}_I = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ k\widetilde{z} & 1 & 0 & 0 & 0 & -kx \\ -k\widetilde{y} & 0 & 1 & 0 & kx & 0 \\ \hline 0 & 0 & 0 & 1 & -k\widetilde{z} & k\widetilde{y} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(\mathcal{M}_Q)_{IJ} = (\rho_y\rho_z)_I{}^K\,\mathcal{M}_{KL}(\rho_z\rho_y)^L{}_J \equiv \begin{pmatrix} G_Q - B_Q G_Q^{-1} B_Q & B_Q G_Q^{-1} \\ -G_Q^{-1} B_Q & G_Q^{-1} \end{pmatrix}$$

The "metric" G_Q and "B-field" B_Q are

$$(G_Q)_{ij} = \frac{1}{1+k^2x^2}\begin{pmatrix} 1+k^2(x^2+\widetilde{y}^2+\widetilde{z}^2) & k\widetilde{z} & -k\widetilde{y} \\ k\widetilde{z} & 1 & 0 \\ -k\widetilde{y} & 0 & 1 \end{pmatrix}$$

$$(B_Q)_{ij} = \frac{1}{1+k^2x^2}\begin{pmatrix} 0 & -k^2x\widetilde{y} & -k^2x\widetilde{z} \\ k^2x\widetilde{y} & 0 & -kx \\ k^2x\widetilde{z} & kx & 0 \end{pmatrix}$$

local $O(3) \times O(3)$ transformation to describe correct form of doubled vielbein

$$(\mathcal{P}'_Q)^A{}_I = h^A{}_B (\mathcal{P}_Q)^B{}_I \equiv \begin{pmatrix} (e_Q)^a{}_i & \mathbf{0}_3 \\ -(e_Q)_a{}^j (B_Q)_{ji} & (e_Q)_a{}^i \end{pmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{k\widetilde{z}}{\sqrt{1+k^2x^2}} & \frac{1}{\sqrt{1+k^2x^2}} & 0 & 0 & 0 & 0 \\ -\frac{k\widetilde{y}}{\sqrt{1+k^2x^2}} & 0 & \frac{1}{\sqrt{1+k^2x^2}} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -k\widetilde{z} & k\widetilde{y} \\ -\frac{k^2x\widetilde{y}}{\sqrt{1+k^2x^2}} & 0 & \frac{kx}{\sqrt{1+k^2x^2}} & 0 & \sqrt{1+k^2x^2} & 0 \\ -\frac{k^2x\widetilde{z}}{\sqrt{1+k^2x^2}} & -\frac{kx}{\sqrt{1+k^2x^2}} & 0 & 0 & 0 & \sqrt{1+k^2x^2} \end{array}\right)$$

$$h^A{}_B = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+k^2x^2}} & 0 & 0 & 0 & \frac{kx}{\sqrt{1+k^2x^2}} \\ 0 & 0 & \frac{1}{\sqrt{1+k^2x^2}} & 0 & -\frac{kx}{\sqrt{1+k^2x^2}} & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{kx}{\sqrt{1+k^2x^2}} & 0 & \frac{1}{\sqrt{1+k^2x^2}} & 0 \\ 0 & -\frac{kx}{\sqrt{1+k^2x^2}} & 0 & 0 & 0 & \frac{1}{\sqrt{1+k^2x^2}} \end{array}\right)$$

- Bianchi identity of doubled vielbein $\mathcal{P}^A = \begin{pmatrix} \mathcal{P}^a \\ \widetilde{\mathcal{P}}_a \end{pmatrix}$ gives a structure constant $t_{AB}{}^C$:

$$\begin{aligned} d\mathcal{P}^1 &= 0\,, & d\mathcal{P}^2 &= \frac{2m}{3}\,\widetilde{\mathcal{P}}_3 \wedge \mathcal{P}^1\,, & d\mathcal{P}^3 &= \frac{2m}{3}\,\mathcal{P}^1 \wedge \widetilde{\mathcal{P}}_2 \\ d\widetilde{\mathcal{P}}_1 &= \frac{2m}{3}\,\widetilde{\mathcal{P}}_2 \wedge \widetilde{\mathcal{P}}_3\,, & d\widetilde{\mathcal{P}}_2 &= 0\,, & d\widetilde{\mathcal{P}}_3 &= 0 \end{aligned}$$

$$\therefore \quad d\mathcal{P}^a = -\frac{1}{3}\,t^{ab}{}_c\,\widetilde{\mathcal{P}}_b \wedge \mathcal{P}^c\,, \qquad d\widetilde{\mathcal{P}}_a = -\frac{1}{3}\,t_a{}^{bc}\,\widetilde{\mathcal{P}}_b \wedge \widetilde{\mathcal{P}}_c$$

Then we can read the structure constant $t^{ab}{}_c \equiv Q^{ab}{}_c$ as

$$[X^a, X^b] = Q^{ab}{}_c\,X^c\,, \qquad [Z_a, X^b] = Q_a{}^{bc}\,Z_c\,, \qquad Q^{12}{}_3 = -m$$

- Periodicity of physical coordinates Y^i and dual ones $\widetilde{Y}_i$:

$$(x, y, z) \sim (x+1, y + k\widetilde{z}, z - k\widetilde{y}) \qquad \widetilde{x} \sim \widetilde{x} + 1$$

$$y \sim y + 1 \qquad (\widetilde{y}, z, \widetilde{x}) \sim (\widetilde{y}+1, z + kx, \widetilde{x} - k\widetilde{z})$$

$$z \sim z + 1 \qquad (\widetilde{z}, \widetilde{x}, y) \sim (\widetilde{z}+1, \widetilde{x} + k\widetilde{y}, y - kx)$$

$$\mathrm{d}s^2 = (\mathrm{d}x)^2 + \frac{1}{1+k^2x^2}\Big[(\mathrm{d}y + k\widetilde{z}\,\mathrm{d}x)^2 + (\mathrm{d}z - k\widetilde{y}\,\mathrm{d}x)^2\Big]$$

This periodic shift yields a β-trsf: duality trsf. --→ globally nongeometric

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$$
\begin{aligned}
(x, y, z) &\sim (x + 1, y + k\widetilde{z}, z - k\widetilde{y}) & \widetilde{x} &\sim \widetilde{x} + 1 \\
y &\sim y + 1 & (\widetilde{y}, z, \widetilde{x}) &\sim (\widetilde{y} + 1, z + kx, \widetilde{x} - k\widetilde{z}) \\
z &\sim z + 1 & (\widetilde{z}, \widetilde{x}, y) &\sim (\widetilde{z} + 1, \widetilde{x} + k\widetilde{y}, y - kx)
\end{aligned}
$$

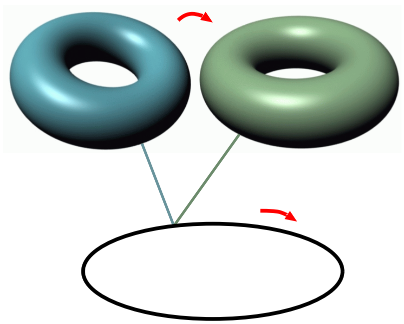
$$
\mathrm{d}s^2 = (\mathrm{d}x)^2 + \frac{1}{1 + k^2 x^2}\Big[(\mathrm{d}y + k\widetilde{z}\,\mathrm{d}x)^2 + (\mathrm{d}z - k\widetilde{y}\,\mathrm{d}x)^2\Big]
$$

This periodic shift yields a β-trsf: duality trsf. --→ globally nongeometric

However, imposing the self-duality constraint,

we see that this duality transformation is interpreted as T-duality on fibred T^2

in terms of only the physical coordinate objects --→ locally geometric



- Doubled vielbein by T-duality along (x, y, z)-directions:

$$(\mathcal{P}_R)^A{}_I = (\rho_x \rho_y \rho_z)^A{}_B \, \mathcal{P}^B{}_J (\rho_z \rho_y \rho_x)^J{}_I = \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & -k\widetilde{z} & k\widetilde{y} \\ 0 & 1 & 0 & k\widetilde{z} & 0 & -k\widetilde{x} \\ 0 & 0 & 1 & -k\widetilde{y} & k\widetilde{x} & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(\mathcal{M}_R)_{IJ} = (\rho_x \rho_y \rho_z)_I{}^K \, \mathcal{M}_{KL} (\rho_z \rho_y \rho_x)^L{}_J \equiv \begin{pmatrix} G_R - B_R G_R^{-1} B_R & B_R G_R^{-1} \\ -G_R^{-1} B_R & G_R^{-1} \end{pmatrix}$$

The "metric" G_R and "B-field" B_R can be read from the doubled metric as

$$(G_R)_{ij} = \chi \begin{pmatrix} 1 + k^2 \widetilde{x}^2 & k^2 \widetilde{x}\widetilde{y} & k^2 \widetilde{z}\widetilde{x} \\ k^2 \widetilde{x}\widetilde{y} & 1 + k^2 \widetilde{y}^2 & k^2 \widetilde{y}\widetilde{z} \\ k^2 \widetilde{z}\widetilde{x} & k^2 \widetilde{y}\widetilde{z} & 1 + k^2 \widetilde{z}^2 \end{pmatrix}, \qquad (B_R)_{ij} = \chi \begin{pmatrix} 0 & -k\widetilde{z} & k\widetilde{y} \\ k\widetilde{z} & 0 & -k\widetilde{x} \\ -k\widetilde{y} & k\widetilde{x} & 0 \end{pmatrix}$$

$$\chi = \frac{1}{1 + k^2 \widetilde{x}^2 + k^2 \widetilde{y}^2 + k^2 \widetilde{z}^2}$$

local $O(3) \times O(3)$ transformation to describe correct form of doubled vielbein

$$
(\mathcal{P}'_R)^A{}_I = h^A{}_B (\mathcal{P}_R)^B{}_I \equiv \begin{pmatrix} (e_R)^a{}_i & \mathbf{0}_3 \\ -(e_R)_a{}^j (B_R)_{ji} & (e_R)_a{}^i \end{pmatrix}
$$

$$
= \left(\begin{array}{ccc:ccc}
\chi(1+k^2\widetilde{x}^2) & \chi(k^2\widetilde{x}\widetilde{y}+k\widetilde{z}) & \chi(k^2\widetilde{z}\widetilde{x}-k\widetilde{y}) & 0 & 0 & 0 \\
\chi(k^2\widetilde{x}\widetilde{y}-k\widetilde{z}) & \chi(1+k^2\widetilde{y}^2) & \chi(k^2\widetilde{y}\widetilde{z}+k\widetilde{x}) & 0 & 0 & 0 \\
\chi(k^2\widetilde{z}\widetilde{x}+k\widetilde{y}) & \chi(k^2\widetilde{y}\widetilde{z}-k\widetilde{x}) & \chi(1+k^2\widetilde{z}^2) & 0 & 0 & 0 \\
\hdashline
-k^2\chi(\widetilde{y}^2+\widetilde{z}^2) & \chi(k^2\widetilde{x}\widetilde{y}+k\widetilde{z}) & \chi(k^2\widetilde{z}\widetilde{x}-k\widetilde{y}) & 1 & k\widetilde{z} & -k\widetilde{y} \\
\chi(k^2\widetilde{x}\widetilde{y}-k\widetilde{z}) & -k^2\chi(\widetilde{x}^2+\widetilde{z}^2) & \chi(k^2\widetilde{y}\widetilde{z}+k\widetilde{x}) & -k\widetilde{z} & 1 & k\widetilde{x} \\
\chi(k^2\widetilde{z}\widetilde{x}+k\widetilde{y}) & \chi(k^2\widetilde{y}\widetilde{z}-k\widetilde{x}) & -k^2\chi(\widetilde{x}^2+\widetilde{y}^2) & k\widetilde{y} & -k\widetilde{x} & 1
\end{array}\right)
$$

$$
h^A{}_B = \chi \left(\begin{array}{ccc:ccc}
1+k^2\widetilde{x}^2 & k^2\widetilde{x}\widetilde{y}+k\widetilde{z} & k^2\widetilde{z}\widetilde{x}-k\widetilde{y} & -k^2(\widetilde{y}^2+\widetilde{z}^2) & k^2\widetilde{x}\widetilde{y}+k\widetilde{z} & k^2\widetilde{z}\widetilde{x}-k\widetilde{y} \\
k^2\widetilde{x}\widetilde{y}-k\widetilde{z} & 1+k^2\widetilde{y}^2 & k^2\widetilde{y}\widetilde{z}+k\widetilde{x} & k^2\widetilde{x}\widetilde{y}-k\widetilde{z} & -k^2(\widetilde{z}^2+\widetilde{x}^2) & k^2\widetilde{y}\widetilde{z}+k\widetilde{x} \\
k^2\widetilde{z}\widetilde{x}+k\widetilde{y} & k^2\widetilde{y}\widetilde{z}-k\widetilde{x} & 1+k^2\widetilde{z}^2 & k^2\widetilde{z}\widetilde{x}+k\widetilde{y} & k^2\widetilde{y}\widetilde{z}-k\widetilde{x} & -k^2(\widetilde{x}^2+\widetilde{y}^2) \\
\hdashline
-k^2(\widetilde{y}^2+\widetilde{z}^2) & k^2\widetilde{x}\widetilde{y}+k\widetilde{z} & k^2\widetilde{z}\widetilde{x}-k\widetilde{y} & 1+k^2\widetilde{x}^2 & k^2\widetilde{x}\widetilde{y}+k\widetilde{z} & k^2\widetilde{z}\widetilde{x}-k\widetilde{y} \\
k^2\widetilde{x}\widetilde{y}-k\widetilde{z} & -k^2(\widetilde{z}^2+\widetilde{x}^2) & k^2\widetilde{y}\widetilde{z}+k\widetilde{x} & k^2\widetilde{x}\widetilde{y}-k\widetilde{z} & 1+k^2\widetilde{y}^2 & k^2\widetilde{y}\widetilde{z}+k\widetilde{x} \\
k^2\widetilde{z}\widetilde{x}+k\widetilde{y} & k^2\widetilde{y}\widetilde{z}-k\widetilde{x} & -k^2(\widetilde{x}^2+\widetilde{y}^2) & k^2\widetilde{z}\widetilde{x}+k\widetilde{y} & k^2\widetilde{y}\widetilde{z}-k\widetilde{x} & 1+k^2\widetilde{z}^2
\end{array}\right)
$$

- Bianchi identity of doubled vielbein $\mathcal{P}^A = \begin{pmatrix} \mathcal{P}^a \\ \widetilde{\mathcal{P}}_a \end{pmatrix}$ gives a structure constant $t_{AB}{}^C$:

$$
\begin{aligned}
\mathrm{d}\mathcal{P}^1 &= \frac{2m}{3}\,\widetilde{\mathcal{P}}_2 \wedge \widetilde{\mathcal{P}}_3\,, & \mathrm{d}\mathcal{P}^2 &= \frac{2m}{3}\,\widetilde{\mathcal{P}}_3 \wedge \widetilde{\mathcal{P}}_1\,, & \mathrm{d}\mathcal{P}^3 &= \frac{2m}{3}\,\widetilde{\mathcal{P}}_1 \wedge \widetilde{\mathcal{P}}_2 \\
\mathrm{d}\widetilde{\mathcal{P}}_1 &= 0\,, & \mathrm{d}\widetilde{\mathcal{P}}_2 &= 0\,, & \mathrm{d}\widetilde{\mathcal{P}}_3 &= 0
\end{aligned}
$$

$$
\therefore \quad \mathrm{d}\mathcal{P}^a = -\frac{m}{3}\,t^{abc}\,\widetilde{\mathcal{P}}_b \wedge \widetilde{\mathcal{P}}_c\,, \qquad \mathrm{d}\widetilde{\mathcal{P}}_a = 0
$$

Then we can read the structure constant $t^{abc} \equiv R^{abc}$ as

$$
[X^a, X^b] = R^{abc}\,Z_c\,, \qquad R^{123} = -m
$$

- Periodicity of physical coordinates Y^i and dual coordinates $\widetilde{Y}_i$:

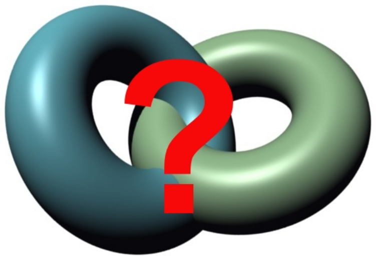
$$\begin{aligned} x &\sim x+1 & (\widetilde{x}, y, z) &\sim (\widetilde{x}+1, y+k\widetilde{z}, z-k\widetilde{y}) \\ y &\sim y+1 & (\widetilde{y}, z, x) &\sim (\widetilde{y}+1, z+k\widetilde{x}, x-k\widetilde{z}) \\ z &\sim z+1 & (\widetilde{z}, x, y) &\sim (\widetilde{z}+1, x+k\widetilde{y}, y-k\widetilde{x}) \end{aligned}$$

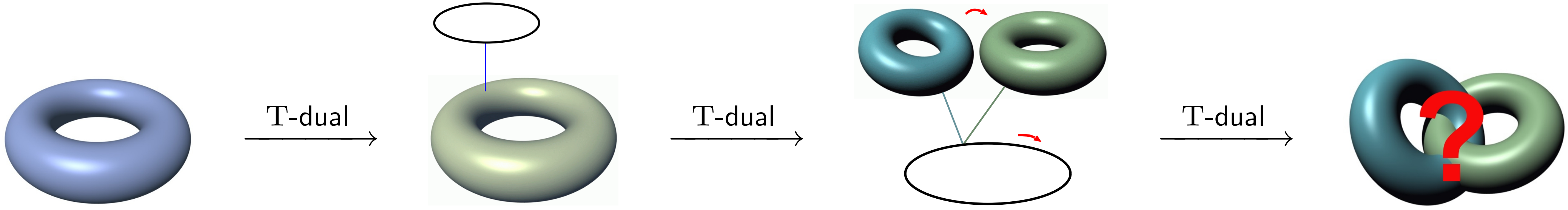
$$\mathrm{d}s^2 = \frac{1}{1+k^2\widetilde{x}^2+k^2\widetilde{y}^2+k^2\widetilde{z}^2}\Big[(\mathrm{d}x)^2+(\mathrm{d}y)^2+(\mathrm{d}z)^2+k^2(\widetilde{x}\,\mathrm{d}x+\widetilde{y}\,\mathrm{d}y+\widetilde{z}\,\mathrm{d}z)^2\Big]$$

This periodic shift yields a β-trsf: duality trsf. --→ globally nongeometric

The self-duality constraint does not yield a well-defined projection

--→ locally nongeometric





h_{abc} — flat-torus

$f^a{}_{bc}$ — nilmanifold

$Q^{ab}{}_c$ — T-fold — stringy

R^{abc} — nongeometric — stringy

T-duality in the presence of B-field generates geometric/nongeometric backgrounds.

They also have to be investigated as low energy stringy geometries.

Extended formalism can proceed the analysis.

Generalized geometry would also know the existence of Q- and R-fluxes.

M. Graña, J. Louis, D. Waldram hep-th/0612237 M. Graña, R. Minasian, M. Petrini, D. Waldram arXiv:0807.4527

- Start from scalar moduli matrix in supergravity on $\mathcal{M}_d$
- Introduce doubled space $\mathcal{M}_{2d}$ induced by B-field
- Perform $\mathbb{T}$-duality transformations
- Evaluate Lie algebra and geometries

Extend to $\mathbb{U}$-fold endowed with $\mathbb{U}$-duality transformation (hidden symmetry)

? Supersymmetry on doubled geometry ?

? Investigate quantum aspects of the doubled sigma model ?

Summary and Discussions

Here we have studied two typical extensions of compactified geometry:

generalized geometry and doubled formalism

- Generalized geometry provides the most general descriptions of the Kähler potentials and the superpotentials.

- Doubled formalism indicates the origin of nongeometric backgrounds which appears via T-duality transformations

Next we should...

- Find a way to analyze dimensions of moduli spaces
- Find relations between generalized geometry and doubled formalism
- Find application to moduli stabilization, landscape of flux vacua, etc.
- Include D-branes (and orientifold planes) into generalized/doubled geometries

C. Albertsson, TK, R.A. Reid-Edwards "D-branes and doubled geometry," arXiv:0806.1783

Appendix

Compactification Ansatz for the ten-dimensional spacetime:

$$\mathcal{M}_{1,9} \;=\; \mathcal{M}_{1,3} \times_{\mathrm{W}} \mathcal{M}_6$$

$$\mathrm{d}s^2_{1,9} \;=\; \mathcal{G}_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N \;=\; \mathrm{e}^{2A} g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \mathcal{G}_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j$$

Maximal symmetry of $\mathcal{M}_{1,3} \to \langle \text{fermions} \rangle = 0$

Supersymmetric vacuum $\leftrightarrow \langle \delta(\text{fermions}) \rangle = 0$

$$\delta \begin{pmatrix} \Psi^1_M \\ \Psi^2_M \end{pmatrix} = D_M \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix} - \frac{1}{96}\mathrm{e}^{-\phi}\Big(\gamma_M{}^{PQR}\mathcal{H}_{PQR} - 9\gamma^{PQ}\mathcal{H}_{MPQ}\Big)\mathcal{P}\begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix}$$

$$- \sum_n \frac{1}{64n!}\mathrm{e}^{\frac{5-n}{4}\phi}\Big[(n-1)\gamma_M{}^{N_1\cdots N_n} - n(9-n)\delta_M{}^{N_1}\gamma^{N_2\cdots N_n}\Big]\mathcal{F}_{N_1\cdots N_n}\mathcal{P}_n \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix}$$

	n	$\mathcal{P}$	$\mathcal{P}_n$
IIA	$0, 2, 4, 6, 8$	γ_{11}	$(\gamma_{11})^{n/2}\sigma^1$
IIB	$1, 5, 9$	$-\sigma^3$	$i\sigma^2$
	$3, 7$		σ^1

question: $\mathcal{P}_n$ in IIA

$(\gamma_{11})^{n/2}$? $\quad (\gamma_{11})^{n/2}\sigma^1$?

hep-th/0103233 $\quad$ hep-th/0505264

hep-th/0602241 $\quad$ hep-th/0509003

$\vdots \quad \vdots$

Decomposition of Lorentz symmetry:

$$Spin(1,9) \to Spin(1,3) \times Spin(6) = SL(2,\mathbb{C}) \times SU(4)$$
$$\mathbf{16}_1 = (\mathbf{2},\mathbf{4})_1 \oplus (\overline{\mathbf{2}},\overline{\mathbf{4}})_1, \quad \mathbf{16}_2 = (\mathbf{2},\overline{\mathbf{4}})_2 \oplus (\overline{\mathbf{2}},\mathbf{4})_2$$

Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\begin{cases} \epsilon^1_{\rm IIA} &= \xi^1_+ \otimes (a\eta^1_+) + \xi^1_- \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\rm IIA} &= \xi^2_+ \otimes (\overline{b}\eta^2_-) + \xi^2_- \otimes (b\eta^2_+) \end{cases} \qquad \begin{cases} \epsilon^1_{\rm IIB} &= \xi^1_+ \otimes (a\eta^1_+) + \xi^1_- \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\rm IIB} &= \xi^2_+ \otimes (b\eta^2_+) + \xi^2_- \otimes (\overline{b}\eta^2_-) \end{cases}$$

Set $SU(3)$ invariant spinor η^A_+ s.t. $D^{(T)}\eta^A_+ = 0$ $(A = 1, 2)$:

spacetime $\mathcal{M}_{1,3}$	compactified space $\mathcal{M}_6$
$\mathcal{N} = 2$: (ξ^1_+, ξ^2_+)	a pair of $SU(3)$ (η^1_+, η^2_+)
$\downarrow$	$\downarrow$
$\mathcal{N} = 1$: $(\xi^1_+ = \xi^2_+ = \xi_+)$	a single $SU(3)$ $(\eta^1_+ = \eta^2_+ = \eta_+)$

back to spinor decompositions

NS-NS fields in ten-dimensional spacetime are expanded as

$$\begin{aligned}
\phi(x,y) &= \varphi(x) \\
\mathcal{G}_{m\overline{n}}(x,y) &= iv^a(x)(\omega_a)_{m\overline{n}}(y), \quad \mathcal{G}_{mn}(x,y) \;=\; i\overline{z}^k(x)\left(\frac{(\overline{\chi}_k)_{m\overline{pq}}\Omega^{\overline{pq}}{}_n}{||\Omega||^2}\right)(y) \\
\mathcal{B}_2(x,y) &= B_2(x) + b^a(x)\omega_a(y)
\end{aligned}$$

R-R fields in type IIA are

$$\begin{aligned}
\mathcal{C}_1(x,y) &= C_1^0(x) \\
\mathcal{C}_3(x,y) &= C_1^a(x)\omega_a(y) + \xi^K(x)\alpha_K(y) - \widetilde{\xi}_K(x)\beta^K(y)
\end{aligned}$$

R-R fields in type IIB are

$$\begin{aligned}
\mathcal{C}_0(x,y) &= C_0(x) \\
\mathcal{C}_2(x,y) &= C_2(x) + c^a(x)\omega_a(y) \\
\mathcal{C}_4(x,y) &= V_1^K(x)\alpha_K(y) + \rho_a(x)\widetilde{\omega}^a(y)
\end{aligned}$$

cohomology class	basis	
$H^{(1,1)}$	ω_a	$a = 1,\ldots,h^{(1,1)}$
$H^{(0)} \oplus H^{(1,1)}$	$\omega_A = (1,\omega_a)$	$A = 0,1,\ldots,h^{(1,1)}$
$H^{(2,2)}$	$\widetilde{\omega}^a$	$a = 1,\ldots,h^{(1,1)}$
$H^{(2,1)}$	χ_k	$k = 1,\ldots,h^{(2,1)}$
$H^{(3)}$	(α_K,β^K)	$K = 0,1,\ldots,h^{(2,1)}$

Four-dimensional type IIA $\mathcal{N} = 2$ ungauged supergravity action of bosonic fields is

$$S^{(4)}_{\rm IIA} = \int_{\mathcal{M}_{1,3}} \Big(-\frac{1}{2} R * \mathbf{1} + \frac{1}{2} {\rm Re}\mathcal{N}_{AB} F^A \wedge F^B + \frac{1}{2} {\rm Im}\mathcal{N}_{AB} F^A \wedge * F^B - G_{a\bar{b}}\, {\rm d}t^a \wedge * {\rm d}\bar{t}^{\bar{b}} - h_{uv}\, {\rm d}q^u \wedge * {\rm d}q^v \Big)$$

gravity multiplet	$(g_{\mu\nu}, C_1^0)$	1
vector multiplet	(C_1^a, v^a, b^a)	$a = 1, \ldots, h^{(1,1)}$
hypermultiplet	$(z^k, \xi^k, \widetilde{\xi}_k)$	$k = 1, \ldots, h^{(2,1)}$
tensor multiplet	$(B_2, \varphi, \xi^0, \widetilde{\xi}_0)$	1

Various functions in the actions:

$$B + iJ = (b^a + iv^a)\, \omega_a = t^a \omega_a \qquad K^{\sf KS} = -\log \Big(\tfrac{4}{3} \int_{\mathcal{M}_6} J \wedge J \wedge J \Big)$$

$$\mathcal{K}_{abc} = \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge \omega_c \qquad \mathcal{K}_{ab} = \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge J = \mathcal{K}_{abc} v^c$$

$$\mathcal{K}_a = \int_{\mathcal{M}_6} \omega_a \wedge J \wedge J = \mathcal{K}_{abc} v^b v^c \qquad \mathcal{K} = \int_{\mathcal{M}_6} J \wedge J \wedge J = \mathcal{K}_{abc} v^a v^b v^c$$

$${\rm Re}\mathcal{N}_{AB} = \begin{pmatrix} -\frac{1}{3}\mathcal{K}_{abc} b^a b^b b^c & \frac{1}{2}\mathcal{K}_{abc} b^b b^c \\ \frac{1}{2}\mathcal{K}_{abc} b^b b^c & -\mathcal{K}_{abc} b^c \end{pmatrix} \qquad {\rm Im}\mathcal{N}_{AB} = -\frac{\mathcal{K}}{6} \begin{pmatrix} 1 + 4G_{ab} b^a b^b & -4G_{ab} b^b \\ -4G_{ab} b^b & 4G_{ab} \end{pmatrix}$$

$$G_{a\bar{b}} = \frac{3}{2} \frac{\int \omega_a \wedge * \omega_b}{\int J \wedge J \wedge J} = \partial_{t^a} \overline{\partial}_{\bar{t}^{\bar{b}}} K^{\sf KS}$$

Four-dimensional type IIB $\mathcal{N} = 2$ ungauged supergravity action of bosonic fields is

$$S^{(4)}_{\rm IIB} = \int_{\mathcal{M}_{1,3}} \Big(-\frac{1}{2} R * \mathbf{1} + \frac{1}{2} {\rm Re}\mathcal{M}_{KL} F^K \wedge F^L + \frac{1}{2} {\rm Im}\mathcal{M}_{KL} F^K \wedge * F^L - G_{k\bar{l}}\, {\rm d}z^k \wedge * {\rm d}\bar{z}^{\bar{l}} - h_{pq}\, {\rm d}\widetilde{q}^p \wedge * {\rm d}\widetilde{q}^q \Big)$$

gravity multiplet	$(g_{\mu\nu}, V^0_1)$	1
vector multiplet	(V^k_1, z^k)	$k = 1, \ldots, h^{(2,1)}$
hypermultiplet	(v^a, b^a, c^a, ρ_a)	$a = 1, \ldots, h^{(1,1)}$
tensor multiplet	(B_2, C_2, φ, C_0)	1

Various functions in the actions:

$$\Omega = \mathcal{Z}^K \alpha_K - \mathcal{F}_K \beta^K \qquad z^k = \mathcal{Z}^K / \mathcal{Z}^0 \qquad \mathcal{F}_{KL} = \partial_L \mathcal{F}_K$$

$$K^{\sf CS} = -\log \Big(i \int_{\mathcal{M}_6} \Omega \wedge \overline{\Omega} \Big) \qquad G_{k\bar{l}} = - \frac{\int \chi_k \wedge \overline{\chi}_{\bar{l}}}{\int \Omega \wedge \overline{\Omega}} = \partial_{z^k} \overline{\partial}_{\overline{z}^{\bar{l}}} K^{\sf CS}$$

$$\mathcal{M}_{KL} = \overline{\mathcal{F}}_{KL} + 2i \frac{({\rm Im}\mathcal{F})_{KM} \mathcal{Z}^M ({\rm Im}\mathcal{F})_{LN} \mathcal{Z}^N}{\mathcal{Z}^N ({\rm Im}\mathcal{F})_{NM} \mathcal{Z}^M}$$

back to CY Moduli

Information from Killing spinor eqs. with torsion $D^{(T)}\eta_\pm = 0$ ($^\exists$complex Weyl $\eta_\pm$)

- Invariant p-forms on $SU(3)$-structure manifold:

a real two-form $J_{ij} = \mp 2i\, \eta_\pm^\dagger\, \gamma_{ij}\, \eta_\pm$

a holomorphic three-form $\Omega_{ijk} = -2i\, \eta_-^\dagger\, \gamma_{ijk}\, \eta_+$

$$\mathrm{d}J = \frac{3}{2}\mathrm{Im}(\overline{\mathcal{W}}_1\Omega) + \mathcal{W}_4 \wedge J + \mathcal{W}_3 \qquad \mathrm{d}\Omega = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}}_5 \wedge \Omega$$

- Five classes of (intrinsic) torsion are given as

component	interpretation	$SU(3)$-representation
$\mathcal{W}_1$	$J \wedge \mathrm{d}\Omega$ or $\Omega \wedge \mathrm{d}J$	$\mathbf{1} \oplus \mathbf{1}$
$\mathcal{W}_2$	$(\mathrm{d}\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
$\mathcal{W}_3$	$(\mathrm{d}J)_0^{2,1} + (\mathrm{d}J)_0^{1,2}$	$\mathbf{6} \oplus \overline{\mathbf{6}}$
$\mathcal{W}_4$	$J \wedge \mathrm{d}J$	$\mathbf{3} \oplus \overline{\mathbf{3}}$
$\mathcal{W}_5$	$(\mathrm{d}\Omega)^{3,1}$	$\mathbf{3} \oplus \overline{\mathbf{3}}$

In case of heterotic string, see, for instance, K. Becker, M. Becker, J.-X. Fu, L.-S. Tseng, S.-T. Yau hep-th/0604137

T. Kimura, P. Yi hep-th/0605247 etc.

- Vanishing torsion classes in special $SU(3)$-structure manifolds:

complex	hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = 0$
	balanced	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = 0$
	special hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	Kähler	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	conformally Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 3\mathcal{W}_4 + 2\mathcal{W}_5 = 0$
almost complex	symplectic	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	nearly Kähler	$\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	almost Kähler	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	quasi Kähler	$\mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	semi Kähler	$\mathcal{W}_4 = \mathcal{W}_5 = 0$
	half-flat	$\mathrm{Im}\mathcal{W}_1 = \mathrm{Im}\mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$

Four-dimensional $\mathcal{N}=1$ Minkowski vacua in type IIA hep-th/0509003

IIA	$a=0$ or $b=0$ (type A)	$a=be^{i\beta}$ (type BC)	
1	$\mathcal{W}_1 = H_3^{(1)} = 0$		
	$F_0^{(1)} = \mp F_2^{(1)} = F_4^{(1)} = \mp F_6^{(1)}$	$F_{2n}^{(1)} = 0$	
8		generic β	$\beta = 0$
	$\mathcal{W}_2 = F_2^{(8)} = F_4^{(8)} = 0$	$\mathrm{Re}\mathcal{W}_2 = \mathrm{e}^{\phi} F_2^{(8)}$ $\mathrm{Im}\mathcal{W}_2 = 0$	$\mathrm{Re}\mathcal{W}_2 = \mathrm{e}^{\phi} F_2^{(8)} + \mathrm{e}^{\phi} F_4^{(8)}$ $\mathrm{Im}\mathcal{W}_2 = 0$
6	$\mathcal{W}_3 = \mp *_6 H_3^{(6)}$	$\mathcal{W}_3 = H_3^{(6)}$	
3	$\overline{\mathcal{W}}_5 = 2\mathcal{W}_4 = \mp 2i H_3^{(3)} = \overline{\partial}\phi$ $\overline{\partial}A = \overline{\partial}a = 0$	$F_2^{(3)} = 2i\overline{\mathcal{W}}_5 = -2i\overline{\partial}A = \frac{2i}{3}\overline{\partial}\phi$ $\mathcal{W}_4 = 0$	

type A	NS-flux only (common to IIA, IIB, heterotic) $\mathcal{W}_1 = \mathcal{W}_2 = 0$, $\mathcal{W}_3 \neq 0$: complex
type BC	RR-flux only $\mathcal{W}_1 = \mathrm{Im}\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$, $\mathrm{Re}\mathcal{W}_2 \neq 0$, $\mathcal{W}_5 \neq 0$: symplectic

For $\mathcal{N}=1$ AdS_4 vacua: hep-th/0403049 hep-th/0407263 hep-th/0412250 hep-th/0502154 hep-th/0609124 , etc..

Four-dimensional $\mathcal{N}=1$ Minkowski vacua in type IIB hep-th/0509003

back to Questions

IIB	$a=0$ or $b=0$ (type A)	$a=\pm ib$ (type B)	$a=\pm b$ (type C)	(type ABC)
1	$\mathcal{W}_1 = F_3^{(1)} = H_3^{(1)} = 0$			
8	$\mathcal{W}_2 = 0$			
6	$F_3^{(6)} = 0$ $\mathcal{W}_3 = \pm * H_3^{(6)}$	$\mathcal{W}_3 = 0$ $\mathrm{e}^{\phi} F_3^{(6)} = \mp * H_3^{(6)}$	$H_3^{(6)} = 0$ $\mathcal{W}_3 = \pm \mathrm{e}^{\phi} * F_3^{(6)}$	$(***)$
3	$\overline{\mathcal{W}}_5 = 2\mathcal{W}_4 = \mp 2iH_3^{(3)} = 2\bar{\partial}\phi$ $\bar{\partial}A = \bar{\partial}a = 0$	$\mathrm{e}^{\phi}F_5^{(3)} = \frac{2i}{3}\overline{\mathcal{W}}_5 = i\mathcal{W}_4$ $= -2i\bar{\partial}A = -4i\bar{\partial}\log a$ $\bar{\partial}\phi = 0$ F: $\mathrm{e}^{\phi}F_1^{(3)} = 2\mathrm{e}^{\phi}F_5^{(3)}$ $= i\overline{\mathcal{W}}_5 = i\mathcal{W}_4 = i\bar{\partial}\phi$	$\mathrm{e}^{\phi}F_3^{(3)} = 2i\overline{\mathcal{W}}_5 = -2i\bar{\partial}A$ $= -4i\bar{\partial}\log a = -i\bar{\partial}\phi$	$(***)$

type A	NS-flux only (common to IIA, IIB, heterotic) $\mathrm{d}J \pm iH_3$ is (2,1)-primitive $\mathcal{W}_1 = \mathcal{W}_2 = 0$: complex
type B	both NS- and RR-flux $G_3 = F_3 - i\mathrm{e}^{-\phi}H_3 = -i *_6 G_3$ is (2,1)-primitive $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$, $2\mathcal{W}_5 = 3\mathcal{W}_4 = -6\bar{\partial}A$: conformally CY
type C	RR-flux only (S-dual of type A) $\mathrm{d}(\mathrm{e}^{-\phi}J) \pm iF_3$ is (2,1)-primitive $\mathcal{W}_1 = \mathcal{W}_2 = 0$: complex
type ABC	(skip detail...)

- on a single $SU(3)$:

a real two-form	$J_{ij} = \mp 2i\, \eta_\pm^\dagger \gamma_{ij}\, \eta_\pm$
a complex three-form	$\Omega_{ijk} = -2i\, \eta_-^\dagger \gamma_{ijk}\, \eta_+$

- on a pair of $SU(3)$:

two real vectors	$(v - iv')^i = \eta_+^{1\dagger} \gamma^i\, \eta_-^2$
(J^A, Ω^A)	$J^1 = j + v \wedge v' \quad \Omega^1 = w \wedge (v + iv')$ $J^2 = j - v \wedge v' \quad \Omega^2 = w \wedge (v - iv')$ (j, w): local $SU(2)$-invariant forms

If $\eta_+^1 \neq \eta_+^2$ globally, a pair of $SU(3)$ is reduced to global single $SU(2)$ w/ (j, w, v, v')

If $\eta_+^1 = \eta_+^2$ globally, a pair of $SU(3)$ is reduced to a single global $SU(3)$ w/ (J, Ω)

$$\eta_+^2 \;=\; c_\parallel \eta_+^1 + c_\perp (v + iv')^i\, \gamma_i\, \eta_-^1 \qquad |c_\parallel|^2 + |c_\perp|^2 = 1$$

cf.) a pair of $SU(3)$ on $T\mathcal{M}_6 \;\sim\;$ an $SU(3) \times SU(3)$ on $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$

back to pure spinors

A configuration of six-torus T^6 in the presence of H-flux in five-brane solution:

$$\to \begin{cases} \mathrm{d}s^2 = \boxed{\mathrm{d}s^2_{\mathbb{R}^{1,2}} + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}y^3)^2} + V\Big\{\mathrm{d}\xi^2 + (\mathrm{d}x^3)^2 + (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2\Big\} \\ H_3 = *_4 \mathrm{d}V = \lambda\, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}x^3 \\ \mathrm{e}^{2\phi} = V = \lambda \xi \end{cases}$$

A configuration of six-torus T^6 in the presence of H-flux in five-brane solution:

$$\to \begin{cases} \mathrm{d}s^2 = \boxed{\mathrm{d}s^2_{\mathbb{R}^{1,2}} + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}y^3)^2} + V\Big\{\mathrm{d}\xi^2 + (\mathrm{d}x^3)^2 + (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2\Big\} \\ H_3 = *_4 \mathrm{d}V = \lambda\, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}x^3 \\ \mathrm{e}^{2\phi} = V = \lambda\xi \end{cases}$$

Perform T-duality along all x^i-directions with respect to the Buscher's rule:

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\mathbb{R}^{1,2}} + (\mathrm{d}\widetilde{x}^1)^2 + (\mathrm{d}\widetilde{x}^2)^2 + (\mathrm{d}y^3)^2 + V^{-1}(\mathrm{d}\widetilde{x}^3 - \lambda y^1 \mathrm{d}y^2)^2 + V\Big\{\mathrm{d}\xi^2 + (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2\Big\}$$

$$\widetilde{H}_3 = 0 \qquad \mathrm{e}^{2\widetilde{\phi}} = 1$$

A configuration of six-torus T^6 in the presence of H-flux in five-brane solution:

$$\rightarrow \begin{cases} \mathrm{d}s^2 = \boxed{\mathrm{d}s^2_{\mathbb{R}^{1,2}} + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}y^3)^2} + V\Big\{\mathrm{d}\xi^2 + (\mathrm{d}x^3)^2 + (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2\Big\} \\ H_3 = *_4 \mathrm{d}V = \lambda\, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}x^3 \\ \mathrm{e}^{2\phi} = V = \lambda\xi \end{cases}$$

Perform T-duality along all x^i-directions with respect to the Buscher's rule:

$$\boxed{\begin{gathered} \mathrm{d}s^2 = \mathrm{d}s^2_{\mathbb{R}^{1,2}} + (\mathrm{d}\widetilde{x}^1)^2 + (\mathrm{d}\widetilde{x}^2)^2 + (\mathrm{d}y^3)^2 + V^{-1}(\mathrm{d}\widetilde{x}^3 - \lambda y^1 \mathrm{d}y^2)^2 + V\Big\{\mathrm{d}\xi^2 + (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2\Big\} \\ \widetilde{H}_3 = 0 \qquad \mathrm{e}^{2\widetilde{\phi}} = 1 \end{gathered}}$$

Choose $\quad e^1 = \mathrm{d}\widetilde{x}^1 + i\sqrt{V}\mathrm{d}y^1 \qquad e^2 = \mathrm{d}\widetilde{x}^2 + i\sqrt{V}\mathrm{d}y^2 \qquad e^3 = \frac{1}{\sqrt{V}}(\mathrm{d}\widetilde{x}^3 - \lambda y^1 \mathrm{d}y^2) + i\mathrm{d}y^3$

Two- and three-forms: $J = -i\delta_{m\overline{n}}\, e^m \wedge \overline{e}^{\overline{n}}$ and $\Omega \equiv e^1 \wedge e^2 \wedge e^3$ with

$$\mathrm{d}J = -\frac{2\lambda}{\sqrt{V}}\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 \neq 0 \quad \text{and} \quad J \wedge \mathrm{d}J = 0$$

$$\mathrm{d}\Omega = -\frac{\lambda}{\sqrt{V}}\mathrm{d}\widetilde{x}^1 \wedge \mathrm{d}\widetilde{x}^2 \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2 \qquad \text{i.e.,} \quad \mathrm{Re}\,\mathrm{d}\Omega \neq 0 \;\; \text{and} \;\; \mathrm{Im}\,\mathrm{d}\Omega = 0$$

This is a (torsionful) half-flat manifold and Entrance Gate to doubled formalism