KEK-TH-941, OU-HET 466, hep-th/0402054
since: October 16th, 2003
submitted on February, 2004
last modified: April 25, 2006

Zero-mode Spectrum of Eleven-dimensional Theory on the Plane-wave Background

Tetsuji Kimura

Theory Division, Institute of Particle and Nuclear Studies,
High Energy Accelerator Research Organization (KEK)
Tsukuba, Ibaraki 305-0801, Japan

tetsuji@post.kek.jp

and

Department of Physics, Graduate School of Science, Osaka University
Toyonaka, Osaka 560-0043, Japan

t-kimura@het.phys.sci.osaka-u.ac.jp

A Dissertation Presented to the Faculty of Osaka University
in Candidacy for the Degree of Doctor of Science

Recommended for Acceptance by the Department of Physics

This doctoral thesis is dedicated to my family.

Acknowledgements

First I would like to thank my advisor, Professor Kiyoshi Higashijima, for introducing me to various wonderful topics and problems in quantum field theory, and for encouraging me even when I left Osaka University and when I studied in the Theory Division, Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK) for the last one year of my graduate studies. I thank him for everything.

I would like to thank Professor Satoshi Iso for his warm hospitality and encouraging me during my stay in KEK. I would like to also thank Dr. Kentaroh Yoshida, Dr. Machiko Hatsuda, Dr. Makoto Sakaguchi and Dr. Yasuhiro Sekino for advising me on various topics around supergravities and M-theory.

I would like to express my gratitude to Professor Hiroshi Itoyama, Professor Takahiro Kubota, Professor Nobuyoshi Ohta, Professor Norisuke Sakai, Professor Asato Tsuchiya, Dr. Kazuyuki Furuuchi, Dr. Koichi Murakami, Dr. Muneto Nitta, Dr. Dan Tomino, Dr. Hiroshi Umetsu and Dr. Naoto Yokoi for introducing and discussing various topics and problems around string theories and field theories. I also thank Naoyuki Kawahara, Toshiharu Maeda, Hidenori Terachi and Yastoshi Takayama for discussing in the superstring theory seminar in KEK.

I am grateful for Professor Tohru Eguchi, Professor Hiroaki Kanno, Professor Toshio Nakatsu, Professor Yuji Sugawara, Professor Yukinori Yasui, Dr. Hiroyuki Fuji, Dr. Takahiro Masuda, Dr. Takao Suyama, Dr. Takashi Yokono, Tomoyuki Fujita, Yutaka Ookouchi, Kazuhiro Sakai and Kensuke Tsuda to introduce a lot of interesting topics among geometry and topological strings. Some unpublished studies are inspired by discussions and warm advice.

Thanks all members of high energy theory group in Osaka University and all members of theory division in KEK for useful discussions, comments and encouragement.

Finally, I would like to thank all of my friends and my family for their encouragement and powerful support.

This thesis is supported in part by the Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists (No.15-03926).

Abstract

In this doctoral thesis we study zero-mode spectra of Matrix theory and eleven-dimensional supergravity on the plane-wave background. This background is obtained via the Penrose limit of $AdS_4 \times S^7$ and $AdS_7 \times S^4$. The plane-wave background is a maximally supersymmetric spacetime supported by non-vanishing constant four-form flux in eleven-dimensional spacetime. First, we discuss the Matrix theory on the plane-wave background suggested by Berenstein, Maldacena and Nastase. We construct the Hamiltonian, 32 supercharges and their commutation relations. We discuss a spectrum of one specific supermultiplet which represents the center of mass degrees of freedom of N D0-branes. This supermultiplet would also represent a superparticle of the eleven-dimensional supergravity in the large-N limit. Second, we study the linearized supergravity on the plane-wave background in eleven dimensions. Fixing the bosonic and fermionic fields in the light-cone gauge, we obtain the spectrum of physical modes. We obtain the fact that the energies of the states in Matrix theory completely correspond to those of fields in supergravity. Thus, we find that the Matrix theory on the plane-wave background contains the zero-mode spectrum of the eleven-dimensional supergravity completely. Through this result, we can argue the Matrix theory on the plane-wave as a candidate of quantum extension of eleven-dimensional supergravity, or as a candidate which describes M-theory.

Contents

Eleven-dimensional supergravity remains an enigma. It is hard to believe that its existence is just an accident, but it is difficult at the present time to state a compelling conjecture for what its role may be in the scheme of things.

— M.B. Green, J.H. Schwarz and E. Witten, "*Superstring Theory*".

Chapter I

Introduction

Supergravity

Eleven-dimensional (Lorentzian) spacetime is the maximal spacetime in which one can formulate a consistent supersymmetric multiplet including fields with spin less than two[1]. Nahm first recognized this fact in his classification and representation of supersymmetry algebra [108]. Not so long after this understanding, Cremmer, Julia and Scherk realized that supergravity not only permits up to seven extra dimensions from four dimensions but in fact takes its simplest and most elegant form [30]. The unique supergravity in eleven-dimensional spacetime contains a graviton g_{MN}, a gravitino Ψ_M and a three-form gauge field C_{MNP} with 44, 128 and 84 on-shell degrees of freedom, respectively. The theory was regarded not only as a candidate for the fundamental theory including quantum gravity but also as a mathematically important tool to derive a four-dimensional supergravity with extended supersymmetries via *dimensional reduction*. The research interests in those days were to find a (supersymmetric) grand unified theory which gives gauge groups greater than $SU(3) \times SU(2) \times U(1)$, and to analyze the hidden symmetries of extended supergravities in four dimensions [29, 46]. In this context, eleven-dimensional supergravities on some non-trivially curved spacetimes (in particular, the product space of four-dimensional anti-de Sitter spaces AdS_4 and seven-dimensional Einstein spaces such as round or squashed S^7, or the product space of seven-dimensional anti-de Sitter space AdS_7 and four-dimensional Einstein space) were also investigated via *Kaluza-Klein mechanism* [145, 59, 53, 57, 112]. Now we can read a lot of important works of supergravities in diverse dimensions in the book edited by Salam and Sezgin [122]. We can also study the review of supergravity from the reports written by van Nieuwenhuizen [143] and by Duff, Nilsson and Pope [58].

Although the eleven-dimensional supergravity is intrinsically important theory as we introduced above, this theory has some serious problems as the fundamental *field* theory: In eleven dimensions, we cannot impose Weyl condition on the $SO(10,1)$ Dirac spinor because of odd-dimensional spacetime. So we cannot make four-dimensional *chiral* field theory via Kaluza-Klein mechanism, i.e., via the *smooth* compactifications of eleven-dimensional spacetime [145][2]. Moreover, the eleven-dimensional supergravity is non-renormalizable in perturbation. Although ten-dimensional supergravities are also

[1] If the spacetime metric has two negative signatures, one could formally construct the supersymmetric theory in *twelve*-dimensions in which the supermultiplet would contain the fields "spin" less than two. This is because the Majorana-Weyl spinor with 32 real degrees of freedom is the irreducible representation of spinors in such "spacetime". As you know the "F-theory" will be formulated in such twelve dimensions [142, 125], but this theory may not have a field theory realization. Bars has been studying the *two-time physics* in order to understand the field theory in such a specific spacetime [15, 16, 13, 14].

[2] But, performing an *orbifold* compactification one can obtain supersymmetric chiral field theories in four-dimensional spacetime [84, 1, 4].

non-renormalizable, they had barely survived because ten-dimensional supergravities could be regarded as the low energy effective theory of ten-dimensional *superstrings*, which are *renormalizable* as perturbation theories. As you know Salam also stated below in the introduction of the proceedings of the Trieste Spring School 1986 [42]:

> *"Supergravity is dead. Long live supergravity in the context of superstrings". This seemed to be the motto of the Fourth Spring School on Supergravity and Supersymmetry which was held at the International Centre for Theoretical Physics at Trieste between 7 – 15 April 1986.*

Through the above recognition, the eleven-dimensional supergravity was abandoned in the middle eighties.

Super p-branes

Theories of supersymmetric extended objects in diverse dimensions are mysterious. In the early eighties, Green and Schwarz constructed supersymmetric one-dimensional extended objects (called the "Green-Schwarz (GS) superstrings") in ten-dimensional spacetime [70]. Moreover it was shown that the GS superstrings also live *classically* in $D = 3, 4$ and 6 dimensions. In the case of spatially two-dimensional objects (the membranes), Bergshoeff, Sezgin and Townsend showed that the supermembrane can classically propagate in $D = 4, 5, 7$ and 11 dimensions [21, 22]. Thus people wondered which p-branes can exist in D-dimensional spacetime (p denotes the spatial dimensions of extended objects). A simple way to understand this question is to consider the numbers of boson and fermion degrees of freedom *on the d-dimensional worldvolume* of extended objects ($d = p + 1$) [2]. If the numbers of boson and fermion degrees of freedom are equal, we can classically discuss the p-brane in D-dimensional spacetime. Here let us explain the way of counting of the numbers of boson and fermion degrees of freedom in the Green-Schwarz type theory [54]. As a p-brane moves through D-dimensional spacetime, its trajectory is described by the functions $X^M(\sigma^i)$, where X^M represent not only the spacetime coordinates but also the scalar functions on the worldvolume ($M = 0, 1, \cdots, D-1$), and σ^i denote the d-dimensional worldvolume coordinates ($i = 0, 1, \cdots, d-1$). Choosing the static gauge $X^\mu(\sigma) = \sigma^\mu$ ($\mu = 0, 1, \cdots, d-1$), we find that the number of on-shell bosonic degrees of freedom is

$$N_{\mathrm{B}}^{\mathrm{scalar}} \ = \ D - d \, . \tag{I.1}$$

In order to describe the super p-brane we should count the number of fermionic degrees of freedom on the worldvolume. Let us introduce anticommuting fermionic coordinates $\theta^\alpha(\sigma)$ in the D-dimensional *spacetime*. We can impose the κ-symmetry on the fermionic coordinates, which implies that half of the fermionic degrees of freedom are redundant and may be gauged away from the physical degrees of freedom. The net result is that the theory exhibits a d-dimensional *worldvolume* supersymmetry whose number of fermionic generators is half of the generators in the original *spacetime* supersymmetry. Let M be the minimal number of real components of the minimal spinor and N be the number of supersymmetry of D-dimensional spacetime, and let m and n be the corresponding quantities in d-dimensional worldvolume (see Table I.1).

dimension (D or d)	irreducible spinor	minimal number (M or m)	supersymmetry (N or n)
2	Majorana-Weyl	1	$1, 2, \cdots, 32$
3	Majorana	2	$1, 2, \cdots, 16$
4	Majorana or Weyl	4	$1, 2, \cdots, 8$
5	Dirac	8	$1, 2, 3, 4$
6	Weyl	8	$1, 2, 3, 4$
7	Dirac	16	$1, 2$
8	Majorana or Weyl	16	$1, 2$
9	Majorana	16	$1, 2$
10	Majorana-Weyl	16	$1, 2$
11	Majorana	32	1

Table I.1: *The minimal number of fermion in D-dimensional (Lorentzian) spacetime and d-dimensional (Lorentzian) worldvolume. We also describe the number of supersymmetry.*

Since the κ-symmetry always halves the number of fermionic degrees of freedom and on-shell condition also halves it again, we can write the number of on-shell fermionic degrees of freedom as

$$N_{\mathrm{F}} \;=\; \frac{1}{2}mn \;=\; \frac{1}{4}MN \, . \tag{I.2}$$

Worldvolume supersymmetry demands $N_{\mathrm{B}}^{\mathrm{scalar}} = N_{\mathrm{F}}$, hence

$$D - d \;=\; \frac{1}{2}mn \;=\; \frac{1}{4}MN \, . \tag{I.3}$$

Notice that this relation is satisfied except for the superstring $d = 2$, in which left- and right-moving modes should be treated independently. In the case of the superstring, the following relation is obeyed:

$$D - 2 \;=\; n \;=\; \frac{1}{2}MN \, . \tag{I.4}$$

On the worldvolume, bosons and fermions subject to (I.3) or (I.4) belong to a scalar supermultiplet of the worldvolume supersymmetry. The solutions of scalar multiplets are categorized into four compositions via division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ [128, 2, 62]; for example, the GS superstrings in $D = 3, 4, 6$ and 10 dimensions belong to the $\mathcal{R}$-, $\mathcal{C}$-, $\mathcal{H}$- and $\mathcal{O}$-sequence, respectively [22].

We can consider other possibilities on the worldvolume supersymmetry. If vectors also live on the worldvolume, the number of the on-shell bosonic degrees of freedom $N_{\rm B}^{\rm vector}$ is

$$N_{\rm B}^{\rm vector} \ = \ D - d + (d-2) \ = \ D - 2 \, . \tag{I.5}$$

Thus the matching condition (I.3) replaces

$$D - 2 \ = \ \frac{1}{2} mn \ = \ \frac{1}{4} MN \, . \tag{I.6}$$

In this case there lives a supersymmetric vector multiplet on the worldvolume. The case of existence of an antisymmetric tensor field is also considerable. We summarize the results of the possibilities of super p-branes in various spacetime dimensions in Table I.2, which is called the *Brane Scan* [54].

$D \uparrow$											
11			**S**			**T**					
10	**V**	**S**/**V**	**V**	**V**	**V**	**S**/**V**	**V**	**V**	**V**	**V**	
9	**S**				**S**						
8				**S**							
7			**S**			**T**					
6	**V**	**S**/**V**	**V**	**S**/**V**	**V**	**V**					
5	**S**		**S**								
4	**V**	**S**/**V**	**S**/**V**	**V**							
3	**S**/**V**	**S**/**V**	**V**								
2	**S**										
	1	2	3	4	5	6	7	8	9	10	11 $d \rightarrow$

Table I.2: *The brane scan, where the spacetime dimensions D are plotted vertically and the worldvolume dimensions d of p-branes ($d = p + 1$) are plotted horizontally. Note that* **S**, **V** *and* **T** *denote scalar, vector and antisymmetric tensor multiplets. The colored symbols of scalar multiplets such as* **S**, **S**, **S** *and* **S** *represent the solutions of $\mathcal{R}$-, $\mathcal{C}$-, $\mathcal{H}$- and $\mathcal{O}$-sequences, respectively.*

Most of the super p-branes in Table I.2 are interpreted as solitons rather than fundamental extended objects. Here we use the word *solitons* to mean any such *non-singular* lumps of field energy which solve the (supergravity) field equations, which have finite mass per unit p-volume and which are prevented from dissipating by some topological conservation law. We can understand that only the super p-branes in the $\mathcal{O}$-sequences are *fundamental* objects, which are described by singular configurations with δ-function sources at the spacetime locations of p-branes. Moreover we know that only the super p-branes in the $\mathcal{O}$-sequences are quantum consistent objects, which do not have Lorentz anomalies in the light-cone gauge [12, 17]. The other super p-branes can be regarded as the solitons, for example, super p-branes of vector multiplets in ten dimensions are interpreted as Dirichlet p-branes (Dp-branes), which carry the Ramond-Ramond charges and which are solitonic non-perturbative objects in type IIA/IIB string theories, etc [119].

Supermembrane

In eleven-dimensional spacetime, there exists a supergraviton (point particle) [30], a supermembrane ($p = 2$ in the $\mathcal{O}$-sequence) as a fundamental object [60], and a super fivebrane as a solitonic, dual object of the supermembrane [78]. The supermembrane couples to a three-form gauge field C_3 electrically via

$$\int C_3$$

and the fivebrane couples to C_3 magnetically. Supergraviton, supermembrane and super fivebrane appear in the eleven-dimensional supersymmetry algebra [137]. The anticommutator of two supersymmetry generators Q_α is schematically given by

$$\{Q_\alpha, Q_\beta\} \;=\; (C\widehat{\Gamma}_M)_{\alpha\beta} P^M + (C\widehat{\Gamma}_{MN})_{\alpha\beta} Z^{MN} + (C\widehat{\Gamma}_{MNPQR})_{\alpha\beta} Z^{MNPQR} \;,$$

where $\widehat{\Gamma}_M$ is a Dirac gamma matrix in eleven-dimensional spacetime and C is a charge conjugation matrix; $\widehat{\Gamma}_{M_1 M_2 \cdots M_n}$ are antisymmetrized products of Dirac gamma matrices. We see that the right hand side involves not only the momentum P^M of the superparticle but also the two-form central charge Z^{MN} and five-form central charge Z^{MNPQR}, which are charges of supermembrane and super fivebrane, respectively.

Here we introduce a short review of the supermembrane [21, 22, 138, 41]. The supermembrane action is defined by the Green-Schwarz type Lagrangian $\mathcal{L}_0$ and Wess-Zumino term $\mathcal{L}_{\text{WZ}}$ as

$$\mathcal{L} \;=\; \mathcal{L}_0 + \mathcal{L}_{\text{WZ}} \;, \tag{I.7a}$$

$$\mathcal{L}_0 \;=\; -\sqrt{-g(Z)}\;, \qquad \mathcal{L}_{\rm WZ} \;=\; \frac{1}{6}\epsilon^{ijk}\,\Pi_i^{\underline{A}}\,\Pi_j^{\underline{B}}\,\Pi_k^{\underline{C}}\,C_{\underline{ABC}}(Z)\;, \tag{I.7b}$$

where $Z^{\underline{M}}(\sigma) = \{X^M(\sigma), \theta^\alpha(\sigma)\}$ are eleven-dimensional superspace embedding coordinates (θ is a fermionic coordinate denoted by $SO(10,1)$ Majorana spinor) and σ^i ($i = 0,1,2$) are worldvolume coordinates; $\Pi_i^{\underline{A}} = \partial Z^{\underline{M}}/\partial\sigma^i \widehat{E}_{\underline{M}}{}^{\underline{A}}$ are pullbacks of the superspace coordinates to the membrane worldvolume coordinates and $C_{\underline{ABC}}$ denotes the three-form superfield[3]. Note that $g(Z)$ is a determinant of the worldvolume metric, and this is represented by the spacetime background metric g_{MN} as

$$g(Z(\sigma)) \;=\; \det\{\Pi_i^A\,\Pi_j^B\,\eta_{AB}\}\;, \qquad g_{MN} \;=\; \eta_{AB}\,e_M{}^A\,e_N{}^B\;.$$

Note that $\widehat{E}_{\underline{M}}{}^{\underline{A}}$ is a supervielbein. In the flat superspace case, the supervielbein and a three-form superfield $C_{\underline{ABC}}$ are given by

$$\begin{aligned}
\widehat{E}_M{}^A &= \delta_M^A\;, & E_M{}^a &= 0\;,\\
E_\alpha{}^a &= \delta_\alpha^a\;, & E_\alpha{}^A &= -(\overline{\theta}\widehat{\Gamma}^A)_\alpha\;,\\
C_{MN\alpha} &= (\overline{\theta}\widehat{\Gamma}_{MN})_\alpha\;, & C_{M\alpha\beta} &= (\overline{\theta}\widehat{\Gamma}_{MN})_{(\alpha}(\overline{\theta}\widehat{\Gamma}^N)_{\beta)}\;,\\
C_{\alpha\beta\gamma} &= (\overline{\theta}\widehat{\Gamma}_{MN})_{(\alpha}(\overline{\theta}\widehat{\Gamma}^M)_\beta(\overline{\theta}\widehat{\Gamma}^N)_{\gamma)}\;, & C_{MNP} &= 0\;.
\end{aligned}$$

On general curved background [20], the supervielbeins and three-form gauge field become so complicated that we have only a few solutions of curved spaces such as $AdS_4\times S^7$, $AdS_7\times S^4$ and their continuously deformed ones.

As in the case of Green-Schwarz superstring, the supermembrane action also has a reparametrization invariance and fermionic κ-symmetry invariance. In order to fix these local gauge symmetries we can take the light-cone gauge

$$X^+(\tau) \;=\; \tau\;, \qquad \widehat{\Gamma}^+\theta \;=\; 0\;.$$

Although we fix the above gauge symmetries in the supermembrane action, there is a residual gauge symmetry such as diffeomorphism on the membrane surface. Thus we rewrite the supermembrane action (I.7) in the flat spacetime background as a gauge theory action [44]:

$$w^{-1}\mathcal{L} \;=\; \frac{1}{2}D_\tau X^I D_\tau X^I + \frac{i}{2}\Psi^\dagger D_\tau\Psi - \frac{1}{4}\{X^I, X^J\}^2 + \frac{i}{2}\Psi^\dagger\gamma^I\{X^I,\Psi\}\;, \tag{I.8}$$

where Ψ is an $SO(9)$ Majorana spinor satisfying the reality condition $\Psi^\dagger = \Psi^T$ and γ^I are $SO(9)$ Dirac's gamma matrix[4] with (flat) spacetime indices $I = 1,2,\cdots,9$; the bracket $\{*,*\}$ is the Lie

[3] The convention about indices as follows. Curved space indices are denoted by $\underline{M} = \{M,\alpha\}$, whereas tangent space indices are $\underline{A} = \{A,a\}$. Here M, A refer to commuting and α, a to anticommuting coordinates.

[4] Definitions are described in section A.4.

bracket defined in terms of an arbitrary function $w(\sigma^r)$ of worldvolume spatial coordinates σ^r $(r = 1, 2)$ as

$$\{A, B\} \;=\; \frac{1}{w}\epsilon^{rs}\partial_r A\,\partial_s B \;,$$

with $\partial_r = \partial/\partial\sigma^r$ and $\epsilon^{12} = 1$. This system has, as mentioned above, a residual gauge symmetry called the "area preserving diffeomorphism" (APD) and we define the covariant derivative of this gauge symmetry as

$$D_\tau X^I \;=\; \partial_\tau X^I - \{\omega, X^I\} \;,$$

where ω is a gauge field of this symmetry. In 1988, de Wit, Hoppe and Nicolai argued that the supermembrane Lagrangian (I.8) might be written down as a supersymmetric quantum mechanical theory in terms of the following "matrix regularization" in order to analyze quantum properties of supermembrane:

$$X^I(\sigma) \;\to\; X^I(\tau) \;, \qquad \Psi(\sigma) \;\to\; \Psi(\tau) \;, \qquad \int \mathrm{d}^2\sigma\, w(\sigma) \;\to\; \mathrm{Tr} \;, \qquad \{A, B\} \;\to\; -i[A, B] \;.$$

Via this matrix regularization procedure, the supermembrane action is written in terms of the $N \times N$ matrix variables X^I and Ψ as

$$L \;=\; \mathrm{Tr}\Big\{\frac{1}{2}D_\tau X^I D_\tau X^I + \frac{i}{2}\Psi^\dagger D_\tau \Psi + \frac{1}{4}[X^I, X^J]^2 + \frac{1}{2}\Psi^\dagger \gamma^I [X^I, \Psi]\Big\} \tag{I.9}$$

with covariant derivative $D_\tau X^I = \partial_\tau X^I + i[\omega, X^I]$.

Type IIA superstring theory emerges via double dimensional reductions of supermembrane theory in the eleven-dimensional spacetime [56]. Moreover, it is believed that all the Dp-branes in type IIA string theory emerge in various reductions from the extended objects such as supermembrane and super fivebrane in the eleven-dimensional theory. Thus one may think that the eleven-dimensional theory is the most fundamental theory including gravity. But, unfortunately, we have not completely understood the supermembrane yet because of a lot of problems: the difficulty of the classification of three dimensional topologies, the interpretation of the Hilbert space [45], the zero mode spectrum of supermembranes [48, 40, 67], etc. In order to go beyond these difficulties, a lot of scientists have been studying by using various methods.

Here let us introduce one of these difficulties; a supermembrane instability problem. When de Wit, Hoppe and Nicolai showed that the regularized supermembrane could be described in terms of supersymmetric quantum mechanics, most people thought that the quantized supermembrane would have a *discrete spectrum* of states. In the case of string theory, the spectrum of states in the Hilbert space of string can be put into one-to-one correspondence with elementary particle states in the spacetime. It is crucial that the massless spectrum contains a "graviton" and that there is a mass gap separating the massive excitations from massless states. However, for the supermembrane theory (and also for the super p-brane theory as $p \geq 2$), the spectrum does not seem to have these important properties. We call this problem the *membrane instability problem*.

This problem is explained simply at the classical level [133]. Consider a supermembrane whose energy is given by the area of the membrane times a constant tension T. Such a membrane can have a lot of long *narrow* spikes at very low cost in energy. If the spike is roughly cylindrical and has a radius r and length L, the energy of this spike is $2\pi rLT$. For a spike with large L but a small $r \ll 1/TL$, the energy cost is very small but the spike is very long. This situation shows that a membrane will tend to have many fluctuations of this type, making it difficult to conceive of the membrane as single object which is well localized in spacetime. Note that the string theory does not have this type of problems because a long spike in a string always has energy *proportional* to the length of the string. In the quantum supermembrane theory the above process can also occur without energy loss because of the existence of flat directions protected by the supersymmetry (the quantum bosonic membrane theory is cured because the flat directions rise via quantum corrections). This phenomenon occurs in any *quantum supersymmetric* p-brane theories ($p \geq 2$). By virtue of this phenomenon, the supermembrane theory has a continuous spectrum and it is very difficult to distinguish the zero-modes from the other excited states [48, 40, 67].

Owing to the above serious problem, the supermembrane theory has not been investigated more than the superstring theories. On the other hand, the superstring theories have been well studied since 1984, the "first string revolution year", in terms of of some keywords such as the "anomaly free", "mass gap", "derivation of GUTs", and so on. Furthermore we have been re-investigating (super)string theories since 1995, the "second revolution year", with the keyword "duality".

Superstrings, Dualities and M-theory

Since the first string revolution year, five superstrings have been studied as perturbatively consistent theories. They are all anomaly free and live in ten-dimensional spacetime. These five theories are introduced in the glossary of the Polchinski's book [120] as:

> Type IIA superstring theory: *a theory of closed oriented superstrings. The right-movers and left-movers transform under separate spacetime supersymmetries, which have opposite*

chiralities.

Type IIB superstring theory: *a theory of closed oriented superstrings. The right-movers and left-movers transform under separate spacetime supersymmetries, which have the same chirality.*

Type I superstring theory: *the theory of open and closed unoriented superstrings, which is consistent only for the gauge group $SO(32)$. The right-movers and left-movers, being related by the open string boundary condition, transform under the same spacetime supersymmetry.*

Heterotic $E_8 \times E_8$ or $SO(32)$ superstring theory: *a string with different constraint algebras acting on the left- and right-moving fields. The case of phenomenological interest has a $(0, 1)$ superconformal constraint algebra, with spacetime supersymmetry acting only on the right-movers and with gauge group $E_8 \times E_8$ or $SO(32)$.*

These superstring theories have ten-dimensional supergravities as the massless excitation modes of superstring theory in the low energy limit, as mentioned by Salam. The field contents of these superstring theories are summarized in Table I.3 and I.4:

	sectors	fields	supersymmetry
type IIA	NS-NS	$g_{MN}(35)$, $B_{MN}(28)$, $\phi(1)$	32
	NS-R	$\Psi_M(56)$, $\psi(8)$	
	R-NS	$\widetilde{\Psi}_M(56)$, $\widetilde{\psi}(8)$	
	R-R	$C_1(8)$, $C_3(56)$	
type IIB	NS-NS	$g_{MN}(35)$, $B_{MN}(28)$, $\phi(1)$	32
	NS-R	$\Psi_M(56)$, $\psi(8)$	
	R-NS	$\Psi_M(56)$, $\psi(8)$	
	R-R	$C_0(1)$, $C_2(28)$, $C_4^+(35)$	

Table I.3: *Field contents of type IIA/IIB superstring theory in ten-dimensional spacetime.*

Note that in all superstring theories there exists the supergravity multiplet which contains graviton g_{MN}, Kalb-Ramond field B_{MN}, dilaton ϕ, gravitino Ψ_M and dilatino ψ. There exist various dimensional Ramond-Ramond fields C_{p+1}, which couple to Dp-branes in type IIA or type IIB string theory. On the other hand, type I and heterotic string theory have gauge supermultiplets containing gauge potential A_M and gaugino λ in the adjoint representations.

In the first five years from 1984, the heterotic $E_8 \times E_8$ string theory was regarded as a candidate of the *theory of everything*, i.e., a candidate of the fundamental grand unified theory. The heterotic

	sectors	fields	supersymmetry
type I	NS-NS	$g_{MN}(35), \qquad \phi(1)$	16
	NS-R	$\Psi_M(56), \qquad \psi(8)$	
	R-NS	(reflections of NS-R)	
	R-R	$B_{MN}(28)$	
	NS	$A_M(8 \times 496)$ of $SO(32)$ gauge group	
	R	$\lambda(8 \times 496)$	
heterotic	boson	$g_{MN}(35), \qquad B_{MN}(28), \qquad \phi(1)$	16
	fermion	$\Psi_M(56), \qquad \psi(8)$	
	gauge boson	$A_M(8 \times 496)$ of $SO(32)$ or $E_8 \times E_8$	
	gauge fermion	$\lambda(8 \times 496)$	

Table I.4: *Field contents of type I/heterotic superstring theory in ten-dimensional spacetime.*

$E_8 \times E_8$ theory has enough large gauge symmetry. Via Calabi-Yau compactification mechanism [27], one could obtain four-dimensional quantum consistent field theory, with E_6 gauge group and four supercharges. Surprisingly, we could also obtain the generation numbers from the geometric data of Calabi-Yau. Since Maldacena have found that the AdS/CFT correspondence in 1997 [101, 3], people have studied some exact solutions for four-dimensional gauge theories via gauge/gravity dualities [73, 95, 94, 115, 114, 86]. In order to find new configurations and new phenomena in superstrings or supergravities, they engineered new (non-)compact manifolds with special holonomies [31, 32, 33, 80, 96, 4, 88, 34, 81, 89, 82]. Unfortunately, however, they found *tremendously many vacua* from such compactifications because we could compactify superstrings in terms of *any* Calabi-Yau manifolds, i.e., because we could not tell that some Calabi-Yau manifolds are more special than others. Thus the string theorists wondered whether string theories might or might not predict any dynamics in four dimensions. But they have studied around superstring theories in order to achieve the *theory of everything*...

By virtue of the sting theorists' inexhaustible studies, one found some important properties among string theories: the above five superstring theories are not distinct theories but they are closely related to one another via Dirichlet branes, which we now regard as the solitonic extended objects and as the sources of Ramond-Ramond fields in string theories, and via perturbative and non-perturbative dualities such as T-duality, S-duality, and so on. These observations leads to the postulate of an underlying fundamental theory, called **M-theory** [68, 85, 146, 119, 83, 123, 139]. This situation is schematically represented by Figure I.1.

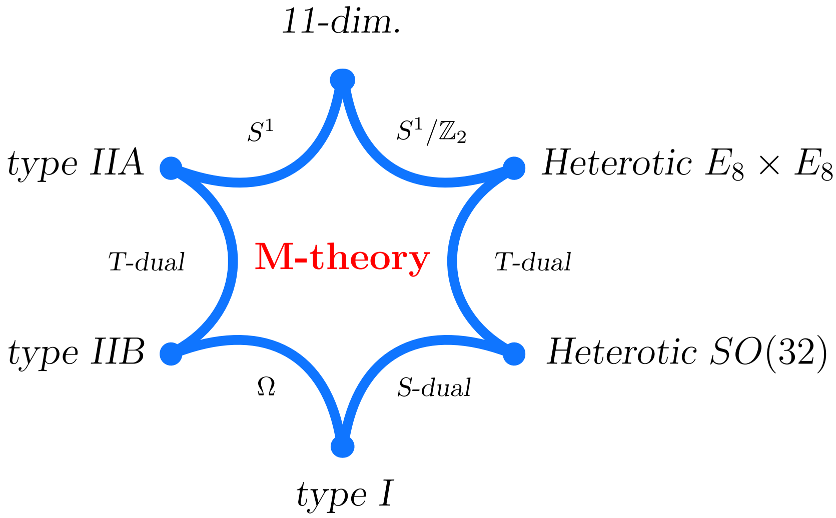

Figure I.1: *The duality web among five superstring theories and eleven-dimensional theory.*

We discuss a very rough explanation for the string duality web described in Figure I.1. First, performing the worldsheet parity projection (Ω projection) and introducing an appropriate orientifold plane in type IIB string theory, we obtain the closed string sector of type I string theory: When we compactify one direction to a circle of radius R and take T-duality to this circle in type IIA (or IIB) string theory, we obtain type IIB (or type IIA) string theory on nine-dimensional spacetime plus one circle of radius α'/R. We also connect heterotic string theory with gauge group $E_8 \times E_8$ to heterotic string with gauge group $SO(32)$ via T-duality: S-duality is a duality under which the coupling constant of a quantum theory changes non-trivially, including the case of strong-weak duality. Via S-duality we can connect heterotic $SO(32)$ string theory to type I string theory. Type IIB string theory is invariant under the S-duality transformation. Performing S-duality to type IIA string, i.e., taking the strong coupling limit of type IIA string, we may reach an unknown eleven-dimensional theory whose low energy effective theory is the eleven-dimensional supergravity: Performing compactification the eleven-dimensional theory on $S^1/\mathbb{Z}_2$, we obtain heterotic $E_8 \times E_8$ string theory: Furthermore, if we compactify some string theory on nontrivial compact manifolds, for instance, $K3$ surface and Calabi-Yau three-fold, we find deeper relations among these string theories.

There is a substantial piece of evidence that eleven-dimensional quantum theory, i.e., M-theory, might underlie type IIA string theory in the strong coupling limit. The first evidence is the existence of the dilaton field in the low energy action (see Table I.3). When an eleven-dimensional gravitational theory is compactified on x^{10}-directions, the component of the metric $g_{10,10}$ behaves as a scalar field in the lower dimensional theory. Furthermore this scalar field enters the lower dimensional action in the same way that the dilaton does. This suggests that the dilaton in type IIA string theory really emerges via local compactification of higher dimensional theory, say, via local compactification of eleven-dimensional theory. The second piece of evidence is the existence of Ramond-Ramond one-form field in type IIA string theory. Massless fields in type IIA string theory also appears via dimensional reduction of eleven-dimensional supergravity. This dimensional reduction keeps only the $p_{10} = 0$ states (where p_{10} denotes the Kaluza-Klein momentum in the compactified direction), but type IIA string theory has also states of $p_{10} \neq 0$ in the form of N *D0-branes* and their bound states. In this situation a D0-brane mass m_0 is given by

$$m_0 \ = \ \frac{1}{R} \ = \ \frac{1}{\mathrm{g}\ell_s} \ , \tag{I.10}$$

where R is the radius of compactified direction, g is the type IIA string coupling and ℓ_s is the string length. The Kaluza-Klein momentum p_{10} is represented as $p_{10} = Nm_0$. Furthermore we know that the D0-branes couple to Ramond-Ramond one-form gauge field in type IIA string theory via $\int C_1$. Thus the D0-brane analysis is much important to understand the mysterious properties of eleven-dimensional theory, the M-theory properties.

Through the above string dualities, a simple and intriguing model was proposed in order to define a microscopic description of M-theory.

Matrix Theory

In 1996, Banks, Fischler, Shenker and Susskind proposed that the degrees of freedom of M-theory in the infinite momentum frame could be described in terms of D0-branes and that all dynamics of M-theory in this frame are described by the system of the low energy effective theory of N D0-branes in the large-N limit [10]. Furthermore, in 1997, Susskind refined the proposal by conjecturing that for all finite N the quantum theory describes the sector of N units of momentum of M-theory with discrete light-cone quantization (DLCQ) [130]. We refer the ideas of Banks, Fischler, Shenker and Susskind and Susskind's refinement to the "BFSS conjecture" and the model described by the matrix variables is called the "Matrix theory" (for the review lectures, see, for instance, [24, 8, 23, 131, 9, 132, 133].).

Matrix theory is defined in the framework of type IIA string theory. In this framework the string coupling is weak and the D0-brane mass becomes infinitely heavy as in (I.10). Thus the Lagrangian of this theory should be described by the *non-relativistic limit* of N D0-brane system. The relativistic effective theory of D-brane system is described by Dirac-Born-Infeld (DBI) action [36, 100]. In the non-relativistic limit this action reduces to (0 + 1)-dimensions of the ten-dimensional $U(N)$ super Yang-Mills theory:

$$L = \frac{(2\pi\alpha')^2}{2\mathrm{g}\ell_s}\mathrm{Tr}\Big\{\dot{X}^I\dot{X}^I + \frac{1}{2}[X^I, X^J]^2 + \Psi^\dagger\big(i\dot{\Psi} + \gamma^I[X^I, \Psi]\big)\Big\}\ ,$$

where we set the gauge potential $A_0 = 0$. The bosonic fields X^I, which have dimensions of $(\mathrm{mass})^1$, and fermionic fields Ψ, the mass dimensions 3/2, are described as $N \times N$ matrix variables. This Lagrangian gives the same Hamiltonian as the one of matrix-regularized supermembrane theory via an appropriate field rescaling[5]!

While the BFSS conjecture is based on a different viewpoint from the matrix-regularized supermembrane theory, the Matrix theory provides us a lot of new interpretations for the supermembrane in M-theory. Here we introduce a few piece of important evidence. One is that the Hilbert space of the matrix quantum mechanics contains multiple particle states. This observation resolves the problem of the continuous spectrum and the membrane instability problem in the supermembrane theory [45]. It is natural to think of the Matrix theory as a *second quantized theory* from the point of view of the target space. Another evidence is the fact that quantum effects in the Matrix theory give rise to long-range interactions between a pair of quanta, i.e., a pair of D0-branes. These interactions have precisely the structure expected from the light-front supergravity. There are lectures around this topic written by Taylor [131, 132, 133].

Although the Matrix theory has been well studied in various works and there are many non-trivial results to check the above arguments, there still exist serious question which have not been understood: *Can we formulate the Matrix theory on curved spacetime backgrounds without any inconsistency?* Well-defined construction of Matrix theory on (arbitrary) curved spacetime background is one of the most interesting and mysterious subjects because we would like to understand whether the Matrix theory is a fundamental description of M-theory through various relations (or correspondences) between matrix model and supermembrane theory. There are a lot of attempts around the Matrix theory on curved background [52, 126, 124, 51]. In particular, Taylor and Van Raamsdonk discussed the Matrix theory on weakly curved spacetime background [134, 135, 136] but it is still difficult to analyze the Matrix theory on curved spaces.

In the end of the last century, one ten-dimensional spacetime background was discovered as a specific limit of the product space of anti-de Sitter space and the Einstein space which is a well-known background in supergravity [79, 64]. This specific spacetime is the "plane-wave background" as the

[5]There is one relation in eleven-dimensional spacetime such as $R = \mathrm{g}^{2/3}\ell_{11}$, where ℓ_{11} is the Planck length of eleven dimensions.

"Penrose limit" of the $AdS_5 \times S^5$ spacetime which appears in the near horizon limit of D3-brane in type IIB theory. This plane-wave background is so useful that the study on the "AdS/CFT correspondence" has been developed rapidly [19].

There is also such a specific spacetime in eleven dimensions. This eleven-dimensional spacetime background was first discovered by Kowalski-Glikman [97, 28] and was obtained as the Penrose limit of $AdS_4 \times S^7$ or $AdS_7 \times S^4$ backgrounds which appear in the near horizon limit of M2-brane or M5-brane, respectively [64]. This eleven-dimensional plane-wave background is also useful to analyze Matrix theory on non-trivially curved background. Although there is no tunable parameter in the flat background, we can introduce one tunable mass parameter μ from the constant four-form flux on the plane-wave. Thus we can perform a Matrix perturbation theory for M-theory on such a specific background!

In this doctoral thesis, we will investigate a zero-mode spectrum included in Matrix theory on the plane-wave background and will compare this to the massless spectrum in the eleven-dimensional supergravity on the same background. This task should be an intrinsic work for Matrix theory on curved background because Matrix theory on curved spacetime must also include the superparticle subject to the eleven-dimensional supergravity as in the case of flat spacetime background.

Organization

The subjects of the doctoral thesis are organized as follows:

In **section II** we will review the Matrix theory on the plane-wave background proposed by Berenstein, Maldacena and Nastase. Introducing the construction procedure of this matrix model, we will construct the Hamiltonian and the supercharges of 32 local supersymmetry on the plane-wave. There we will discuss only the $U(1)$ part of the system, i.e., the center of mass degrees of freedom of N D0-branes which corresponds to the superparticles. We will construct the supermultiplet including the ground state and will read the energy spectrum of this multiplet.

In **chapter III** we will analyze the (linearized) supergravity on the same background in eleven dimensions. We will define the light-cone Hamiltonian in terms of the differential operators and argue the Klein-Gordon type field equations. Making bosonic and fermionic fields fluctuate we will obtain the field equations for these fluctuation fields. Since it is difficult to read the correct energies of them, we should combine them in appropriate re-definitions. After these analyses we will obtain the zero-point energy spectrum of these fluctuation and we will compare them with the result obtained in chapter II.

We devote **chapter IV** to the conclusion and discussions for future problems. We will discuss only the superparticles in both Matrix theory and supergravity. In this chapter we will argue the possibilities to study some properties derived from extended objects such as M2-brane and M5-brane in M-theory.

In **appendix A** we will discuss the notation and convention for some variables in the main chapters. In particular we will write down the definitions of Dirac gamma matrices and Majorana spinors in eleven-dimensional Minkowski spacetime. The gamma matrices and spinors in $SO(9)$ Euclidean space and their $SU(4) \times SU(2)$ decomposition rules are also introduced.

In **appendix B** we will discuss the dimensional reduction procedure of ten-dimensional super Yang-Mills theory. The nonabelian D-branes' effective action with non-vanishing background fields will be also discussed. Furthermore we will write down the eleven-dimensional supergravity Lagrangian.

In **appendix C** we will mention the Penrose limit of eleven-dimensional product spaces such as $AdS_4 \times S^7$ and $AdS_7 \times S^4$. We will also argue the geometrical properties of the plane-wave spacetime and its coset construction via the Penrose limit of $AdS_{4(7)} \times S^{7(4)}$ spacetimes.

Chapter II

Matrix Theory on the Plane-wave

On 2002, Berenstein, Maldacena and Nastase proposed the Lagrangian of the DLCQ of Matrix theory on the plane-wave background in a similar way of constructing type IIB superstring Lagrangian on the ten-dimensional plane-wave background [19]. This model is very useful to understand the properties of matrix model on some specific curved spacetime and is now called the "BMN matrix model". Not long after that, Dasgupta, Sheikh-Jabbari and Van Raamsdonk found that the light-cone Hamiltonian of supermembrane on the plane-wave background exactly corresponds to that of BMN matrix model via matrix regularization [38]. Furthermore Sugiyama and Yoshida explained the supersymmetric quantum mechanics of supermembrane theory on the plane-wave in the same way as the quantum mechanics of supermembrane on flat background discussed by de Wit, Hoppe and Nicolai [44, 129]. They started the discussion from the supermembrane Lagrangian as a gauge theory of area preserving diffeomorphism and construct the light-cone Hamiltonian, 32 supercharges, their commutation relations, brane charges and their matrix regularizations. Their results are consistent with the BMN matrix model.

In this chapter we discuss the spectrum of the center of mass degrees of freedom in the BMN matrix model. We describe the Hamiltonian and supercharges in the $N \times N$ matrix representations and study their commutation relations. We also define the ground state of this system and construct the supermultiplet of the $U(1)$ free part of the matrix model in terms of the oscillator method as discussed by Dasgupta, Sheikh-Jabbari and Van Raamsdonk [38, 39], Kim and Plefka [91], Kim and Park [90], and Nakayama, Sugiyama and Yoshida [109].

II.1 Derivation of Lagrangian

In this section we construct the Lagrangian of the discrete light-cone quantization (DLCQ) of Matrix theory on the plane-wave background which was suggested by Berenstein, Maldacena and Nastase [19]. Let us first consider the action for single D0-brane on the plane-wave and next expand this action to the non-abelian matrix model via various techniques.

The single D0-brane action would be described as the superparticle action moving in the eleven-dimensional plane-wave background in the Green-Schwarz formalism, where we use superspace coordinates and supervielbeins of spacetime background. Here we write the superparticle action

$$S \;=\; \int \mathrm{d}t\, e^{-1}(t) \Big\{ \frac{1}{2} \eta_{AB}\, \Pi_t^A\, \Pi_t^B \Big\} \;=\; \int \mathrm{d}t \Big\{ -\Pi_t^+\, \Pi_t^- + \frac{1}{2} \Pi_t^I\, \Pi_t^I \Big\} \,. \tag{II.1.1}$$

Note that $\Pi_t^A = \partial_t Z^{\underline{M}} E_{\underline{M}}{}^A$ are pullbacks from the eleven-dimensional curved spacetime[1] spanned by

[1]The index I runs from 1 to 9 in the tangent space.

the superspace coordinates $Z^{\underline{M}} = (X^M, \theta^\alpha)$ to the worldline coordinate t, and the supervielbeins are denoted by $E_{\underline{M}}{}^A$; the einbein of the worldline "metric" is denoted by $e(t)$ and we can choose $e(t) = 1$ because of the existence of diffeomorphism of one-dimensional worldline. As discussed in appendix C.1, the plane-wave background is the Penrose limit of the $AdS_{4(7)} \times S^{7(4)}$ spacetime. Thus we can describe the supervielbein on the plane-wave as the Penrose limit of the $AdS_{4(7)} \times S^{7(4)}$ supervielbein and we obtain them by substituting the geometrical variables of the plane-wave (C.1.3) into the supervielbein on the $AdS_{4(7)} \times S^{7(4)}$ background (C.2.11).

The superparticle action (II.1.1) has a fermionic gauge symmetry called the κ-symmetry which the Green-Schwarz superstring action also has. This κ-symmetry should be gauge-fixed by choosing the fermionic light-cone gauge (This procedure is adopted when we obtain the superstring in $AdS_5 \times S^5$ and its Penrose limit [104]). Here we can choose the following fermionic gauge-fixing

$$\widehat{\Gamma}^{+}\theta \ = \ 0 \tag{II.1.2}$$

which is equivalent to the condition $\overline{\theta}\widehat{\Gamma}^{+} = 0$. Under this condition the fermionic matrix $\mathcal{M}^2$ in the supervielbein (C.2.11) vanishes and we can simply write the components of supervielbein and the pullback

$$\begin{aligned}
\Pi^{+} \ &= \ \mathrm{d}X^{+} \, , \qquad \Pi^{I} \ = \ \mathrm{d}X^{I} \, , \\
\Pi^{-} \ &= \ \mathrm{d}X^{-} - \frac{1}{2} G_{++}\, \mathrm{d}X^{+} + \overline{\theta}\widehat{\Gamma}^{-}\mathrm{d}\theta - \frac{\mu}{4} e^{+} \overline{\theta}\widehat{\Gamma}^{-}\widehat{\Gamma}^{123}\theta \, ,
\end{aligned}$$

where μ is a parameter included in the plane-wave metric discussed in appendix C.1. Thus the superparticle action is rewritten as[2]

$$S \ = \ \int \mathrm{d}t \Big\{ \frac{1}{2} \sum_{I=1}^{9} (\partial_t X^I)^2 - \overline{\theta}\widehat{\Gamma}^{-}\partial_t \theta - \frac{1}{2}\Big[\Big(\frac{\mu}{3}\Big)^2 \sum_{\widetilde{I}=1}^{3} (X^{\widetilde{I}})^2 + \Big(\frac{\mu}{6}\Big)^2 \sum_{I'=4}^{9} (X^{I'})^2 \Big] + \frac{\mu}{4} \overline{\theta}\widehat{\Gamma}^{-}\widehat{\Gamma}^{123}\theta \Big\} \, , \tag{II.1.3}$$

where we also choose the bosonic light-cone gauge fixing $X^+ = t$, $\partial_t X^- = 0$[3]. Note that the $SO(10,1)$ Majorana spinor θ can be represented by the $SO(9)$ Majorana spinor Ψ because of the fermionic light-cone gauge fixing (II.1.2)

$$\theta \ \equiv \ \frac{1}{2^{3/4}} \begin{pmatrix} 0 \\ \Psi \end{pmatrix} \, , \qquad \overline{\theta} \ = \ \theta^T C \ = \ \frac{1}{2^{3/4}} \Big(-\Psi^T \, , \, 0 \Big) \, .$$

[2] From now on we use the relation $\widehat{\Gamma}^{123} = \widehat{\Gamma}^{123}$ because these directions are flat on the plane-wave background (see the plane-wave metric in appendix C.1).

[3] Here we do not mention the strict definitions of variables.

Utilizing the $SO(9)$ Majorana spinor Ψ, we reduce the θ bilinear terms in (II.1.3) to the following:

$$-\overline{\theta}\widehat{\Gamma}^{-}\partial_t\theta \;=\; \frac{i}{2}\Psi^\dagger\partial_t\Psi\;, \qquad \frac{\mu}{4}\overline{\theta}\widehat{\Gamma}^{-}\widehat{\Gamma}^{123}\theta \;=\; -\frac{i\,\mu}{8}\Psi^\dagger\gamma^{123}\Psi\;.$$

Definitions of the $SO(9)$ gamma matrices γ^I are summarized in appendix A.4. Thus we write down the superparticle action as follows:

$$S \;=\; \int \mathrm{d}t\Big\{\frac{1}{2}\sum_{I=1}^{9}(\partial_t X^I)^2 + \frac{i}{2}\Psi^\dagger\partial_t\Psi - \frac{1}{2}\Big[\Big(\frac{\mu}{3}\Big)^2\sum_{\widetilde{I}=1}^{3}(X^{\widetilde{I}})^2 + \Big(\frac{\mu}{6}\Big)^2\sum_{I'=4}^{9}(X^{I'})^2\Big] - \frac{i\,\mu}{8}\Psi^\dagger\gamma^{123}\Psi\Big\}\;. \tag{II.1.4}$$

Let us consider the supersymmetry invariance of the action (II.1.4) and generalize it to the multi-superparticle action, i.e., N D0-branes' action represented by non-abelian $U(N)$ gauge symmetry group. First we look for the supersymmetry transformation of the type

$$\begin{aligned} \delta X^I \;&\equiv\; \Psi^\dagger\gamma^I\,\epsilon(t)\;, \\ \delta\Psi \;&\equiv\; b\,\partial_t X^I\gamma^I\,\epsilon(t) + \mu\,X^I\gamma^I M_I'\,\epsilon(t)\;, \\ \epsilon(t) \;&=\; \exp\big(\mu M t\big)\,\epsilon_0\;, \end{aligned} \tag{II.1.5}$$

where b is a numerical constant and ϵ_0 is a constant $SO(9)$ Majorana spinor; M and M_I' are matrix valued parameters. We will determine the values of these variables via properties of the invariance of action S under the supersymmetry of type (II.1.5). The invariance of the action under the supersymmetry transformations (II.1.5) leads to the following equation:

$$\begin{aligned} 0 \;=\; &\int\mathrm{d}t\Big\{\big(1-bi\big)\partial_t X^I(\partial_t\Psi)^\dagger\gamma^I\epsilon\Big\} \\ &+\mu\int\mathrm{d}t\Big\{\partial_t X^I\Psi^\dagger\gamma^I M\epsilon + i\,\partial_t X^I\,\Psi^\dagger\gamma^I M_I'\epsilon - b\frac{i}{4}\partial_t X^I\,\Psi^\dagger\gamma^{123}\gamma^I\epsilon\Big\} \\ &+\mu^2\int\mathrm{d}t\Big\{i\,X^I\,\Psi^\dagger\gamma^I\big(M_I'M\big)\epsilon - \frac{1}{9}X^{\widetilde{I}}\Psi^\dagger\gamma^{\widetilde{I}}\epsilon - \frac{1}{36}X^{I'}\Psi^\dagger\gamma^{I'}\epsilon - \frac{i}{4}X^I\Psi^\dagger\gamma^{123}\gamma^I M_I'\epsilon\Big\}\;. \end{aligned} \tag{II.1.6}$$

From now on we omit summation symbols with respect to the spacetime coordinates. We consider the invariance (II.1.6) order by order with respect to the parameter μ. The terms of order μ^0 determine the constant as b in the supersymmetry transformation (II.1.5) as $b=-i$. The terms of order μ^1 in (II.1.6) give the equations

$$M + iM_{\widetilde{I}}' - \frac{1}{4}\gamma^{123} \;=\; 0\;, \qquad M + iM_{I'}' + \frac{1}{4}\gamma^{123} \;=\; 0\;,$$

and the terms of order μ^2 in (II.1.6) leads to

$$iM_{\widetilde{I}}'M - \frac{1}{9} - \frac{i}{4}\gamma^{123}M_{\widetilde{I}}' \;=\; 0\;, \qquad iM_{I'}'M - \frac{1}{36} + \frac{i}{4}\gamma^{123}M_{I'}' \;=\; 0\;.$$

Then we obtain the values of unknown parameters M and M'_I as

$$M \;=\; -\frac{1}{12}\gamma^{123}\,, \qquad iM'_{\widetilde{I}} \;=\; \frac{1}{3}\gamma^{123}\,, \qquad iM'_{I'} \;=\; -\frac{1}{6}\gamma^{123}\,. \tag{II.1.7}$$

The extension to the non-abelian theory is obvious; besides the usual commutator terms which are present in the Lagrangian and supersymmetry transformation rules in flat spaces, we have an extra coupling of order μ. Indeed, it was found that a term $F_{tIJK}\mathrm{Tr}(X^I X^J X^K)$ should be included in the action for N D0-branes in constant Ramond-Ramond field strength [140, 106, 107] (see also appendix B.2). In our case, the coupling is

$$F_{+\widetilde{I}\widetilde{J}\widetilde{K}}\mathrm{Tr}(X^{\widetilde{I}}X^{\widetilde{J}}X^{\widetilde{K}}) \;=\; -\mu\,\epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}}\,\mathrm{Tr}(X^{\widetilde{I}}X^{\widetilde{J}}X^{\widetilde{K}})\,.$$

Thus the action is written in terms of $N \times N$ matrix valued fields X^I and Ψ

$$\begin{aligned} S \;=\; \int \mathrm{d}t\,\mathrm{Tr}\Big\{ & \frac{1}{2}(\partial_t X^I)^2 + \frac{i}{2}\Psi^\dagger\partial_t\Psi - \frac{1}{2}\Big[\Big(\frac{\mu}{3}\Big)^2 (X^{\widetilde{I}})^2 + \Big(\frac{\mu}{6}\Big)^2 (X^{I'})^2\Big] \\ & - \frac{i\,\mu}{8}\Psi^\dagger\gamma^{123}\Psi + d\,\mu\,g\,\epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}}\,(X^{\widetilde{I}}X^{\widetilde{J}}X^{\widetilde{K}}) + \frac{1}{4}g^2\,[X^I,X^J]^2 + \frac{1}{2}\,g\,\Psi^\dagger\gamma^I[X^I,\Psi]\Big\}\,. \end{aligned} \tag{II.1.8}$$

We explain newly introduced terms in the above action from the viewpoint of the dimensional reduction of ten-dimensional $U(N)$ super Yang-Mills as in appendix B.1. The matrix valued fields X^I and Ψ, whose mass dimensions are $-1/2$ and 0, are not only the the adjoint representations of $U(N)$ gauge group but also the dynamical variables in ten-dimensional SYM. The parameter g is the Yang-Mills coupling with mass dimensions 3/2. The quartic term $\frac{1}{4}g^2[X^I,X^J]^2$ can be derived from the reduction of the field strength of $U(N)$ gauge potential. We obtain the three-point vertex term $\frac{1}{2}g\Psi^\dagger\gamma^I[X^I,\Psi]$ from the covariant derivative of fermion $D_M\Psi = \partial_M\Psi + ig[A_M,\Psi]$ in super Yang-Mills. Notice that although the fermion Ψ is the $SO(9)$ Majorana spinor in our derivation, we can also regard this as the $SO(9,1)$ Majorana-Weyl spinor in ten dimensions.

The supersymmetry transformations of this system should be extended as

$$\begin{aligned} \delta X^I &\;=\; \Psi^\dagger\gamma^I\epsilon(t)\,, \\ \delta\Psi &\;=\; -i\partial_t X^I\gamma^I\epsilon(t) - \frac{i\mu}{3}X^{\widetilde{I}}\gamma^{\widetilde{I}}\gamma^{123}\epsilon(t) + \frac{i\mu}{6}\,X^{I'}\gamma^{I'}\gamma^{123}\epsilon(t) + \frac{1}{2}\,g\,[X^I,X^J]\gamma^{IJ}\epsilon(t)\,, \\ \epsilon(t) &\;=\; \exp\big(-\frac{\mu}{12}\gamma^{123}t\big)\epsilon_0\,. \end{aligned} \tag{II.1.9}$$

Last, we introduce the gauge potential A_t as an auxiliary matrix variable of this system and rewrite the derivative ∂_t to the covariant derivative $D_t X^I = \partial_t X^I + ig[A_t, X^I]$:

$$\begin{aligned} S \;=\; \int \mathrm{d}t\,\mathrm{Tr}\Big\{ & \frac{1}{2}D_t X^I\,D_t X^I + \frac{i}{2}\Psi^\dagger D_t\Psi - \frac{1}{2}\Big[\Big(\frac{\mu}{3}\Big)^2 (X^{\widetilde{I}})^2 + \Big(\frac{\mu}{6}\Big)^2 (X^{I'})^2\Big] \\ & - \frac{i\,\mu}{8}\Psi^\dagger\gamma^{123}\Psi - \frac{i\,\mu}{3}g\,\epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}}\,X^{\widetilde{I}}X^{\widetilde{J}}X^{\widetilde{K}} + \frac{1}{4}g^2\,[X^I,X^J]^2 + \frac{1}{2}\,g\,\Psi^\dagger\gamma^I[X^I,\Psi]\Big\}\,. \end{aligned} \tag{II.1.10}$$

Here we can interpret that the covariant derivative D_tX^I comes from the dimensional reduction of field strength

$$F_{MN} \;=\; \partial_M A_N - \partial_N A_M + ig[A_M, A_N]$$

in the ten-dimensional $U(N)$ super Yang-Mills Lagrangian. This action (II.1.10) is also obtained by the matrix regularization of the supermembrane on the plane-wave under the appropriate rescaling of some variables [38, 129].

Now let us re-define the field variables in order for the compatibility of the description of the nonabelian Dirac-Born-Infeld type Lagrangian discussed in appendix B.2. Combining the Yang-Mills coupling g and field variables

$$gX^I \;\equiv\; X'^I\,, \qquad gA_t \;\equiv\; A'_t\,, \qquad g\Psi \;\equiv\; \Psi'\,,$$

we rewrite the action (II.1.10) as

$$\begin{aligned} S \;=\; \frac{1}{g^2}\int \mathrm{d}t\,\mathrm{Tr}\Big\{&\frac{1}{2}D_tX'^I\,D_tX'^I + \frac{i}{2}\Psi'^\dagger D_t\Psi' - \frac{1}{2}\Big[\Big(\frac{\mu}{3}\Big)^2(X'^{\widetilde{I}})^2 + \Big(\frac{\mu}{6}\Big)^2(X'^{I'})^2\Big] \\ &- \frac{i\,\mu}{8}\Psi'^\dagger\gamma^{123}\Psi' - \frac{i\,\mu}{3}\epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}}\,X'^{\widetilde{I}}X'^{\widetilde{J}}X'^{\widetilde{K}} + \frac{1}{4}\,[X'^I, X'^J]^2 + \frac{1}{2}\,\Psi'^\dagger\gamma^I[X'^I, \Psi']\Big\}\,. \end{aligned} \tag{II.1.11}$$

Note that the mass dimensions of X' and Ψ' are 1 and 3/2, respectively. But, for simplicity, we omit the prime symbol in field variables. We also rewrite the Yang-Mills coupling g in terms of the D0-brane mass (or tension) m_0 and the Regge constant α' as $g^{-2} = (2\pi\alpha')^2 m_0$. (We will discuss this relation in appendix B.2.) From the viewpoint of DLCQ with compactification $x^- \sim x^- + 2\pi R$, the D0-brane mass is represented in terms of R as $m_0 = 1/R$. Thus, we can write the overall factor of the action (II.1.11) is

$$\frac{1}{g^2} \;=\; \frac{(2\pi\alpha')^2}{R}\,.$$

In the next section we will construct the Hamiltonian, supercharges and their commutation relations in terms of the conventions adopted by Dasgupta, Sheikh-Jabbari and Van Raamsdonk [38]. We will also analyze one specific spectrum.

II.2 Hamiltonian, Supercharges and their Commutation Relations

We would like to study the zero-mode spectrum of this matrix model. Before starting a discussion, we must prepare some operators such as Hamiltonian, supercharges, and the commutation relations between them. Here we review such preliminary discussed by Dasgupta, Sheikh-Jabbari and Van Raamsdonk [38]. Now we rewrite the matrix model Lagrangian (II.1.11) via the following rescaling[4]:

$$\begin{aligned} t &= R^{2/3}\tau \,, \qquad A_t = R^{-2/3}A_\tau \,, \qquad \mu = R^{-2/3}\widetilde{\mu} \,, \\ X^I &= R^{1/3}\widetilde{X}^I \,, \qquad \Psi = R^{1/2}\widetilde{\Psi} \end{aligned} \tag{II.2.1}$$

and $2\pi\alpha' \equiv 1$. Under the above rescaling, the Matrix Theory Lagrangian describing the DLCQ of M-theory on the plane-wave background [38] is given by

$$\begin{aligned} S &= \int \mathrm{d}\tau \, \mathcal{L} \,, \\ \mathcal{L} &= \mathrm{Tr}\Big\{ \frac{1}{2R} \widetilde{D}_\tau \widetilde{X}^I \, \widetilde{D}_\tau \widetilde{X}^I + \frac{i}{2} \widetilde{\Psi}^\dagger \widetilde{D}_\tau \widetilde{\Psi} + \frac{R}{2} \widetilde{\Psi}^\dagger \gamma^I [\widetilde{X}^I, \widetilde{\Psi}] + \frac{R}{4} [\widetilde{X}^I, \widetilde{X}^J]^2 \Big\} \\ &\quad + R\,\mathrm{Tr}\Big\{ - \frac{1}{2} \Big[\Big(\frac{\widetilde{\mu}}{3R} \Big)^2 (\widetilde{X}^{\widetilde{I}})^2 + \Big(\frac{\widetilde{\mu}}{6R} \Big)^2 (\widetilde{X}^{I'})^2 \Big] - \frac{i\,\widetilde{\mu}}{3R} \epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}} \, \widetilde{X}^{\widetilde{I}} \widetilde{X}^{\widetilde{J}} \widetilde{X}^{\widetilde{K}} - \frac{i\,\widetilde{\mu}}{8R} \widetilde{\Psi}^\dagger \gamma^{123} \widetilde{\Psi} \Big\} \,, \end{aligned} \tag{II.2.2}$$

where the covariant derivative $\widetilde{D}_\tau \widetilde{X}^I$ is given by $\widetilde{D}_\tau \widetilde{X}^I = \partial_\tau \widetilde{X}^I + i[A_\tau, \widetilde{X}^I]$. For simplicity, we omit the tildes written above the rescaled variables. Performing Legendre transformation, we obtain the Hamiltonian of this system. We define the canonical momenta of X^I and Ψ in terms of the right-derivative:

$$(P_I)_{kl} = \frac{\partial}{\partial(\partial_\tau X^I)_{lk}} \mathcal{L} = \frac{1}{R} (D_\tau X^I)_{kl} \,, \qquad (S)_{kl} = \frac{\partial}{\partial(\partial_\tau \Psi)_{lk}} \mathcal{L} = \frac{i}{2} (\Psi^\dagger)_{kl} \,,$$

where k and l are indices of $N \times N$ matrices. Thus the Hamiltonian is described as

$$\begin{aligned} H &= \mathrm{Tr}\{P_I \partial_\tau X^I\} + \mathrm{Tr}\{S \partial_\tau \Psi\} - \mathcal{L} \\ &= R\,\mathrm{Tr}\Big\{ \frac{1}{2}(P_I)^2 - \frac{1}{2} \Psi^\dagger \gamma^I [X^I, \Psi] - \frac{1}{4} [X^I, X^J]^2 \\ &\qquad + \frac{1}{2} \Big[\Big(\frac{\mu}{3R} \Big)^2 (X^{\widetilde{I}})^2 + \Big(\frac{\mu}{6R} \Big)^2 (X^{I'})^2 \Big] + \frac{i\,\mu}{3R} \epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}} \, X^{\widetilde{I}} X^{\widetilde{J}} X^{\widetilde{K}} + \frac{i\,\mu}{8R} \Psi^\dagger \gamma^{123} \Psi \Big\} \,, \end{aligned} \tag{II.2.3}$$

where we solved some Dirac constraints and substituted them into the Hamiltonian, or simply, wrote down this Hamiltonian under the gauge $A_\tau = 0$.

As for the case of flat spacetime, the $U(1)$ part of the theory (i.e., the free part describing the center of mass degrees of freedom) decouples from the $SU(N)$ part (the interaction part of the theory). On the plane-wave background, the $U(1)$ sector is described by the harmonic oscillator Hamiltonian with bosonic oscillators in the $SO(3)$ directions of mass $\mu/3$ and in the $SO(6)$ directions of mass $\mu/6$ as well

[4] The re-definition (II.2.1) is somewhat complicated and looks like strange. Of course we can discuss the same investigation without this re-definition. But we will analyze the system described by the action (II.2.2), the same representation as [38], where Dasgupta, Sheikh-Jabbari and Van Raamsdonk suggested the perturbation of the BMN matrix model.

as 8 fermionic oscillators of mass $\mu/4$. Thus unlike the flat spacetime case, the different polarization states have different masses.

Here we pick up the symmetry algebra of this Matrix theory and provide explicit expressions for the bosonic generators in terms of the matrix variables X^I and P_I. The bosonic generators include the harmonic oscillators a^I, the Hamiltonian H, the light-cone momentum P^+ (a central terms of the algebra) and the rotation generators of $SO(3)\times SO(6)$ symmetry $\Sigma^{\widetilde{I}\widetilde{J}}$ and $\Sigma^{I'J'}$, respectively. These variables satisfy the following algebra [19, 38]

$$\begin{aligned}
[a^{\widetilde{I}}, a^{\dagger\widetilde{J}}] &= P^+\,\delta^{\widetilde{I}\widetilde{J}}\,, & [a^{I'}, a^{\dagger J'}] &= P^+\,\delta^{I'J'}\,,\\
[H, a^{\widetilde{I}}] &= -\frac{\mu}{3}a^{\widetilde{I}}\,, & [H, a^{I'}] &= -\frac{\mu}{6}a^{I'}\,,\\
[\Sigma^{\widetilde{I}\widetilde{J}}, a^{\widetilde{K}}] &= i\big(\delta^{\widetilde{J}\widetilde{K}}\,a^{\widetilde{I}} - \delta^{\widetilde{I}\widetilde{K}}\,a^{\widetilde{J}}\big)\,, & [\Sigma^{I'J'}, a^{K'}] &= i\big(\delta^{J'K'}\,a^{I'} - \delta^{I'K'}\,a^{J'}\big)\,,
\end{aligned}\tag{II.2.4}$$
$$\begin{aligned}
i[\Sigma^{\widetilde{I}\widetilde{J}}, \Sigma^{\widetilde{K}\widetilde{L}}] &= \delta^{\widetilde{I}\widetilde{K}}\Sigma^{\widetilde{J}\widetilde{L}} + \delta^{\widetilde{J}\widetilde{L}}\Sigma^{\widetilde{I}\widetilde{K}} - \delta^{\widetilde{I}\widetilde{L}}\Sigma^{\widetilde{J}\widetilde{K}} - \delta^{\widetilde{J}\widetilde{K}}\Sigma^{\widetilde{I}\widetilde{L}}\,,\\
i[\Sigma^{I'J'}, \Sigma^{K'L'}] &= \delta^{I'K'}\Sigma^{J'L'} + \delta^{J'L'}\Sigma^{I'K'} - \delta^{I'L'}\Sigma^{J'K'} - \delta^{J'K'}\Sigma^{I'L'}\,.
\end{aligned}$$

Note that the harmonic oscillators $a^{\dagger I}$ and a^I are creation and annihilation operators corresponding to the decoupled $U(1)$ part of the theory which describes the center of mass degrees of freedom (a particle) in a harmonic potential. These generators are realized by the Matrix theory variables X^I, P^I and $\psi_{i\alpha}$:

$$\begin{aligned}
P^+ &= \frac{1}{R}\mathrm{Tr}(\mathbf{1})\,,\\
a^{\widetilde{I}} &= \frac{1}{\sqrt{R}}\mathrm{Tr}\Big(\sqrt{\frac{\mu}{6R}}X^{\widetilde{I}} + i\sqrt{\frac{3R}{2\mu}}P^{\widetilde{I}}\Big)\,, \qquad a^{I'} = \frac{1}{\sqrt{R}}\mathrm{Tr}\Big(\sqrt{\frac{\mu}{12R}}X^{I'} + i\sqrt{\frac{3R}{\mu}}P^{I'}\Big)\,,\\
\Sigma^{\widetilde{I}\widetilde{J}} &= \mathrm{Tr}\Big(P^{\widetilde{I}}X^{\widetilde{J}} - P^{\widetilde{J}}X^{\widetilde{I}} - i\epsilon^{\widetilde{I}\widetilde{J}\widetilde{K}}\,\psi^{\dagger i\alpha}\,(\sigma^{\widetilde{K}})_\alpha{}^\beta\,\psi_{i\beta}\Big)\,,\\
\Sigma^{I'J'} &= \mathrm{Tr}\Big(P^{I'}X^{J'} - P^{J'}X^{I'} - \frac{1}{2}\psi^{\dagger i\alpha}\,(\mathsf{g}^{I'J'})_i{}^j\,\psi_{j\alpha}\Big)\,.
\end{aligned}$$

Notice that we have already used the $SU(4)\times SU(2)$ decomposition rule with respect to the fermionic variables $\psi_{i\alpha}$ discussed in appendix A.5; the gamma matrix in the last equation is defined as $\mathsf{g}^{I'J'} = \frac{1}{2}\{\mathsf{g}^{I'}(\mathsf{g}^{J'})^\dagger - \mathsf{g}^{J'}(\mathsf{g}^{I'})^\dagger\}$. These generators expressed by the matrix variables satisfy the algebra (II.2.4) via the (anti-)commutation relations

$$\begin{aligned}
[X^{\widetilde{I}}_{kl}, P^{\widetilde{J}}_{mn}] &= i\,\delta^{\widetilde{I}\widetilde{J}}\,\delta_{kn}\,\delta_{lm}\,, \qquad [X^{I'}_{kl}, P^{J'}_{mn}] = i\,\delta^{I'J'}\,\delta_{kn}\,\delta_{lm}\,,\\
\{(\psi^{\dagger i\alpha})_{kl}, (\psi_{j\beta})_{mn}\} &= \delta^i_j\,\delta^\alpha_\beta\,\delta_{kn}\,\delta_{lm}\,.
\end{aligned}$$

These (anti-)commutation relations are also introduced when one discuss the quantum mechanics of regularized supermembrane theory in the light-cone gauge [129].

The 32 components of the $SO(10,1)$ spacetime supersymmetry decompose into two 16 components supersymmetry in the light-cone gauge. One supersymmetry is linearly realized and the other nonlinearly realized as we shall discuss now. As discussed in the previous section, the Matrix theory Lagrangian (II.2.2) has the invariance of nonlinearly realized supersymmetry transformation. We rewrite the rescaled transformation rule of (II.1.9):

$$
\begin{aligned}
\delta_\epsilon X^I &= \sqrt{R}\,\Psi^\dagger\gamma^I\epsilon(\tau)\,, \qquad \delta_\epsilon\omega \;=\; \sqrt{R}\,\Psi^\dagger\epsilon(\tau)\,,\\
\delta_\epsilon\Psi &= \sqrt{R}\Big(-\frac{i}{R}D_\tau X^I\gamma^I\epsilon(\tau)+\frac{1}{2}[X^I,X^J]\gamma^{IJ}\epsilon(\tau)-\frac{i\,\mu}{3R}X^{\tilde I}\gamma^{\tilde I}\gamma^{123}\epsilon(\tau)+\frac{i\,\mu}{6R}X^{I'}\gamma^{I'}\gamma^{123}\epsilon(\tau)\Big)\,,\\
\epsilon(\tau) &= \exp\Big(-\frac{\mu}{12}\gamma^{123}\tau\Big)\epsilon_0\,.
\end{aligned}
$$

We call this symmetry the "dynamical supersymmetry" whose supercharges are written by

$$
Q \;=\; \sqrt{R}\,\mathrm{Tr}\Big\{P^I\gamma^I\Psi-\frac{i}{2}[X^I,X^J]\gamma^{IJ}\Psi-\frac{\mu}{3R}X^{\tilde I}\gamma^{\tilde I}\gamma^{123}\Psi-\frac{\mu}{6R}X^{I'}\gamma^{I'}\gamma^{123}\Psi\Big\}\,. \tag{II.2.5}
$$

The Lagrangian (II.2.2) also has a linearly realized supersymmetry whose transformation rule is

$$
\begin{aligned}
\delta_\eta X^I \;=\; 0\,, \qquad \delta_\eta\omega \;=\; 0\,, \qquad \delta_\eta\Psi \;=\; \frac{1}{\sqrt{R}}\eta(\tau)\,,\\
\eta(\tau) \;=\; \exp\Big(\frac{\mu}{4}\gamma^{123}\tau\Big)\eta_0\,,
\end{aligned}
$$

where the $SO(9)$ Majorana spinor η_0 is constant. This supersymmetry is called the "kinematical supersymmetry" whose supercharge is realized as

$$
q \;=\; \frac{1}{\sqrt{R}}\mathrm{Tr}(\Psi)\,. \tag{II.2.6}
$$

Note that the dynamical supersymmetry acts on the $SU(N)$ interaction part of theory whereas the kinematical supersymmetry acts only on the free $U(1)$ part. In addition, the kinematical supercharges generate the overall polarization states. Between the dynamical and kinematical supersymmetries there are some nontrivial relation as follows [38]:

$$
\begin{aligned}
\{Q_\alpha,Q_\beta\} &= 2\delta_{\alpha\beta}H+\frac{\mu}{3}\big(\gamma^{\tilde I\tilde J}\gamma^{123}\big)_{\alpha\beta}\Sigma^{\tilde I\tilde J}-\frac{\mu}{3}\big(\gamma^{I'J'}\gamma^{123}\big)_{\alpha\beta}\Sigma^{I'J'}\,,\\
\{Q_\alpha,q_\beta\} &= -\sqrt{\frac{2\mu}{3}}\Big(\Big\{\frac{1}{2}\big(1-i\gamma^{123}\big)\gamma^{\tilde I}\Big\}_{\alpha\beta}a^{\tilde I}-\Big\{\frac{1}{2}\big(1+i\gamma^{123}\big)\gamma^{\tilde I}\Big\}_{\alpha\beta}a^{\dagger\tilde I}\Big)\\
&\quad+\sqrt{\frac{\mu}{3}}\Big(\Big\{\frac{1}{2}\big(1-i\gamma^{123}\big)\gamma^{I'}\Big\}_{\alpha\beta}a^{\dagger I'}-\Big\{\frac{1}{2}\big(1+i\gamma^{123}\big)\gamma^{I'}\Big\}_{\alpha\beta}a^{\dagger I'}\Big)\,,\\
\{q_\alpha,q_\beta\} &= \delta_{\alpha\beta}P^+\,.
\end{aligned} \tag{II.2.7}
$$

Unlike the flat spacetime case, the commutation relations between the Hamiltonian and supercharges do not vanish:

$$
[H,Q_\alpha] \;=\; \frac{\mu}{12}\big(i\gamma^{123}Q\big)_\alpha\,, \qquad [H,q_\alpha] \;=\; -\frac{\mu}{4}\big(i\gamma^{123}q\big)_\alpha\,. \tag{II.2.8}
$$

Thus different members of a multiplet of supersymmetric states generated by acting with supercharges will have different energies, although the energy differences will still be exactly determined by the supersymmetry algebra (II.2.8).

II.3 Spectrum of the Ground State Supermultiplet

In this section we discuss a supermultiplet generated by the kinematical supercharges, which is the $U(1)$ part of the theory including the ground state. We would like to compare the supermultiplet of the $U(1)$ free sector in the Matrix theory on the plane-wave background with the massless spectrum of eleven-dimensional linearized supergravity on the plane-wave background [93] which will be discussed in chapter III. For later convenience, we express the $SO(9)$ Majorana spinor supercharge q in terms of the $SU(4)\times SU(2)$ representation (for the decomposition rule, see appendix A.5). And we construct the states labeled by the $SU(4)$ indices $i=1,2,\cdots,4$ and the $SU(2)$ indices $\alpha=1,2$. Under the decomposition rules the supercharges are represented as follows:

$$\begin{aligned}
Q_{i\alpha} &= \sqrt{R}\,\mathrm{Tr}\Big\{-\Big(P^{\widetilde{I}}+\frac{i\,\mu}{3R}X^{\widetilde{I}}\Big)(\sigma^{\widetilde{I}})_\alpha{}^\beta\,\psi_{i\beta}+\Big(P^{I'}-\frac{i\,\mu}{6R}X^{I'}\Big)(\mathrm{g}^{I'})_{ij}\,\epsilon_{\alpha\beta}\,\psi^{\dagger j\beta} \\
&\qquad +\frac{1}{2}[X^{\widetilde{I}},X^{\widetilde{J}}]\epsilon^{\widetilde{I}\widetilde{J}\widetilde{K}}\,(\sigma^{\widetilde{K}})_\alpha{}^\beta\,\psi_{i\beta}-\frac{i}{2}[X^{\widetilde{I}},X^{\widetilde{J}}](\mathrm{g}^{I'J'})_i{}^j\,\psi_{j\alpha} \\
&\qquad +i[X^{\widetilde{I}},X^{J'}](\sigma^{\widetilde{I}})_\alpha{}^\beta(\mathrm{g}^{I'})_{ij}\,\epsilon_{\beta\gamma}\,\psi^{\dagger j\gamma}\Big\}\,, \\
q_{i\alpha} &= \frac{1}{\sqrt{R}}\mathrm{Tr}(\psi_{i\alpha})\,.
\end{aligned}$$

The algebras (II.2.7) and (II.2.8) are also rewritten as

$$\{Q^{\dagger i\alpha}\,,\,Q_{j\beta}\} \;=\; 2\delta^i_j\,\delta^\alpha_\beta\,H+\frac{\mu}{3}\epsilon^{\widetilde{I}\widetilde{J}\widetilde{K}}\,(\sigma^{\widetilde{K}})_\beta{}^\alpha\,\delta^i_j\,\Sigma^{\widetilde{I}\widetilde{J}}+\frac{i\,\mu}{6}\delta^\alpha_\beta\,(\mathrm{g}^{I'J'})_j{}^i\,\Sigma^{I'J'}\,, \tag{II.3.1a}$$

$$\{q_{i\alpha}\,,\,Q_{j\beta}\} \;=\; -i\sqrt{\frac{\mu}{3}}\,(\mathrm{g}^{I'})_{ij}\,\epsilon_{\alpha\beta}\,a^{I'}\,,\qquad \{q^{\dagger i\alpha}\,,\,Q_{j\beta}\} \;=\; -i\sqrt{\frac{2\mu}{3}}\,(\sigma^{\widetilde{I}})_\beta{}^\alpha\,\delta^i_j\,a^{\widetilde{I}\dagger}\,, \tag{II.3.1b}$$

$$\{q^{\dagger i\alpha}\,,\,q_{j\beta}\} \;=\; \delta^\alpha_\beta\,\delta^i_j\,P^+\,, \tag{II.3.1c}$$

$$[H\,,\,Q_{i\alpha}] \;=\; \frac{\mu}{12}Q_{i\alpha}\,,\qquad [H\,,\,q_{i\alpha}] \;=\; -\frac{\mu}{4}q_{i\alpha}\,. \tag{II.3.1d}$$

Here we define the ground state $|\,\Lambda\,\rangle$ annihilated by supercharges of the kinematical supersymmetry with arbitrary indices i and α:

$$q_{i\alpha}|\,\Lambda\,\rangle \;=\; 0 \qquad \text{for all}\;\; i,\alpha\,.$$

Starting from this ground state we construct the bosonic and fermionic states generated by the kinematical supercharges $q^{\dagger i\alpha}$. These states are also eigenstates of the Hamiltonian because of the commutation relation $[H,q^{\dagger i\alpha}]=\frac{\mu}{4}q^{\dagger i\alpha}$. Since it is somewhat difficult to display the supersymmetric states in terms of the supercharges themselves, we introduce the Young Tableaux

$$q^{\dagger i\alpha} \sim (\square, \square) \ .$$

The first and second boxes $\square$ in the right hand side indicate the Young Tableaux of $SU(4)$ and $SU(2)$ fundamental representations, respectively. Since we define the ground state $|\,\Lambda\,\rangle$ as a singlet with respect to the action on the supercharge $q_{i\alpha}$, we label this state as

$$|\,\Lambda\,\rangle = \Big|\, 1 \ , \ 1 \Big\rangle \ . \tag{II.3.2}$$

We find that the energy of this state is zero by using the commutation relation (II.3.1d)[5]. The "first floor" is generated by acting the kinematical supercharge $q^{\dagger i\alpha}$:

$$(\square, \square) \otimes \Big|\, 1 \ , \ 1 \Big\rangle = \Big|\, \square \ , \ \square \Big\rangle \ . \tag{II.3.3}$$

The energy of the first floor is evaluated to $\mu/4$. The "second floor" is also generated by the supercharge acting on the first floor:

$$(\square, \square) \otimes \Big|\, \square \ , \ \square \Big\rangle = \Big|\, \begin{smallmatrix}\square\\\square\end{smallmatrix} \ , \ \begin{smallmatrix}\square\\\square\end{smallmatrix} \Big\rangle \oplus \Big|\, \begin{smallmatrix}\square\\\square\end{smallmatrix} \ , \ \square\square \Big\rangle \oplus \Big|\, \square\square \ , \ \begin{smallmatrix}\square\\\square\end{smallmatrix} \Big\rangle \oplus \Big|\, \square\square \ , \ \square\square \Big\rangle \ . \tag{II.3.4}$$

Notice that the generators $q^{\dagger i\alpha}$ is a fermionic charge. Thus the states symmetric with respect to the supercharges, are forbidden as a member of the supermultiplet and these terms are written by gray color. In the same way we obtain the "third floor" as

$$\begin{aligned} &(\square, \square) \otimes \Big\{ \Big|\, \begin{smallmatrix}\square\\\square\end{smallmatrix} \ , \ \square\square \Big\rangle \oplus \Big|\, \square\square \ , \ \begin{smallmatrix}\square\\\square\end{smallmatrix} \Big\rangle \Big\} \\ &\quad = \Big|\, \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \ , \ \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \Big\rangle \oplus \Big|\, \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \ , \ \square\square\square \Big\rangle \oplus \Big|\, \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \ , \ \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \Big\rangle \oplus \Big|\, \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \ , \ \square\square\square \Big\rangle \oplus \Big|\, \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \ , \ \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \Big\rangle \oplus \Big|\, \square\square\square \ , \ \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \Big\rangle \ . \end{aligned} \tag{II.3.5}$$

Here the two $\Big|\, \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \ , \ \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \Big\rangle$ states are generated from different states in the second floor. In this case these states are linearly combined and only the antisymmetrized combination is chosen as a member of supermultiplet (because of the fermionic generators).

The states in the higher "floors" are also described in terms of the Young Tableaux. Since we generate the states by using fermionic supercharges $q^{\dagger i\alpha}$, the highest state is generated when we act eight supercharges on the ground state and the process will stop. The ninth supercharge annihilate the highest state. Here we continue to generate the other states:

[5] Notice that the parameter μ is rescaled in (II.2.1). But since the "time" variable τ is also rescaled, the Hamiltonian (II.2.3) is defined the rescaled-time-evolution operator. Thus we can obtain the energy eigenvalues of the states with correct mass dimensions.

Fourth floor:

$$
\begin{aligned}
(\square,\square)\otimes\Big\{\Big|\,\yng(1,1,1),\ \yng(3)\Big\rangle \oplus \Big|\,\yng(2,1),\ \yng(2,1)\Big\rangle\Big\}
&= \Big|\,\yng(1,1,1,1),\ \yng(3,1)\Big\rangle \oplus \Big|\,\yng(1,1,1,1),\ \yng(4)\Big\rangle \oplus \Big|\,\yng(2,1,1),\ \yng(3,1)\Big\rangle \oplus \Big|\,\yng(2,1,1),\ \yng(4)\Big\rangle \\
&\oplus \Big|\,\yng(2,1,1),\ \yng(2,2)\Big\rangle \oplus \Big|\,\yng(2,1,1),\ \yng(3,1)\Big\rangle \oplus \Big|\,\yng(2,2),\ \yng(2,2)\Big\rangle \oplus \Big|\,\yng(2,2),\ \yng(3,1)\Big\rangle \\
&\oplus \Big|\,\yng(3,1),\ \yng(2,2)\Big\rangle \oplus \Big|\,\yng(3,1),\ \yng(3,1)\Big\rangle
\end{aligned}
\tag{II.3.6}
$$

Fifth floor:

$$
\begin{aligned}
(\square,\square)\otimes\Big\{\Big|\,\yng(1,1,1,1),\ \yng(4)\Big\rangle \oplus \Big|\,\yng(2,1,1),\ \yng(3,1)\Big\rangle \oplus \Big|\,\yng(2,2),\ \yng(2,2)\Big\rangle\Big\}
&= \Big|\,\yng(2,1,1,1),\ \yng(4,1)\Big\rangle \oplus \Big|\,\yng(2,1,1,1),\ \yng(5)\Big\rangle \oplus \Big|\,\yng(2,1,1,1),\ \yng(3,2)\Big\rangle \oplus \Big|\,\yng(2,1,1,1),\ \yng(4,1)\Big\rangle \\
&\oplus \Big|\,\yng(2,2,1),\ \yng(3,2)\Big\rangle \oplus \Big|\,\yng(2,2,1),\ \yng(4,1)\Big\rangle \oplus \Big|\,\yng(3,1,1),\ \yng(3,2)\Big\rangle \oplus \Big|\,\yng(3,1,1),\ \yng(4,1)\Big\rangle \\
&\oplus \Big|\,\yng(2,2,1),\ \yng(3,2)\Big\rangle \oplus \Big|\,\yng(3,2),\ \yng(3,2)\Big\rangle
\end{aligned}
\tag{II.3.7}
$$

Sixth floor:

$$
\begin{aligned}
(\square,\square)\otimes\Big\{\Big|\,\yng(2,1,1,1),\ \yng(4,1)\Big\rangle \oplus \Big|\,\yng(2,2,1),\ \yng(3,2)\Big\rangle\Big\}
&= \Big|\,\yng(2,2,1,1),\ \yng(4,2)\Big\rangle \oplus \Big|\,\yng(2,2,1,1),\ \yng(5,1)\Big\rangle \oplus \Big|\,\yng(3,1,1,1),\ \yng(4,2)\Big\rangle \oplus \Big|\,\yng(3,1,1,1),\ \yng(5,1)\Big\rangle \\
&\oplus \Big|\,\yng(2,2,1,1),\ \yng(3,3)\Big\rangle \oplus \Big|\,\yng(2,2,1,1),\ \yng(4,2)\Big\rangle \oplus \Big|\,\yng(2,2,2),\ \yng(3,3)\Big\rangle \oplus \Big|\,\yng(2,2,2),\ \yng(4,2)\Big\rangle \\
&\oplus \Big|\,\yng(3,2,1),\ \yng(3,3)\Big\rangle \oplus \Big|\,\yng(3,2,1),\ \yng(4,2)\Big\rangle
\end{aligned}
\tag{II.3.8}
$$

Seventh floor:

$$
\begin{aligned}
(\square,\square)\otimes\Big\{\Big|\,\yng(2,2,1,1),\ \yng(4,2)\Big\rangle \oplus \Big|\,\yng(2,2,2),\ \yng(3,3)\Big\rangle\Big\}
&= \Big|\,\yng(2,2,2,1),\ \yng(4,3)\Big\rangle \oplus \Big|\,\yng(2,2,2,1),\ \yng(5,2)\Big\rangle \oplus \Big|\,\yng(3,2,1,1),\ \yng(4,3)\Big\rangle \oplus \Big|\,\yng(3,2,1,1),\ \yng(5,2)\Big\rangle \\
&\oplus \Big|\,\yng(2,2,2,1),\ \yng(4,3)\Big\rangle \oplus \Big|\,\yng(3,2,2),\ \yng(4,3)\Big\rangle
\end{aligned}
\tag{II.3.9}
$$

Eighth floor:

$$
\begin{aligned}
(\square,\square)\otimes\Big|\,\yng(2,2,2,1),\ \yng(4,3)\Big\rangle
&= \Big|\,\yng(2,2,2,2),\ \yng(4,4)\Big\rangle \oplus \Big|\,\yng(2,2,2,2),\ \yng(5,3)\Big\rangle \oplus \Big|\,\yng(3,2,2,1),\ \yng(4,4)\Big\rangle \oplus \Big|\,\yng(3,2,2,1),\ \yng(5,3)\Big\rangle
\end{aligned}
\tag{II.3.10}
$$

The state in the "eighth floor" is the highest state which is annihilated by ninth supercharge. Thus we find that the supermultiplet contains the above states from the ground state $|\,\Lambda\,\rangle$ to the highest state $\left|\,\ydiagram{2,2,2,2}\,,\ \ydiagram{4,4}\,\right\rangle$. The energy eigenvalues of the above states are also obtained by the commutation relations (II.2.8). We summarize the members of supermultiplet in Table II.1.

N-th Floor	$SU(4)\times SU(2)$ Representations	Energy Eigenvalues
8	$(\mathbf{1},\mathbf{1})$	2μ
7	$(\overline{\mathbf{4}},\mathbf{2})$	$7\mu/4$
6	$(\overline{\mathbf{6}},\mathbf{3})$ $(\overline{\mathbf{10}},\mathbf{1})$	$3\mu/2$
5	$(\mathbf{4},\mathbf{4})$ $(\overline{\mathbf{20}},\mathbf{2})$	$5\mu/4$
4	$(\mathbf{1},\mathbf{5})$ $(\mathbf{15},\mathbf{3})$ $(\mathbf{20'},\mathbf{1})$	μ
3	$(\overline{\mathbf{4}},\mathbf{4})$ $(\mathbf{20},\mathbf{2})$	$3\mu/4$
2	$(\mathbf{6},\mathbf{3})$ $(\mathbf{10},\mathbf{1})$	$\mu/2$
1	$(\mathbf{4},\mathbf{2})$	$\mu/4$
ground state	$(\mathbf{1},\mathbf{1})$	0

Table II.1: *The simplest multiplet grouped into irreducible representations of* $SU(4)\times SU(2)$ *on each Floor of equal energies.*

If the Matrix theory conjecture [10] is correct (and if M-theory conjecture [146] is also correct) even on curved spacetime background, the resulting spectrum should correspond to the massless spectrum of eleven-dimensional supergravity, because the Matrix theory is proposed as a candidate of the well-defined description of M-theory, whose low energy effective theory is eleven-dimensional supergravity. Thus, in the next chapter, we will construct the supermultiplet of the ground state in eleven-dimensional supergravity and compare it to the result obtained here.

Note that we have considered only this $U(1)$ free sector of the Matrix theory. The remaining $SU(N)$ sector, which describes the interactions among N D0-branes from the viewpoint of type IIA string theory, would also describe the M-branes dynamics from the M-theory point of view [19, 38, 102]. It is quite interesting to investigate these dynamics in the supergravity side. But since this topic is beyond the scope of this doctoral thesis, we would like to consider this in the future.

Chapter III

Eleven-dimensional Supergravity Revisited

In this chapter, we discuss the eleven-dimensional supergravity on the plane-wave background. Eleven-dimensional supergravity Lagrangian with full interaction terms was discovered by Cremmer, Julia and Scherk [30]. But, for simplicity and later convenience, we describe the Lagrangian and classical field equations without the terms derived from spacetime torsion. With this formulation we make all the bosonic/fermionic fields fluctuate around classical field equations and construct linearized field equations. From the linearized field equations we study the zero point energy spectrum on the plane-wave background and compare with the zero-mode spectrum of the Matrix theory on the plane-wave.

III.1 Supergravity Lagrangian

As mentioned in chapter I, the eleven-dimensional supergravity is one of the simplest model in supersymmetric field theories because there are a few number of bosonic and fermionic fields

$$\begin{aligned}
e_M{}^{A} &: \text{ vielbein ,} \qquad E_A{}^{M} : \text{ inverse vielbein} \\
\Psi_M &: \text{ gravitino (vectorial Majorana spinor)} \\
C_{MNP} &: \text{ three-form gauge field} \\
\omega_M{}^{AB} &: \text{ spin connection}
\end{aligned}$$

The number of on-shell degrees of freedom of the vielbein (graviton), gravitino and three-form gauge field are 44, 128 and 84, respectively. Notice that the spin connection is independent of the vielbein in the first order formalism, but it is expressed by the vielbein in the second order formalism. By using these fields we describe the on-shell Lagrangian (up to torsion) [30]

$$\begin{aligned}
S &= \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\,\mathcal{L}\,, \\
\mathcal{L} &= e\,\mathcal{R} - \frac{1}{2}e\,\overline{\Psi}_M\widehat{\Gamma}^{MNP}D_N(\omega)\Psi_P - \frac{1}{48}e\,F_{MNPQ}\,F^{MNPQ} \\
&\quad - \frac{1}{192}e\,\overline{\Psi}_M\,\widetilde{\Gamma}^{MNPQRS}\,\Psi_N F_{PQRS} - \frac{1}{(144)^2}\,\varepsilon^{MNPQRSUVWXY}\,F_{MNPQ}\,F_{RSUV}\,C_{WXY}\,,
\end{aligned} \tag{III.1.1}$$

where $e = \det(e_M{}^{A}) = \sqrt{-\det g_{MN}}$ and $\widehat{\Gamma}_M$ is the gamma matrix defined in appendix A.2; the eleven-dimensional gravitational constant is κ; the rank six matrix $\widetilde{\Gamma}^{MNPQRS}$ is defined by

$$\widetilde{\Gamma}^{MNPQRS} = \widehat{\Gamma}^{MNPQRS} + 12g^{M[P}\widehat{\Gamma}^{QR}g^{S]N}\,.$$

The Lagrangian (III.1.1) contains two types of covariant derivatives explicitly or implicitly. One is the covariant derivative for general coordinate transformations denoted by ∇_M, and the other is the covariant derivative for local Lorentz transformations denoted by D_M. They are defined by the affine connection Γ^R_{MN} and the spin connection $\omega_M{}^{AB}$, for instance, as

$$\nabla_M A_N \;=\; \partial_M A_N - \Gamma^P_{NM} A_P \,, \qquad D_N \Psi_P \;=\; \partial_N \Psi_P - \frac{i}{2} \omega_N{}^{AB} \Sigma_{AB} \Psi_P \,.$$

Note that Σ_{AB} are the generators of the Lorentz algebra in the tangent space. The covariant derivative ∇_M does not appear in the Lagrangian explicitly but the Einstein-Hilbert term (the scalar curvature) is the contraction of Riemann tensor, which is defined by the commutator of the covariant derivative ∇_M. The precise definitions are described in appendix A.6. We mention that the Lagrangian (III.1.1) is defined up to torsion contributions because the terms derived from the torsion do not contribute to the analysis of the linearized supergravity in this thesis. We should consider such a contribution to the Lagrangian in order to discuss full nonlinear supergravity. (We will prepare the full supergravity Lagrangian in appendix B.3.) In addition, this Lagrangian is invariant under the local supersymmetry transformation with fermionic parameter $\varepsilon(x)$:

$$\begin{aligned} \delta e_M{}^A \;&=\; \frac{1}{2} \overline{\varepsilon} \widehat{\Gamma}^A \Psi_M \,, \qquad & \delta C_{MNP} \;&=\; -\frac{3}{2} \overline{\varepsilon} \widehat{\Gamma}_{[MN} \Psi_{P]} \,, \\ \delta \Psi_M \;&=\; 2 D_M \varepsilon + 2 F_{NPQR} T_M{}^{NPQR} \varepsilon \,, \qquad & T_M{}^{NPQR} \;&=\; \frac{1}{288} \Big(\widehat{\Gamma}_M{}^{NPQR} - 8 \delta_M^{[N} \widehat{\Gamma}^{PQR]} \Big) \,, \end{aligned}$$

where we also neglect the higher order contribution with respect to torsion.

Classical Field Equations

Varying g_{MN}, Ψ_M and C_{MNP}, we obtain classical field equations from the Lagrangian (III.1.1):

$$0 \;=\; \frac{1}{2} g_{MN} \mathcal{R} - \mathcal{R}_{MN} - \frac{1}{96} g_{MN} F_{PQRS} F^{PQRS} + \frac{1}{12} F_{MPQR} F_N{}^{PQR} \,, \tag{III.1.2a}$$

$$0 \;=\; \widehat{\Gamma}^{MNP} D_N \Psi_P + \frac{1}{96} \widetilde{\Gamma}^{MNPQRS} \Psi_N F_{PQRS} \,, \tag{III.1.2b}$$

$$0 \;=\; \nabla^Q \big\{ e F_{QMNP} \big\} - \frac{18}{(144)^2} \, g_{MZ} \, g_{NK} \, g_{PL} \, \varepsilon^{ZKLQRSUVWXY} F_{QRSU} F_{VWXY} \,. \tag{III.1.2c}$$

Note that we neglect the gravitino quadratic term $\overline{\Psi}_M \widetilde{\Gamma}^{MNPQRS} \Psi_N$ which does not contribute to the linearized field equations for fluctuation fields which we will calculate in later discussions. From the classical field equation for the metric g_{MN}, we find that some relations among curvatures and four-form flux F_{MNPQ}. Contracting curved spacetime indices in (III.1.2a), we obtain the equations

$$\mathcal{R}_{MN} \;=\; -\frac{1}{144} g_{MN} \, F_{PQRS} \, F^{PQRS} + \frac{1}{12} F_{MPQR} \, F_N{}^{PQR} \,, \tag{III.1.3a}$$

$$\mathcal{R} \;=\; \frac{1}{144} F_{PQRS} \, F^{PQRS} \,. \tag{III.1.3b}$$

Thus we find that if there are some non-vanishing constant four-form flux in eleven-dimensional spacetime, the spacetime have nontrivial curvature and will be compactified. This mechanism derived from the existence of constant flux is called "spontaneous compactification", and the assumption of a constant flux is called the "Freund-Rubin Ansatz" [66, 59].

From now on we consider field equations for fluctuation fields in terms of equations (III.1.3a), (III.1.2b) and (III.1.2c).

Fluctuations

Let us consider equations of motion of fluctuation fields in eleven-dimensional spacetime. We make fields fluctuate around the eleven-dimensional spacetime background:

$$\begin{aligned} g_{MN} &= \mathring{g}_{MN} + h_{MN} \, , \qquad g^{MN} = \mathring{g}^{MN} + \widetilde{h}^{MN} \, , \\ \Psi_M &= 0 + \psi_M \, , \\ F_{MNPQ} &= \mathring{F}_{MNPQ} + \mathcal{F}_{MNPQ} \, , \qquad \mathcal{F}_{MNPQ} = 4\partial_{[M}\mathcal{C}_{NPQ]} \, . \end{aligned} \tag{III.1.4}$$

In order to preserve the Lorentz invariance in the tangent space, we assume that the gravitino background field vanishes. The fluctuation of the inverse metric $\widetilde{h}^{MN}$ is represented by $\widetilde{h}^{MN} = -\mathring{g}^{MP}\,\mathring{g}^{NQ}\,h_{PQ} = -h^{MN}$. Under the above expansions, we calculate fluctuations of the determinant of vielbein e, affine connection Γ^P_{MN}, Ricci tensor $\mathcal{R}_{MN}$ and scalar curvature $\mathcal{R}$ as follows[1]:

$$\begin{aligned} \delta e &= \frac{1}{2} e\, g^{MN}\, h_{MN} \, , \\ \delta\Gamma^M_{NP} &= \frac{1}{2} g^{MR}\big(\nabla_N h_{PR} + \nabla_P h_{NR} - \nabla_R h_{NP}\big) \, , \\ \delta\mathcal{R}_{MN} &= -\frac{1}{2}\Big\{\nabla_N\nabla_M h_P{}^P - \nabla_N\nabla^P h_{MP} - \nabla_M\nabla^P h_{NP}\Big\} + \frac{1}{2}\widehat{\Delta} h_{MN} \, , \\ \delta\mathcal{R} &= -h_{PQ}\, g^{MP}\, g^{NQ}\, \mathcal{R}_{MN} + \mathring{g}^{MN}\, \delta\mathcal{R}_{MN} \, . \end{aligned}$$

Note that the above covariant derivative ∇_M is written in terms of the classical affine connection $\Gamma^P_{MN} = \frac{1}{2} g^{PR}(\partial_M g_{NR} + \partial_N g_{MR} - \partial_R g_{MN})$ because the plane-wave background, on which we analyze the physical modes, is torsion free; the operator $\widehat{\Delta}$ is called the Lichnerowicz operator which acts on the rank two symmetric tensor h_{MN} below [59]:

$$\widehat{\Delta} h_{MN} = -\nabla_P\nabla^P h_{MN} - 2R_{MPNQ}\, h^{PQ} + \mathcal{R}_M{}^P\, h_{PN} + \mathcal{R}_N{}^P\, h_{PM} \, .$$

[1] From now on we omit the circle in (III.1.4), which is the symbol of classical background.

In terms of these expressions we derive the following linearized field equations for fluctuation fields from the classical field equations (III.1.2):

$$\begin{aligned}\delta\mathcal{R}_{MN} &= -\frac{1}{2}\Big\{\nabla_N\nabla_M h_P{}^P - \nabla_N\nabla^P h_{MP} - \nabla_M\nabla^P h_{NP}\Big\} + \frac{1}{2}\widehat{\Delta} h_{MN} \\ &= -\frac{1}{144}h_{MN}\,F_{PQRS}\,F^{PQRS} - \frac{1}{72}g_{MN}\,F^{PQRS}\,\mathcal{F}_{PQRS} + \frac{1}{36}h^{PU}\,g_{MN}\,F_{PQRS}\,F_U{}^{QRS} \\ &\quad + \frac{1}{12}\Big(\mathcal{F}_{MPQR}\,F_N{}^{PQR} + \mathcal{F}_{NPQR}\,F_M{}^{PQR}\Big) - \frac{1}{4}h^{PU}\,F_{MPQR}\,F_{NU}{}^{QR}\ , \end{aligned} \tag{III.1.5a}$$

$$0 = \widehat{\Gamma}^{MNP}\,D_N\psi_P + \frac{1}{96}\widetilde{\Gamma}^{MNPQRS}\,F_{PQRS}\,\psi_N\ , \tag{III.1.5b}$$

$$\begin{aligned}0 &= e\Big\{\frac{1}{2}h_U{}^U\,g^{QR} - h^{QR}\Big\}\nabla_R F_{QMNP} + e\,\nabla^Q\mathcal{F}_{QMNP} \\ &\quad - e\Big\{F_{SMNP}\Big(\nabla^Q h_Q{}^S - \frac{1}{2}\partial^S h_Q{}^Q\Big) + F_{QSNP}\nabla^Q h_M{}^S + F_{QMSP}\nabla^Q h_N{}^S + F_{QMNS}\nabla^Q h_P{}^S\Big\} \\ &\quad - \frac{1}{576}\,\varepsilon^{ZKLQRSUVWXY}\,\mathcal{F}_{QRSU}\,g_{MZ}\,g_{NK}\,g_{PL}\,F_{VWXY} \\ &\quad - \frac{18}{(144)^2}\,\varepsilon^{ZKLQRSUVWXY}\big(h_{MZ}\,g_{NK}\,g_{PL} + h_{NK}\,g_{MZ}\,g_{PL} + h_{PL}\,g_{MZ}\,g_{NK}\big)F_{QRSU}\,F_{VWXY}\ . \end{aligned} \tag{III.1.5c}$$

Notice that the gray-colored terms do not contribute to the equations under the Freund-Rubin ansatz which we will assume on the plane-wave background in the next section.

III.2 Plane-wave Background

In the previous section we defined the supergravity Lagrangian and derived the classical field equations from it. Furthermore we made fields fluctuate around general classical backgrounds. Since the main theme of this section is to investigate the spectrum of fluctuation fields around the plane-wave background, we introduce the geometrical variables on this specific spacetime

$$\begin{aligned}\mathrm{d}s^2 &= -2\mathrm{d}x^+\mathrm{d}x^- + G_{++}\cdot(\mathrm{d}x^+)^2 + \sum_{I=1}^{9}(\mathrm{d}x^I)^2\ , \\ G_{++} &= -\Big[\Big(\frac{\mu}{3}\Big)^2\sum_{\widetilde{I}=1}^{3}(x^{\widetilde{I}})^2 + \Big(\frac{\mu}{6}\Big)^2\sum_{I'=4}^{9}(x^{I'})^2\Big]\ .\end{aligned} \tag{III.2.1}$$

Under this background we can set the constant four-form flux as the Freund-Rubin ansatz

$$F_{123+} = \mu \neq 0\ .$$

In our consideration, no contributions from torsion are included, i.e., the affine connection is symmetric under lower indices: $\Gamma^P_{MN} = \Gamma^P_{NM}$. The components of vielbein, affine connection, spin connection and their curvature tensors are described below (see also appendix C.1):

$$\begin{aligned} e_+{}^+ \ &= \ e_-{}^- \ = \ 1 \, , \qquad e_+{}^- \ = \ -\frac{1}{2} G_{++} \, , \\ E_+{}^+ \ &= \ E_-{}^- \ = \ 1 \, , \qquad E_+{}^- \ = \ \frac{1}{2} G_{++} \, , \\ \omega_+{}^{I-} \ &= \ \frac{1}{2} \partial_I G_{++} \, , \\ \Gamma^I_{++} \ &= \ \Gamma^-_{+I} \ = \ -\frac{1}{2} \partial_I G_{++} \, , \\ R^I{}_{+J+} \ &= \ -\frac{1}{2} \partial_I \partial_J G_{++} \, , \qquad \mathcal{R}_{++} \ = \ \frac{1}{2} \mu^2 \, , \qquad \mathcal{R} \ = \ 0 \, . \end{aligned} \tag{III.2.2}$$

Note that this background is almost flat but non-trivial curvature tensor which is proportional to the constant parameter μ. This constant comes from the non-vanishing constant flux F_{123+}. Substituting (III.2.2) into the field equations for fluctuations (III.1.5), we will discuss the linearized supergravity and its spectrum on the plane-wave background.

III.3 Light-cone Hamiltonian on the Plane-wave

Now let us discuss the Hamiltonian and its energy eigenvalue. We need to calculate and solve field equations for fluctuation modes around the plane-wave background in the next section. Then we will encounter Klein-Gordon type equations of motion and have to evaluate its energy spectrum.

We shall consider a Klein-Gordon type equation of motion for a field $\phi(x)$:

$$\big(\Box + \alpha \, \mu \, i \partial_- \big) \phi(x^+, x^-, x^I) \ = \ 0 \, , \tag{III.3.1}$$

where α is an arbitrary numerical constant and x^+ is an evolution parameter. The d'Alembertian $\Box$ on the plane-wave background is given by

$$\begin{aligned} \Box \ &= \ -\nabla^P \nabla_P \ = \ -\partial^P \partial_P \\ &= \ -\frac{1}{\sqrt{-g}} \partial_M \big(\sqrt{-g} g^{MN} \partial_N \big) \ = \ 2 \partial_+ \partial_- + G_{++} \cdot (\partial_-)^2 - (\partial_K)^2 \, . \end{aligned}$$

The above Klein-Gordon type field equation will appear later as equations of motion of fluctuation fields. Fourier transformed expression of $\phi(x)$

$$\phi(x^+, x^-, x^I) \ = \ \int \frac{\mathrm{d} p_- \mathrm{d}^9 p_I}{\sqrt{(2\pi)^{10}}} \, \mathrm{e}^{i (p_- x^- + p_I x^I)} \, \widetilde{\phi}(x^+, p_-, p_I)$$

leads to the following expression:

$$0 \;=\; 2p_- \, i\partial_+ - \widetilde{G}_{++} \cdot (p_-)^2 + (p_I)^2 - \alpha\, \mu\, p_- \ ,$$

where $\widetilde{G}_{++}$ is defined by

$$\widetilde{G}_{++} \;\equiv\; \Big(\frac{\mu}{3}\Big)^2 \sum_{\widetilde{I}=1}^{3} (\partial_{p_{\widetilde{I}}})^2 + \Big(\frac{\mu}{6}\Big)^2 \sum_{I'=4}^{9} (\partial_{p_{I'}})^2 \ .$$

By rewriting the above equation and defining the Hamiltonian $H = i\partial_+$, we obtain the explicit expression for the Hamiltonian

$$H \;=\; \frac{1}{-2p_-} \big\{ (p_I)^2 - \widetilde{G}_{++} \cdot (p_-)^2 - \alpha\, \mu\, p_- \big\} \ .$$

The energy spectrum of this Hamiltonian can be derived via the standard technique of harmonic oscillators. Now we define "creation/annihilation" operators

$$\begin{aligned}
a^{\widetilde{I}} \;&\equiv\; \frac{1}{\sqrt{2\widetilde{m}}} \big\{ p_{\widetilde{I}} + \widetilde{m} \partial_{p_{\widetilde{I}}} \big\} \ , &
\overline{a}^{\widetilde{I}} \;&\equiv\; \frac{1}{\sqrt{2\widetilde{m}}} \big\{ p_{\widetilde{I}} - \widetilde{m} \partial_{p_{\widetilde{I}}} \big\} \ , &
\widetilde{m} \;&\equiv\; -\frac{1}{3} \mu\, p_- \ , \\
a^{I'} \;&\equiv\; \frac{1}{\sqrt{2m'}} \big\{ p_{I'} + m' \partial_{p_{I'}} \big\} \ , &
\overline{a}^{I'} \;&\equiv\; \frac{1}{\sqrt{2m'}} \big\{ p_{I'} - m' \partial_{p_{I'}} \big\} \ , &
m' \;&\equiv\; -\frac{1}{6} \mu\, p_- \ ,
\end{aligned}$$

whose commutation relations are represented by

$$[a^{\widetilde{I}}, \overline{a}^{\widetilde{J}}] \;=\; \delta^{\widetilde{I}\widetilde{J}} \ , \qquad [a^{I'}, \overline{a}^{J'}] \;=\; \delta^{I'J'} \ , \qquad [a^{\widetilde{I}}, \overline{a}^{J'}] \;=\; [a^{I'}, \overline{a}^{\widetilde{J}}] \;=\; 0 \ .$$

Thus we express the Hamiltonian in terms of the above oscillators:

$$H \;=\; \frac{1}{3} \mu \sum_{\widetilde{I}} \overline{a}^{\widetilde{I}} a^{\widetilde{I}} + \frac{1}{6} \mu \sum_{I'} \overline{a}^{I'} a^{I'} + \frac{1}{2} \mu \, (2 + \alpha) \ .$$

Note that the last term implies the zero point energy E_0 of the system, which is represented by

$$E_0 \;=\; \frac{1}{2} \mu\, \mathcal{E}_0(\phi) \ , \qquad \mathcal{E}_0(\phi) \;=\; 2 + \alpha \ . \tag{III.3.2}$$

In the next section, we will use $\mathcal{E}_0$ to evaluate the energy of the zero-modes of fluctuation fields.

After the above setup, we will discuss the physical spectrum of fluctuation fields around the plane-wave background. First we will take the light-cone gauge for fluctuation fields and reduce field equations of them. After field re-definition we will discuss the zero-point energy and the number of physical degrees of freedom.

III.4 Field Equations for Fluctuations on the Plane-wave Background

We discuss the spectrum of fluctuation fields on the plane-wave background. In order to consider the spectrum of the physical fields we take the light-cone gauge fixing as follows:

$$h_{-M} = 0 \qquad h^{+M} = 0 \qquad \mathcal{C}_{-MN} = 0 \qquad \psi_{-} = 0 \,. \tag{III.4.1}$$

We write all the field equations for fluctuation fields h_{MN}, ψ_M and $\mathcal{C}_{MNP}$ on the plane-wave background (III.2.1) and (III.2.2) under the light-cone gauge-fixing condition (III.4.1). First, the following field equations are derived from (III.1.5a):

$$\begin{aligned}
0 \;=\;& \frac{1}{2}\Big\{ \nabla_{+}\nabla_{+} h_{P}{}^{P} - \nabla_{+}\nabla^{P} h_{+P} - \nabla_{+}\nabla^{P} h_{+P} - \Box\, h_{++} \Big\} - \Big(\frac{\mu}{3}\Big)^2 h_{\widetilde{K}\widetilde{K}} - \Big(\frac{\mu}{6}\Big)^2 h_{L'L'} \\
& - \frac{1}{3}\mu\, G_{++}\, \partial_{-}\mathcal{C}_{123} - \mu\, \mathcal{F}_{+123} - \frac{1}{2}\mu^2\, h_{\widetilde{L}\widetilde{L}} \,, && \text{(III.4.2a)} \\
0 \;=\;& \Big\{ \partial_{-}\partial_{+} h_{P}{}^{P} - \partial_{-}\partial^{P} h_{+P} \Big\} - \frac{1}{3}\mu\, \partial_{-}\mathcal{C}_{123} \,, && \text{(III.4.2b)} \\
0 \;=\;& \Big\{ \nabla_{\widetilde{I}}\nabla_{+} h_{P}{}^{P} - \partial_{\widetilde{I}}\partial^{P} h_{+P} - \partial_{+}\partial^{P} h_{\widetilde{I}P} - \Box\, h_{+\widetilde{I}} \Big\} + \frac{1}{2}\mu\, \epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}}\, \partial_{-}\mathcal{C}_{+\widetilde{J}\widetilde{K}} \,, && \text{(III.4.2c)} \\
0 \;=\;& \Big\{ \nabla_{I'}\nabla_{+} h_{P}{}^{P} - \partial_{I'}\partial^{P} h_{+P} - \partial_{+}\partial^{P} h_{I'P} - \Box\, h_{+I'} \Big\} - \frac{1}{6}\mu\, \epsilon_{\widetilde{J}\widetilde{K}\widetilde{L}} \mathcal{F}_{I'\widetilde{J}\widetilde{K}\widetilde{L}} \,, && \text{(III.4.2d)} \\
0 \;=\;& \partial_{-}\partial_{-} h_{P}{}^{P} \,, && \text{(III.4.2e)} \\
0 \;=\;& \partial_{I}\partial_{-} h_{P}{}^{P} - \partial_{-}\partial^{P} h_{IP} \,, && \text{(III.4.2f)} \\
0 \;=\;& \Big\{ \partial_{\widetilde{J}}\partial_{\widetilde{I}} h_{P}{}^{P} - \partial_{\widetilde{J}}\partial^{P} h_{\widetilde{I}P} - \partial_{\widetilde{I}}\partial^{P} h_{\widetilde{J}P} - \Box\, h_{\widetilde{I}\widetilde{J}} \Big\} + \frac{4}{3}\mu\, \delta_{\widetilde{I}\widetilde{J}}\, \partial_{-}\mathcal{C}_{123} \,, && \text{(III.4.2g)} \\
0 \;=\;& \Big\{ \partial_{J'}\partial_{\widetilde{I}} h_{P}{}^{P} - \partial_{J'}\partial^{P} h_{\widetilde{I}P} - \partial_{\widetilde{I}}\partial^{P} h_{J'P} - \Box\, h_{\widetilde{I}J'} \Big\} + \frac{1}{2}\mu\, \epsilon_{\widetilde{I}\widetilde{K}\widetilde{L}}\, \partial_{-}\mathcal{C}_{J'\widetilde{K}\widetilde{L}} \,, && \text{(III.4.2h)} \\
0 \;=\;& \Big\{ \partial_{J'}\partial_{I'} h_{P}{}^{P} - \partial_{J'}\partial^{P} h_{I'P} - \partial_{I'}\partial^{P} h_{J'P} - \Box\, h_{I'J'} \Big\} - \frac{2}{3}\mu\, \delta_{I'J'}\, \partial_{-}\mathcal{C}_{123} \,. && \text{(III.4.2i)}
\end{aligned}$$

The next four equations are the components of field equations (III.1.5b):

$$\begin{aligned}
0 \;=\;& \widehat{\Gamma}^{+NP} D_{N}\psi_{P} \,, && \text{(III.4.3a)} \\
0 \;=\;& \widehat{\Gamma}^{-NP} D_{N}\psi_{P} + \frac{1}{4}\mu\, \widehat{\Gamma}^{+-123I'}\, \psi_{I'} + \frac{1}{8}\mu\, \epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}} \widehat{\Gamma}_{\widetilde{I}\widetilde{J}}\, \psi_{\widetilde{K}} \,, && \text{(III.4.3b)} \\
0 \;=\;& \widehat{\Gamma}^{\widetilde{I}NP} D_{N}\psi_{P} - \frac{1}{4}\mu\, \widehat{\Gamma}^{+123} \big(\delta_{\widetilde{I}\widetilde{J}} - \widehat{\Gamma}_{\widetilde{I}} \widehat{\Gamma}_{\widetilde{J}} \big) \psi_{\widetilde{J}} \,, && \text{(III.4.3c)} \\
0 \;=\;& \widehat{\Gamma}^{I'NP} D_{N}\psi_{P} + \frac{1}{4}\mu\, \widehat{\Gamma}^{+123} \big(\delta_{I'J'} - \widehat{\Gamma}_{I'} \widehat{\Gamma}_{J'} \big) \psi_{J'} \,. && \text{(III.4.3d)}
\end{aligned}$$

Finally, we write the components of field equations (III.1.5c) under the light-cone gauge fixing:

$$\begin{aligned}
0 \;=\;& \partial_{-}\partial^{Q}\mathcal{C}_{Q+I} \,, && \text{(III.4.4a)} \\
0 \;=\;& \partial^{Q}\mathcal{F}_{Q+\widetilde{I}\widetilde{J}} - \partial_{K} G_{++} \partial_{-}\mathcal{C}_{K\widetilde{I}\widetilde{J}}
\end{aligned}$$

$$
\begin{aligned}
& -\mu\,\epsilon_{\widetilde{I}\widetilde{J}\widetilde{L}}\Big(\partial^Q h_{Q\widetilde{L}} - \frac{1}{2}\partial_{\widetilde{L}} h_{KK}\Big) + \mu\,\epsilon_{\widetilde{I}\widetilde{J}\widetilde{L}}\partial^+ h_{+\widetilde{L}} - \mu\,\epsilon_{\widetilde{J}\widetilde{K}\widetilde{L}}\partial_{\widetilde{K}} h_{\widetilde{I}\widetilde{L}} + \mu\,\epsilon_{\widetilde{I}\widetilde{K}\widetilde{L}}\partial_{\widetilde{K}} h_{\widetilde{J}\widetilde{L}} \,, && \text{(III.4.4b)} \\
0 &= \partial^Q \mathcal{F}_{Q+\widetilde{I}J'} - \partial_K G_{++}\partial_- \mathcal{C}_{K\widetilde{I}J'} + \mu\,\epsilon_{\widetilde{I}\widetilde{K}\widetilde{L}}\partial_{\widetilde{K}} h_{J'\widetilde{L}} \,, && \text{(III.4.4c)} \\
0 &= \partial^Q \mathcal{F}_{Q+I'J'} - \partial_K G_{++}\partial_- \mathcal{C}_{KI'J'} + \frac{1}{24}\mu\,\varepsilon^{I'J'Q'R'S'U'}\mathcal{F}_{Q'R'S'U'} \,, && \text{(III.4.4d)} \\
0 &= -\partial_- \partial^Q \mathcal{C}_{QIJ} \,, && \text{(III.4.4e)} \\
0 &= \partial^Q \mathcal{F}_{Q\widetilde{I}\widetilde{J}\widetilde{K}} - \frac{1}{2}\mu\,\epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}}\partial^+ h_{LL} + \mu\,\epsilon_{\widetilde{J}\widetilde{K}\widetilde{L}}\partial^+ h_{\widetilde{I}\widetilde{L}} - \mu\,\epsilon_{\widetilde{I}\widetilde{K}\widetilde{L}}\partial^+ h_{\widetilde{J}\widetilde{L}} + \mu\,\epsilon_{\widetilde{I}\widetilde{J}\widetilde{L}}\partial^+ h_{\widetilde{K}\widetilde{L}} \,, && \text{(III.4.4f)} \\
0 &= \partial^Q \mathcal{F}_{Q\widetilde{I}\widetilde{J}K'} + \mu\,\epsilon_{\widetilde{I}\widetilde{J}\widetilde{L}}\partial^+ h_{K'\widetilde{L}} \,, && \text{(III.4.4g)} \\
0 &= \partial^Q \mathcal{F}_{Q\widetilde{I}J'K'} \,, && \text{(III.4.4h)} \\
0 &= \partial^Q \mathcal{F}_{QI'J'K'} + \frac{1}{6}\mu\,\varepsilon^{I'J'K'R'S'U'}\partial_- \mathcal{C}_{R'S'U'} \,. && \text{(III.4.4i)}
\end{aligned}
$$

These equations are somewhat complicated and one might wonder whether these equations can be solved explicitly. But, we can obtain some constraints from the above equations, and we will be able to solve the other equations completely when we substitute the constraints into the equations!

Physical Modes of Bosonic Fields

Now let us derive a physical spectrum of the bosonic fields under the light-cone gauge-fixing: $h_{-M} = \mathcal{C}_{-MN} = 0$. All we have to do is to analyze physical modes in linearized field equations. First, we find a traceless condition

$$
0 \;=\; h_M{}^M \;=\; h_{II} \qquad \text{(III.4.5)}
$$

from the field equation (III.4.2e). This condition, which the graviton h_{MN} should satisfy, is derived from the light-cone gauge-fixing $h_{-M} = 0$. Substituting (III.4.5) into (III.4.2f) leads to the divergence free condition for the graviton field $\partial^M h_{IM} = 0$ and we can rewrite h_{I+} as

$$
h_{I+} \;=\; \frac{1}{\partial_-}\partial_J h_{IJ} \,.
$$

Thus we find that h_{I+} is non-dynamical. Moreover, we obtain another constraint

$$
\partial^M h_{+M} \;=\; \frac{1}{3}\mu\mathcal{C}_{123} \,,
$$

which leads to the expression for h_{++} from the equation (III.4.2b) as

$$
h_{++} \;=\; \frac{1}{(\partial_-)^2}\partial_I\partial_J h_{IJ} + \frac{1}{3\partial_-}\mu\,\mathcal{C}_{123} \,.
$$

In contrast to the IIB supergravity case [105], h_{++} includes the term proportional to μ. The appearance of this term is characteristic of our case[2]. In the same way, we can read off the following condition from the equation (III.4.4a): $\partial_J \mathcal{C}_{+IJ} = 0$. The equation (III.4.4e) leads to the divergence free condition $\partial^M \mathcal{C}_{MIJ} = 0$ and the expression for the field $\mathcal{C}_{+IJ}$ is

$$\mathcal{C}_{+IJ} \;=\; \frac{1}{\partial_-}\partial_K \mathcal{C}_{IJK} \; .$$

We find that $\mathcal{C}_{+IJ}$ is also non-dynamical.

Under the light-cone gauge-fixing conditions and the above mentioned conditions for the non-dynamical modes, we can reduce field equations for h_{MN} and $\mathcal{C}_{MNP}$ as follows:

$$\text{equation (III.4.2g)}: \qquad 0 \;=\; \Box\, h_{\widetilde{I}\widetilde{J}} - \frac{2}{3}\mu\, \delta_{\widetilde{I}\widetilde{J}} \partial_- \mathcal{C} \, , \tag{III.4.6a}$$

$$\text{equation (III.4.2h)}: \qquad 0 \;=\; \Box\, h_{\widetilde{I}J'} - \mu\, \partial_- \mathcal{C}_{\widetilde{I}J'} \, , \tag{III.4.6b}$$

$$\text{equation (III.4.2i)}: \qquad 0 \;=\; \Box\, h_{I'J'} + \frac{1}{3}\mu\, \delta_{I'J'} \partial_- \mathcal{C} \, , \tag{III.4.6c}$$

$$\text{equation (III.4.4f)}: \qquad 0 \;=\; \Box\, \mathcal{C} + 2\mu\, \partial_- h_{\widetilde{I}\widetilde{I}} \, , \tag{III.4.6d}$$

$$\text{equation (III.4.4g)}: \qquad 0 \;=\; \Box\, \mathcal{C}_{\widetilde{I}J'} + \mu\, \partial_- h_{\widetilde{I}J'} \, , \tag{III.4.6e}$$

$$\text{equation (III.4.4h)}: \qquad 0 \;=\; \Box\, \mathcal{C}_{\widetilde{I}J'K'} \, , \tag{III.4.6f}$$

$$\text{equation (III.4.4i)}: \qquad 0 \;=\; \Box\, \mathcal{C}_{I'J'K'} - \frac{1}{6}\mu\, \varepsilon^{I'J'K'W'X'Y'} \partial_- \mathcal{C}_{W'X'Y'} \, , \tag{III.4.6g}$$

where $\varepsilon^{I'J'K'W'X'Y'}$ is the $SO(6)$ invariant tensor density (or equivalently, the Levi-Civita symbol) whose normalization is $\varepsilon^{456789} = \varepsilon_{456789} = 1$. Note that we wrote the above equations in terms of the following two quantities defined by

$$\mathcal{C}_{\widetilde{I}J'} \;\equiv\; \frac{1}{2}\epsilon_{\widetilde{I}\widetilde{K}\widetilde{L}} \mathcal{C}_{\widetilde{K}\widetilde{L}J'} \, , \qquad \mathcal{C} \;\equiv\; 2\mathcal{C}_{123} \, ,$$

where we introduced the $SO(3)$ invariant tensor density (or equivalently, Levi-Civita symbol) $\epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}}$ ($\epsilon_{123} = \epsilon^{123} = 1$).

Now let us solve the above reduced equations of motion for fluctuation modes, and derive the zero-mode energy spectrum and degrees of freedom of bosonic fields. We consider the field $\mathcal{C}_{\widetilde{I}J'K'}$. From the above equation (III.4.6f), we find that this field does not couple to the other fields. So the zero point energy $\mathcal{E}_0(\mathcal{C}_{\widetilde{I}J'K'})$ and degrees of freedom $\mathcal{D}(\mathcal{C}_{\widetilde{I}J'K'})$ are given by

$$\mathcal{E}_0(\mathcal{C}_{\widetilde{I}J'K'}) \;=\; 2 \, , \qquad \mathcal{D}(\mathcal{C}_{\widetilde{I}J'K'}) \;=\; 45 \, . \tag{III.4.7}$$

[2]The spectrum of type IIA string theory and linearized supergravity is studied in [98]. In this case h_{++} contains the additional term proportional to μ.

Next, we consider $SO(3)\times SO(6)$ tensor fields $h_{\tilde{I}J'}$ and $\mathcal{C}_{\tilde{I}J'}$ coupled to each other. In order to diagonalize these coupled fields, we define two complex fields $H_{\tilde{I}J'}$ and $\overline{H}_{\tilde{I}J'}$ as

$$H_{\tilde{I}J'} \;=\; h_{\tilde{I}J'} + i\mathcal{C}_{\tilde{I}J'}\,, \qquad \overline{H}_{\tilde{I}J'} \;=\; h_{\tilde{I}J'} - i\mathcal{C}_{\tilde{I}J'}\,.$$

By using these fields, (III.4.6b) and (III.4.6e) can be rewritten as

$$0 \;=\; \big(\Box + \mu\, i\partial_-\big)H_{\tilde{I}J'}\,, \qquad 0 \;=\; \big(\Box - \mu\, i\partial_-\big)\overline{H}_{\tilde{I}J'}\,.$$

Thus the zero point energies and degrees of freedom of $H_{\tilde{I}J'}$ and $\overline{H}_{\tilde{I}J'}$ are given by

$$\mathcal{E}_0(H_{\tilde{I}J'}) \;=\; 3\,, \qquad \mathcal{E}_0(\overline{H}_{\tilde{I}J'}) \;=\; 1\,, \qquad \mathcal{D}(H_{\tilde{I}J'}) \;=\; \mathcal{D}(\overline{H}_{\tilde{I}J'}) \;=\; 18\,. \tag{III.4.8}$$

Then we will solve the field equations (III.4.6a), (III.4.6c) and (III.4.6d) concerning $h_{\tilde{I}\tilde{J}}$, $h_{I'J'}$ and $\mathcal{C}$. Since these fields are coupled to one another, we have to diagonalize these fields in order to solve the equations. Hence let us introduce the following fields:

$$\begin{aligned} h^{\perp}_{\tilde{I}\tilde{J}} \;&\equiv\; h_{\tilde{I}\tilde{J}} - \frac{1}{3}\delta_{\tilde{I}\tilde{J}}h_{\tilde{K}\tilde{K}}\,, \qquad & h^{\perp}_{I'J'} \;&\equiv\; h_{I'J'} - \frac{1}{6}\delta_{I'J'}h_{K'K'}\,, \\ h \;&\equiv\; h_{\tilde{K}\tilde{K}} + i\mathcal{C}\,, & \overline{h} \;&\equiv\; h_{\tilde{K}\tilde{K}} - i\mathcal{C}\,. \end{aligned}$$

Note that $h^{\perp}_{\tilde{I}\tilde{J}}$ and $h^{\perp}_{I'J'}$ are transverse modes and two complex scalar fields h and $\overline{h}$ are trace modes. In this re-definition we find $\Box h^{\perp}_{\tilde{I}\tilde{J}} = 0$, and so its energy and degrees of freedom are given by

$$\mathcal{E}_0(h^{\perp}_{\tilde{I}\tilde{J}}) \;=\; 2\,, \qquad \mathcal{D}(h^{\perp}_{\tilde{I}\tilde{J}}) \;=\; 5\,. \tag{III.4.9}$$

Since we also find $\Box h^{\perp}_{I'J'} = 0$, we obtain the energy and degrees of freedom of $h^{\perp}_{I'J'}$:

$$\mathcal{E}_0(h^{\perp}_{I'J'}) \;=\; 2\,, \qquad \mathcal{D}(h^{\perp}_{I'J'}) \;=\; 20\,. \tag{III.4.10}$$

Similarly the field equations for h and $\overline{h}$ are described by

$$\big(\Box + 2\mu\, i\partial_-\big)h \;=\; 0\,, \qquad \big(\Box - 2\mu\, i\partial_-\big)\overline{h} \;=\; 0\,.$$

Thus the energies and degrees of freedom of them are

$$\mathcal{E}_0(h) \;=\; 4\,, \qquad \mathcal{E}_0(\overline{h}) \;=\; 0\,, \qquad \mathcal{D}(h) \;=\; \mathcal{D}(\overline{h}) \;=\; 1\,. \tag{III.4.11}$$

Finally we consider (III.4.6g) by decomposing $\mathcal{C}_{I'J'K'}$ into self-dual part and anti-self-dual part as follows: $\mathcal{C}_{I'J'K'} \equiv \mathcal{C}^{\oplus}_{I'J'K'} + \mathcal{C}^{\ominus}_{I'J'K'}$, where $\mathcal{C}^{\oplus}_{I'J'K'}$ is a self-dual part and $\mathcal{C}^{\ominus}_{I'J'K'}$ is an anti-self-dual part. These are defined by, respectively,

$$\mathcal{C}^{\oplus}_{I'J'K'} \;=\; \frac{i}{3!}\varepsilon^{I'J'K'W'X'Y'}\mathcal{C}^{\oplus}_{W'X'Y'}\,, \qquad \mathcal{C}^{\ominus}_{I'J'K'} \;=\; -\frac{i}{3!}\varepsilon^{I'J'K'W'X'Y'}\mathcal{C}^{\ominus}_{W'X'Y'}\,.$$

Due to this decomposition, the field equations of them are expressed as

$$(\Box + \mu\, i\partial_-)\mathcal{C}^{\oplus}_{I'J'K'} \ = \ 0 \,, \qquad (\Box - \mu\, i\partial_-)\mathcal{C}^{\ominus}_{I'J'K'} \ = \ 0 \,,$$

and hence we find the energies and degrees of freedom of $\mathcal{C}^{\oplus}_{I'J'K'}$ and $\mathcal{C}^{\ominus}_{I'J'K'}$:

$$\mathcal{E}_0(\mathcal{C}^{\oplus}_{I'J'K'}) \ = \ 3 \,, \qquad \mathcal{E}_0(\mathcal{C}^{\ominus}_{I'J'K'}) \ = \ 1 \,, \qquad \mathcal{D}(\mathcal{C}^{\oplus}_{I'J'K'}) \ = \ \mathcal{D}(\mathcal{C}^{\ominus}_{I'J'K'}) \ = \ 10 \,. \tag{III.4.12}$$

Now we have fully solved the field equations for bosonic fluctuations and have derived the spectrum of h_{MN} and $\mathcal{C}_{MNP}$. The resulting spectrum is splitting with a certain energy difference in contrast to the flat case. We summarize the spectrum of bosonic fields in Table III.1:

energy $\mathcal{E}_0$	bosonic fields	degrees of freedom
4	h	1
3	$H_{\widetilde{I}J'}$ $\quad$ $\mathcal{C}^{\oplus}_{I'J'K'}$	$18+10$
2	$h^{\perp}_{\widetilde{I}\widetilde{J}}$ $\quad$ $\mathcal{C}_{\widetilde{I}J'K'}$ $\quad$ $h^{\perp}_{I'J'}$	$5+45+20$
1	$\overline{H}_{\widetilde{I}J'}$ $\quad$ $\mathcal{C}^{\ominus}_{I'J'K'}$	$18+10$
0	$\overline{h}$	1

Table III.1: *Zero point energy spectrum of the bosonic fields in eleven-dimensional supergravity on the plane-wave background.*

Physical Modes of Fermionic Fields

Let us solve the field equations of the fluctuations of gravitino imposed the light-cone gauge-fixing condition $\psi_- = 0$. First, we consider the equation (III.4.3b), which is rewritten as

$$\widehat{\Gamma}^N D_N \psi_- - \widehat{\Gamma}^N D_- \psi_N \ = \ J_- - \frac{1}{9}\widehat{\Gamma}_- \widehat{\Gamma}_N J^n \,. \tag{III.4.13}$$

Note that we represent the field equations (III.4.3) as

$$\widehat{\Gamma}^{MNP} D_N \psi_P \ = \ J^M \,,$$

where J^M in the right hand side of the above equation is described by

$$J^+ \ = \ -J_- \ = \ 0 \,, \qquad J^- \ = \ -\frac{1}{4}\mu\, \widehat{\Gamma}^{+-123I'} \psi_{I'} - \frac{1}{8}\mu\, \epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}}\, \widehat{\Gamma}_{\widetilde{I}\widetilde{J}} \psi_{\widetilde{K}} \,,$$

$$J^{\widetilde{I}} \ = \ \frac{1}{4}\mu\, \widehat{\Gamma}^{+123}\big(\delta_{\widetilde{I}\widetilde{J}} - \widehat{\Gamma}_{\widetilde{I}}\widehat{\Gamma}_{\widetilde{J}}\big)\psi_{\widetilde{J}} \,, \qquad J^{I'} \ = \ -\frac{1}{4}\mu\, \widehat{\Gamma}^{+123}\big(\delta_{I'J'} - \widehat{\Gamma}_{I'}\widehat{\Gamma}_{J'}\big)\psi_{J'} \,.$$

Using the above variables and the properties of the plane-wave background (III.2.2), we simplify the equation (III.4.13) as

$$\widehat{\Gamma}^M \psi_M \;=\; 0 \,. \tag{III.4.14}$$

This constraint is the condition which the on-shell gravitino should obey. Next we consider the equation (III.4.3a), which is rewritten as

$$0 \;=\; g^{P+}\widehat{\Gamma}^N D_N \psi_P - g^{PN}\widehat{\Gamma}^+ D_N \psi_P + \frac{1}{2}\big(\widehat{\Gamma}^+\widehat{\Gamma}^N - \widehat{\Gamma}^N\widehat{\Gamma}^+\big)\widehat{\Gamma}^P D_N \psi_P \,. \tag{III.4.15}$$

We find that the first and third term are deleted by light-cone gauge-fixing and (III.4.14). Thus we can reduce (III.4.15) to $0 = \widehat{\Gamma}^+(-\partial_- \psi_+ + \partial_I \psi_I)$. So we obtain the divergence free condition for the gravitino such as $\partial^M \psi_M = 0$, which is also the condition that the on-shell gravitino should satisfy. Thus we see that ψ_+ is expressed by the other fields

$$\psi_+ \;=\; \frac{1}{\partial_-}\partial_I \psi_I$$

and we find that this component of the gravitino is a non-dynamical field.

Here we shall reduce (III.4.3c) to

$$0 \;=\; \widehat{\Gamma}^+\Big(\partial_+ + \frac{1}{2}G_{++}\partial_-\Big)\psi^{\oplus}_{\widetilde{I}} + \widehat{\Gamma}^- \partial_- \psi^{\ominus}_{\widetilde{I}} + \widehat{\Gamma}^K \partial_K (\psi^{\oplus}_{\widetilde{I}} + \psi^{\ominus}_{\widetilde{I}}) - \frac{1}{4}\mu \widehat{\Gamma}^{+123}(\delta_{\widetilde{I}\widetilde{J}} - \widehat{\Gamma}_{\widetilde{I}}\widehat{\Gamma}_{\widetilde{J}})\psi^{\oplus}_{\widetilde{J}} \,, \tag{III.4.16}$$

where we decomposed gravitino as $\psi_{\widetilde{I}} \equiv \psi^{\oplus}_{\widetilde{I}} + \psi^{\ominus}_{\widetilde{I}}$. The $\psi^{\oplus}_{\widetilde{I}}$ and $\psi^{\ominus}_{\widetilde{I}}$ are defined as

$$\psi^{\oplus}_{\widetilde{I}} \;\equiv\; -\frac{1}{2}\widehat{\Gamma}^-\widehat{\Gamma}^+\psi_{\widetilde{I}} \,, \qquad \psi^{\ominus}_{\widetilde{I}} \;\equiv\; -\frac{1}{2}\widehat{\Gamma}^+\widehat{\Gamma}^-\psi_{\widetilde{I}} \,,$$

which satisfy the projection conditions: $\widehat{\Gamma}^-\psi^{\oplus}_{\widetilde{I}} = \widehat{\Gamma}^+\psi^{\ominus}_{\widetilde{I}} = 0$. When we act $\widehat{\Gamma}^+$ on (III.4.16) from the left, $\psi^{\ominus}_{\widetilde{I}}$ can be expressed in terms $\psi^{\oplus}_{\widetilde{I}}$ as follows:

$$\psi^{\ominus}_{\widetilde{I}} \;=\; \frac{1}{2\partial_-}\widehat{\Gamma}^+\widehat{\Gamma}^K \partial_K \psi^{\oplus}_{\widetilde{I}} \,. \tag{III.4.17}$$

Thus $\psi^{\ominus}_{\widetilde{I}}$ is not independent of $\psi^{\oplus}_{\widetilde{I}}$. Similarly, when we act $\widehat{\Gamma}^-$ on (III.4.16) from the left and utilize (III.4.17), we obtain the following equation:

$$0 \;=\; \Box \psi^{\oplus}_{\widetilde{I}} - \frac{1}{2}\mu \widehat{\Gamma}^{123}(\delta_{\widetilde{I}\widetilde{J}} - \widehat{\Gamma}_{\widetilde{I}}\widehat{\Gamma}_{\widetilde{J}})\partial_- \psi^{\oplus}_{\widetilde{J}} \,. \tag{III.4.18}$$

In order to solve this equation, let us decompose the gravitino fields into the traceless part and the "trace" part with respect to the spacetime indices as follows:

$$\psi^{\oplus\perp}_{\widetilde{I}} \;\equiv\; \Big(\delta_{\widetilde{I}\widetilde{J}} - \frac{1}{3}\widehat{\Gamma}_{\widetilde{I}}\widehat{\Gamma}_{\widetilde{J}}\Big)\psi^{\oplus}_{\widetilde{J}} \,, \qquad \psi^{\oplus\parallel}_{1} \;\equiv\; \widehat{\Gamma}^{\widetilde{I}}\psi^{\oplus}_{\widetilde{I}} \;=\; \widehat{\Gamma}^{\widetilde{I}}\psi^{\oplus}_{\widetilde{I}} \,.$$

We denote the traceless part and the "trace" part to $\psi_{\widetilde{I}}^{\oplus\perp}$ and $\psi_1^{\oplus\parallel}$ and call them the $\widehat{\Gamma}$-transverse mode and the $\widehat{\Gamma}$-parallel mode, respectively. Acting $\widehat{\Gamma}^{\widetilde{I}}$ on (III.4.18) from the left and contracting the index $\widetilde{I}$, we obtain a equation with respect to the $\widehat{\Gamma}$-parallel mode $\psi_1^{\oplus\parallel}$

$$0 \;=\; \Box\, \psi_1^{\oplus\parallel} - \mu \widehat{\Gamma}^{123} \partial_- \psi_1^{\oplus\parallel} \, . \tag{III.4.19}$$

We also obtain a non-trivial equation for the $\widehat{\Gamma}$-transverse mode $\psi_{\widetilde{I}}^{\oplus\perp}$ when we act $(\delta_{\widetilde{K}\widetilde{I}} - \frac{1}{3}\widehat{\Gamma}_{\widetilde{K}}\widehat{\Gamma}_{\widetilde{I}})$ on (III.4.18):

$$0 \;=\; \Box\, \psi_{\widetilde{K}}^{\oplus\perp} - \frac{1}{2}\mu \widehat{\Gamma}^{123} \partial_- \psi_{\widetilde{K}}^{\oplus\perp} \, . \tag{III.4.20}$$

The field equations (III.4.19) and (III.4.20) contain extra factors given by the gamma matrices $\widehat{\Gamma}^{123}$ which prevent us from our obtaining the Klein-Gordon type field equations (III.3.1) for the gravitinos. Thus we decompose $\psi_{\widetilde{I}}^{\oplus\perp}$ and $\psi_1^{\oplus\parallel}$ in terms of the "chiral projection operator" $\frac{1}{2}(1 \pm i\widehat{\Gamma}^{123})$ as follows:

$$\begin{aligned} \psi_{\widetilde{I}\mathrm{R}}^{\oplus\perp} &\equiv \frac{1+i\widehat{\Gamma}^{123}}{2}\psi_{\widetilde{I}}^{\oplus\perp} \, , & \psi_{\widetilde{I}\mathrm{L}}^{\oplus\perp} &\equiv \frac{1-i\widehat{\Gamma}^{123}}{2}\psi_{\widetilde{I}}^{\oplus\perp} \, , \\ \psi_{1\mathrm{R}}^{\oplus\parallel} &\equiv \frac{1+i\widehat{\Gamma}^{123}}{2}\psi_{1}^{\oplus\parallel} \, , & \psi_{1\mathrm{L}}^{\oplus\parallel} &\equiv \frac{1-i\widehat{\Gamma}^{123}}{2}\psi_{1}^{\oplus\parallel} \, . \end{aligned}$$

These variables satisfy the following "chirality" conditions

$$\begin{aligned} i\widehat{\Gamma}^{123}\psi_{\widetilde{I}\mathrm{R}}^{\oplus\perp} &= +\psi_{\widetilde{I}\mathrm{R}}^{\oplus\perp} \, , & i\widehat{\Gamma}^{123}\psi_{\widetilde{I}\mathrm{L}}^{\oplus\perp} &= -\psi_{\widetilde{I}\mathrm{L}}^{\oplus\perp} \, , \\ i\widehat{\Gamma}^{123}\psi_{1\mathrm{R}}^{\oplus\parallel} &= +\psi_{1\mathrm{R}}^{\oplus\parallel} \, , & i\widehat{\Gamma}^{123}\psi_{1\mathrm{L}}^{\oplus\parallel} &= -\psi_{1\mathrm{L}}^{\oplus\parallel} \, . \end{aligned}$$

One can of course write down the above gravitino spinor fields in the $SO(9)$ Majorana spinor representation argued in appendix A.4. But we continue the discussion with the $SO(10,1)$ Majorana spinor representation. Multiplying the "chiral projection operators" $\frac{1}{2}(1 \pm i\widehat{\Gamma}^{123})$ to the field equation for $\widehat{\Gamma}$-parallel mode (III.4.19) on the left, we obtain

$$0 \;=\; \Big(\Box + \mu\, i\partial_-\Big)\psi_{1\mathrm{R}}^{\oplus\parallel} \, , \qquad 0 \;=\; \Big(\Box - \mu\, i\partial_-\Big)\psi_{1\mathrm{L}}^{\oplus\parallel} \, . \tag{III.4.21}$$

It appears that the equations (III.4.21) are the correct field equations for the $\widehat{\Gamma}$-parallel modes. But it is impossible to read the zero-point energy from only these equations. The reason is that the *correct* $\widehat{\Gamma}$-parallel mode is defined by the "trace" part of only the $\psi_{\widetilde{I}}$ mode, which does not include the $\psi_{I'}$ mode. Thus if we would like to obtain the correct informations of this parallel mode, we must also look at the $\widehat{\Gamma}$-parallel mode (i.e., the "trace" part) of $\psi_{I'}$ and combine the field equations for these two $\widehat{\Gamma}$-parallel modes. But we have not look at the field equations for the gravitino $\psi_{I'}$ yet. Thus we will discuss the energies of the $\widehat{\Gamma}$-parallel modes later.

Let us discuss the $\widehat{\Gamma}$-transverse mode here. In the similar way to the $\widehat{\Gamma}$-parallel modes, we obtain the Klein-Gordon type field equations when we perform the "chiral projection" to the field equation for the $\widehat{\Gamma}$-transverse mode (III.4.20) on the left:

$$0 \ = \ \Big(\Box + \frac{1}{2} \mu \, i \partial_- \Big) \psi^{\oplus \perp}_{\widetilde{I} \mathrm{R}} \ , \qquad 0 \ = \ \Big(\Box - \frac{1}{2} \mu \, i \partial_- \Big) \psi^{\oplus \perp}_{\widetilde{I} \mathrm{L}} \ .$$

In the case of these mode we can analyze the energies and the number of degrees of freedom from these equations. We can read off the zero point energies and degrees of freedom of $\psi^{\oplus \perp}_{\widetilde{I} \mathrm{R}}$ and $\psi^{\oplus \perp}_{\widetilde{I} \mathrm{L}}$ from the above equations:

$$\mathcal{E}_0 (\psi^{\oplus \perp}_{\widetilde{I} \mathrm{R}}) \ = \ \frac{5}{2} \ , \qquad \mathcal{E}_0 (\psi^{\oplus \perp}_{\widetilde{I} \mathrm{L}}) \ = \ \frac{3}{2} \ , \qquad \mathcal{D}(\psi^{\oplus \perp}_{\widetilde{I} \mathrm{R}}) \ = \ \mathcal{D}(\psi^{\oplus \perp}_{\widetilde{I} \mathrm{L}}) \ = \ 8 \times (3-1) \ = \ 16 \ . \tag{III.4.22}$$

We will discuss these quantities of $\psi^{\oplus \|}_{1\mathrm{R}}$ and $\psi^{\oplus \|}_{1\mathrm{L}}$ later.

Now we argue the other gravitino fields labeled by curved indices $I' = 4, 5, \cdots, 9$. The decomposition rules for the gravitino fields $\psi_{I'}$ are quite similar to the previous discussions for the $\psi_{\widetilde{I}}$. Let us rewrite the equation (III.4.3d):

$$0 \ = \ \Big\{ \widehat{\Gamma}^{+} \Big(\partial_+ + \frac{1}{2} G_{++} \partial_- \Big) + \widehat{\Gamma}^{-} \partial_- + \widehat{\Gamma}^{K} \partial_K \Big\} \psi_{I'} + \frac{1}{4} \mu \widehat{\Gamma}^{+123} \Big(\delta_{I'J'} - \widehat{\Gamma}_{I'} \widehat{\Gamma}_{J'} \Big) \psi_{J'} \ .$$

In the same way as the case of $\psi_{\widetilde{I}}$, we decompose the gravitino $\psi_{I'}$ into the $\widehat{\Gamma}$-transverse modes and the the $\widehat{\Gamma}$-parallel modes, and we decompose them further in terms of the "chiral projection operators". After these processes we obtain the following field equations:

$$0 \ = \ \Big(\Box - \frac{5}{2} \mu \, i \partial_- \Big) \psi^{\oplus \|}_{2\mathrm{R}} \ , \qquad 0 \ = \ \Big(\Box + \frac{5}{2} \mu \, i \partial_- \Big) \psi^{\oplus \|}_{2\mathrm{L}} \ , \tag{III.4.23a}$$

$$0 \ = \ \Big(\Box - \frac{1}{2} \mu \, i \partial_- \Big) \psi^{\oplus \perp}_{I'\mathrm{R}} \ , \qquad 0 \ = \ \Big(\Box + \frac{1}{2} \mu \, i \partial_- \Big) \psi^{\oplus \perp}_{I'\mathrm{L}} \ , \tag{III.4.23b}$$

where the $\widehat{\Gamma}$-transverse mode and $\widehat{\Gamma}$-parallel mode are defined as

$$\psi^{\oplus}_{I'} \ = \ - \frac{1}{2} \widehat{\Gamma}^{-} \widehat{\Gamma}^{+} \psi_{I'} \ ,$$

$$\psi^{\oplus \perp}_{I'\mathrm{R}} \ = \ \frac{1 + i \widehat{\Gamma}^{123}}{2} \psi^{\oplus \perp}_{I'} \ , \qquad \psi^{\oplus \perp}_{I'\mathrm{L}} \ = \ \frac{1 - i \widehat{\Gamma}^{123}}{2} \psi^{\oplus \perp}_{I'} \ ,$$

$$\psi^{\oplus \|}_{2\mathrm{R}} \ = \ \frac{1 + i \widehat{\Gamma}^{123}}{2} \psi^{\oplus \|}_{2} \ , \qquad \psi^{\oplus \|}_{2\mathrm{L}} \ = \ \frac{1 - i \widehat{\Gamma}^{123}}{2} \psi^{\oplus \|}_{2} \ .$$

We find the energy and the number of degrees of freedom for the $\widehat{\Gamma}$-transverse modes from (III.4.23b):

$$\mathcal{E}_0 (\psi^{\oplus \perp}_{I'\mathrm{R}}) \ = \ \frac{3}{2} \ , \qquad \mathcal{E}_0 (\psi^{\oplus \perp}_{I'\mathrm{L}}) \ = \ \frac{5}{2} \ , \qquad \mathcal{D}(\psi^{\oplus \perp}_{I'\mathrm{R}}) \ = \ \mathcal{D}(\psi^{\oplus \perp}_{I'\mathrm{L}}) \ = \ 8 \times (6-1) \ = \ 40 \ . \tag{III.4.24}$$

By the same discussion on $\psi^{\oplus \|}_{1\mathrm{R}}$ and $\psi^{\oplus \|}_{1\mathrm{L}}$ in the previous analysis, it is also impossible to read the correct energies and the number of degrees of freedom for the $\widehat{\Gamma}$-parallel modes from only the equation

(III.4.23a). But when we summarize the equations for the $\widehat{\Gamma}$-parallel modes for $\psi_{\widetilde{I}}$ (III.4.21) and the equations for the $\widehat{\Gamma}$-parallel modes for $\psi_{I'}$ (III.4.23a), we can obtain the correct field equations for them. Thus we perform a linear combination of (III.4.21) and (III.4.23a), and define new $\widehat{\Gamma}$-parallel modes as

$$\psi_{\rm R}^{\oplus\parallel} \ \equiv \ \frac{2}{5}\psi_{\rm 1R}^{\oplus\parallel} - \psi_{\rm 2R}^{\oplus\parallel} \, , \qquad \psi_{\rm L}^{\oplus\parallel} \ \equiv \ \frac{2}{5}\psi_{\rm 1L}^{\oplus\parallel} - \psi_{\rm 2L}^{\oplus\parallel} \, .$$

Then, by the on-shell gravitino condition (III.4.14), we find that the re-defined fermions satisfy the equations

$$0 \ = \ \Big(\Box - \frac{3}{2} \mu \, i \partial_- \Big) \psi_{\rm R}^{\oplus\parallel} \, , \qquad 0 \ = \ \Big(\Box + \frac{3}{2} \mu \, i \partial_- \Big) \psi_{\rm L}^{\oplus\parallel} \, .$$

Thus the zero point energies and the number of degrees of freedom of them are represented by

$$\mathcal{E}_0(\psi_{\rm R}^{\oplus\parallel}) \ = \ \frac{1}{2} \, , \qquad \mathcal{E}_0(\psi_{\rm L}^{\oplus\parallel}) \ = \ \frac{7}{2} \, , \qquad \mathcal{D}(\psi_{\rm R}^{\oplus\parallel}) \ = \ \mathcal{D}(\psi_{\rm L}^{\oplus\parallel}) \ = \ 8 \, . \tag{III.4.25}$$

Now we have fully solved the field equations for fermionic fluctuations, and have derived the spectrum of gravitino on the plane-wave. As a result, we have found that the spectrum is splitting with a certain energy difference in the same manner with the spectrum of bosons. Summarizing (III.4.22), (III.4.24) and (III.4.25), we obtain the spectrum of gravitino as in Table III.2:

energy $\mathcal{E}_0$	fermionic fields		degrees of freedom
7/2	$\psi_{\rm L}^{\oplus\parallel}$		8
5/2	$\psi_{\widetilde{I}{\rm R}}^{\oplus\perp}$	$\psi_{I'{\rm L}}^{\oplus\perp}$	$16 + 40$
3/2	$\psi_{\widetilde{I}{\rm L}}^{\oplus\perp}$	$\psi_{I'{\rm R}}^{\oplus\perp}$	$16 + 40$
1/2	$\psi_{\rm R}^{\oplus\parallel}$		8

Table III.2: *Zero point energy spectrum of fermionic fields in eleven-dimensional supergravity on the plane-wave background.*

III.5 Result

Until the previous sections we constructed the field equations for fluctuation fields of linearized supergravity and calculated the zero point energies of fluctuations. We summarize the results of spectrum of fluctuation fields in Table III.3.

energy E_0	bosonic/fermionic fields	degrees of freedom
2μ	h	1
$7\mu/4$	$\psi_{\mathrm{L}}^{\oplus\parallel}$	8
$3\mu/2$	$H_{\tilde{I}J'}$ $\mathcal{C}^{\ominus}_{I'J'K'}$	$18+10$
$5\mu/4$	$\psi^{\oplus\perp}_{\tilde{I}\mathrm{R}}$ $\psi^{\oplus\perp}_{I'\mathrm{L}}$	$16+40$
μ	$h^{\perp}_{\tilde{I}\tilde{J}}$ $\mathcal{C}_{\tilde{I}J'K'}$ $h^{\perp}_{I'J'}$	$5+45+20$
$3\mu/4$	$\psi^{\oplus\perp}_{\tilde{I}\mathrm{L}}$ $\psi^{\oplus\perp}_{I'\mathrm{R}}$	$16+40$
$\mu/2$	$\overline{H}_{\tilde{I}J'}$ $\mathcal{C}^{\oplus}_{I'J'K'}$	$18+10$
$\mu/4$	$\psi_{\mathrm{R}}^{\oplus\parallel}$	8
0	$\overline{h}$	1

Table III.3: *Zero point energy spectrum of all the physical fields of the linearized supergravity on the plane-wave background.*

The spectrum of the center of mass degrees of freedom of the Matrix theory on the plane-wave background was discussed in the previous chapter (see Table II.1). In that chapter, we found that the energy values of the multiplet starts from zero (the ground state $|\,\Lambda\,\rangle = |\,1,1\,\rangle$) to 2μ (the highest state $\left|\ \ydiagram{2,2,2,2}\,, \ydiagram{4,4}\ \right\rangle = |\,1,1\,\rangle$) at intervals of $\mu/4$ energy values. In comparison with the result of the supergravity discussed in this chapter, we find that the $U(1)$ part spectrum of the Matrix theory on the plane-wave background exactly corresponds to the spectrum of linearized supergravity on the same background!

Chapter IV

Conclusion and Discussions

Conclusion

In this doctoral thesis we have studied Matrix theory and eleven-dimensional supergravity on the plane-wave background.

First we reviewed the Matrix theory on the plane-wave background called the BMN matrix model. We constructed the single D0-brane effective action as a superparticle on the plane-wave, and extended this to N D0-branes' effective action as the non-abelian $U(N)$ matrix model in terms of the Myers' proposition. The resulting action has one non-vanishing parameter μ with mass dimension one. The BMN matrix model has also 32 local supersymmetry which decomposes into the linearly realized supersymmetry called the kinematical supersymmetry and the nonlinearly realized one called the dynamical supersymmetry. We also wrote down the Hamiltonian and momentum operators, $SO(3) \times SO(6)$ rotation operators and supercharges in terms of matrix variables. Unlike the flat space case, there are non-trivial commutation relations between the Hamiltonian and supercharges on the plane-wave. Thus the members in one supermultiplet which is generated by such supercharges have different energies. In this thesis we concentrated only the $U(1)$ free sector of this matrix model, which is the center of mass degrees of freedom of N D0-branes, or the superparticle. States in this part are generated by the kinematical supercharges and we analyzed the energies of supermultiplet including the ground state. The result is summarized in Table II.1. Here let us write down this result again:

N-th Floor	$SU(4) \times SU(2)$ Representations	Energy Eigenvalues
8	$(\mathbf{1}, \mathbf{1})$	2μ
7	$(\overline{\mathbf{4}}, \mathbf{2})$	$7\mu/4$
6	$(\overline{\mathbf{6}}, \mathbf{3})$ $(\overline{\mathbf{10}}, \mathbf{1})$	$3\mu/2$
5	$(\mathbf{4}, \mathbf{4})$ $(\overline{\mathbf{20}}, \mathbf{2})$	$5\mu/4$
4	$(\mathbf{1}, \mathbf{5})$ $(\mathbf{15}, \mathbf{3})$ $(\mathbf{20'}, \mathbf{1})$	μ
3	$(\overline{\mathbf{4}}, \mathbf{4})$ $(\mathbf{20}, \mathbf{2})$	$3\mu/4$
2	$(\mathbf{6}, \mathbf{3})$ $(\mathbf{10}, \mathbf{1})$	$\mu/2$
1	$(\mathbf{4}, \mathbf{2})$	$\mu/4$
ground state	$(\mathbf{1}, \mathbf{1})$	0

The ground state supermultiplet generated by kinematical supercharges.

Next we investigated the eleven-dimensional supergravity on the same background. We prepared the eleven-dimensional supergravity Lagrangian and classical field equations derived from it. They are described up to torsion terms, or quartic terms with respect to gravitino, which do not contribute to the analysis of the spectrum on the plane-wave background. Expanding fields around the plane-wave background and constraining the light-cone gauge-fixing, we obtained equations of motion for fluctuation fields. At first sight these equations seemed to be complicated, however we could obtain the Klein-Gordon type field equations via field re-definitions. From the result of this analysis we found that the fluctuation fields have different zero-point energies as below (see also chapter III):

Zero-point Energy E_0	Bosonic/fermionic Fields	Degrees of Freedom
2μ	h	1
$7\mu/4$	$\psi_{\mathrm{L}}^{\oplus\parallel}$	8
$3\mu/2$	$H_{\tilde{I}J'}$ $\mathcal{C}^{\ominus}_{I'J'K'}$	$18+10$
$5\mu/4$	$\psi^{\oplus\perp}_{\tilde{I}\mathrm{R}}$ $\psi^{\oplus\perp}_{I'\mathrm{L}}$	$16+40$
μ	$h^{\perp}_{\tilde{I}\tilde{J}}$ $\mathcal{C}_{\tilde{I}J'K'}$ $h^{\perp}_{I'J'}$	$5+45+20$
$3\mu/4$	$\psi^{\oplus\perp}_{\tilde{I}\mathrm{L}}$ $\psi^{\oplus\perp}_{I'\mathrm{R}}$	$16+40$
$\mu/2$	$\overline{H}_{\tilde{I}J'}$ $\mathcal{C}^{\oplus}_{I'J'K'}$	$18+10$
$\mu/4$	$\psi_{\mathrm{R}}^{\oplus\parallel}$	8
0	$\overline{h}$	1

Zero point energy spectrum of physical degrees of freedom in supergravity on the plane-wave.

We obtained the energy spectra of the $U(1)$ part of Matrix theory and eleven-dimensional supergravity. Both spectra include the same number of bosonic and fermionic degrees of freedom. This result should be satisfied in all multiplets in any supersymmetric theory. We also obtained the fact that the energies of the states in Matrix theory completely correspond to those of fields in supergravity. Thus, we found that the Matrix theory on the plane-wave background contains the zero-mode spectrum of the eleven-dimensional supergravity completely. We describe the image of the above result in Figure IV.1.

Through this result, we can see the Matrix theory on the plane-wave background as a candidate of a quantum extension of eleven-dimensional supergravity on the same background, or as a candidate of description of yet-unknown theory, i.e., M-theory, on the plane-wave background.

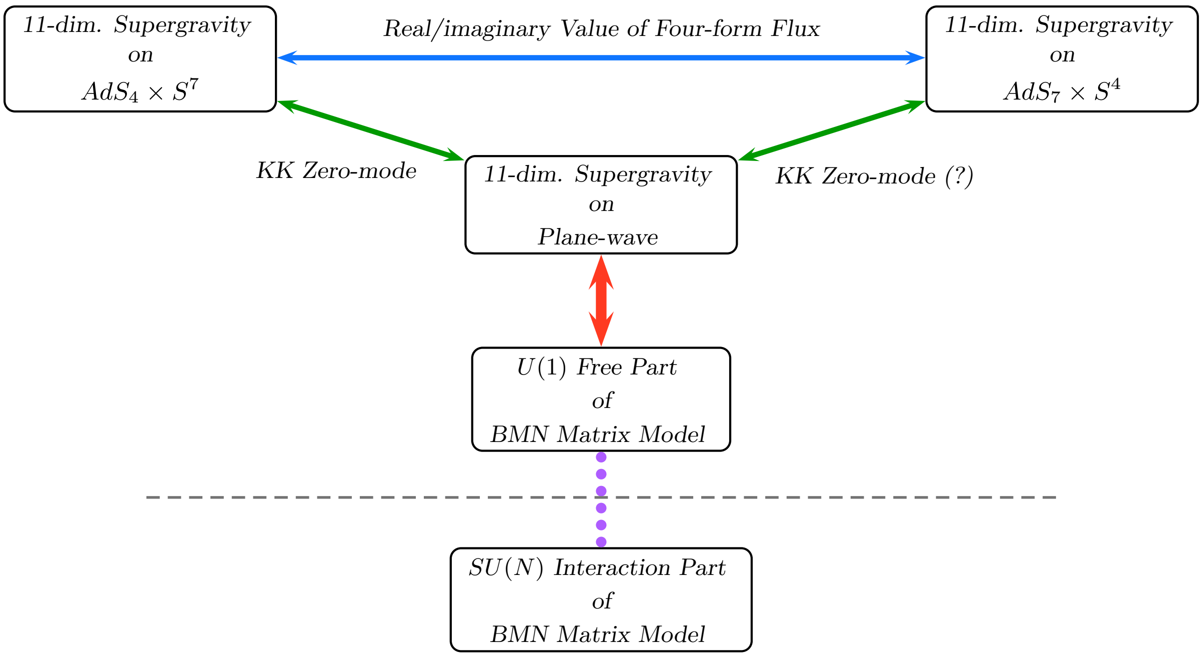

Figure IV.1: *The relationships among the spectrum of the eleven-dimensional supergravity/Matrix theory on the maximally supersymmetric curved background. The $SU(N)$ interaction part in the BMN matrix model is independent of the $U(1)$ part which can be regarded as the superparticle on the plane-wave.*

Discussions and Future Problems

In this thesis we have argued the $U(1)$ free sector in BMN matrix model and fluctuation fields in eleven-dimensional supergravity on the plane-wave. We have found the essential evidence that the BMN matrix model also includes the supergravity on the plane-wave as in the case of the theories on flat background. In order to confirm this evidence more clearly, we must study other kinds of correspondence between the BMN matrix model and supergravity beyond the correspondence of the spectra between them. The next study we should do is to compare graviton scattering amplitudes in those models on the plane-wave background [10, 18, 113, 99]. There are still few direct discussions about interactions of superparticles and scattering amplitudes which should be calculated in both models. In order to argue this topic, vertex operator method seems to be a useful tool as in string theory. There already exists the vertex operator formulation for supergravity in light-cone gauge discussed by Green, Gutperle and Kwon [69], and there also exists the vertex operator formulation for supermembrane or Matrix theory proposed by Dasgupta, Nicolai and Plefka [37]. These formulations can be organized on weakly curved background and we would be able to apply these methods to the analysis on the plane-wave background [134]. Shin and Yoshida studied one-loop quantum corrections of the BMN matrix model on the classical plane-wave background in the framework of path integration [127]. This analysis would give us a helpful information for the graviton scatterings.

There also exist many important tasks which we should work around the physics on the eleven-dimensional plane-wave background. As mentioned in appendix C, the plane-wave background connects, from the purely geometric point of view, to $AdS_4 \times S^7$ and $AdS_7 \times S^4$ via the Penrose limit. We should study the "physics" on the plane-wave also connect to the ones on $AdS_{4(7)} \times S^{7(4)}$ background. We have already understood the properties of the linearized and nonlinear full supergravity on $AdS_{4(7)} \times S^{7(4)}$ background [59, 53, 144, 47, 110, 55]. In fact, Fernando, Günaydin and Pavlyk discussed in this topic via oscillator method [63] and the oscillator modes on the plane-wave connects to the ones on $AdS_{4(7)} \times S^{7(4)}$ consistently. Thus we can trace how the *fluctuation fields* transform and re-define in supergravity on $AdS_{4(7)} \times S^{7(4)}$ through the Penrose limit. When we understand these connections, we will be able to investigate their gauge theory duals, i.e., $\mathrm{AdS}_4/\mathrm{CFT}_3$ and $\mathrm{AdS}_7/\mathrm{CFT}_6$ correspondences [32, 5] as the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [3] in type IIB superstring theory. In particular, we would like to study the strong coupling region of conformal field theory in six dimensions, which would be one of the most mysterious theory in quantum field theory.

Myers' term played a central role in the BMN matrix model construction. Because of the existence of this term, there is one solution that the supermembrane wrapping on the fuzzy two-sphere [19]. How about in the supergravity side? Myers' effect seems to influence the decomposition of three-form gauge fields into self-dual and anti-self-dual part [148].

Where is M-brane configuration? The BMN matrix model has two classical vacua [19]. One is the "fuzzy sphere vacuum" obtained by

$$X^{\widetilde{I}} \;=\; \frac{\mu}{3R} J^{\widetilde{I}} \,, \qquad X^{I'} \;=\; 0 \,,$$

where $J^{\widetilde{I}}$ form a representation of the $SU(2)$ algebra

$$[J^{\widetilde{I}}, J^{\widetilde{J}}] \;=\; i\epsilon^{\widetilde{I}\widetilde{J}\widetilde{K}} J^{\widetilde{K}} \,.$$

In the large N limit this vacuum is related to "giant gravitons" in the plane-wave background which are M2-branes wrapping the two-sphere given by $\sum_{\widetilde{I}}(x^{\widetilde{I}})^2 = (\mathit{constant})$ and classically sitting at a fixed position x^-, but with non-zero momentum p_-. The other vacuum is given by $X^I = 0$ for all $I = 1, 2, \cdots, 9$, which is called the "trivial vacuum". This solution is regarded as giant gravitons which are (transverse) M5-branes wrapping the S^5 given by $\sum_{I'}(x^{I'})^2 = (\mathit{constant})$ in the large N limit. However, this does not appear as a classical solution of the BMN matrix model. This is partly because it is more difficult to describe M5-brane worldvolume theory than to describe M2-brane [147, 6, 7]. As in the case of $\mathcal{N} = 1^*$ super Yang-Mills theory discussed by Polchinski and Strassler [121], it is natural to conjecture that the trivial vacuum in the quantum mechanics theory corresponds to a single large M5-brane. Further discussions are given by Maldacena, Sheikh-Jabbari and Van Raamsdonk [102]. M-brane configurations on the plane-wave background are also studied by Mas and Ramallo [103], etc. Thus it is quite interesting for us to investigate how the M-brane configurations are given in the supergravity on the plane-wave.

In the BMN matrix model, there exist a lot of BPS solutions generated by dynamical and kinematical supercharges [38, 91, 90]. Longitudinal and transverse M-branes should be also BPS states preserving parts of supersymmetry [11]. It is interesting to study the realization of these BPS states in the supergravity side.

In this doctoral thesis I have considered the investigation about the eleven-dimensional theory on the plane-wave background. I have also introduced various *tales* for future works. The work which has been done here seems to be a small one. But *what you take* around these topics would become a *giant step* in M-theory.

I hope that someone gets my message in the thesis!

Appendix A

Convention

A.1 Eleven-dimensional Spacetime

First we should define the signature of spacetime in order to discuss various properties of symmetries, transformation laws and Lagrangian of the system. In this thesis, we adopt the *almost plus* signature to eleven-dimensions Minkowski spacetime: $(-,+,+,\cdots,+)$.

We describe the curved spacetime metric and the tangent space metric as g_{MN} and η_{AB}, respectively. Notice that the capital letters which start from $M,N,P,\cdots$ refer to eleven-dimensional world indices (curved spacetime indices) and the capital letters which start from $A,B,C,\cdots$ denotes eleven-dimensional tangent space indices. Note that the vielbein $e_M{}^A$ and its inverse vielbein $E_A{}^M$ are related to the curved spacetime metric g_{MN} and the tangent space metric η_{AB} as follows:

$$g_{MN} \ = \ e_M{}^A \, e_N{}^B \, \eta_{AB} \, , \qquad \eta_{AB} \ = \ E_A{}^M \, E_B{}^N \, g_{MN} \, .$$

We prepare a character such as $\varepsilon^{MNPQRSUVWXY}$ which makes the three-form gauge field C_{MNP} and its field strength $F_{MNPQ} = 4\partial_{[M}C_{NPQ]}$ couple to each other. This character is an invariant tensor density in eleven-dimensional spacetime (weight $+1$), whose normalization is $\varepsilon^{012\cdots\natural} = 1$.

A.2 Clifford Algebra: $SO(10,1)$ Representation

In chapter III we will use various spinor variables. Thus it is necessary for us to introduce the Clifford algebra and Dirac gamma matrices in order to define various transformations. Here let us define the Clifford algebra and gamma matrices in eleven-dimensional Minkowski spacetime which has $SO(10,1)$ Lorentz symmetry. In the next sections we will decompose them into various representations.

Let us first write down the Clifford algebra and Dirac gamma matrices in eleven-dimensional spacetime

$$\{\widehat{\Gamma}^A, \widehat{\Gamma}^B\} \ = \ 2\eta^{AB} \cdot \mathbf{1}_{32} \, .$$

Note that η^{AB} is the tangent space metric. Hermitian conjugate of the gamma matrices is defined by

$$(\widehat{\Gamma}^A)^\dagger \ = \ \widehat{\Gamma}_A \ = \ -\widehat{\Gamma}^0 \widehat{\Gamma}^A (\widehat{\Gamma}^0)^{-1} \, .$$

Note that the gamma matrices $\widehat{\Gamma}^I$ are Hermitian except for $\widehat{\Gamma}^0$, which is anti-Hermitian. For the convenience we define the following anti-symmetrized products of gamma matrices with unit weight:

$$\widehat{\Gamma}_{A_1 A_2 \cdots A_n} \ \equiv \ \widehat{\Gamma}_{[A_1} \widehat{\Gamma}_{A_2} \cdots \widehat{\Gamma}_{A_n]} \ = \ \frac{1}{n!} \sum_{\sigma} \mathrm{sgn}(\sigma) \, \widehat{\Gamma}_{A_{\sigma_1}} \widehat{\Gamma}_{A_{\sigma_2}} \cdots \widehat{\Gamma}_{A_{\sigma_n}} \, .$$

Utilizing this definition, we write an identity for the gamma matrices:

$$\begin{aligned}
&\widehat{\Gamma}^{A_1 A_2 \cdots A_p} \widehat{\Gamma}_{B_1 B_2 \cdots B_q} \\
&\quad = \sum_{k=0}^{\min(p,q)} (-1)^{\frac{1}{2}k(2p-k-1)} \frac{p!\, q!}{(p-k)!(q-k)!k!} \delta^{[A_1}_{[B_1} \cdots \delta^{A_k}_{B_k} \widehat{\Gamma}^{A_{k+1} \cdots A_p]}{}_{B_{k+1} \cdots B_q]} \ . \qquad \text{(A.2.1)}
\end{aligned}$$

Utilizing these properties, we can define a spinor in eleven-dimensional Minkowski spacetime. In particular, we can define a *Majorana spinor* θ as a irreducible representation of $SO(10,1)$ spinor[1]. Let us define the Dirac conjugate of the spinor θ as

$$\overline{\theta} \ = \ i\theta^\dagger \widehat{\Gamma}^0 \ .$$

Note that the product $\overline{\theta}\theta$ is Hermitian in this definition. In terms of this Dirac conjugate, we describe the Majorana condition of the spinors as

$$\overline{\theta} \ = \ \theta^T C \ ,$$

where C is the charge conjugation matrix. In this thesis this charge conjugation matrix is defined as antisymmetric: $C = -C^{-1} = -C^T$. Under this definition, charge conjugations of the gamma matrices and antisymmetrized gamma matrices are given by the Gauss bracket $[\frac{n+1}{2}] = \{1,1,2,2,3,3,\cdots\}$:

$$C\widehat{\Gamma}^A C^{-1} \ = \ -(\widehat{\Gamma}^A)^T \ , \qquad \text{(A.2.2a)}$$

$$C\widehat{\Gamma}^{A_1 \cdots A_n} C^{-1} \ = \ (-1)^{[\frac{n+1}{2}]} (\widehat{\Gamma}^{A_1 \cdots A_n})^T \ . \qquad \text{(A.2.2b)}$$

A.3 Lorentz Algebra

The Lorentz symmetry on the tangent space is important to describe vectors, tensors, and spinors in curved spacetime via vielbeins and inverse vielbeins. It is also important to understand the dynamics of the theory in the weak coupling limit of gravity. Thus, let us define here the Lorentz algebra in the eleven-dimensional tangent space as

$$i[\Sigma_{AB}, \Sigma_{CD}] \ = \ \eta_{AC}\, \Sigma_{BD} + \eta_{BD}\, \Sigma_{AC} - \eta_{AD}\, \Sigma_{BC} - \eta_{BC}\, \Sigma_{AD} \ ,$$

where the Lorentz generators Σ_{AB} are Hermitian and they are represented by

$$\begin{aligned}
\Sigma_{AB} &= 0 && \text{scalar} \ , \\
(\Sigma_{CD})^A{}_B &= i\big(\delta^A_C\, \eta_{DB} - \delta^A_D\, \eta_{CB}\big) && \text{vector} \ , \\
\Sigma_{AB} &= \frac{i}{2} \widehat{\Gamma}_{AB} && \text{spinor} \ .
\end{aligned}$$

[1] Notice that we denote the gravitino (vectorial $SO(10,1)$ Majorana spinor) as Ψ_M in eleven-dimensional supergravity (see chapter III).

A.4 $SO(9)$ Representation

We defined the Dirac gamma matrices $\widehat{\Gamma}^{A}$ in eleven dimensions in appendix A.2. Performing the fermion light-cone gauge fixing (or κ-symmetry gauge fixing), we decompose these $SO(10,1)$ gamma matrices $\widehat{\Gamma}^{A}$ in terms of 16×16 unit matrix $\mathbf{1}_{16}$ and the $SO(9)$ gamma matrices γ^{I}. First we put the fermionic light-cone gauge fixing on the $SO(10,1)$ Majorana spinor θ:

$$\widehat{\Gamma}^{+}\theta \ = \ 0 \, , \qquad \overline{\theta}\widehat{\Gamma}^{+} \ = \ 0 \, . \tag{A.4.1}$$

By virtue of this constraint 16 degrees of freedom of $SO(10,1)$ Majorana spinor is gauged away and we write down θ by using the $SO(9)$ Majorana spinor Ψ as

$$\theta \ = \ \frac{1}{2^{3/4}} \left(\begin{array}{c} 0 \\ \Psi \end{array} \right) \, , \qquad \overline{\theta} \ = \ i\theta^{\dagger}\widehat{\Gamma}^{0} \ = \ \theta^{T} C \ \equiv \ \frac{1}{2^{3/4}} \Big(-\Psi^{T} \, , \, 0 \Big) \, . \tag{A.4.2}$$

This representation denotes that the $SO(9)$ Majorana spinor Ψ satisfies the reality condition $\Psi^{\dagger} = \Psi^{T}$ explicitly; the normalization of Ψ is defined so as to satisfy the following:

$$-\overline{\theta}\widehat{\Gamma}^{-}\partial\theta \ = \ \frac{i}{2}\Psi^{\dagger}\partial\Psi \, .$$

Under this convention, the charge conjugation matrix C in eleven-dimensional Minkowski spacetime is represented by

$$C \ = \ \left(\begin{array}{cc} 0 & \mathbf{1}_{16} \\ -\mathbf{1}_{16} & 0 \end{array} \right) \, . \tag{A.4.3}$$

Let us express the $SO(10,1)$ gamma matrices in the light-cone directions $\widehat{\Gamma}^{+}$ and $\widehat{\Gamma}^{-}$ in terms of 16×16 matrices

$$\begin{aligned}
\widehat{\Gamma}^{0} \ &= \ \left(\begin{array}{cc} 0 & i\mathbf{1}_{16} \\ i\mathbf{1}_{16} & 0 \end{array} \right) \, , \qquad \widehat{\Gamma}^{10} \ = \ \left(\begin{array}{cc} 0 & -i\mathbf{1}_{16} \\ i\mathbf{1}_{16} & 0 \end{array} \right) \, , \\
\widehat{\Gamma}^{\pm} \ &\equiv \ \frac{1}{\sqrt{2}}(\widehat{\Gamma}^{0} \pm \widehat{\Gamma}^{10}) \, , \qquad \{\widehat{\Gamma}^{+}, \widehat{\Gamma}^{-}\} \ = \ -2 \cdot \mathbf{1}_{32} \, , \\
\widehat{\Gamma}^{+} \ &= \ \sqrt{2} \left(\begin{array}{cc} 0 & 0 \\ i\mathbf{1}_{16} & 0 \end{array} \right) \, , \qquad \widehat{\Gamma}^{-} \ = \ \sqrt{2} \left(\begin{array}{cc} 0 & i\mathbf{1}_{16} \\ 0 & 0 \end{array} \right) \, .
\end{aligned}$$

Next we describe the gamma matrices in the longitudinal directions $\widehat{\Gamma}^{I}$ in terms of the $SO(9)$ gamma matrices γ^{I}

$$\widehat{\Gamma}^{I} \ = \ \left(\begin{array}{cc} -(\gamma^{I})^{T} & 0 \\ 0 & \gamma^{I} \end{array} \right) \, . \tag{A.4.4}$$

Note that γ^I satisfies the Clifford algebra: $\{\gamma^I,\gamma^J\}=2\delta^{IJ}$. Since we define the hermitian conjugation of $\widehat{\Gamma}^I$ as $(\widehat{\Gamma}^I)^\dagger=\widehat{\Gamma}^I$, we obtain the hermitian conjugation of the $SO(9)$ gamma matrices below:

$$(\gamma^I)^\dagger \;=\; \gamma^I\,, \qquad (\gamma^{IJ})^\dagger \;=\; -\gamma^{IJ}\,, \qquad (\gamma^{IJK})^\dagger \;=\; -\gamma^{IJK}\,.$$

A.5 $SU(4)\times SU(2)$ Representation

Let us decompose the $SO(9)$ Majorana spinors Ψ and the gamma matrices in the $SO(9)$ representations into the ones in the $SU(4)\times SU(2)$ representations [38]. The **16** representation of the $SO(9)$ Majorana spinor are split up as

$$\mathbf{16} \;=\; (\mathbf{4},\mathbf{2})\oplus(\overline{\mathbf{4}},\mathbf{2}) \qquad\qquad \Psi \;\to\; \{\psi_{i\alpha}\,,\;\widetilde{\psi}^{j\beta}\}\,,$$

where $\mathbf{4}$ and $\overline{\mathbf{4}}$ are the fundamental and anti-fundamental representations of $SU(4)$ spinor; $\mathbf{2}(=\overline{\mathbf{2}})$ is the fundamental representation of $SU(2)$ spinor. We express the $SU(4)\times SU(2)$ spinor as $\psi_{i\alpha}$ in terms of indices i, the fundamental $SU(4)$ indices, and the fundamental $SU(2)$ indices α. These spinors obey a reality condition, which in the reduced notation becomes simply as $\widetilde{\psi}^{j\beta}=\psi^{\dagger j\beta}$. More concretely we represent the $SO(9)$ Majorana spinor Ψ in terms of $SU(2)\times SU(4)$ representations $\psi_{i\alpha}$ as

$$\Psi \;=\; \left(\begin{array}{c} \psi_{i\alpha} \\ \epsilon_{\alpha\beta}\,\psi^{\dagger i\beta}\end{array}\right)\,, \tag{A.5.1}$$

and we decompose the $SO(9)$ gamma matrices[2] to the direct product of $SU(4)$ and $SU(2)$ gamma matrices

$$\gamma^{\widetilde{I}} \;=\; \left(\begin{array}{cc} -\sigma^{\widetilde{I}}\otimes\mathbf{1}_4 & 0 \\ 0 & \sigma^{\widetilde{I}}\otimes\mathbf{1}_4\end{array}\right)\,, \qquad \gamma^{I'} \;=\; \left(\begin{array}{cc} 0 & \mathbf{1}_2\otimes \mathrm{g}^{I'} \\ \mathbf{1}_2\otimes(\mathrm{g}^{I'})^\dagger & 0\end{array}\right)\,. \tag{A.5.2}$$

Note that the matrices $\sigma^{\widetilde{I}}$ are the ordinary Pauli matrices and the $SU(4)$ gamma matrices $\mathrm{g}^{I'}$ satisfy the Clifford algebra

$$\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}+\sigma^{\widetilde{J}}\sigma^{\widetilde{I}} \;=\; 2\delta^{\widetilde{I}\widetilde{J}}\,, \qquad \mathrm{g}^{I'}(\mathrm{g}^{J'})^\dagger+\mathrm{g}^{J'}(\mathrm{g}^{I'})^\dagger \;=\; 2\delta^{I'J'}\,.$$

A.6 Connections and Curvature Tensors

In this appendix we define the geometrical variables such as connections and their curvature tensors which appear in the eleven-dimensional supergravity Lagrangian. Explicit expressions of these definitions are of important to calculate the fluctuation fields, superspace coset formalism, etc.

[2]The $SO(9)$ gamma matrices and Majorana spinors are defined in appendix A.4.

First let us define the covariant derivative ∇_M for general coordinate transformation by using the affine connection Γ^P_{NM} as

$$\nabla_M A_N \;=\; \partial_M A_N - \Gamma^P_{NM} A_P \,, \tag{A.6.1}$$

where A_P is an arbitrary covariant vector. Riemann curvature tensor $R^R{}_{PMN}$ for the affine connection is defined from the commutator of the covariant derivative as

$$\begin{aligned}
[\nabla_M, \nabla_N] A_P \;&=\; -R^R{}_{PMN} A_R + T^R{}_{MN} A_R \,, \\
R^R{}_{PMN} \;&=\; \partial_M \Gamma^R_{PN} - \partial_N \Gamma^R_{PM} + \Gamma^R_{QM} \Gamma^Q_{PN} - \Gamma^R_{QN} \Gamma^Q_{PM} \,,
\end{aligned} \tag{A.6.2}$$

where $T^R{}_{MN}$ is a torsion coming from the antisymmetric part of the affine connection:

$$\Gamma^R_{[MN]} \;=\; \frac{1}{2}\big(\Gamma^R_{MN} - \Gamma^R_{NM}\big) \;\equiv\; \frac{1}{2} T^R{}_{MN} \,.$$

In addition, let us introduce the Christoffel symbol $\left\{ {R \atop M\,N} \right\}$ in terms of the metric and torsion:

$$\begin{aligned}
\left\{ {R \atop M\,N} \right\} \;&\equiv\; \frac{1}{2} g^{RQ} \big(\partial_M g_{NQ} + \partial_N g_{MQ} - \partial_Q g_{MN}\big) \\
&=\; \Gamma^R_{(MN)} + \frac{1}{2} T_M{}^R{}_N + \frac{1}{2} T_N{}^R{}_M \,,
\end{aligned} \tag{A.6.3}$$

where $\Gamma^R_{(MN)} \equiv \frac{1}{2}(\Gamma^R_{MN} + \Gamma^R_{NM})$ is the symmetric part of the affine connection. Thus the affine connection is written in terms of the Christoffel symbol and torsion:

$$\Gamma^R_{MN} \;=\; \Gamma^R_{(MN)} + \Gamma^R_{[MN]} \;=\; \left\{ {R \atop M\,N} \right\} + K^R{}_{MN} \,, \tag{A.6.4a}$$

$$K^R{}_{MN} \;\equiv\; \frac{1}{2}\big(T^R{}_{MN} + T_{MN}{}^R - T_{NM}{}^R\big) \,. \tag{A.6.4b}$$

Note that the tensor $K^R{}_{MN}$ is called the contorsion. When the torsion vanishes, the affine connection is equal to the Christoffel symbol.

Similarly, we define the covariant derivative D_M for the local Lorentz transformations by using the spin connection $\omega_M{}^{AB}$ as

$$D_M \phi \;=\; \partial_M \phi - \frac{i}{2} \omega_M{}^{AB} \, \Sigma_{AB} \, \phi \,, \tag{A.6.5}$$

where ϕ is an arbitrary field and we write the Lorentz generators as Σ_{AB} whose representations are described in appendix A.2. The curvature tensor $\widetilde{R}^{AB}{}_{MN}$ for the spin connection is defined from the commutator of the covariant derivative as

$$\begin{aligned}
[D_M, D_N] \phi \;&=\; -\frac{i}{2} \widetilde{R}^{AB}{}_{MN} \, \Sigma_{AB} \, \phi \,, \\
\widetilde{R}^{AB}{}_{MN} \;&=\; \partial_M \omega_N{}^{AB} - \partial_N \omega_M{}^{AB} + \omega_M{}^A{}_C \, \omega_N{}^{CB} - \omega_N{}^A{}_C \, \omega_M{}^{CB} \,.
\end{aligned} \tag{A.6.6}$$

We can obtain the curvature tensor $\widetilde{R}^{AB}{}_{MN}$ in terms of vielbein $e_M{}^A$ and spin connection $\omega_M{}^{AB}$. As in the case of the covariant derivative for the affine connection (A.6.2), the torsion term also appears if we write the above commutation relation (A.6.6) in terms of the covariant derivative on the tangent space as $D_A = E_A{}^M D_M$.

We also define the total covariant derivative $\widetilde{D}_M \equiv \nabla_M - \frac{i}{2}\omega^{AB}\Sigma_{AB}$ which contains both the affine connection and the spin connection. By virtue of the total covariant derivative we can simply consider the *vielbein postulate*

$$0 \ = \ \widetilde{D}_P \, e_M{}^A \ = \ \partial_P e_M{}^A - \Gamma^R_{MP} \, e_R{}^A + \omega_P{}^A{}_B \, e_M{}^B \, , \tag{A.6.7}$$

which is equivalent to the equivalent principle for the metric $\nabla_P g_{MN} = 0$. Under this postulate the curvature tensor for the spin connection $\widetilde{R}^{AB}{}_{MN}$ is associated with the curvature for the affine connection $R^R{}_{PMN}$:

$$R^R{}_{PMN} \ = \ \eta_{BC} \, E_A{}^R \, e_P{}^C \, \widetilde{R}^{AB}{}_{MN} \, . \tag{A.6.8}$$

Ricci tensor and scalar curvature are defined below:

$$\mathcal{R}^M{}_N \ = \ g^{PQ} \, R^M{}_{PNQ} \, , \qquad \mathcal{R} \ = \ \mathcal{R}^M{}_M \, .$$

Finally let us introduce the "Cartan's structure equations" from the viewpoint of differential geometry

$$\begin{aligned} \mathrm{d}s^2 \ &= \ g_{MN} \, \mathrm{d}x^M \, \mathrm{d}x^N \ = \ \eta_{AB} \, e^A \, e^B \, , \\ -T^A \ &= \ \mathrm{d}e^A + \omega^A{}_B \wedge e^B \, , \qquad \widetilde{R}^{AB} \ = \ \mathrm{d}\omega^{AB} + \omega^A{}_C \wedge \omega^{CB} \, , \end{aligned}$$

which can be written more explicitly as

$$\begin{aligned} -T^A{}_{MN} \ &= \ \partial_M e_N{}^A - \partial_N e_M{}^A + \omega_M{}^A{}_B \, e_N{}^B - \omega_M{}^A{}_B \, e_N{}^B \, , \\ \widetilde{R}^{AB}{}_{MN} \ &= \ \partial_M \omega_N{}^{AB} - \partial_N \omega_M{}^{AB} + \omega_M{}^A{}_C \, \omega_N{}^{CB} - \omega_N{}^A{}_C \, \omega_M{}^{CB} \, . \end{aligned}$$

Note that T^A is a torsion two-form which vanishes on coset spaces.

Appendix B

Lagrangians

B.1 Matrix Theory Lagrangian: Super Yang-Mills Action

Matrix theory Lagrangian suggested by Banks, Fischler, Shenker and Susskind [10] is described by N D0-branes' effective action , i.e., the dimensionally reduced model of ten-dimensional $U(N)$ super Yang-Mills theory. Thus, in this appendix we derive the Matrix theory Lagrangian on the flat background.

We have not completely understood how to describe the N coincident D-branes' effective action yet. But, in the low energy region, we now believe that the effective action should be described by the dimensionally reduced model of ten-dimensional $U(N)$ super Yang-Mills theory. We will also introduce another derivation of the low energy effective theory of N coincident D-branes' system with/without non-vanishing background fields in appendix B.2.

Here we consider the low energy region of the Dp-branes system. In low energy region, or weak string coupling region, the transverse fluctuations of Dp-branes would freeze. Thus Dp-branes appear as heavily massive solitons in string theory. The resulting modes in Dp-branes are the longitudinal modes moving in $(p+1)$-dimensional hypersurface of Dp-branes, which are the massless excitation modes of open strings. These dynamics could be described by the $(p+1)$-dimensional super Yang-Mills theory with $U(N)$ gauge symmetry. We can obtain this theory Lagrangian from the dimensional reduction to $(p+1)$-dimensions of the ten-dimensional $U(N)$ super Yang-Mills. In this context, we first introduce the ten-dimensional $U(N)$ super Yang-Mills Lagrangian and perform its dimensional reduction procedure.

The Lagrangian of ten-dimensional $U(N)$ super Yang-Mills is described in terms of the $N \times N$ matrix valued gauge fields and $SO(9,1)$ Majorana-Weyl spinors as follows:

$$\begin{aligned} S \ &= \ \int \mathrm{d}^{9+1}x \, \mathcal{L}_{9+1} \ = \ \int \mathrm{d}^{9+1}x \Big\{ -\frac{1}{4}\mathrm{Tr}\big(F_{\mu\nu}F^{\mu\nu}\big) - \mathrm{Tr}\big(\overline{\theta}\Gamma^{\mu}D_{\mu}\theta\big) \Big\} \ , \\ F_{\mu\nu} \ &= \ \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + ig[A_{\mu}, A_{\nu}] \ , \qquad D_{\mu}\theta \ = \ \partial_{\mu}\theta + ig[A_{\mu}, \theta] \ , \end{aligned}$$

where $\mu, \nu = 0, 1, 2, \cdots, 9$. Note that A_{μ} and θ are the $N \times N$ matrix valued dynamical fields whose mass dimensions are 4 and 9/2, respectively. The Yang-Mills coupling constant is denoted by g of mass dimensions -3. They are the $U(N)$ gauge potential and the $SO(9,1)$ spinor (Dirac representation), repetitively. In particular, the spinor θ can be expressed by the irreducible Majorana-Weyl spinor χ (real 16 components) as

$$\theta \ = \ \frac{1}{\sqrt{2}} \left(\begin{array}{c} 0 \\ \chi \end{array} \right) , \qquad \chi^{*} \ = \ \chi \ . \tag{B.1.1}$$

The overall factor is a convention.

It is useful to write down the $SO(9,1)$ Clifford algebra. This algebra corresponds to the one in eleven-dimensions[1]. So the Dirac matrices can be described by the same form of eleven-dimensional ones[2]:

$$\{\Gamma^\mu, \Gamma^\nu\} \;=\; 2\eta^{\mu\nu} \, , \qquad \{\gamma^I, \gamma^I\} \;=\; 2\delta^{IJ} \, , \tag{B.1.2a}$$

$$\Gamma^0 \;=\; \begin{pmatrix} 0 & i\mathbf{1}_{16} \\ i\mathbf{1}_{16} & 0 \end{pmatrix} \, , \qquad \Gamma^I \;=\; \frac{1}{2} \begin{pmatrix} -(\gamma^I)^T + \gamma^I & -i(\gamma^I)^T - i\gamma^I \\ i(\gamma^I)^T + i\gamma^I & -(\gamma^I)^T + \gamma^I \end{pmatrix} \, , \tag{B.1.2b}$$

$$\Gamma \;=\; \begin{pmatrix} \mathbf{1}_{16} & 0 \\ 0 & -\mathbf{1}_{16} \end{pmatrix} \, , \tag{B.1.2c}$$

where Γ is the chirality matrix. The Lorentz indices I runs from 1 to 9, which denotes the spatial directions of ten-dimensional Minkowski spacetime. The gamma matrices γ^I are defined as the generators of $SO(9)$ Clifford algebra. Although the above descriptions (B.1.2) seems somewhat complicated, these are useful expressions to perform the chiral decomposition of $SO(9,1)$ Dirac spinors θ and to construct the Majorana-Weyl spinors χ (B.1.1). Note that we can easily connect the above gamma matrices to the ones described in appendix A.4 in terms of the following unitary rotation

$$\Gamma^\mu \;=\; U\widehat{\Gamma}^\mu U^{-1} \, , \qquad \Gamma \;=\; U\widehat{\Gamma}^{10} U^{-1} \, , \qquad U \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \, .$$

Utilizing the above descriptions (B.1.2), we can easily write down the Lagrangian of ten-dimensional super Yang-Mills in terms of the Majorana-Weyl spinors χ as follows:

$$S \;=\; \int \mathrm{d}^{9+1}x \, \mathcal{L}_{9+1} \;=\; \int \mathrm{d}^{9+1}x \Big\{ -\frac{1}{4}\mathrm{Tr}\big(F_{\mu\nu}F^{\mu\nu}\big) + \frac{i}{2}\mathrm{Tr}\big(\chi^T D_0 \chi\big) - \frac{i}{2}\mathrm{Tr}\big(\chi^T \gamma^I D_I \chi\big) \Big\} \, .$$

Now let us perform the dimensional reduction to (0+1)-dimensional "spacetime". Under this reduction, the gauge potential A_I becomes an $U(N)$ adjoint scalar fields denoted by X^I. The field strength $F_{\mu\nu}$ and the covariant derivative $D_I\chi$ also reduce to

$$\begin{aligned} F_{0I} \;&=\; \partial_0 X^I + ig[A_0, X^I] \;\equiv\; D_0 X^I \, , \qquad F_{IJ} \;=\; ig[X^I, X^J] \, , \\ D_0\chi \;&=\; \partial_0\chi + ig[A_0, \chi] \, , \qquad D_I\chi \;=\; ig[X^I, \chi] \, . \end{aligned}$$

Note that the bosonic fields A_0 and X^I, the fermionic fields χ, and the Yang-Mills coupling g are appropriately rescaled by the volume factor of the reduced nine-dimensional space. Thus the dimensionally

[1] Strictly speaking, the Clifford algebra in eleven-dimensions is defined in the same way as the one in ten dimensions.

[2] In this section we discuss the Lagrangian in the flat Minkowski spacetime. Thus we do not distinguish the curved spacetime indices M and tangent space indices A which are described in the other chapters.

reduced Lagrangian is obtained as

$$\begin{aligned} S &= \int \mathrm{d}^{0+1}x\, \mathcal{L}_{0+1} \\ &= \int \mathrm{d}^{0+1}x\, \mathrm{Tr}\Big\{ \frac{1}{2} D_0 X'^I D_0 X'^I + \frac{1}{4} g'^2 \, [X'^I, X'^J]^2 + \frac{i}{2} \chi'^T D_0 \chi' + \frac{1}{2} g' \, \chi'^T \gamma^I [X'^I, \chi'] \Big\} \, . \end{aligned} \tag{B.1.3}$$

We denote the dimensionally reduced variables to $X' = \sqrt{L^9} X$, $\chi' = \sqrt{L^9} \chi$ and $g' = g/\sqrt{L^9}$, where $\int \mathrm{d}^9 x = L^9$ is a reduced volume. This is the effective Lagrangian of N D0-branes system. In chapter II we use this action in order to write down the action (II.1.10).

We of course start the ten-dimensional Yang-Mills action with rescaled field variables as

$$\begin{aligned} S &= \frac{1}{g^2} \int \mathrm{d}^{9+1}x\, \mathrm{Tr}\Big\{ -\frac{1}{4} \widetilde{F}_{\mu\nu} \widetilde{F}^{\mu\nu} - \overline{\widetilde{\theta}} \Gamma^\mu \widetilde{D}_\mu \widetilde{\theta} \Big\} \\ &= \int \mathrm{d}^{9+1}x\, \mathrm{Tr}\Big\{ -\frac{1}{4} \widetilde{F}_{\mu\nu} \widetilde{F}^{\mu\nu} + \frac{i}{2} \widetilde{\chi}^T D_0 \widetilde{\chi} - \frac{i}{2} \chi^T \gamma^I D_I \widetilde{\chi} \Big\} \, , \end{aligned} \tag{B.1.4}$$

where we performed the following field re-definitions

$$g A_\mu \equiv \widetilde{A}_\mu \, , \qquad F_{\mu\nu} = \frac{1}{g} \widetilde{F}_{\mu\nu} \, , \qquad g\chi = \widetilde{\chi} \, , \qquad D_\mu \chi = \frac{1}{g} \widetilde{D}_\mu \widetilde{\chi} \, .$$

Under the above rescaling, the mass dimensions of rescaled variables in ten-dimensions are $[\widetilde{A}_\mu] = 1$ and $[\widetilde{\chi}] = 3/2$. Now let us perform the dimensional reduction to $(0+1)$-dimensional spacetime:

$$S = \frac{1}{g'^2} \int \mathrm{d}^{0+1}x\, \mathrm{Tr}\Big\{ \frac{1}{2} \widetilde{D}_0 \widetilde{X}^I D_0 \widetilde{X}^I + \frac{1}{4} [\widetilde{X}^I, \widetilde{X}^J]^2 + \frac{i}{2} \widetilde{\chi}^T D_0 \widetilde{\chi} - \frac{i}{2} \chi^T \gamma^I D_I \widetilde{\chi} \Big\} \, . \tag{B.1.5}$$

We find that only the Yang-Mills coupling $g' \equiv g/\sqrt{L^9}$ changes the mass dimensions to $3/2$, and that the mass dimensions of fields $\widetilde{A}_\mu = (\widetilde{A}_0, \widetilde{X}^I)$ and $\widetilde{\chi}$ remain 1 and $3/2$, respectively. Since this phenomenon also occurs in the dimensional reduction to any dimensional spacetime, we sometimes write down the field theory Lagrangian in the same description as (B.1.4). The representation of (II.1.11) is also this type.

These Lagrangians (B.1.3) or (B.1.5) are represented the low energy region of N D0-branes system in flat background. They could be deformed when the non-vanishing background fields turn on. In chapter II, we will construct an effective theory Lagrangian of N D0-branes system on such a non-trivial background.

B.2 Matrix Theory Lagrangian: Dirac-Born-Infeld Type Action

In appendix B.1 we discussed the effective action of N coincident D0-branes system without background fields. Here we consider the effective action for N D-branes with non-vanishing massless

Ramond-Ramond background field strength. The argument presented in this appendix is given by Myers' lecture [107].

As discussed in the previous appendix, we have not completely understood the microscopic description of Dirichlet p-brane(s) (or simply Dp-brane(s)) action yet. But we now believe that the Dp-branes' effective action can be described by the Dirac-Born-Infeld (DBI) type Lagrangian at least in the low energy region [36, 100, 119]. Furthermore, Tseytlin, Myers, and a lot of other people have found that the effective theory of N coincident D-branes system should be added the Chern-Simons term from the viewpoint of T-duality [140, 141, 106, 107]. Thus we introduce a short review of single D-brane effective action and N coincident D-branes' effective action. We sometimes call the latter action the *nonabelian D-branes'* action.

First let us construct a single Dp-brane action. As you know, a Dp-brane is a $(p+1)$-dimensional extended hypersurface in spacetime which supports the endpoints of open string within the framework of perturbative string theory. The massless modes of the open string theory form a supersymmetric $U(1)$ gauge theory with a gauge potential A_a $(a = 0, 1, \cdots, p)$, $9-p$ real scalars X^i $(i = p+1, \cdots, 9)$ and their superpartner fermions. As discussed in the previous appendix, the low energy effective action corresponds to the dimensional reduction of the ten-dimensional $U(1)$ super Yang-Mills theory. However, as usual in string theory, there are higher order $\alpha' = \ell_s^2$ corrections, i.e., the stringy corrections (where ℓ_s is the string length). Due to this stringy corrections, the effective action of Dp-brane is deformed to the DBI form[3]

$$S_{\text{DBI}} = -T_p \int \mathrm{d}^{p+1}\sigma \left(\mathrm{e}^{-\phi} \sqrt{-\det \left\{ \mathrm{P}[G+B]_{ab} + \lambda F_{ab} \right\}} \right) , \tag{B.2.1}$$

where T_p is the Dp-brane tension and λ denotes the inverse of the string tension, i.e., $\lambda = 2\pi\alpha'$. the action (B.2.1) contains the field strength of the gauge potential F_{ab} with dimensions of $(\text{mass})^2$. This DBI action describes the couplings of the Dp-brane to the *massless* Neveu-Schwarz (NS) fields of the bulk closed string as the metric $G_{\mu\nu}$ $(\mu\,\nu = 0, 1, \cdots, 9)$, the dilaton ϕ, and the Kalb-Ramond field $B_{\mu\nu}$. They are all dimensionless fields. The interactions with the *massless* Ramond-Ramond (RR) fields are described by the Wess-Zumino term as

$$S_{\text{WZ}} = \mu_p \int \mathrm{P}\Big[\sum C^{(n)} \, \mathrm{e}^{B} \Big] \, \mathrm{e}^{\lambda F} \, . \tag{B.2.2}$$

Note that $C^{(n)}$ is the n-form RR potential defined as

$$C^{(n)} = \frac{1}{n!} C_{\mu_1 \mu_2 \cdots \mu_n} \, \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \, .$$

[3] In this appendix we ignore contributions from the fermionic fields for simplicity.

The Wess-Zumino term (B.2.2) shows that a Dp-brane is naturally charged under the $(p+1)$-form RR potential with charge μ_p, which relates to the Dp-brane tension as $\mu_p = \pm T_p$ due to the spacetime supersymmetry. Summarizing (B.2.1) and (B.2.2), we describe the single Dp-brane effective action as

$$S_{\mathrm{D}p} \;=\; S_{\mathrm{DBI}} + S_{\mathrm{WZ}} \; . \tag{B.2.3}$$

On the flat spacetime without nontrivial constant background fields (i.e., $G_{\mu\nu} = \eta_{\mu\nu}$ and $B = F = 0$), the leading order of the action (B.2.3) reduces to the $(p+1)$-dimensional $U(1)$ gauge theory action. In the case of $p = 0$, the Yang-Mills coupling g' in (B.1.5) is represented in terms of the D0-brane tension T_0 and the Regge parameter α'

$$\frac{1}{g'^2} \;=\; (2\pi\alpha')^2 T_0 \; .$$

The symbol $\mathrm{P}[\cdots]$ in (B.2.1) and (B.2.2) denotes the pull back of the bulk spacetime tensors to the Dp-brane worldvolume. The DBI action (B.2.1) expresses that the Dp-brane moves dynamically in the spacetime. This dynamics becomes more evident with an explanation of the static gauge. To begin, we employ the spacetime diffeomorphism to set the position of the worldvolume $x^i = 0$. With the worldvolume diffeomorphism, we can match the worldvolume coordinates with the remaining spacetime coordinates as $x^a = \sigma^a$. Then the worldvolume scalar fields X^i play the role of describing the transverse displacements of the Dp-brane through the following identification

$$x^i(\sigma) \;=\; \lambda X^i(\sigma) \; . \tag{B.2.4}$$

With this identification, the general formula for the pullback is written by

$$\mathrm{P}[E]_{ab} \;=\; E_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b} \;=\; E_{ab} + \lambda E_{ib} \partial_a X^i + \lambda E_{aj} \partial_b X^j + \lambda^2 E_{ij} \partial_a X^i \partial_b X^j \; . \tag{B.2.5}$$

Now let us generalize the above effective action for the single Dp-brane (B.2.3) to the N coincident Dp-branes system. As N parallel Dp-branes approach each other, the ground state modes of strings stretching between the different Dp-branes become *massless*. These extra massless states carry the appropriate charges to fill out representations under a $U(N)$ symmetry. Thus the $U(1)^N$ symmetry of the individual Dp-branes enhances to the nonabelian $U(N)$ group for the coincident Dp-branes. The vector A_a becomes a nonabelian gauge potential

$$A_a \;=\; A_a^k T_k \; , \qquad F_{ab} \;=\; \partial_a A_b - \partial_b A_a + i[A_a, A_b] \; , \tag{B.2.6}$$

where T_k are N^2 Hermitian generators of $U(N)$ group with $\mathrm{Tr}(T_k T_l) = N\delta_{kl}$. The scalar fields X^i become also matrix valued transforming in the adjoint of $U(N)$. The covariant derivative of the scalar fields is given by $D_a X^i = \partial_a X^i + i[A_a, X^i]$.

Under the above extension, the DBI action (B.2.1) is generalized to

$$S_{\rm DBI} = -T_p \int \mathrm{d}^{p+1}\sigma \, \mathrm{STr}\Big(\mathrm{e}^{-\phi} \sqrt{\det(Q^i{}_j)} \cdot \sqrt{-\det\left\{ \mathrm{P}[E_{ab} + E_{ai}(Q^{-1}-\delta)^{ij}E_{jb}] + \lambda F_{ab} \right\}} \Big) \,, \qquad \text{(B.2.7)}$$

where $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$ and $Q^i{}_j = \delta^i_j + i\lambda\,[X^i, X^k]\,E_{kj}$. We also generalize the Wess-Zumino term (B.2.2) to

$$S_{\rm WZ} = \mu_p \int \mathrm{STr}\Big(\mathrm{P}\Big[\mathrm{e}^{i\lambda \mathrm{i}_X \mathrm{i}_X} \big(\textstyle\sum C^{(n)} \, \mathrm{e}^{B} \big) \Big] \, \mathrm{e}^{\lambda F} \Big) \,. \qquad \text{(B.2.8)}$$

The symbol STr in (B.2.7) and (B.2.8) denotes the maximally symmetrized trace in which we average over all possible orderings of the matrices in the trace. Furthermore, the symbol i_X in the Wess-Zumino term (B.2.8) denotes the interior product with X^i defined as

$$\mathrm{i}_X \mathrm{i}_X C^{(n)} = \frac{1}{2(n-2)!} \, [X^i, X^j] \, C_{ji\mu_3 \cdots \mu_n} \, \mathrm{d}x^{\mu_3} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \,.$$

Note that acting on forms, the interior product is an anticommuting operator. Thus if the scalar fields X^i are ordinary vector fields v^i, the above equation vanishes: $\mathrm{i}_v \mathrm{i}_v C^{(n)} = 0$.

Now let us consider a specific situation for N coincident D0-branes ($p = 0$). If there is a non-vanishing RR four-form field strength $F^{(4)} = \mathrm{d}C^{(3)}$ in the background, and if the other background fields vanish ($B = C^{(1)} = C^{(5)} = C^{(7)} = C^{(9)} = 0$), the Wess-Zumino term (B.2.8) reduces to

$$\begin{aligned} S_{\rm WZ} &= i\lambda\mu_0 \int \mathrm{Tr}\Big(\mathrm{P}\Big[(\mathrm{i}_X \mathrm{i}_X) \, C^{(3)} \Big] \Big) \\ &= \frac{i\lambda}{2}\mu_0 \int \mathrm{d}t \, \mathrm{Tr}\Big(C_{tij} \, [X^i, X^j] + \lambda C_{ijk} \, D_t X^i \, [X^k, X^j] \Big) \,. \end{aligned} \qquad \text{(B.2.9)}$$

Now we assume that the four-form field strength $F^{(4)}$ can be written in terms of a constant f with dimension of mass:

$$F_{t\widetilde{I}\widetilde{J}\widetilde{K}} = -f \, \epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}} \,,$$

where $\epsilon_{\widetilde{I}\widetilde{J}\widetilde{K}}$ denotes the $SO(3)$ Levi-Civita tensor whose normalization is $\epsilon_{123} = 1$. This assumption is regarded as the Freund-Rubin ansatz. Under this ansatz, we can write the reduced Wess-Zumino term (B.2.9) more simply as

$$S_{\rm WZ} = \frac{i}{3}\lambda^2 \mu_0 \int \mathrm{d}t \, \mathrm{Tr}(X^{\widetilde{I}} X^{\widetilde{J}} X^{\widetilde{K}}) \, F_{t\widetilde{I}\widetilde{J}\widetilde{K}} \,.$$

This term appears in the Lagrangian of the BMN matrix model (II.1.8).

B.3 Supergravity Lagrangian with Full Interactions

In this appendix we introduce the eleven-dimensional supergravity Lagrangian which was described by Cremmer, Julia and Scherk [30]. It is not so difficult to describe the Lagrangian with full interactions including torsion and quartic terms with respect to the gravitino. Here we write the full supergravity Lagrangian below:

$$\begin{aligned} \mathcal{L} \;=\; & e\,\mathcal{R}(e,\omega) - \frac{1}{2}\overline{\Psi}_M \widehat{\Gamma}^{MNP} D_N[\tfrac{1}{2}(\omega + \widehat{\omega})]\Psi_P - \frac{1}{48} e\, F_{MNPQ}\, F^{MNPQ} \\ & - \frac{1}{192} e\, \overline{\Psi}_M \widetilde{\Gamma}^{MNPQRS} \Psi_N \cdot \frac{1}{2}(F + \widehat{F})_{PQRS} \\ & - \frac{1}{(144)^2} \varepsilon^{MNPQRSUVWXY}\, F_{MNPQ}\, F_{RSUV}\, C_{WXY} \;, \end{aligned} \tag{B.3.1}$$

where definitions of $\widehat{\omega}$ and $\widehat{F}_{MNPQ}$ are

$$D_{[M}(\widehat{\omega}) e_{N]}{}^{A} \;=\; \frac{1}{8}\overline{\Psi}_M \widehat{\Gamma}^{A} \Psi_N \;, \qquad \widehat{F}_{MNPQ} \;=\; F_{MNPQ} + \frac{3}{2}\overline{\Psi}_{[M}\widehat{\Gamma}_{NP}\Psi_{Q]} \;.$$

Note that the definition of the covariant derivative D_M is described in appendix A.6. We also the full supersymmetry transformation rules for the vielbein, three-form gauge field and gravitino:

$$\begin{aligned} \delta e_M{}^{A} \;&=\; \frac{1}{2}\overline{\varepsilon}\widehat{\Gamma}^{A}\Psi_M \;, & \delta C_{MNP} \;&=\; -\frac{3}{2}\overline{\varepsilon}\widehat{\Gamma}_{[MN}\Psi_{P]} \;, \\ \delta \Psi_M \;&=\; 2D_M(\widehat{\omega})\varepsilon + 2T_M{}^{NPQR}\varepsilon \widehat{F}_{NPQR} \;, & T_M{}^{NPQR} \;&=\; \frac{1}{288}\Big(\widehat{\Gamma}_M{}^{NPQR} - 8\delta_M^{[N}\widehat{\Gamma}^{PQR]}\Big) \;. \end{aligned}$$

We can see these descriptions in various lectures with respect to higher-dimensional supergravities (for instance, see [143, 53, 58, 41]).

Appendix C

Background Geometry

C.1 Anti-de Sitter Spaces

Anti-de Sitter spaces emerge in the spontaneous compactifications of higher-dimensional supergravities via Kaluza-Klein mechanism. Since the anti-de Sitter space is a maximally symmetric space with negative cosmological constant, we can study supergravity in this spacetime background. In the case of eleven-dimensions, for example, the four- and seven-dimensional anti-de Sitter spaces are derived from the existence of the constant four-form flux and some constraints: the seven- and four-dimensional compactified space should be Einstein spaces and the uncompactified spacetime should be maximally symmetric [53, 57, 58]. In ten-dimensions, $AdS_5 \times S^5$ geometry also appears in type IIB supergravity [92] and type IIB superstring in the near horizon limit of D3-brane via constant self-dual five-form Ramond-Ramond flux [101].

Representations of supersymmetry in anti-de Sitter spaces are discussed by Nicolai [111] and de Wit and Herger [43]. The superalgebra and its unitary representation [111, 43] are discussed in terms of the oscillator method, which is one of the powerful tool to investigate the supermultiplets and their dynamics on the anti-de Sitter space [76, 74, 75, 63].

In this thesis we will argue the (linearized) supergravity on the plane-wave background, which is a specific limit of the product space of anti-de Sitter space and Einstein space, called the "Penrose limit". Next we will discuss the definition of the Penrose limit of the product spaces and will explain a physical meaning.

Penrose Limit of Maximally Supersymmetric Spaces

Let us consider the Penrose limit of the product spaces of anti-de Sitter space and higher-dimensional sphere in eleven-dimensions. Since we would like to consider the maximally supersymmetric spacetime, we concentrate the discussions of the Penrose limit of $AdS_4 \times S^7$ and $AdS_7 \times S^4$ [25].

Since the $AdS_4 \times S^7$ geometry [59] has 32 Killing spinors this spacetime is maximally supersymmetric. This appears as a geometry of the near horizon limit of M2-brane, whose line element is described by the global coordinates

$$\mathrm{d}s^2 \ = \ R_A^2 \big\{ - \cosh^2 \rho \cdot \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2 \rho \cdot \mathrm{d}\Omega_2^2 \big\} + R_S^2 \big\{ \cos^2 \theta \mathrm{d}\varphi^2 + \mathrm{d}\theta^2 + \sin^2 \theta \cdot \mathrm{d}\Omega_5'^{\,2} \big\} \ .$$

We introduce the following coordinates around a null geodesic $\gamma = \{R_S = 2R_A, t = 2\varphi, \rho = \theta = 0\}$:

$$\alpha \ = \ \frac{R_S}{R_A} \ = \ 2 \ , \qquad x^+ \ = \ \frac{1}{2}(t + 2\varphi) \cdot \frac{3}{\mu} \ , \qquad x^- \ = \ R_A^2 (t - 2\varphi) \cdot \frac{\mu}{3} \ ,$$

$$x = R_A \rho \, , \qquad y = 2R_A \theta \, .$$

Notice that we added the rescaling factor $3/\mu$ for later convenience. Performing the large R_A limit and retaining x and y to be finite (the Penrose limit), we obtain the following simple line element

$$\mathrm{d}s^2 = -2\mathrm{d}x^+\mathrm{d}x^- - \Big(\frac{\mu}{3}\Big)^2\Big\{x^2 + \frac{1}{4}y^2\Big\}(\mathrm{d}x^+)^2 + \big\{\mathrm{d}x^2 + x^2\mathrm{d}\Omega_2^2\big\} + \big\{\mathrm{d}y^2 + y^2\mathrm{d}\Omega_5'^2\big\} \, . \qquad \text{(C.1.1)}$$

This metric was constructed by Kowalski-Glikman [97]. Thus this spacetime metric is sometimes called "Kowalski-Glikman (KG) solution" and this spacetime is a maximally supersymmetric solution of the eleven-dimensional supergravity with constant flux $F_{123+} = \mu$ because of the existence of 32 Killing spinors [28].

Let us consider the Penrose limit of another spacetime, i.e., the Penrose limit of the $AdS_7 \times S^4$ spacetime. Since the $AdS_7 \times S^4$ spacetime [118] also has 32 Killing spinors, this is maximally supersymmetric in eleven-dimensions. Moreover it is known that this spacetime appears in the near horizon limit of M5-branes. Now we write down the global coordinates of $AdS_7 \times S^4$

$$\mathrm{d}s^2 = R_A^2\big\{-\cosh^2\rho\cdot\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\cdot\mathrm{d}\Omega_5^2\big\} + R_S^2\big\{\cos^2\theta\mathrm{d}\varphi^2 + \mathrm{d}\theta^2 + \sin^2\theta\cdot\mathrm{d}\Omega_2'^2\big\} \, ,$$

where R_A and R_S are the radius of AdS_7 and S^4, respectively. In order to perform the Penrose limit we take the following constraints:

$$\alpha = \frac{R_S}{R_A} = \frac{1}{2} \, , \qquad x^+ = \frac{1}{2}\Big(t + \frac{1}{2}\varphi\Big)\cdot\frac{6}{\mu} \, , \qquad x^- = R_A^2\Big(t - \frac{1}{2}\varphi\Big)\cdot\frac{\mu}{6} \, ,$$
$$x = R_A\rho \, , \qquad y = \frac{1}{2}R_A\theta$$

around a null geodesic $\gamma = \{R_A = 2R_S, t = \frac{1}{2}\varphi,\ \theta = \rho = 0\}$. Taking $R_A \to \infty$ and remaining the coordinates $(x,\, y)$ to be finite, and exchanging the coordinate labels between x and y, we obtain the same line element as the Penrose limit of $AdS_4 \times S^7$

$$\mathrm{d}s^2 = -2\mathrm{d}x^+\mathrm{d}x^- - \Big(\frac{\mu}{3}\Big)^2\Big\{x^2 + \frac{1}{4}y^2\Big\}(\mathrm{d}x^+)^2 + \big\{\mathrm{d}x^2 + x^2\mathrm{d}\Omega_2^2\big\} + \big\{\mathrm{d}y^2 + y^2\mathrm{d}\Omega_5'^2\big\} \, .$$

We can easily find that the vanishing limit of the parameter μ of the Penrose limit of $AdS_{4(7)} \times S^{7(4)}$ is the flat Minkowski spacetime, which is of course maximally supersymmetric. Here we draw the relations of the four maximally supersymmetric spacetime in eleven-dimensions in Figure C.1:

So far we discussed the procedure of the Penrose limit of the product spaces such as $AdS_4 \times S^7$ and $AdS_7 \times S^4$. We can also argue the Penrose limit of other spaces, for example, the $AdS_4 \times$ squashed S^7, the $AdS_4 \times Q^{1,1,1}$, the $AdS_4 \times N^{0,1,0}$, and so on, which also reduce to the above Kowalski-Glikman

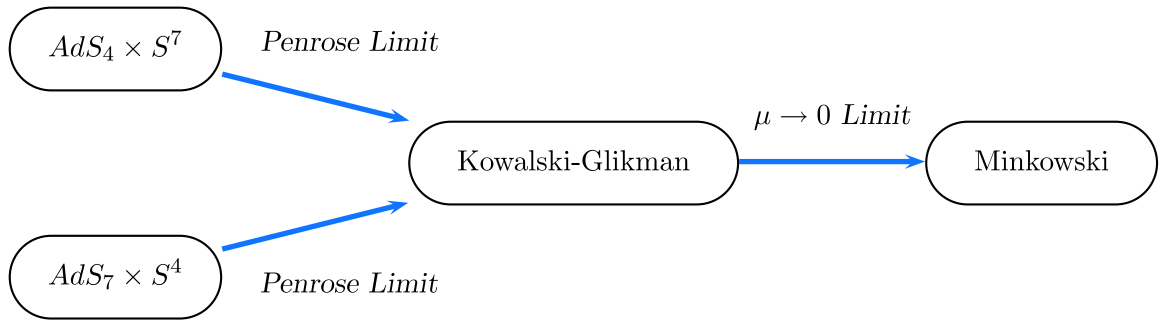

Figure C.1: *The relationships among the maximally supersymmetric spacetimes in eleven-dimensions.*

solution (C.1.1) via Penrose limit [77]. Now it is important to explain the physical meaning of the procedure of the Penrose limit. Let us quote a famous paragraph from the Penrose's lecture [117] (see also [26]):

> *There is a 'physical' interpretation of the above mathematical procedure, which is the following. We envisage a succession of observers travelling in the space-time M whose world lines approach the null geodesic γ more and more closely; so we picture these observers as travelling with greater and greater speeds, approaching that of light. As their speeds increase they must correspondingly recalibrate their clocks to run faster and faster (assuming that all space-time measurements are referred to clock measurements in the standard way), so that in the limit the clocks measure the affine parameter x^0 along γ. (Without clock recalibration a degenerate space-time metric would result.) In the limit the observers measure the space-time to have the plane wave structure W_γ.*

In other words, the Penrose limit can be understood as a boost followed by a commensurate uniform rescaling of the coordinates in such a way that the affine parameter along the null geodesic remains invariant. The obtained spacetime backgrounds are called the "plane-wave" or "pp-wave" backgrounds, where the term "pp-wave" is the abbreviation of "*plane fronted gravitational wave with parallel rays*".

Geometrical Variables on the Plane-wave

Here we discuss several properties of the maximally supersymmetric plane-wave background. As discussed above, this solution was found by Kowalski-Glikman [97, 28] and often called the KG solution. This is the unique plane-wave type solution preserving maximal 32 supersymmetries in eleven dimensions. The metric of this background is given by (C.1.1) as

$$\begin{aligned} \mathrm{d}s^2 &= -2\mathrm{d}x^+\mathrm{d}x^- + G_{++}\cdot(\mathrm{d}x^+)^2 + \sum_{I=1}^{9}(\mathrm{d}x^I)^2 \,, \\ G_{++} &= -\Big[\Big(\frac{\mu}{3}\Big)^2\sum_{\widetilde{I}=1}^{3}(x^{\widetilde{I}})^2 + \Big(\frac{\mu}{6}\Big)^2\sum_{I'=4}^{9}(x^{I'})^2\Big] \,, \end{aligned} \tag{C.1.2}$$

which is equipped with the constant four-form flux

$$F_{123+} = \mu \neq 0 \,.$$

In our consideration the contribution from torsion is not included, i.e., affine connection is symmetric under lower indices: $\Gamma^P_{MN} = \Gamma^P_{NM}$. For the metric on the KG solution (C.1.2), we obtain the following variables:

$$\begin{aligned} e_+{}^{+} &= e_-{}^{-} = 1 \,, \qquad e_+{}^{-} = -\frac{1}{2}G_{++} \,, \\ E_{+}{}^{+} &= E_{-}{}^{-} = 1 \,, \qquad E_{+}{}^{-} = \frac{1}{2}G_{++} \,, \\ \omega_+{}^{I-} &= \frac{1}{2}\partial_I G_{++} \,, \\ \Gamma^I_{++} &= \Gamma^-_{+I} = -\frac{1}{2}\partial_I G_{++} \,, \\ R^I{}_{+J+} &= -\frac{1}{2}\partial_I\partial_J G_{++} \,, \qquad \mathcal{R}_{++} = \frac{1}{2}\mu^2 \,, \qquad \mathcal{R} = 0 \,. \end{aligned} \tag{C.1.3}$$

C.2 Coset Construction

Here we discuss the coset construction of product spaces of anti-de Sitter and sphere, in particular $AdS_4 \times S^7$ and $AdS_7 \times S^4$ spacetimes, which lead to the KG solution in the Penrose limit. In this construction we define supervielbeins to all order in θ, the superspace coordinates ($SO(10,1)$ Majorana spinor coordinates) in eleven dimensions [49, 50, 41].

Superalgebra

Let us consider the superalgebra of the plane-wave in terms of the Penrose limit of $AdS_4 \times S^7$ spacetime superalgebra. Thus we first prepare the superalgebras of $AdS_{4(7)} \times S^{7(4)}$. This spacetime is solutions of the eleven-dimensional supergravity with a constant four-form flux given by

$$F_{\widetilde{M}\widetilde{N}\widetilde{P}\widetilde{Q}} = f\, e\, E^{-1}_{\widetilde{M}\widetilde{N}\widetilde{P}\widetilde{Q}} \,. \tag{C.2.1}$$

Note that the indices $\widetilde{M}, \widetilde{N}, \cdots$ are the indices expanded to the four-dimensional spacetime directions and the variable $e = \sqrt{|\det g_{MN}|}$ denotes to the square root of the determinant of the metric in the eleven-dimensional curved spacetime. The $E^{-1}_{\widetilde{M}\widetilde{N}\widetilde{P}\widetilde{Q}}$ is an invariant tensor density in four dimensions (weight -1) whose normalization is defined by $E^{-1}_{0123} = 1$. The constant f decides the property of spacetime: if f is real and non-vanishing, we can obtain the $AdS_4 \times S^7$ spacetime and if f is non-zero pure imaginary, $AdS_7 \times S^4$ spacetime appears. Of course we obtain the flat spacetime when we choose $f = 0$[1]. Under this setup[2], the Riemann tensors of four- and seven-dimensional spaces are given by equations of motion of eleven-dimensional supergravity (III.1.3a):

$$R_{\widetilde{M}\widetilde{N}\widetilde{P}\widetilde{Q}} = -\frac{1}{9} f^2 \big(g_{\widetilde{M}\widetilde{P}}\, g_{\widetilde{N}\widetilde{Q}} - g_{\widetilde{M}\widetilde{Q}}\, g_{\widetilde{N}\widetilde{P}} \big) \qquad \text{four-dimensional space} , \tag{C.2.2a}$$

$$R_{M'N'P'Q'} = \frac{1}{36} f^2 \big(g_{M'P'}\, g_{N'Q'} - g_{M'Q'}\, g_{N'P'} \big) \qquad \text{seven-dimensional space} , \tag{C.2.2b}$$

where $M', N', \cdots$ denotes the seven-dimensional space indices. In this configuration the superalgebra of $AdS_{4(7)} \times S^{7(4)}$ can be written down in terms of Hermitian generators $\{P_A, \Sigma_{AB}\}$ and fermionic generators $Q_{aa'}$ below (see the lecture note written by de Wit [41]):

$$[P_{\widetilde{A}}, P_{\widetilde{B}}] = \frac{i}{9} f^2\, \Sigma_{\widetilde{A}\widetilde{B}} , \qquad [P_{A'}, P_{B'}] = -\frac{i}{36} f^2\, \Sigma_{A'B'} , \tag{C.2.3a}$$

$$[P_{\widetilde{A}}, \Sigma_{\widetilde{B}\widetilde{C}}] = i \big(\eta_{\widetilde{A}\widetilde{B}}\, P_{\widetilde{C}} - \eta_{\widetilde{A}\widetilde{C}}\, P_{\widetilde{B}} \big) , \qquad [P_{A'}, \Sigma_{B'C'}] = i \big(\eta_{A'B'}\, P_{C'} - \eta_{A'C'}\, P_{B'} \big) , \tag{C.2.3b}$$

$$i[\Sigma_{\widetilde{A}\widetilde{B}}, \Sigma_{\widetilde{C}\widetilde{D}}] = \eta_{\widetilde{A}\widetilde{C}}\, \Sigma_{\widetilde{B}\widetilde{D}} + \eta_{\widetilde{B}\widetilde{D}}\, \Sigma_{\widetilde{A}\widetilde{C}} - \eta_{\widetilde{A}\widetilde{D}}\, \Sigma_{\widetilde{B}\widetilde{C}} - \eta_{\widetilde{B}\widetilde{C}}\, \Sigma_{\widetilde{A}\widetilde{D}} , \tag{C.2.3c}$$

$$i[\Sigma_{A'B'}, \Sigma_{C'D'}] = \eta_{A'C'}\, \Sigma_{B'D'} + \eta_{B'D'}\, \Sigma_{A'C'} - \eta_{A'D'}\, \Sigma_{B'C'} - \eta_{B'C'}\, \Sigma_{A'D'} , \tag{C.2.3d}$$

$$[P_{\widetilde{A}}, Q_{aa'}] = -\frac{i}{6} f\, (\gamma_{\widetilde{A}} \gamma_5)_a{}^b\, Q_{ba'} , \qquad [P_{A'}, Q_{aa'}] = -\frac{i}{12} f\, (\Gamma_{A'})_{a'}{}^{b'}\, Q_{ab'} , \tag{C.2.3e}$$

$$[\Sigma_{\widetilde{A}\widetilde{B}}, Q_{aa'}] = -\frac{i}{2} (\gamma_{\widetilde{A}\widetilde{B}})_a{}^b\, Q_{ba'} , \qquad [\Sigma_{A'B'}, Q_{aa'}] = -\frac{i}{2} (\Gamma_{A'B'})_{a'}{}^{b'}\, Q_{ab'} , \tag{C.2.3f}$$

$$\begin{aligned} \{Q_{aa'}, Q_{bb'}\} = & -C'_{a'b'} \Big\{ -2i\, (\gamma_{\widetilde{A}} C)_{ab}\, P^{\widetilde{A}} + \frac{i}{6} f\, (\gamma_{\widetilde{A}\widetilde{B}} \gamma_5 C)_{ab}\, \Sigma^{\widetilde{A}\widetilde{B}} \Big\} \\ & - (\gamma_5 C)_{ab} \Big\{ -2i\, (\Gamma_{A'} C')_{a'b'}\, P^{A'} - \frac{i}{3} f\, (\Gamma_{A'B'} C')_{a'b'}\, M^{A'B'} \Big\} . \end{aligned} \tag{C.2.3g}$$

Notice that the indices $\widetilde{A}$ and B' are the indices of four- and seven-dimensional tangent spaces; indices (a, b) are the spinor indices in four-dimensional space and (a', b') are the spinor indices in seven-dimensional space. The matrices $\gamma_{\widetilde{A}}$, γ_5 and C_{ab} are Dirac gamma matrices and charge conjugation

[1] We assume that the spacetime is (maximally) symmetric.

[2] This setup is called the "Freund-Rubin ansatz" [66]. This situation can be derived under some simple assumption [65].

matrix in four-dimensional space and $\Gamma_{A'}$, $C'_{a'b'}$ are gamma matrices and charge conjugation matrix in seven-dimensional space. We can rewrite the above superalgebra (C.2.3) in the language of eleven-dimensional spacetime

$$
\begin{aligned}
[P_{\widetilde{A}}, P_{\widetilde{B}}] &= \frac{i}{9} f^2 \Sigma_{\widetilde{A}\widetilde{B}} , \qquad [P_{A'}, P_{B'}] = -\frac{i}{36} f^2 \Sigma_{A'B'} , \\
[P_{\widetilde{A}}, \Sigma_{\widetilde{B}\widetilde{C}}] &= i\big(\eta_{\widetilde{A}\widetilde{B}} P_{\widetilde{C}} - \eta_{\widetilde{A}\widetilde{C}} P_{\widetilde{B}}\big) , \\
[P_{A'}, \Sigma_{B'C'}] &= i\big(\eta_{A'B'} P_{C'} - \eta_{A'C'} P_{B'}\big) , \\
i[\Sigma_{\widetilde{A}\widetilde{B}}, \Sigma_{\widetilde{C}\widetilde{D}}] &= \eta_{\widetilde{A}\widetilde{C}} \Sigma_{\widetilde{B}\widetilde{D}} + \eta_{\widetilde{B}\widetilde{D}} \Sigma_{\widetilde{A}\widetilde{C}} - \eta_{\widetilde{A}\widetilde{D}} \Sigma_{\widetilde{B}\widetilde{C}} - \eta_{\widetilde{B}\widetilde{C}} \Sigma_{\widetilde{A}\widetilde{D}} , \\
i[\Sigma_{A'B'}, \Sigma_{C'D'}] &= \eta_{A'C'} \Sigma_{B'D'} + \eta_{B'D'} \Sigma_{A'C'} - \eta_{A'D'} \Sigma_{B'C'} - \eta_{B'C'} \Sigma_{A'D'} , \\
[P_A, \overline{Q}] &= i\overline{Q}\, T_A{}^{BCDE} F_{BCDE} , \qquad [\Sigma_{AB}, \overline{Q}] = \frac{i}{2}\overline{Q}\,\widehat{\Gamma}_{AB} , \\
\{Q, \overline{Q}\} &= 2i\,\widehat{\Gamma}_A P^A - \frac{i}{144}\{\widehat{\Gamma}^{ABCDEF} F_{CDEF} + 24\,\widehat{\Gamma}_{CD} F^{ABCD}\}\Sigma_{AB} ,
\end{aligned}
\tag{C.2.4}
$$

where P_A and Σ_{AB} are bosonic Hermitian generators and fermionic generators Q are $SO(10,1)$ Majorana spinors[3]. The symbol $T_A{}^{BCDE}$ is described by the gamma matrices as

$$
T_A{}^{BCDE} = \frac{1}{288}\Big(\widehat{\Gamma}_A{}^{BCDE} - 8\delta_A^{[B}\widehat{\Gamma}^{CDE]}\Big) .
$$

On the geometry of the plane-wave background this superalgebra is also satisfied because the plane-wave is continuously connected to $AdS_{4(7)} \times S^{7(4)}$ geometries.

We will construct supervielbeins on the $AdS_{4(7)} \times S^{7(4)}$ and on the plane-wave by utilizing this superalgebra (C.2.4). The supervielbeins are important to construct Lagrangians of supermembranes and Matrix theory on the AdS background and the plane-wave background, which are discussed in chapter II.

Coset Space Representatives and Supervielbeins

Here we construct the supervielbeins on the $AdS_{4(7)} \times S^{7(4)}$ background and the plane-wave background of them. First we define a (super)representative $L(Z)$ on the backgrounds

$$
L(Z) = \ell(x)\cdot\widehat{L}(\theta) , \qquad \ell(x) = \exp(i\,x^A P_A) \quad \text{and} \quad \widehat{L}(\theta) = \exp(i\,\overline{\theta} Q) ,
$$

where $Z = (x^A, \theta)$ are the tangent space coordinates of eleven-dimensional curved spacetime; the bosonic generators P_A are Hermitian and the fermionic generators Q are the $SO(10,1)$ Majorana

[3] We define the Dirac conjugate of the Majorana spinor as $\overline{Q} = iQ^\dagger\widehat{\Gamma}^0 = Q^T C$. Thus the product of two Majorana spinors has the following properties: $\overline{\theta}Q = -Q^T C^T\theta = Q^T C\theta = \overline{Q}\theta$ and $(\overline{\theta}Q)^\dagger = -iQ^\dagger(\widehat{\Gamma}^0)^\dagger\theta = iQ^\dagger\widehat{\Gamma}^0\theta = \overline{Q}\theta = \overline{\theta}Q$.

spinors. In this definition the representative is unitary. Utilizing this representative we define a "super" Maurer-Cartan one-form α in the same way as bosonic one-form

$$\alpha \;=\; i^{-1}\, L^{-1} \mathrm{d}L \;=\; \widetilde{E} + \widetilde{\Omega}\,. \tag{C.2.5}$$

Here we introduce a supervielbein $\widetilde{E}$ and super H-connection $\widetilde{\Omega}$ which are expanded by the bosonic and fermionic generators

$$\widetilde{E} \;=\; \widehat{E}^A\, P_A + \overline{Q}\widehat{E}\,, \qquad \widetilde{\Omega} \;=\; \frac{1}{2}\widehat{\Omega}^{AB}\, \Sigma_{AB}\,. \tag{C.2.6}$$

Note that we also refer the components $\widehat{E}^A$, $\widehat{E}$ and $\widehat{\Omega}^{AB}$ to supervielbeins and super H-connections. Maurer-Cartan one-form (C.2.5) satisfies the following relation

$$\mathrm{d}\alpha + i\, \alpha \wedge \alpha \;=\; 0\,. \tag{C.2.7}$$

Utilizing the equations (C.2.5) and (C.2.6), we obtain the "super" Cartan's structure equations

$$\begin{aligned}
0 \;&=\; \mathrm{d}\widetilde{\Omega} + i\, \widetilde{\Omega} \wedge \widetilde{\Omega} + \frac{i}{2}\widehat{E}^A \wedge \widehat{E}^B [P_A, P_B] \\
&\quad + \frac{1}{288}\overline{\widehat{E}}\Big\{ \widehat{\Gamma}^{ABCDEF}\, F_{CDEF} + 24\, \widehat{\Gamma}_{CD}\, F^{ABCD} \Big\} \widehat{E}\, \Sigma_{AB}\,, \\
0 \;&=\; \mathrm{d}\widehat{E}^A - \widehat{\Omega}^A{}_B \wedge \widehat{E}^B - \overline{\widehat{E}}\, \widehat{\Gamma}^A \wedge \widehat{E}\,, \\
0 \;&=\; \mathrm{d}\widehat{E} - \widehat{E}^A \wedge T_A{}^{BCDE}\, \widehat{E}\, F_{BCDE} - \frac{1}{4}\widehat{\Omega}^{AB} \wedge \widehat{\Gamma}_{AB}\, \widehat{E}\,.
\end{aligned} \tag{C.2.8}$$

Substituting the superalgebra (C.2.4) into the above super Cartan's structure equations (C.2.8), we solve the supervielbeins and H-connections

$$\widehat{E}^A \;=\; e^A + \mathcal{O}(\theta^2)\,, \qquad \widehat{\Omega}^{AB} \;=\; -\omega^{AB} + \mathcal{O}(\theta^2)\,. \tag{C.2.9}$$

Here we wrote down the solutions up to fermionic contributions. The vielbeins e^A and the spin connections ω^{AB} are obtained such that they satisfy the Riemann tensors (C.2.2). It is somewhat difficult to solve the equations (C.2.8) with all the fermionic contributions. Thus we introduce a trick proposed by Kallosh, Rahmfeld and Rajaraman [87]. We rescale the fermionic coordinates θ to $t\theta$ with one arbitrary parameter $t \in [0,1]$ which we put to unity at the end. Taking the derivative with respect to this parameter t of the Maurer-Cartan one-form (C.2.5) leads to first order differential equations for supervielbeins $\widehat{E}$ and H-connection $\widehat{\Omega}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{E} + \widetilde{\Omega}) \;=\; \mathrm{d}\overline{\theta}Q + i\, (\widetilde{E} + \widetilde{\Omega})\overline{\theta}Q - i\, \overline{\theta}Q(\widetilde{E} + \widetilde{\Omega})\,. \tag{C.2.10}$$

The left-hand side and right-hand side of this equation is calculated respectively:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{E} + \widetilde{\Omega}) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\widehat{E}^A P_A + \overline{Q}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{E} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Omega}^{AB}\Sigma_{AB}\,,$$

$$
\begin{aligned}
(\widetilde{E}+\widetilde{\Omega})\bar{\theta}Q-\bar{\theta}Q(\widetilde{E}+\widetilde{\Omega}) \;=\;& i\overline{Q}T_A{}^{BCDE}\theta\, F_{BCDE}\,\widehat{E}^A+\frac{i}{4}\widehat{\Omega}^{AB}\overline{Q}\widehat{\Gamma}_{AB}\theta \\
&-\bar{\theta}\Big[2i\widehat{\Gamma}^A P_A-\frac{i}{144}\{\widehat{\Gamma}^{ABCDEF}F_{CDEF}+24\,\widehat{\Gamma}_{CD}\,F^{ABCD}\}\Sigma_{AB}\Big]\widehat{E}\,.
\end{aligned}
$$

Note that we substituted the superalgebra (C.2.4) into the above equations. Summarizing the equations in terms of the supersymmetry generators P_A, Σ_{AB} and Q, we find that a couple of first-order differential equations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\widehat{E}^A \;&=\; 2\bar{\theta}\widehat{\Gamma}^A\widehat{E}\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\widehat{E} \;&=\; \mathrm{d}\theta-\widehat{E}^A\,T_A{}^{BCDE}\theta\,F_{BCDE}-\frac{1}{4}\widehat{\Omega}^{AB}\,\widehat{\Gamma}_{AB}\theta\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Omega}^{AB} \;&=\; -\frac{1}{72}\bar{\theta}\Big\{\widehat{\Gamma}^{ABCDEF}\,F_{CDEF}+24\,\widehat{\Gamma}_{CD}\,F^{ABCD}\Big\}\widehat{E}\,.
\end{aligned}
$$

Since these equations have a structure of coupled harmonic oscillators with respect to $\widehat{E}^A$ and $\widehat{\Omega}^{AB}$, we can solve these completely as

$$
\begin{aligned}
\widehat{E}^A(x,\theta) \;&=\; e^A+\bar{\theta}\widehat{\Gamma}^A D\theta+2\sum_{n=1}^{15}\frac{1}{(2n+2)!}\bar{\theta}\,\widehat{\Gamma}^A\mathcal{M}^{2n}\,D\theta\,,\\
\widehat{E}(x,\theta) \;&=\; D\theta+\sum_{n=1}^{16}\frac{1}{(2n+1)!}\mathcal{M}^{2n}D\theta\,,\\
\widehat{\Omega}^{AB}(x,\theta) \;&=\; -\omega^{AB}-\frac{1}{72}\sum_{n=0}^{15}\frac{1}{(2n+2)!}\bar{\theta}\{\widehat{\Gamma}^{ABCDEF}\,F_{CDEF}+24\,\widehat{\Gamma}_{CD}\,F^{ABCD}\}\mathcal{M}^{2n}D\theta\,, \qquad \text{(C.2.11)}\\
D\theta \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}\widehat{E}\Big|_{t=0} \;=\; \mathrm{d}\theta-e^A T_A{}^{BCDE}\theta\,F_{BCDE}+\frac{1}{4}\omega^{AB}\widehat{\Gamma}_{AB}\theta\,,\\
\mathcal{M}^2 \;&=\; -2\big(T_A{}^{BCDE}\,\theta\big)\,F_{BCDE}\,\big(\bar{\theta}\widehat{\Gamma}^A\big)\\
&\quad+\frac{1}{288}\big(\widehat{\Gamma}_{AB}\,\theta\big)\Big(\bar{\theta}\Big[\widehat{\Gamma}^{ABCDEF}\,F_{CDEF}+24\,\widehat{\Gamma}_{CD}\,F^{ABCD}\Big]\Big)\,.
\end{aligned}
$$

Notice that the coordinate θ is the anticommuting $SO(10,1)$ Majorana spinor and we put the free parameter t to unity. These variables correctly represents the superspaces of the $AdS_{4(7)}\times S^{7(4)}$. In chapter II we use these variables on the plane-wave which is continuously related to $AdS_{4(7)}\times S^{7(4)}$ as discussed in appendix C.1.

References

[1] B.S. Acharya, "*On Realising $\mathcal{N} = 1$ Super Yang-Mills in M-theory*", hep-th/0011089.

[2] A. Achúcarro, J.M. Evans, P.K. Townsend and D.L. Wiltshire, "*Super p-branes*", Phys. Lett. B198 (1987) 441.

[3] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, "*Large N Field Theories, String Theory and Gravity*", Phys. Rept. 323 (2000) 183, hep-th/9905111.

[4] M. Atiyah and E. Witten, "*M-theory Dynamics on a Manifold of G_2 Holonomy*", Adv. Theor. Math. Phys. 6 (2003) 1, hep-th/0107177.

[5] H. Awata and S. Hirano, "*AdS_7/CFT_6 Correspondence and Matrix Model of M5-branes*", Adv. Theor. Math. Phys. 3 (1999) 147, hep-th/9812218.

[6] I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D.P. Sorokin and M. Tonin, "*Covariant Action for the Super-five-brane of M-theory*", Phys. Rev. Lett. 78 (1997) 4332, hep-th/9701149.

[7] I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D.P. Sorokin and M. Tonin, "*On the Equivalence of Different Formulations of the M-theory Five-brane*", Phys. Lett. B408 (1997) 135, hep-th/9703127.

[8] T. Banks, "*Matrix Theory*", Nucl. Phys. Proc. Suppl. 67 (1998) 180, hep-th/9710231.

[9] T. Banks, "*TASI Lectures on Matrix Theory*", lectures given at Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 99): Strings, Branes, and Gravity, Boulder, Colorado, 1999, hep-th/9911068.

[10] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, "*M-theory as a Matrix Model: a Conjecture*", Phys. Rev. D55 (1997) 5112, hep-th/9610043.

[11] T. Banks, N. Seiberg and S.H. Shenker, "*Branes from Matrices*", Nucl. Phys. B490 (1997) 91, hep-th/9612157.

[12] I. Bars, "*First Massive Level and Anomalies in the Supermembrane*", Nucl. Phys. B308 (1988) 462.

[13] I. Bars, C. Deliduman and D. Minic, "*Lifting M-theory to Two-time Physics*", Phys. Lett. B457 (1999) 275, hep-th/9904063.

[14] I. Bars, C. Deliduman, A. Pasqua and B. Zumino, "*A Hidden Twelve-dimensional Super-Poincaré Symmetry in Eleven Dimensions*", hep-th/0312147.

[15] I. Bars and C. Kounnas, "*Theories with Two Times*", Phys. Lett. B402 (1997) 25, hep-th/9703060.

[16] I. Bars and C. Kounnas, "*String and Particle with Two Times*", Phys. Rev. D56 (1997) 3664, hep-th/9705205.

[17] I. Bars and C.N. Pope, "*Anomalies in Super p-branes*", Class. Quant. Grav. 5 (1988) 1157.

[18] K. Becker, M. Becker, J. Polchinski and A.A. Tseytlin, "*Higher Order Graviton Scattering in M(atrix) Theory*", Phys. Rev. D56 (1997) 3174, hep-th/9706072.

[19] D. Berenstein, J. Maldacena and H. Nastase, "*Strings in Flat Space and PP-waves from $\mathcal{N} = 4$ Super Yang Mills*", JHEP 0204 (2002) 013, hep-th/0202021.

[20] E. Bergshoeff, M.J. Duff, C.N. Pope and E. Sezgin, "*Compactifications of the Eleven-dimensional Supermembrane*", Phys. Lett. B224 (1989) 71.

[21] E. Bergshoeff, E. Sezgin and P.K. Townsend, "*Supermembranes and Eleven-dimensional Supergravity*", Phys. Lett. B189 (1987) 75.

[22] E. Bergshoeff, E. Sezgin and P.K. Townsend, "*Properties of the Eleven-dimensional Supermembrane Theory*", Ann. Phys. 185 (1988) 330.

[23] D. Bigatti and L. Susskind, "*Review of Matrix Theory*", prepared for NATO Advanced Study Institute on Strings, Branes and Dualities, Cargese, France, 1997, he-th/9712072.

[24] A. Bilal, "*M(atrix) Theory: a Pedagogical Introduction*", Fortsch. Phys. 47 (1999) 5, hep-th/9710136.

[25] M. Blau, J. Figueroa-O'Farrill, C.M. Hull and G. Papadopoulos, "*Penrose Limits and Maximal Supersymmetry*", Class. Quant. Grav. 19 (2002) L87, hep-th/0201081.

[26] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, "*Penrose Limits, Supergravity and Brane Dynamics*", Class. Quant. Grav. 19 (2002) 4753, hep-th/0202111.

[27] P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, "*Vacuum Configurations for Superstrings*", Nucl. Phys. B258 (1985) 46.

[28] P.T. Chrusciel and J. Kowalski-Glikman, "*The Isometry Group and Killing Spinors for the PP-wave Spacetime in $D = 11$ Supergravity*", Phys. Lett. B149 (1984) 107.

[29] E. Cremmer and B. Julia, "*The $SO(8)$ Supergravity*", Nucl. Phys. B159 (1979) 141.

[30] E. Cremmer, B. Julia and J. Scherk, "*Supergravity Theory in Eleven-dimensions*", Phys. Lett. B76 (1978) 409.

[31] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, "*Ricci-flat Metrics, Harmonic Forms and Brane Resolutions*", Commun. Math. Phys. 232 (2003) 457, hep-th/0012011.

[32] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, "*Hyper-Kähler Calabi Metrics, L^2 Harmonic Forms, Resolved M2-branes, and AdS_4/CFT_3 Correspondence*", Nucl. Phys. B617 (2001) 151, hep-th/0102185.

[33] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, "*New Complete Non-compact $Spin(7)$ Manifolds*", Nucl. Phys. B620 (2002) 29, hep-th/0103155.

[34] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, "*Cohomogeneity One Manifolds of $Spin(7)$ and G_2 Holonomy*", Phys. Rev. D65 (2002) 106004, hep-th/0108245.

[35] M. Cvetič, H. Lü and C.N. Pope, "*Consistent Kaluza-Klein Sphere Reductions*", Phys. Rev. D62 (2000) 064028, hep-th/0003286.

[36] J. Dai, R.G. Leigh and J. Polchinski, "*New Connections between String Theories*", Mod. Phys. Lett. A4 (1989) 2073.

[37] A. Dasgupta, H. Nicolai and J. Plefka, "*Vertex Operators for the Supermembrane*", JHEP 0005 (2000) 007, hep-th/0003280.

[38] K. Dasgupta, M.M. Sheikh-Jabbari and M. Van Raamsdonk, "*Matrix Perturbation Theory for M-theory on a PP-wave*", JHEP 0205 (2002) 056, hep-th/0205185.

[39] K. Dasgupta, M.M. Sheikh-Jabbari and M. Van Raamsdonk, "*Protected Multiplets of M-theory on a Plane Wave*", JHEP 0209 (2002) 021, hep-th/0207050.

[40] B. de Wit, "*Supersymmetric Quantum Mechanics, Supermembranes and Dirichlet Particles*", Nucl. Phys. Proc. Suppl. 56B (1997) 76, hep-th/9701169.

[41] B. de Wit, "*Supermembranes and Super Matrix Models*", lectures given at 6th Hellenic School and Workshop on Elementary Particle Physics: Corfu, Greece, 1998, Quantum Aspects of Gauge Theories, Supersymmetry and Unification, hep-th/9902051.

[42] B. de Wit, P. Fayet and M.T. Grisaru, ed., "*Supersymmetry, Supergravity, Superstrings '86*", proceedings of the Trieste Spring School, Italy, 1986, World Scientific.

[43] B. de Wit and I. Herger, "*Anti-de Sitter Supersymmetry*", Lect. Notes Phys. 541 (2000) 79, hep-th/9908005.

[44] B. de Wit, Hoppe and H. Nicolai, "*On the Quantum Mechanics of Supermembranes*", Nucl. Phys. B305 (1988) 545.

[45] B. de Wit, M. Lüscher and H. Nicolai, "*The Supermembrane is Unstable*", Nucl. Phys. B320 (1989) 135.

[46] B. de Wit and H. Nicolai, "$\mathcal{N} = 8$ *Supergravity*", Nucl. Phys. B208 (1982) 323.

[47] B. de Wit and H. Nicolai, "*The Consistency of the* S^7 *Truncation in* $d = 11$ *Supergravity*", Nucl. Phys. B281 (1987) 211.

[48] B. de Wit and H. Nicolai, "*Supermembranes: a Fond Farewell?*", Contribution to Conf. on Supermembranes and Physics in $2+1$-Dimensions, Trieste, Italy, 1989.

[49] B. de Wit, K. Peeters and J. Plefka, "*Superspace Geometry for Supermembrane Backgrounds*", Nucl. Phys. B532 (1998) 99, hep-th/9803209.

[50] B. de Wit, K. Peeters, J. Plefka and A. Sevrin, "*The M-theory Two-brane in* $AdS_4 \times S^7$ *and* $AdS_7 \times S^4$", Phys. Lett. B443 (1998) 153, hep-th/9808052.

[51] M.R. Douglas and H. Ooguri, "*Why Matrix Theory is Hard*", Phys. Lett. B425 (1998) 71, hep-th/9710178.

[52] M.R. Douglas, H. Ooguri and S.H. Shenker, "*Issues in M(atrix) Theory Compactification*", Phys. Lett. B402 (1997) 36, hep-th/9702203.

[53] M.J. Duff, "*Supergravity, the Seven-sphere, and Spontaneous Symmetry Breaking*", Nucl. Phys. 219 (1983) 389.

[54] M.J. Duff, "*Supermembranes*", lectures given at Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality, Boulder, Colorado, 1996, hep-th/9611203.

[55] M.J. Duff, "*TASI Lectures on Branes, Black Holes and Anti-de Sitter Space*", talk delivered at the Theoretical Advanced Study Institute on Strings, Branes and Gravity, Boulder, Colorado, 1999, hep-th/9912164.

[56] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, "*Superstrings in $D = 10$ from Supermembranes in $D = 11$*", Phys. Lett. B191 (1987) 70.

[57] M.J. Duff, B.E.W. Nilsson, C.N. Pope and N. Warner, "*On the Consistency of the Kaluza-Klein Ansatz*", Phys. Lett. B149 (1984) 90.

[58] M.J. Duff, B.E.W. Nilsson and C.N. Pope, "*Kaluza-Klein Supergravity*", Phys. Rept. 130 (1986) 1.

[59] M.J. Duff and C.N. Pope, "*Kaluza-Klein Supergravity and the Seven Sphere*", ICTP/82/83-7, lectures given at September School on Supergravity and Supersymmetry '82, Trieste, Italy, 1982.

[60] M.J. Duff and K.S. Stelle, "*Multi-membrane Solutions of $D = 11$ Supergravity*", Phys. Lett. B253 (1991) 113.

[61] H.D. Elvang, "*The M(atrix) Model of M-theory: 1- and 2-loop Calculations of Graviton-Graviton Scattering in M(atrix) Theory*", Cand. Scient. Thesis, the Niels Bohr Institute.

[62] J.M. Evans, "*Supersymmetric Yang-Mills Theories and Division Algebras*", Nucl. Phys. B298 (1988) 92.

[63] S. Fernando, M. Günaydin and O. Pavlyk, "*Spectra of PP-wave Limits of M-/Superstring Theory on $AdS_p \times S^q$ Spaces*", JHEP 0210 (2002) 007, hep-th/0207175.

[64] J. Figueroa-O'Farrill and G. Papadopoulos, "*Homogeneous Fluxes, Branes and a Maximally Supersymmetric Solution of M-theory*", JHEP 0108 (2001) 036, hep-th/0105308.

[65] J. Figueroa-O'Farrill and G. Papadopoulos, "*Maximally Supersymmetric Solutions of Ten- and Eleven-dimensional Supergravities*", JHEP 0303 (2003) 048, hep-th/0211089.

[66] P.G.O. Freund and M.A. Rubin, "*Dynamics of Dimensional Reduction*", Phys. Lett. B97 (1980) 233.

[67] J. Fröhlich and J. Hoppe, "*On Zero-mass Ground States in Super-membrane Matrix Models*", Commun. Math. Phys. 191 (1998) 613, hep-th/9701119.

[68] A. Giveon, M. Porrati and E. Rabinovici, "*Target Space Duality in String Theory*", Phys. Rept. 244 (1994) 77, hep-th/9401139.

[69] M.B. Green, M. Gutperle and H. Kwon, "*Light-cone Quantum Mechanics of the Eleven-dimensional Superparticle*", JHEP 9908 (1999) 012, hep-th/9907155.

[70] M.B. Green and J.H. Schwarz, "*Covariant Description of Superstrings*", Phys. Lett. B180 (1984) 370.

[71] M.B. Green and J.H. Schwarz, "*Anomaly Cancellation in Supersymmetric $D = 10$ Gauge Theory and Superstring Theory*", Phys. Lett. B149 (1984) 117.

[72] M.B. Green, J.H. Schwarz and E. Witten, "*Superstring Theory*", Cambridge University Press.

[73] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, "*Gauge Theory Correlators from Non-critical String Theory*", Phys. Lett. B428 (1998) 105, hep-th/9802109.

[74] M. Günaydin and D. Minic, "*Singletons, Doubletons and M-theory*", Nucl. Phys. B523 (1998) 145, hep-th/9802047.

[75] M. Günaydin, D. Minic and M. Zagermann, "*Novel Supermultiplets of $SU(2,2|4)$ and the AdS_5/CFT_4 Duality*", Nucl. Phys. B544 (1999) 737, hep-th/9810226.

[76] M. Günaydin, P. van Nieuwenhuizen and N.P. Warner, "*General Construction of the Unitary Representations of Anti-de Sitter Superalgebras and the Spectrum of the S^4 Compactification of Eleven-dimensional Supergravity*", Nucl. Phys. B255 (1985) 63.

[77] U. Gürsoy, C. Núñez and M. Schvellinger, "*RG Flows from $Spin(7)$, CY 4-fold and HK Manifolds to AdS, Penrose Limits and PP-waves*", JHEP 0206 (2002) 015, hep-th/0203124.

[78] R. Güven, "*Black p-brane Solutions of $D = 11$ Supergravity Theory*", Phys. Lett. B276 (1992) 49.

[79] R. Güven, "*Plane Wave Limits and T-duality*", Phys. Lett. B482 (2000) 255, hep-th/0005061.

[80] K. Higashijima, T. Kimura and M. Nitta, "*Supersymmetric Nonlinear Sigma Models on Ricci-flat Kähler Manifolds with $O(N)$ Symmetry*", Phys. Lett. B515 (2001) 421, hep-th/0104184.

[81] K. Higashijima, T. Kimura and M. Nitta, "*Gauge Theoretical Construction of Non-compact Calabi-Yau Manifolds*", Ann. Phys. 296 (2002) 346, hep-th/0110216.

[82] K. Higashijima, T. Kimura and M. Nitta, "*Calabi-Yau Manifolds of Cohomogeneity One as Complex Line Bundles*", Nucl. Phys. B645 (2002) 438, hep-th/0202064.

[83] P. Hořava and E. Witten, "*Heterotic and Type I String Dynamics from Eleven-dimensions*", Nucl. Phys. B460 (1996) 506, hep-th/9510209.

[84] P. Hořava and E. Witten, "*Eleven-dimensional Supergravity on a Manifold with Boundary*", Nucl. Phys. 475 (1996) 94, hep-th/9603142.

[85] C.M. Hull and P.K. Townsend, "*Unity of Superstring Dualities*", Nucl. Phys. B438 (1995) 109, hep-th/9410167.

[86] S. Imai and T. Yokono, "*Comments on Orientifold Projection in the Conifold and $SO \times USp$ Duality Cascade*", Phys. Rev. D65 (2002) 066007, hep-th/0110209.

[87] R. Kallosh, J. Rahmfeld and A. Rajaraman, "*Near Horizon Superspace*", JHEP 9809 (1998) 002, hep-th/9805217.

[88] H. Kanno and Y. Yasui, "*On $Spin(7)$ Holonomy Metric Based on $SU(3)/U(1)$*", J. Geom. Phys. 43 (2002) 293, hep-th/0108226.

[89] H. Kanno and Y. Yasui, "*On $Spin(7)$ Holonomy Metric Based on $SU(3)/U(1)$, II*", J. Geom. Phys. 43 (2002) 310, hep-th/0111198.

[90] N. Kim and J. Park, "*Superalgebra for M-theory on a PP-wave*", Phys. Rev. D66 (2002) 106007, hep-th/0207061.

[91] N. Kim and J. Plefka, "*On the Spectrum of PP-wave Matrix Theory*", Nucl. Phys. B643 (2002) 31, hep-th/0207034.

[92] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, "*Mass Spectrum of Chiral Ten-dimensional $\mathcal{N} = 2$ Supergravity on S^5*", Phys. Rev. D32 (1985) 389.

[93] T. Kimura and K. Yoshida, "*Spectrum of Eleven-dimensional Supergravity on a PP-wave Background*", Phys. Rev. D68 (2003) 125007, hep-th/0307193.

[94] I.R. Klebanov and M.J. Strassler, "*Supergravity and a Confining Gauge Theory: Duality Cascades and χSB-resolution of Naked Singularities*", JHEP 0008 (2000) 052, hep-th/0007191.

[95] I.R. Klebanov and E. Witten, "*Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity*", Nucl. Phys. B536 (1998) 199, hep-th/9807080.

[96] Y. Konishi and M. Naka, "*Coset Construction of Spin(7), G_2 Gravitational Instantons*", Class. Quant. Grav. 18 (2001) 5521, hep-th/0104208.

[97] J. Kowalski-Glikman, "*Vacuum States in Supersymmetric Kaluza-Klein Theory*", Phys. Lett. B134 (1984) 194.

[98] O. Kwon and H. Shin, "*Type IIA Supergravity Excitations in Plane-wave Background*", Phys. Rev. D68 (2003) 046007, hep-th/0303153.

[99] H.K. Lee and X. Wu, "*Two-graviton Interaction in PP-wave Background in Matrix Theory and Supergravity*", Nucl. Phys. B665 (2003) 153, hep-th/0301246.

[100] R.G. Leigh, "*Dirac-Born-Infeld Action from Dirichlet Sigma Model*", Mod. Phys. Lett. A4 (1989) 2767.

[101] J. Maldacena, "*The Large N Limit of Superconformal Field Theories and Supergravity*", Adv. Theor. Math. Phys. 2 (1998) 231, [Int. J. Theor. Phys. 38 (1999) 1113], hep-th/9711200.

[102] J. Maldacena, M. Sheikh-Jabbari and M. Van Raamsdonk, "*Transverse Fivebranes in Matrix Theory*", JHEP 0301 (2003) 038, hep-th/0211139.

[103] J. Mas and A.V. Ramallo, "*Supersymmetric Intersections of M-branes and PP-waves*", JHEP 0305 (2003) 021, hep-th/0303193.

[104] R.R. Metsaev, "*Type IIB Green-Schwarz Superstring in Plane Wave Ramond-Ramond Background*", Nucl. Phys. B625 (2002) 70, hep-th/0112044.

[105] R.R. Metsaev and A.A. Tseytlin, "*Exactly Solvable Model of Superstring in Ramond-Ramond Plane Wave Background*", Phys. Rev. D65 (2002) 126004, hep-th/0202109.

[106] R.C. Myers, "*Dielectric-branes*", JHEP 9912 (1999) 022, hep-th/9910053.

[107] R.C. Myers, "*Nonabelian Phenomena on D-branes*", Class. Quant. Grav. 20 (2003) S347, hep-th/0303072.

[108] W. Nahm, "*Supersymmetries and their Representations*", Nucl. Phys. B135 (1978) 149.

[109] N. Nakayama, K. Sugiyama and K. Yoshida, "*Ground State of Supermembrane on PP-wave*", Phys. Rev. D68 (2003) 026001, hep-th/0209081.

[110] H. Nastase, D. Vaman and P. van Nieuwenhuizen, "*Consistency of the $AdS_7 \times S^4$ Reduction and the Origin of Self-duality in Odd Dimensions*", Nucl. Phys. B581 (2000) 179, hep-th/9911238.

[111] H. Nicolai, "*Representations of Supersymmetry in Anti-de Sitter Space*", presented at Spring School on Supergravity and Supersymmetry '84, Trieste, Italy, 1984.

[112] N. Ohta, "*Classical Stability of M^{pqr} in Eleven-dimensional Supergravity*," Z. Phys. C26 (1985) 535.

[113] Y. Okawa and T. Yoneya, "*Multi-Body Interactions of D-particles in Supergravity and Matrix Theory*", Nucl. Phys. B538 (1999) 67, hep-th/9806108.

[114] L.A. Pando Zayas and A.A. Tseytlin, "*3-branes on Spaces with $\mathbb{R} \times S^2 \times S^3$ Topology*", Phys. Rev. D63 (2001) 086006, hep-th/0101043.

[115] G. Papadopoulos and A.A. Tseytlin, "*Complex Geometry of Conifolds and 5-brane Wrapped on 2-sphere*", Class. Quant. Grav. 18 (2001) 1333, hep-th/0012034.

[116] J. Park, "*Supersymmetric Objects in the M-theory on a PP-wave*", JHEP 0210 (2002) 032, hep-th/0208161.

[117] R. Penrose, "*Any Spacetime has a Plane Wave as a Limit*", in Differential Geometry and Relativity, pp. 271, Reidel, Dordrecht, 1976.

[118] K. Pilch, P. van Nieuwenhuizen and P.K. Townsend, "*Compactification of $d = 11$ Supergravity on S^4 (or $11 = 7 + 4$, too)*", Nucl. Phys. B242 (1984) 377.

[119] J. Polchinski, "*Dirichlet-branes and Ramond-Ramond Charges*", Phys. Rev. Lett. 75 (1995) 4724, hep-th/9510017.

[120] J. Polchinski, "*String Theory*", Cambridge University Press.

[121] J. Polchinski and M.J. Strassler, "*The String Dual of a Confining Four-dimensional Gauge Theory*", hep-th/0003136.

[122] A. Salam and E. Sezgin, ed., "*Supergravities in Diverse Dimensions*", North-Holland.

[123] J.H. Schwarz, "*The Power of M-theory*", Phys. Lett. B367 (1996) 97, hep-th/9510086.

[124] N. Seiberg, "*Why is the Matrix Model Correct?*", Phys. Rev. Lett. 79 (1997) 3577, hep-th/9710009.

[125] A. Sen, "*F-theory and Orientifolds*", Nucl. Phys. B475 (1996) 562, hep-th/9605150.

[126] A. Sen, "*D0-branes on T^n and Matrix Theory*", Adv. Theor. Math. Phys. 2 (1998) 51, hep-th/9709220.

[127] H. Shin and K. Yoshida, "*One-loop Flatness of Membrane Fuzzy Sphere Interaction in Plane-wave Matrix Model*", Nucl. Phys. B679 (2004) 99, hep-th/0309258.

[128] G. Sierra, "*An Application to the Theories of Jordan Algebras and Freudenthal Triple Systems to Particles and Strings*", Class. Quant. Grav. 4 (1987) 227.

[129] K. Sugiyama and K. Yoshida, "*Supermembrane on the PP-wave Background*", Nucl. Phys. B644 (2002) 113, hep-th/0206070.

[130] L. Susskind, "*Another Conjecture about M(atrix) Theory*", hep-th/9704080.

[131] W. Taylor IV, "*Lectures on D-branes, Gauge Theory and M(atrices)*", talk given at 2nd Trieste Conference on Duality in String Theory, Trieste, Italy, 1997, hep-th/9801182.

[132] W. Taylor IV, "*The M(atrix) Model of M-theory*", lectures given at NATO Advanced Study Institute on Quantum Geometry, Akureyri, Iceland, 1999, hep-th/0002016.

[133] W. Taylor IV, "*M(atrix) Theory: Matrix Quantum Mechanics as a Fundamental Theory*", Rev. Mod. Phys. 73 (2001) 419, hep-th/0101126.

[134] W. Taylor IV and M. Van Raamsdonk, "*Supergravity Currents and Linearized Interactions for Matrix Theory Configurations with Fermionic Backgrounds*", JHEP 9904 (1999) 013, hep-th/9812239.

[135] W. Taylor IV and M. Van Raamsdonk, "*Multiple D0-branes in Weakly Curved Backgrounds*", Nucl. Phys. B558 (1999) 63, hep-th/9904095.

[136] W. Taylor IV and M. Van Raamsdonk, "*Multiple Dp-branes in Weak Background Fields*", Nucl. Phys. B573 (2000) 703, hep-th/9910052.

[137] P.K. Townsend, "*p-brane Democracy*", in Particles, Strings and Cosmology, hep-th/9507048.

[138] P.K. Townsend, "*Three Lectures on Supermembranes*", published in Trieste School: Superstrings '88, 1988, PRINT-88-0732 (CAMBRIDGE).

[139] P.K. Townsend, "*Four Lectures on M-theory*", talk given at Summer School in High-energy Physics and Cosmology, Trieste, Italy, 1996, hep-th/9612121.

[140] A.A. Tseytlin, "*On Non-Abelian Generalization of the Born-Infeld Action in String Theory*", Nucl. Phys. B501 (1997) 41, hep-th/9701125.

[141] A.A. Tseytlin, "*Born-Infeld Action, Supersymmetry and String Theory*", hep-th/9908105.

[142] C. Vafa, "*Evidence for F-theory*", Nucl. Phys. B469 (1996) 403, hep-th/9602022.

[143] P. van Nieuwenhuizen, "*Supergravity*", Phys. Rept. 68 (1981) 189.

[144] P. van Nieuwenhuizen, "*General Theory of Coset Manifolds and Antisymmetric Tensors Applied to Kaluza-Klein Supergravity*", ITP-SB-84-57, published in Trieste School 1984, Supersymmetry and Supergravity '84.

[145] E. Witten, "*Search for a Realistic Kaluza-Klein Theory*", Nucl. Phys. B186 (1981) 412.

[146] E. Witten, "*String Theory Dynamics in Various Dimensions*", Nucl. Phys. B443 (1995) 85, hep-th/9503124.

[147] E. Witten, "*Five-brane Effective Action in M-theory*", J. Geom. Phys. 22 (1997) 103, hep-th/9610234.

[148] K. Yoshida, in private communications.