Microscopic derivation of (non-)relativistic second-order hydrodynamics from Boltzmann Equation

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based on work done with
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Strangeness and charm in hadrons and dense matter
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1. Introduction: Geometrical formulation of reduction of dynamics
2. Renormalization group method for constructing the asymptotic invariant/attractive manifold
3. Application of the RG method for derivation of the 2\textsuperscript{nd}-order (non-)relativistic hydrodynamics with quantum statistics
4. Example: cold fermionic atoms and validity test of the relaxation-time (BGK) approximation
5. Brief summary and concluding remarks
Introduction

Separation of scales in the time evolution of a physical system

Liouville ➔ Kinetic (Boltzmann eq.) ➔ Fluid dyn.

Hamiltonian (i) (ii)

One-body dist. fn. Hydrodynamic variables, \( \mathbf{u}^\mu, T, n \) and so on

Slower dynamics with fewer variables

(i) From Liouville (BBGKY) to Boltzmann (Bogoliubov) The relaxation time of the \( s \)-body distribution function \( F_s (s>1) \) should be short and hence slaving variables of \( F_1 \). The reduced dynamics is described solely with the one-body distribution function \( F_1 \) as the coordinate of the attractive manifold. N.N. Bogoliubov, in “Studies in Statistical Mechanics”, (J. de Boer and G. E. Uhlenbeck, Eds.) vol2, (North-Holland, 1962)

(ii) Boltzmann to hydrodynamics (Hilbert, Chapman-Enskog, Bogoliubov)

After some time, the one-body distribution function is asymptotically well described by local temperature \( T(x) \), density \( n(x) \), and the flow velocity \( \mathbf{u} \), i.e., the hydrodynamic variables

(iii) Langevin to Fokker-Planck equation,

(iv) Critical dynamics as described by TDGL etc…..
Geometrical image of reduction of dynamical systems

(including Hydrodynamic limit of Boltzmann equation)

\textbf{n-dimensional dynamical system:}

\[ \frac{dX}{dt} = F(X) \]

\[ \dim X = n \]

\[ \dim M = m \leq n \]

eg.

\[ X = f(r, p) \]; distribution function in the phase space (infinite dimensions)

\[ s = \{u^\mu, T, n\} \]; the hydrodynamic quantities or conserved quantities for 1\textsuperscript{st}-order eq.
The problems listed above may be formulated as a construction of an asymptotic invariant/attractive manifold with possible space-time coarse-graining, and it may be interpreted as a geometrical resolution to Hilbert’s 6th problem, which is based on a similarity of geometry and physics.


We adopt the Renormalization Group method (Chen et al, 1995; T.K. (1995)) to construct the attractive/invariant manifolds and extend it so as to incorporate excited modes as well as the would-be zero modes as the slow/collective variables and thereby derive the second-order hydrodynamics as the mesoscopic dynamics.
The talk is based on the following work done with Tsumura, Kikuchi and K. Ohnishi;

K. Tsumura, K. Ohnishi and TK, PLB46 (2007), 134:
The original. 1st-order eq.

Tsumura, Kikuchi and TK, Physica D336 (2016), 1;
The doublet scheme with application to derivation of second-order non-rel hydro in classical statistics

Tsumura, Kikuchi, TK, PRD92 (2015);
Quantum and relativistic with single component

Kikuchi, Tsumura and TK, PRC92 (2015);
Quantum and relativistic with multiple reactive species

Quantum and non-rel with application to cold fermionic gas.

Tsumura and T.K., EPJA 48 (2012), 162: A Review
Use of envelopes of a family of curves/surfaces:

-- RG eq. as the envelope eq.--

T.K. ('95)

Let \( \{C_\tau\}_\tau \) be a family of curves parametrized by \( \tau \) in the \( x-y \) plane;

\[
C_\tau : F(x, y, \tau, C(\tau)) = 0
\]

E: The envelope of \( C_\tau \)

\[
E: G(x, y) = 0
\]

\[
G(x, y) = F(x, y, C(x))
\]

The envelop equation: \( \frac{dF}{d\tau_0} = 0 \)  \( \leftrightarrow \) RG eq.

the solution is inserted to \( F \) with the condition

\[
\tau_0 = x_0
\]

the tangent point
Resummation of seemingly divergent pert. series and extracting slow dynamics by the envelope/RG eq.

A simple example: the dumped oscillator!

\[
\frac{d^2 x}{dt^2} + \epsilon \frac{dx}{dt} + x = 0,
\]

\[
x(t) = \tilde{A} \exp\left(-\frac{\epsilon}{2}t\right) \sin\left(\sqrt{1 - \frac{\epsilon^2}{4}}t + \bar{\theta}\right),
\]

\[
x(t, t_0) = x_0(t, t_0) + \epsilon x_1(t, t_0) + \epsilon^2 x_2(t, t_0) + \ldots,
\]

\[
\ddot{x}_0 + x_0 = 0, \quad \ddot{x}_{n+1} + x_{n+1} = -\dot{x}_n.
\]

\[
x(t_0, t_0) = W(t_0).
\]

\[
W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \epsilon^2 W_2(t_0) + \ldots,
\]

\[
x_0(t, t_0) = A(t_0) \sin(t + \theta(t_0)), \quad W_0(t_0) = x_0(t_0, t_0) = A(t_0) \sin(t_0 + \theta(t_0)).
\]

\[
x_1(t, t_0) = -\frac{A}{2} \cdot (t - t_0) \sin(t + \theta), \quad W_1(t_0) = 0
\]

a secular term appears, invalidating P.T.
\[ x_2(t) = \frac{A}{8} \{(t - t_0)^2 \sin(t + \theta) - (t - t_0) \cos(t + \theta)\}, \quad W_2(t_0) = 0 \]

Secular terms appear again!

Collecting the terms, we have

\[ x(t, t_0) = A \sin(t + \theta) - \epsilon \frac{A}{2} (t - t_0) \sin(t + \theta) \]
\[ + \epsilon^2 \frac{A}{8} \{(t - t_0)^2 \sin(t + \theta) - (t - t_0) \cos(t + \theta)\} \]

With I.C.:

\[ W(t_0) = W_0(t_0) = A(t_0) \sin(t_0 + \theta(t_0)) \]

\[ A(t_0), \phi(t_0) \equiv t_0 + \theta(t_0) \]

The secular terms invalidate the pert. theory, like the log-divergence in QFT!

\[ \{C_{t_0}\}_{t_0} : \quad \{x(t, t_0)\}_{t_0} \quad x_E(t) = x(t, t) = W(t). \]
\[ \frac{dx(t, t_0)}{dt_0} = 0, \quad t_0 = t. \quad \rightarrow \quad A(t_0) \text{ and } \theta(t_0) \]
\[ x_2(t) = \frac{A}{8} \left\{ (t - t_0)^2 \sin(t + \theta) - (t - t_0) \cos(t + \theta) \right\}, \quad W_2(t_0) = 0 \]

Secular terms appear again!

Collecting the terms, we have

\[ x(t, t_0) = A \sin(t + \theta) - \epsilon \frac{A}{2} (t - t_0) \sin(t + \theta) \]
\[ + \epsilon^2 \frac{A}{8} \left\{ (t - t_0)^2 \sin(t + \theta) - (t - t_0) \cos(t + \theta) \right\} \]

With I.C.:

\[ W(t_0) = W_0(t_0) = A(t_0) \sin(t_0 + \theta(t_0)) \]

; parameterized by the functions,

\[ A(t_0), \phi(t_0) \equiv t_0 + \theta(t_0) \]

Let us try to construct the envelope function of the set of locally divergent functions, parameterized by \( t_0 \)!

\[ \{C_{t_0}\}_{t_0} : \quad \{x(t, t_0)\}_{t_0} \quad x_E(t) = x(t, t) = W(t). \]

\[ \frac{dx(t, t_0)}{dt_0} = 0, \quad t_0 = t. \quad \rightarrow \quad A(t_0) \text{ and } \theta(t_0) \]
\[ \frac{dA}{dt_0} + \epsilon A = 0, \quad \frac{d\theta}{dt_0} + \frac{\epsilon^2}{8} = 0, \]

\[ A(t_0) = \bar{A}e^{-\epsilon t_0/2}, \quad \theta(t_0) = -\frac{\epsilon^2}{8}t_0 + \bar{\theta}, \]

Extracted the amplitude and phase equations, separately!

\[ x_E(t) = x(t, t) = W_0(t) = \bar{A}\exp\left(-\frac{\epsilon}{2}t\right)\sin\left((1 - \frac{\epsilon^2}{8})t + \bar{\theta}\right), \]

\[ \sqrt{1 - \epsilon^2/4} = 1 - \epsilon^2/8 + O(\epsilon^4) \]

The envelop function \( x_E(t) = W_0(t) \) an approximate but global solution in contrast to the pertubative solutions which have secular terms and valid only in local domains.

Notice also the resummed nature!
RG analysis of Van der Pol eq. with a limit cycle

\[ \ddot{x} + x = \epsilon (1 - x^2) \dot{x} \]

\[ \left( \frac{d^2}{dt^2} + 1 \right) \tilde{x}_0 = 0, \quad \tilde{x}_0(t; t_0) = A(t_0) \cos(t + \theta(t_0)) \quad \tilde{x}_0(t_0; t_0) = A(t_0) \cos(t_0 + \theta(t_0)) \]

\[ \left( \frac{d^2}{dt^2} + 1 \right) \tilde{x}_1 = -A(t_0) \left( 1 - \frac{A(t_0)^2}{4} \right) \sin(t + \theta(t_0)) + \frac{A(t_0)^3}{4} \sin 3(t + \theta(t_0)) \]

\[ \tilde{x}_1(t; t_0) = (t - t_0) \frac{A(t_0)}{2} \left( 1 - \frac{A(t_0)^2}{4} \right) \sin(t + \theta(t_0)) - \frac{A(t_0)^3}{32} \sin 3(t + \theta(t_0)) \quad \tilde{x}_1(t_0; t_0) = -\frac{A(t_0)^3}{32} \sin 3(t_0 + \theta(t_0)) \]

\[ \frac{d\tilde{x}}{dt_0} \bigg|_{t_0=t} = 0 \]

\[ \dot{A} = \epsilon \frac{A}{2} \left( 1 - \frac{A^2}{4} \right), \quad \dot{\theta} = 0. \]
(a) perturbative solution

`Exact' numerical solution

RG improved solution in 1\textsuperscript{st} order perturbation

Let us take the following $n$-dimensional equation:

$$\frac{dX}{dt} = F(X, t), \quad (B.11)$$

where $n$ may be infinity. Let $X(t) = W(t)$ be an yet unknown exact solution to Eq. (B.11), and we try to solve the equation with the initial condition at $t = \forall t_0$;

$$X(t = t_0) = W(t_0). \quad (B.12)$$

Then, the solution may be written as $X(t; t_0, W(t_0))$.

$$X(t; t_0, W(t_0)) = X(t; t'_0, W(t'_0)).$$

Taking the limit $t'_0 \to t_0$, we have

$$\frac{dX}{dt_0} = \frac{\partial X}{\partial t_0} + \frac{\partial X}{\partial W} \frac{dW}{dt_0} = 0.$$

Pert. Theory:

$X(t; t_0, W(t_0))$ and $X(t; t'_0, W(t'_0))$ may be valid only for $t \sim t_0$ and $t \sim t'_0$,

$$t_0 < t < t'_0 \quad \text{(or} \quad t'_0 < t < t_0\text{)}$$

$$\frac{dX}{dt_0} \bigg|_{t_0=t} = \frac{\partial X}{\partial t_0} \bigg|_{t_0=t} + \frac{\partial X}{\partial W} \frac{dW}{dt_0} \bigg|_{t_0=t} = 0, \quad t_0 = t$$

with the RG equation!
Let $X(t; t_0)$ is an approximate solution to Eq.(B.11) around $t \sim t_0$;

$$\frac{dX(t; t_0)}{dt} \simeq F(X(t; t_0), t).$$

Then, we have

$$\frac{dW(t)}{dt} = \frac{\partial X(t; t_0)}{\partial t} \bigg|_{t_0=t} + \frac{\partial X(t; t_0)}{\partial t_0} \bigg|_{t_0=t}$$

$$= \frac{\partial X(t; t_0)}{\partial t} \bigg|_{t_0=t}
\simeq F(X(t; t_0), t) \bigg|_{t_0=t},
= F(W(t), t),$$

showing that our envelope function satisfies the original equation (B.11) in the global domain uniformly.
Eg. RG reduction of a generic equation with zero modes  

\[ \partial_t u = Au + \epsilon F(u), \quad \dim u = n \quad |\epsilon| < 1 \]

\[ AU_i = 0, \quad (i = 1, 2, \ldots, m). \]

We suppose that other eigenvalues have negative real parts;

\[ AU_\alpha = \lambda_\alpha U_\alpha, \quad (\alpha = m + 1, m + 2, \ldots, n), \]

where \( \Re \lambda_\alpha < 0. \) One may assume without loss of generality that \( u_i \)'s and \( U_\alpha \)'s are linearly independent.

\[ P \text{ the projection onto the kernel } \ker A \quad P + Q = 1 \]

Perturbative expansion around arbitrary time \( t_0 \) in the asymptotic regime

\[ u(t; t_0) = u_0(t; t_0) + \epsilon u_1(t; t_0) + \epsilon^2 u_2(t; t_0) + \cdots \]

With the initial value at \( t_0 \):

\[ W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \epsilon^2 W_2(t_0) + \cdots, \]

\[ = W_0(t_0) + \rho(t_0), \]

\[ (\partial_t - A)u_0 = 0, \]

\[ (\partial_t - A)u_1 = F(u_0), \]

\[ (\partial_t - A)u_2 = F'(u_0)u_1, \quad (F'(u_0)u_1) = \sum_{j=1}^{n} \frac{\partial(F'(u_0))}{\partial (u_0)_j} (u_1)_j \]
Since we are interested in the asymptotic state as \( t \to \infty \), we may assume that the lowest-order initial value belongs to \( \ker A \):

\[
W_0(t_0) = \sum_{i=1}^{m} C_i(t_0)U_i = W_0[C].
\]

\[
\mathbf{u}_0(t; t_0) = e^{(t-t_0)A}W_0(t_0) = \sum_{i=1}^{m} C_i(t_0)U_i.
\]

Parameterized with \( n \) variables, instead of \( m \)!

1\textsuperscript{st}-order solution reads

\[
\mathbf{u}_1(t; t_0) = e^{(t-t_0)A}[\mathbf{W}_1(t_0) + A^{-1}QF(W_0(t_0))] + (t-t_0)PF(W_0(t_0)) - A^{-1}QF(W_0(t_0)).
\]

The would-be rapidly changing terms can be eliminated by the choice:

\[
\mathbf{W}_1(t_0) = -A^{-1}QF(W_0(t_0)), \quad PW_1(t_0) = 0
\]

Then, the secular term appears only in the P space;

\[
\mathbf{u}_1(t; t_0) = (t - t_0)PF - A^{-1}QF.
\]

Unperturbed manifold \( \mathbf{M}_0 \)

A deformation of the manifold \( \mathbf{\rho} \)
Deformed (invariant) slow manifold:

\[
M_1 = \{ u | u = W_0 - \epsilon A^{-1} QF(W_0) \}
\]

\[ u(t; t_0) = W_0 + \epsilon \{ (t - t_0) PF - A^{-1} QF \} \]

A set of locally divergent functions parameterized by \( t_0 \)!

The RG/E equation \( \frac{\partial u}{\partial t_0} \bigg|_{t_0=t} = 0 \) gives the envelope, which is globally valid:

\[
\dot{W}_0(t) = \epsilon PF(W_0(t)),
\]

which is reduced to an \( m \)-dimensional coupled equation,

\[
\dot{C}_i(t) = \epsilon \langle \tilde{U}_i, F(W_0[C]) \rangle, \quad (i = 1, 2, \cdots, m).
\]

The global solution (the invariant manifold):

\[
u(t) = u(t; t_0 = t) = \sum_{i=1}^{m} C_i(t) U_i - \epsilon A^{-1} QF(W_0[C]).
\]

We have derived the invariant manifold and the slow dynamics on the manifold by the RG method.

It can be shown that the so-constructed global sol. satisfies the original eq. in a global domain up to the order with which the local sol.'s are constructed.

Extensions

a) $A$ is not semi-simple with Jordan cell

b) Higher orders.

c) PD equations;

Layered pulse dynamics for TDGL and Non-lin. Schrödinger.

See also, T.K., Jpn. J. Ind. Appl. Math. 14 ('97), 51

Y. Hatta and T.K., Ann. Phys. ('00)

d) Reduction of stochastic equation with several variables;

Liouville to Boltzmann, Langevin to Focker-Plank:
Further reduction of F-P with hierarchy of time scales.


e) Discrete systems

T.K. and J. Matsukidaira, Phys. Rev. E57 ('98), 4817

f) Derivation of hydrodynamic limit of Boltzmann eq. in classical/quantum (non) relativistic (reactive multicomponent) systems

Remark

The (arbitrary) initial value (in the asymptotic region) play an essential role in the RG method. An intimate similarity of the method with the holographic AdS/CFT method is indicated; see for example,

Basics about Rel. Hydrodynamics

1. The fluid dynamic equations as conservation (balance) equations

\[
\begin{align*}
\partial_\mu N_i^\mu &\equiv 0 \, , \, i = 1, \ldots, n \, , \\
\partial_\mu T^{\mu\nu} &\equiv 0 \, , \, \nu = 0, \ldots, 3 \, .
\end{align*}
\]

local conservation of charges
local conservation of energy-mom.

2. Tensor decomposition and choice of frame

\[ u^\mu \, ; \textbf{arbitrary normalized time-like vector} \quad u \cdot u = 1 \]

Def. space-like projection

\[ \Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu \, , \, \Delta^{\mu\nu} u_\nu = 0 \, , \, \Delta^{\mu\alpha} \Delta^\nu_\alpha = \Delta^{\mu\nu} \]

space-like vector

\[ N_i^\mu = n_i u^\mu + v_i^\mu \, ; \text{net density of charge } i \text{ in the Local Rest Frame} \]

space-like traceless tensor

\[ T^{\mu\nu} = \epsilon u^\mu u^\nu - p \Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} \]

\[ n_i = N_i \cdot u \, ; \text{net flow in LRF} \]

\[ v_i^\mu = \Delta_{\nu}^\mu N_i^\nu \]

\[ \epsilon = u_\mu T^{\mu\nu} u_\nu \, ; \text{energy density in LRF} \]

\[ p = -\frac{1}{3} T^{\mu\nu} \Delta_{\mu\nu} \, ; \text{isotropic pressure in LRF} \]

\[ q^\mu = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta \, ; \text{heat flow in LRF} \]

\[ \pi^{\mu\nu} = \left[ \frac{1}{2} \left( \Delta_{\alpha}^\mu \Delta^\nu_{\beta} + \Delta_{\beta}^\mu \Delta^\nu_{\alpha} \right) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right] T^{\alpha\beta} \, ; \text{stress tensor in LRF} \]
Define $U^\mu$ so that it has a physical meaning.

A. Particle frame (Eckart frame)

$$u^\mu_E \equiv \frac{N^\mu_i}{\sqrt{N_i \cdot N_i}} \quad ; \text{parallel to particle current of } i \quad \rightarrow \quad 0 = N^\mu_i \Delta_{\mu\nu} = \nu^\mu_i$$

B. Energy frame (Landau-Lifshitz frame)

$$u^\mu_L \equiv \frac{T^\mu_{\nu} u^\nu_L}{\sqrt{u^\alpha_L T^\alpha_{\beta} T^\beta_{\gamma} u^\gamma_L}} \quad ; \text{flow of the energy-momentum density} \quad \rightarrow \quad q^\mu = 0$$

$$T^\mu_{\nu} u^\nu = \varepsilon u^\mu + q^\mu$$

$$N^\mu_i = n_i u^\mu + \nu^\mu_i$$,

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p \Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}$$
Typical hydrodynamic equations for a viscous fluid
--- Choice of the frame and ambiguities in the form ---

Fluid dynamics = a system of balance equations

\[ \partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu N^\mu = 0. \]

**energy–momentum:** \( T^{\mu\nu} \) **number:** \( N^\mu \)

**Eckart eq.:**
- No dissipation in the number flow;
- Describing the flow of matter with transport coefficients:
  - \( \varsigma \); Bulk viscosity,
  - \( \eta \); Shear viscosity
  - \( \lambda \); Heat conductivity

\[ \delta T^{\mu\nu} = u^\mu T \lambda \left( \frac{1}{T} \nabla^\nu T - D u^\nu \right) + u^\nu T \lambda \left( \frac{1}{T} \nabla_\mu T - D u^\mu \right) + 2 \eta \frac{1}{2} \left( \nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} \nabla \cdot u \right) + \zeta \Delta^{\mu\nu} \nabla \cdot u \]

\[ \delta N^\mu = 0. \]

--- Involving time-like derivative ---.

**Landau–Lifshits**
- No dissipation in energy flow

\[ \delta T^{\mu\nu} = 2 \eta \frac{1}{2} \left( \nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} \nabla \cdot u \right) + \zeta \Delta^{\mu\nu} \nabla \cdot u \]

\[ \delta N^\mu = -\lambda \frac{n T}{\epsilon + p} \left( \frac{1}{T} \nabla^\mu T - \frac{1}{\epsilon + p} \nabla_\mu p \right) \]

--- Involving only space-like derivatives ---.

\[ \delta T^{\mu\nu} u_\nu = 0, \quad u_\mu \delta N^\mu = 0 \]

\[ D \equiv u^\mu \partial_\mu \nabla_\nu \equiv \Delta^{\mu\nu} \partial_\nu \]

\[ \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \equiv \Delta^{\mu\nu}, \]
Acausality problem

Fluctuations around the equilibrium:

\[ \varepsilon = \varepsilon_0 + \delta \varepsilon(t, x) \quad u^\mu = (1, 0) + \delta u^\mu(t, x) \]

Linearized equation:

\[
(\varepsilon + p) Du^y - \nabla^y p + \Delta^y_\nu \partial_\mu \Pi^{\mu\nu} = (\varepsilon_0 + p_0) \partial_t \delta u^y + \partial_x \Pi^{xy}
\]

\[
\Pi^{xy} = \eta \left( \nabla^x u^y + \nabla^y u^x \right) + \left( \zeta - \frac{2}{3} \eta \right) \Delta^{xy} \nabla_\alpha u^\alpha = -\eta_0 \partial_x \delta u^y
\]

\[
\partial_t \delta u^y - \frac{\eta_0}{\varepsilon_0 + p_0} \partial_x^2 \delta u^y = 0
\]

Diffusion equation!

The signal runs with an infinite speed.
Non-local thermodynamics (Maxwell-Cattaneo) → Mueller-Israel-Stewart


Telegrapher’s equation

\[ \tau_\pi \partial_t \Pi^{xy} + \Pi^{xy} = -\eta_0 \partial_x \delta u^y \]

\[ \partial_t \delta u^y + \frac{1}{\epsilon_0 + p_0} \partial_x \pi^{xy} = 0, \quad \tau_\pi \partial_t \pi^{xy} + \pi^{xy} = -\eta_0 \partial_x \delta u^y \]

\[ \left[ \partial_t^2 + \frac{\partial_t}{\tau_\pi} - \frac{\nu}{\tau_\pi} \partial_x^2 \right] G(x, x') = \frac{1}{\tau_\pi} \delta^2(x - x') \]

\[ G(x, x') = \theta(t - t') \theta \left( \frac{(t - t')^2 \nu}{\tau_\pi} - (x - x')^2 \right) \frac{e^{-\frac{t - t'}{2\tau_\pi}}}{\sqrt{4\nu \tau_\pi}} I_0 \left( \sqrt{\frac{(t - t')^2}{4\tau_\pi^2} - \frac{(x - x')^2}{4\nu \tau_\pi}} \right) \]

Diffusion Eq. vs. Maxwell-Cattaneo

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<th>Excluded by causality</th>
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\( x/\nu = 10 \)
Compatibility of the definition of the flow and the LRF

In the kinetic approach, one needs a matching condition.

Seemingly plausible ansatz are;

\[\epsilon \equiv u_\mu T^{\mu \nu} u_\nu = \epsilon_0 \equiv u_\mu T_0^{\mu \nu} u_\nu\]
\[n \equiv u \cdot N = n_0 \equiv u \cdot N_0\]

Is this always correct, irrespective of the frames?

Particle frame is the same local equilibrium state as the energy frame?

Note that the entropy density \(S(x)\) and the pressure \(P(x)\) etc can be quite different from those in the equilibrium.

Eg. the bulk viscosity

Local equilibrium \(\rightarrow\) No dissipation!

Distribution function in LRF:

\[f_0(k, x) = \frac{g}{(2\pi)^3} \left[\exp\{y_0(k, x)\} \pm 1\right]^{-1} \quad y_0(k, x) = [k \cdot u(x) - \mu(x)]/T(x)\]

Non-local distribution function;

\[f(k, x) = \frac{g}{(2\pi)^3} \left[\exp\{y(k, x)\} \pm 1\right]^{-1}\]

\[y(k, x) \simeq y_0(k, x) + \varepsilon_1(x) + k \cdot \varepsilon_2(x) + k_\mu k_\nu \varepsilon_3^{\mu \nu}(x)\]

D. H. Rischke, nucl-th/9809044
The problem of causality:

\[ C_v \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} \]

Fourier's law;

\[ q = -\lambda \frac{\partial T}{\partial x} \]

Then

\[ C_v \frac{\partial T}{\partial t} = \lambda \nabla^2 T \]

Causality is broken; the signal propagate with an infinite speed.

Modification;

\[ \tau_q \frac{\partial}{\partial t} q(t, x) + q(t, x) = -\lambda \frac{\partial}{\partial x} T(t, x) \]

Extended thermodynamics

Nonlocal thermodynamics

Memory effects; i.e., non-Markovian

Derivation (Israel-Stewart): Grad's 14-moments method

+ ansats so that Landau/Eckart eq.'s are derived.

Problematic
The problems:

- Foundation of Grad’s 14 moments method
- ad-hoc constraints on $\delta T^{\mu \nu}$ and $\delta N^\mu$ consistent with the underlying dynamics?
Relativistic Boltzmann equation

\[ p^\mu \partial_\mu f_p(x) = C[f]_p(x), \]

\[ C[f]_p(x) = \frac{1}{2!} \int dp_1 dp_2 dp_3 \omega(p, p_1|p_2, p_3) \]
\[ \times ((1 + a f_p(x))(1 + a f_{p_1}(x)) f_{p_2}(x) f_{p_3}(x)
\]
\[ - f_p(x) f_{p_1}(x)(1 + a f_{p_2}(x))(1 + a f_{p_3}(x))), \]
\[ dp \equiv d^3p/[(2\pi)^3 p^0] \]

Symm. property of the transition probability:

\[ \omega(p, p_1|p_2, p_3) = \omega(p_2, p_3|p, p_1) = \omega(p_1, p|p_3, p_2) = \omega(p_3, p_2|p_1, p) \]

--- (1)

Energy-mom. conservation:

\[ \omega(p, p_1|p_2, p_3) \propto \delta^4(p + p_1 - p_2 - p_3) \]

--- (2)

Owing to (1),

\[ \int dp \varphi_p C[f]_p = \frac{1}{2!} \frac{1}{4} \int dp dp_1 dp_2 dp_3 \omega(p, p_1|p_2, p_3) \]
\[ \times (\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3}) \]
\[ \times ((1 + a f_p)(1 + a f_{p_1}) f_{p_2} f_{p_3} - f_p f_{p_1} (1 + a f_{p_2})(1 + a f_{p_3})) \]

--- (3)

Collision Invariant \( \varphi_p(x) : \)

\[ \int dp \frac{1}{p^0} C[f]_p = 0, \int dp \frac{1}{p^0} p^\mu C[f]_p = 0 \]

Eq.’s (3) and (2) tell us that the general form of a collision invariant;

\[ \varphi_p(x) = \alpha(x) + p^\mu \beta_\mu(x), \]

which can be x-dependent!
Local equilibrium distribution

The entropy current:

\[ S^\mu = - \int dp \, p^\mu \left[ f_p \ln f_p - \frac{(1 + a f_p) \ln(1 + a f_p)}{a} \right] \]

\[ \partial_\mu S^\mu = - \int dp \, C[f] \ln \left[ \frac{f_p}{1 + a f_p} \right] \]

Conservation of entropy \[ \ln(f_p/(1 + a f_p)) = \alpha(x) + p^\mu \beta_\mu(x). \]

\[ f_p(x) = \frac{1}{e^{(p^\mu u_\mu - \mu)/T} - a} \equiv f_p^{eq} \]

i.e., the local equilibrium distribution \( f_n \); 

Remark:

Owing to the energy-momentum conservation, the collision integral also vanishes for the local equilibrium distribution \( f_n \);

\[ C[f_p^{eq}](x) = 0. \]
Previous attempts to derive the dissipative hydrodynamics as a reduction of the dynamics

unique but non-covariant form and hence not Landau either Eckart!

Here, 

**In the covariant formalism, in a unified way and systematically derive dissipative rel. hydrodynamics at once!**

Cf. Chapman-Enskog method to derive Landau and Eckart eq.’s; see, eg, de Groot et al (‘80)
Introduction of the macroscopic frame vector


Ansatz of the origin of the dissipation= the spatial inhomogeneity, leading to Navier-Stokes in the non-rel. case.

\( \alpha_p^\mu \) would become a macro flow-velocity and will be identified with \( u^\mu \)

\[
\tau \equiv \alpha_p^\mu x_\mu, \quad \sigma^\mu \equiv \left( g^{\mu\nu} - \frac{\alpha_p^\mu \alpha_p^\nu}{\alpha_p^2} \right) x_\nu \equiv \Delta_p^{\mu\nu} x_\nu \equiv \nabla^\mu \quad \nabla^\mu
\]

\[
\frac{\partial}{\partial \tau} = \frac{1}{\alpha_p^2} \frac{\partial \sigma^\mu}{\partial \sigma_\mu} \equiv D, \quad \text{time-like derivative}
\]

\[
\Delta_p^{\mu\nu} \frac{\partial}{\partial \sigma_\nu} = \Delta_p^{\mu\nu} \partial_\nu \equiv \nabla^\mu \quad \text{space-like derivative}
\]

Rewrite the Boltzmann equation as,

\[
\frac{\partial}{\partial \tau} f_p(\tau, \sigma) = \frac{1}{p \cdot \alpha_p} C[f]_p(\tau, \sigma) - \frac{1}{p \cdot \alpha_p} p \cdot \nabla f_p(\tau, \sigma)
\]

Only spatial inhomogeneity leads to dissipation.
Solution by the perturbation theory

\( \frac{\partial}{\partial \tau} \tilde{f}_p^{(0)} = \frac{1}{p \cdot a_p} C[f]_p \bigg|_{f = \tilde{f}^{(0)}} \)

We seek for a slow solution in the asymptotic regime:

\[ \frac{\partial}{\partial \tau} \tilde{f}_p^{(0)} = 0 \quad \Rightarrow \quad \frac{1}{p \cdot a_p} C[f]_p \bigg|_{f = \tilde{f}^{(0)}} = 0 \]

\[ \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = f_p^{eq}(\sigma; \tau_0) \]

written in terms of the hydrodynamic variables. Asymptotically, the solution can be written solely in terms of the hydrodynamic variables.

- **Five conserved quantities**
  - \( T(\sigma; \tau_0), \mu(\sigma; \tau_0), u_\mu(\sigma; \tau_0) \)
  - \( u^\mu(\sigma; \tau_0) u_\mu(\sigma; \tau_0) = 1 \)\n
- **Reduced degrees of freedom**
  - \( m = 5 \)

- **0th invariant manifold**
  - \( f_p^{(0)}(\tau_0, \sigma) = f_p^{eq}(\sigma; \tau_0) \)
  - \( f^{(0)}(\tau_0) = f^{eq} \)

- **Local equilibrium**
The lin. op. $L$ has good properties:

1. Self-adjoint

2. Semi-negative definite

3. $L \varphi_0^\alpha = 0 \quad \varphi_0^\alpha = \begin{cases} p^\mu & \alpha = \mu, \\ m & \alpha = 4 \end{cases}$

$L$ has 5 zero modes and other eigenvalues are negative.
Metric is given in terms of The zero modes:

\[
\eta_{\alpha\beta} = \langle \varphi_0^\alpha, \varphi_0^\beta \rangle
\]

1st order solution:

\[
f^{(1)}(\tau, \sigma; \tau_0) = \text{fix}_{\text{fix}}\left[ e^\hat{L}(\tau - \tau_0)\Psi + (\tau - \tau_0)P_0F_0 + (e^\hat{L}(\tau - \tau_0) - 1)\hat{L}^{-1}Q_0F_0 \right]
\]

With the initial value: \( f^{(1)}(\sigma; \tau_0) = f^{(1)}(\tau = \tau_0, \sigma; \tau_0) = \text{fix}_{\text{fix}}\Psi \)

which is yet to be determined.

In the case of the 1st-order (N-S) equation:

\[
e^\hat{L}(\tau - \tau_0)\hat{L}^{-1}Q_0F_0
\]

can be cancelled out by a choice of the initial value \( \Psi = -\hat{L}^{-1}Q_0F_0 \)

1st-order (Landau) equation

Tsumura, Ohnishi and TK, PLB46(2007);

Envelope/RG eq. \( \frac{d\tilde{f}(\tau; \tau_0)}{d\tau_0} \bigg|_{\tau_0=\tau} = 0 \)

Note: we can assume that \( P_0\Psi = 0 \).

because possible zero modes can be renormalized into the zero-th sol.
Resultant 1st-order Hydrodynamic equation

\[ \partial_\mu J^{\mu\alpha}_{\text{hydro}} = 0, \quad J^{\mu\alpha}_{\text{hydro}} \equiv \int \frac{d\rho p^\mu}{\rho} \varphi_{0p} \tilde{f}_p^{\text{eq}} \left( 1 + \tilde{f}_p^{\text{eq}} \psi_p \right) \]

\[ = \left\{ \begin{array}{l}
eq u^\mu u^\nu - (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu}, \quad \alpha = \nu, \\
u u^\mu + J^\mu, \quad \alpha = 4. \end{array} \right. \]

\[ \Pi = -\zeta \theta \quad J^\mu = \lambda \frac{T^2}{\hbar^2} \nabla^\mu \frac{\mu}{T} \quad \pi^{\mu\nu} = 2\eta \sigma^{\mu\nu} \]

\[ \zeta = -\frac{1}{T} \langle \hat{\Pi}, \hat{L}^{-1} \hat{\Pi} \rangle \quad \lambda = \frac{1}{3T^2} \langle \hat{J}^\mu, \hat{L}^{-1} \hat{J}_\mu \rangle \quad \eta = -\frac{1}{10T} \langle \hat{\pi}^{\mu\nu}, \hat{L}^{-1} \hat{\pi}_{\mu\nu} \rangle \]

\[ = \frac{1}{T} \int_0^\infty ds \langle \hat{\Pi}_p(0), \hat{\Pi}(s) \rangle \quad = -\frac{1}{3T^2} \int_0^\infty ds \langle \hat{J}_p^\mu(0), \hat{J}_\mu(s) \rangle \quad = \frac{1}{10T} \int_0^\infty ds \langle \hat{\pi}_{\mu\nu}(0), \hat{\pi}_{\mu\nu}(s) \rangle \]

\[ \langle \hat{\Pi}_p(s), \hat{J}_p^\mu(s), \hat{\pi}_{\mu\nu}^p(s) \rangle \equiv \int dq \left[ e^{\hat{L}_q} \right]_{\rho \rho} \langle \hat{\Pi}_q, \hat{J}_q^\mu, \hat{\pi}_{\mu\nu}^q \rangle \]

\[ (\hat{\Pi}_p, \hat{J}_p^\mu, \hat{\pi}_{\mu\nu}^p) = \frac{1}{p \cdot u} (\Pi_p, J_p^\mu, \pi_{\mu\nu}^p) \]

\[ \Pi_p \equiv (p \cdot u)^2 \left[ \frac{1}{3} - \frac{\partial P}{\partial e} \right]_n + (p \cdot u) \left. \frac{\partial P}{\partial n} \right|_e - \frac{1}{3} m^2, \]

\[ J_p^\mu \equiv -\Delta^{\mu\nu} p_\nu ((p \cdot u) - h) \quad \pi_{\mu\nu}^p \equiv \Delta^{\mu\nu\rho\sigma} p_{\rho} p_{\sigma}, \quad h \equiv (e + P)/n \]

\[ \Delta^{\mu\nu\rho\sigma} \equiv 1/2 (\Delta^{\mu\rho} \Delta^{\nu\sigma} + \Delta^{\mu\sigma} \Delta^{\nu\rho} - 2/3 \Delta^{\mu\nu} \Delta^{\rho\sigma}) \]
Transport coefficients for a recative flow

First-order transport coefficients

\[
\zeta = \frac{1}{T} \int_0^\infty ds \langle \hat{\Pi}(0), \hat{\Pi}(s) \rangle \\
\eta = \frac{1}{10T} \int_0^\infty ds \langle \hat{\pi}^{\mu\nu}(0), \hat{\pi}^{\mu\nu}(s) \rangle \\
\lambda_{AB} = -\frac{1}{3T^2} \int_0^\infty ds \langle \hat{J}_A^\mu(0), \hat{J}_B,\mu(s) \rangle
\]

\[
\langle \psi, \chi \rangle \equiv \sum_{k=1}^N \int dp_k (p_k \cdot u) f_{k,p_k}^{eq} \bar{f}_{k,p_k}^{eq} \psi_{k,p_k} \chi_{k,p_k}
\]

First-order transport coefficients have the same expressions as those of Chapman-Enskog method and consistent with the field theoretic calculation based on Green-Kubo formula.

Jeon, PRD 52, 3591 (1995); Hidaka and TK, PRD 83, 076004 (2011)
Derivation of 2\textsuperscript{nd}-order solution: How to deform the invariant manifold so as to incorporate excited modes

\[ \tilde{f}^{(1)}(\tau, \sigma; \tau_0) \approx f^{\text{eq}} f^{\text{eq}} \left[ \Psi + (\tau - \tau_0) \hat{L} \Psi + (\tau - \tau_0) P_0 F_0 + (\tau - \tau_0) Q_0 F_0 \right] \]

\( \tau - \tau_0 : \text{small} \)

- \( \hat{L} \Psi \) and \( Q_0 F_0 \) should belong to a common vector space.
- The \( P_1 \) space is spanned by independent components of \( \hat{L} \Psi \) and \( \Psi \).

\( \Psi \) and \( \hat{L}^{-1} Q_0 F_0 \) should belong to the same vector subspace.

Now an explicit calculations give

\[ [\hat{L}^{-1} Q_0 F_0]_p = \frac{1}{T} \left[ [\hat{L}^{-1} \hat{H}]_p (-\nabla \cdot u) - [\hat{L}^{-1} \hat{J}^\mu]_p \frac{T}{p} \nabla_\mu \frac{\mu}{T} + [\hat{L}^{-1} \hat{\pi}^{\mu \nu}]_p \Delta_{\mu \nu \rho \sigma} \nabla^\rho u^\sigma \right] \]

Here the microscopic dissipative currents are given by

\[
(\Pi_p, \hat{J}_p^\mu, \hat{\pi}^{\mu \nu}_p) = \frac{1}{p \cdot u} (\Pi_p, J_p^\mu, \pi_p^{\mu \nu})
\]

\[ h \equiv (e + P)/n, \]

\[ \Delta_{\mu \nu \rho \sigma} \equiv 1/2(\Delta^\mu_\rho \Delta^\nu_\sigma + \Delta^\mu_\sigma \Delta^\nu_\rho - 2/3 \Delta^\mu_\nu \Delta^\rho_\sigma) \]

\[ \Pi_p \equiv (p \cdot u)^2 \left[ \frac{1}{3} - \frac{\partial P}{\partial e} \right]_n + (p \cdot u) \left. \frac{\partial P}{\partial n} \right|_e - \frac{1}{3} m^2, \]

\[ J_p^\mu \equiv -\Delta^{\mu \nu} p_\nu ((p \cdot u) - h), \]

\[ \pi_p^{\mu \nu} \equiv \Delta^{\mu \nu \rho \sigma} p_\rho p_\sigma. \]
Second-order perturbative eq.

\[
\frac{\partial}{\partial \tau} \tilde{f}^{(2)}(\tau) = f^{eq} \tilde{f}^{eq} L(f^{eq} \tilde{f}^{eq})^{-1} \tilde{f}^{(2)}(\tau) + f^{eq} \tilde{f}^{eq} K(\tau - \tau_0)
\]

\[
K(\tau - \tau_0) \equiv F^{(1)}(\tau) + \frac{1}{2} B[\tilde{f}^{(1)}, \tilde{f}^{(1)}](\tau),
\]

\[
B[\chi, \psi]_{k,p_k;m,q_m;n,r_n} \equiv -(f^{eq}_{k,p_k} \tilde{f}^{eq}_{k,p_k})^{-1}
\]

\[
\times \frac{\delta^2}{\delta f_{m,q_m} \delta f_{n,r_n}} \left( \frac{1}{p_k \cdot u} \sum_{l=1}^{N} C_{kl}[f_{k,p_k}] \right) \bigg|_{f=f^{eq}} \chi_{m,q_m}^{(1)} \psi_{n,r_n}^{(1)}
\]

Second-order perturbative solution

\[
\tilde{f}^{(2)}(\tau) = f^{eq} \tilde{f}^{eq} \left[ (\tau - \tau_0) P_0 + (\tau - \tau_0) G(s)^{-1} P_1 G(s) Q_0 - (1 + (\tau - \tau_0) \partial / \partial s) Q_1 G(s) Q_0 \right] K(s) \bigg|_{s=0}
\]

``Initial'' value at arbitrary time \( \tau = \tau_0 \)

\[
f^{(2)}(\tau_0) = -f^{eq} \tilde{f}^{eq} Q_1 G(s) Q_0 K(s) \bigg|_{s=0}
\]

The perturbative calculation finished.
Hydrodynamic eq. through the RG equation

\[ \tilde{f}(\tau = \tau_0; \tau_0) = f^G(\tau_0) \Rightarrow f^G(\tau) \]

\[ \frac{d\tilde{f}(\tau; \tau_0)}{d\tau_0} \bigg|_{\tau_0=\tau} = 0 \quad \left( \frac{dC}{dt} = G(C) \right) \]

**Second-order hydrodynamics**

- Fast modes
- Quasi-slow modes
- Slow modes

Projection onto
- \( P_0 \)-space
- \( P_1 \)-space

Eq. of continuity:

\[ (T(\tau), \mu_A(\tau), u^\mu(\tau)) \]

Eq. of relaxation:

\[ (\Pi(\tau), J_A^\mu(\tau), \pi^{\mu\nu}(\tau)) \]

Second-order multi-component quantum hydrodynamic Equation

Hydrodynamic equation

\[ \partial_{\mu} T^{\mu \nu} = 0 \]
\[ \partial_{\mu} N_{A}^{\mu} = 0 \]

\[ T^{\mu \nu} = \varepsilon u^{\mu} u^{\nu} - (P - \Pi) \Delta^{\mu \nu} + \pi^{\mu \nu} \]

\[ N_{A}^{\mu} = n_{A} u^{\mu} + J_{A}^{\mu} \]

\[ \Delta^{\mu \nu} = g^{\mu \nu} - u^{\mu} u^{\nu} \]
\[ \nabla_{\mu} = \Delta^{\mu \nu} \partial_{\nu} \]
\[ \Delta^{\mu \nu \rho \sigma} = \frac{1}{2} \left( \Delta^{\mu \rho \nu} \Delta^{\sigma} + \Delta^{\mu \sigma} \Delta^{\nu \rho} - \frac{2}{3} \Delta^{\mu \nu} \Delta^{\rho \sigma} \right) \]
\[ \theta = \nabla \cdot u \]
\[ \sigma^{\mu \nu} = \Delta^{\mu \nu \rho \sigma} \nabla_{\rho} u_{\sigma} \]

\[ \Pi = -\xi \theta - \gamma \frac{\partial}{\partial t} \Pi - \sum_{a=1}^{M} \ell_{a}^{2} \nabla \cdot J_{a} \]
\[ + \kappa_{\Pi \Pi} \Pi \theta + \sum_{A=1}^{M} \kappa_{\Pi A}^{(1) A} J_{A, \rho} \nabla^{\rho} T + \sum_{A, B=1}^{M} \kappa_{\Pi B}^{(2) AB} J_{A, \rho} \nabla^{\rho} \frac{\mu_{B}}{T} + \kappa_{\Pi \Pi} \pi_{\rho \sigma} \sigma^{\rho \sigma} \]
\[ + b_{\Pi \Pi} \Pi^{2} + \sum_{A, B=1}^{M} b_{AB}^{2} J_{A, \rho} J_{B, \rho} + b_{\Pi \Pi} \pi_{\rho \sigma} \pi_{\rho \sigma} \]

\[ J_{A}^{\mu} = \sum_{B=1}^{M} \lambda_{AB} \frac{T^{2}}{\hbar^{2}} \nabla_{\rho} \frac{\mu_{B}}{T} - \sum_{B=1}^{M} \tau_{AB}^{J} \Delta^{\mu \rho} \frac{\partial}{\partial \tau} J_{B, \rho} + \ell_{A}^{J} \nabla_{\rho} \Pi - \ell_{A}^{J} \Delta^{\mu \rho} \nabla_{\nu} \pi_{\rho \nu} \]
\[ + \kappa_{J}^{(1) A} J_{A} \nabla^{\mu} T + \sum_{B=1}^{M} \kappa_{J B}^{(2) AB} J_{A} \nabla^{\mu} \frac{\mu_{B}}{T} + \sum_{B=1}^{M} \kappa_{J B}^{(1) AB} J_{A} \mu_{B} + \sum_{B=1}^{M} \kappa_{J B}^{(2) AB} J_{A, \rho} \sigma_{\mu \rho} \]
\[ + \kappa_{J}^{(3) AB} J_{B, \rho} \omega_{\mu \rho} + \kappa_{J}^{(1) A} \pi_{\rho \sigma} \nabla_{\rho} T + \sum_{B=1}^{M} \kappa_{J B}^{(2) AB} \pi_{\rho \sigma} \nabla_{\rho} \frac{\mu_{B}}{T} \]
\[ + \sum_{B=1}^{M} b_{J B}^{AB} \Pi J_{B}^{\mu} + \sum_{B=1}^{M} b_{J B}^{AB} J_{B, \rho} \pi_{\rho \mu} \]

\[ \pi^{\mu \nu} = 2\eta \sigma^{\mu \nu} - \gamma \frac{\partial}{\partial \tau} \pi_{\rho \sigma} - \sum_{a=1}^{M} \rho_{a}^{J} \nabla_{a} \mu_{a} + \sum_{a=1}^{M} \ell_{a}^{J} \nabla_{a} \mu_{a} \]
\[ + \kappa_{\Pi \Pi} \Pi \sigma^{\mu \nu} + \sum_{A=1}^{M} \kappa_{\Pi A}^{(1) A} J_{A}^{(\mu \nu)} T + \sum_{A, B=1}^{M} \kappa_{\Pi B}^{(2) AB} J_{A}^{(\mu \nu)} \frac{\mu_{B}}{T} \]
\[ + \kappa_{\Pi A}^{(1) A} \pi_{\mu \nu} + \kappa_{\Pi B}^{(2) AB} \pi_{\mu \nu} + \kappa_{\Pi A}^{(3) A} \pi_{\mu \nu} \pi_{\mu \nu} + \kappa_{\Pi B}^{(2) AB} \pi_{\mu \nu} \pi_{\mu \nu} \]
\[ + b_{\Pi \Pi} \Pi \mu_{\nu} + \sum_{A, B=1}^{M} b_{AB}^{2} J_{A}^{(\mu \nu)} J_{B}^{(\mu \nu)} + b_{\Pi \Pi} \pi_{\mu \nu} \pi_{\mu \nu} \]

Dissipative relaxation times

\[ \tau_{\Pi} = \frac{\int_0^\infty ds s \langle \hat{\Pi}(0), \hat{\Pi}(s) \rangle}{\int_0^\infty ds \langle \hat{\Pi}(0), \hat{\Pi}(s) \rangle} = \frac{\int_0^\infty ds s R_{\Pi}(s)}{\int_0^\infty ds R_{\Pi}(s)} \]

\[ \tau_{\pi} = \frac{\int_0^\infty ds s \langle \hat{\pi}_{\mu\nu}(0), \hat{\pi}_{\mu\nu}(s) \rangle}{\int_0^\infty ds \langle \hat{\pi}_{\rho\sigma}(0), \hat{\pi}_{\rho\sigma}(s) \rangle} = \frac{\int_0^\infty ds s R_J(s)}{\int_0^\infty ds R_J(s)} \]

\[ \tau_{\pi} = -\frac{\langle \hat{\pi}_{\mu\nu}, \hat{L}^{-2} \hat{\pi}_{\mu\nu} \rangle}{\langle \hat{\pi}_{\rho\sigma}, \hat{L}^{-1} \hat{\pi}_{\rho\sigma} \rangle} = \frac{\int_0^\infty ds s R_{\pi}(s)}{\int_0^\infty ds R_{\pi}(s)} \]

\[ R_{\Pi}(s) \equiv \frac{1}{T} \langle \hat{\Pi}(0), \hat{\Pi}(s) \rangle, \]

\[ R_J(s) \equiv -\frac{1}{3T^2} \langle \hat{J}^\mu(0), \hat{J}_\mu(s) \rangle, \]

\[ R_{\pi}(s) = \frac{1}{10T} \langle \hat{\pi}_{\mu\nu}(0), \hat{\pi}_{\mu\nu}(s) \rangle. \]

For a reactive case,

\[ \tau_{jAB}^{AB} = \sum_{C=1}^M \left( \int_0^\infty ds s \langle \hat{j}^\mu(0), \hat{j}_\mu(s) \rangle \right)_{AC} \left( \int_0^\infty ds \langle \hat{j}^\mu(0), \hat{j}_\mu(s) \rangle \right)_{CE}^{-1} \]

Kikuchi, Tsumura, TK, PRC(2015)

Correlation times!, which are different for the (respective) dissipative quantities.

c.f. Israel–Stewart 14 moment formulae:

\[ \tau_{\Pi}^{IS} = -\frac{\langle \Pi, \Pi \rangle}{\langle \Pi, \hat{L}\Pi \rangle}, \quad \tau_J^{IS} = \frac{\langle J^\mu, J_\mu \rangle}{\langle J_\rho, \hat{L}J_\rho \rangle}, \quad \tau_{\pi}^{IS} = -\frac{\langle \pi_{\mu\nu}, \pi_{\mu\nu} \rangle}{\langle \pi_{\rho\sigma}, \hat{L}\pi_{\rho\sigma} \rangle} \]

Comparison with other methods:

<table>
<thead>
<tr>
<th></th>
<th>RG</th>
<th>Israel-Stewart</th>
<th>Denicol et al.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta )</td>
<td>1.27</td>
<td>1.2</td>
<td>1.267</td>
</tr>
<tr>
<td>( \tau_\pi )</td>
<td>1.66</td>
<td>1.8</td>
<td>2</td>
</tr>
</tbody>
</table>

G. Denicol, H. Niemi, E. Molnar, D. Rischke, PRD 85 (2012); An elaborated moments expansion with 41 moments.
Properties of resultant hydrodynamic eq,

Following properties are proved:  

- **Causality** (Propagating velocities of fluctuation of hydrodynamic variables do not exceed the light speed)

- **Stability** (Equilibrium state is stable for any perturbation)

- **Positive definiteness of the entropy production rate**

- **Onsager’s reciprocal theorem**

\[
J_1^\mu = \lambda_{11} \frac{T^2}{\hbar^2} \nabla^\mu \frac{\mu_1}{T} + \lambda_{12} \frac{T^2}{\hbar^2} \nabla^\mu \frac{\mu_2}{T} + \cdots
\]

\[
\lambda_{AB} = -\frac{1}{3T^2} \int_0^\infty ds \langle \hat{J}_A^\mu(0), \hat{J}_{B,\mu}(s) \rangle
\]

\[
J_2^\mu = \lambda_{21} \frac{T^2}{\hbar^2} \nabla^\mu \frac{\mu_1}{T} + \lambda_{22} \frac{T^2}{\hbar^2} \nabla^\mu \frac{\mu_2}{T} + \cdots
\]

Indicating that our way of solution respect the fundamental property of Boltzmann equation that the microscopic process is time-reversal invariant.
Derivation of Second-order hydrodynamic equations: Similarity of rel H-I with (unitary) cold atomic gas

Unitary Cold Atomic Gas

Expanding gas behaves \textit{hydrodynamically}.

Problem

- Two regions: hydrodynamic core and dilute corona
- How to describe the transition between these regions
- Consider a relaxation of dissipative currents


Strongly correlated quantum fluid

\[ \frac{\eta}{s} \gtrsim \frac{\hbar}{4\pi k_B} \]

nearly perfect fluid

From Boltzmann eq. with mean field to hydrodynamic eq.


Boltzmann eq.

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \mathbf{F} \cdot \nabla_p \right) f_p(t, \mathbf{x}) = C[f]_p(t, \mathbf{x})
\]

\[
\mathbf{F} = -\nabla E_p(\mathbf{x}) = -\nabla V(\mathbf{x})
\]

\[
\epsilon : \text{measure of the inhomogeneity of fluid}
\]

\[
\left( \frac{\partial}{\partial t} + \epsilon \mathbf{v} \cdot \nabla + \epsilon \mathbf{F} \cdot \nabla_p \right) f_p(t, \mathbf{x}) = C[f]_p(t, \mathbf{x})
\]

To which the RG method is applied to obtain the 2nd-order hydrodynamic equations, together with the microscopic expressions of the transport coefficients and relaxation times.
Shear viscosity

S-wave scattering

\[
\frac{d\sigma}{d\Omega} = \frac{1}{(1/a_s)^2 + q^2}
\]

scattering length dependence

scattering length

relative momentum

Microscopic expressions:

\[
\eta = \frac{1}{10T} \int_0^\infty ds \langle \hat{\pi}^{ij}(0), \hat{\pi}^{ij}(s) \rangle
\]

\[
= -\frac{1}{10T} \langle \hat{\pi}^{ij}, L^{-1}\hat{\pi}^{ij} \rangle
\]

\[
L_{pq} \equiv \frac{\delta}{\delta f_q} C[f]_p(t) \bigg|_{f=f^{eq}}
\]

\[
\langle \psi, \chi \rangle \equiv \int_p f_p^{eq}(1+af_p^{eq})\psi_p\chi_p
\]

\[
\hat{\pi}^{ij}_p(s) \equiv [e^{sL}\hat{\pi}^{ij}]_p
\]
Viscous relaxation times from kinetic theory

Boltzmann eq. \( \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \mathbf{F} \cdot \nabla_p \right) f_p(t, \mathbf{x}) = C[f]_p(t, \mathbf{x}) \quad L_{pq} \equiv \frac{\delta}{\delta f_q} C[f]_p(t) \bigg|_{f = f_{eq}} \)

Microscopic expressions

\[
\begin{align*}
\tau_{\pi}^{\text{exact}} &= \frac{1}{10T \eta} \langle \hat{\pi}^{ij}, L^{-2} \hat{\pi}^{ij} \rangle \\
\tau_{J}^{\text{exact}} &= \frac{1}{3T^2 \lambda} \langle \hat{J}^i, L^{-2} \hat{J}^i \rangle
\end{align*}
\]

\[
\begin{align*}
\frac{Dn}{Dt} &= -n \nabla \cdot \mathbf{u}, \\
mn \frac{Du^i}{Dt} &= -\nabla^i P + n F^i + \nabla^j \pi^{ij}, \\
Tn \frac{Ds}{Dt} &= \sigma^{ij} \pi^{ij} + \nabla \cdot \mathbf{J} \\
\pi^{ij} &= \eta \sigma^{ij} - \tau_{\pi} \frac{D}{Dt} \pi^{ij} + \cdots \\
J^i &= \lambda \nabla^i T - \tau_{J} \frac{D}{Dt} J^i + \cdots
\end{align*}
\]
Test of reliability of the Relaxation–time approximation (RTA)

(BGK)


\( \eta^\text{exact} = -\frac{1}{10T} \langle \hat{\pi}^{ij}, L^{-1} \hat{\pi}^{ij} \rangle \quad \lambda^\text{exact} = -\frac{1}{3T^2} \langle \hat{j}^i, L^{-1} \hat{j}^i \rangle \)

Our exact expressions

\[ \tau^\text{exact}_\pi = \frac{1}{10T \eta} \langle \hat{\pi}^{ij}, L^{-2} \hat{\pi}^{ij} \rangle \]
\[ \tau^\text{exact}_j = \frac{1}{3T^2 \lambda} \langle \hat{j}^i, L^{-2} \hat{j}^i \rangle \]

(Improved) RTA


\[ \tilde{\tau}^\text{RTA}_\pi = \frac{\eta^\text{exact}}{P} \]
\[ \tilde{\tau}^\text{RTA}_j = \frac{12mT \lambda^\text{exact}}{(7Q - 75P^2/n)} \]

Relaxation–time approximation (RTA or BGK)


\[ C[f]_p(t, x) \sim \frac{f(t, x) - f^{eq}(x)}{\tau} \]
\[ \tau^\text{RTA}_\pi = \tau^\text{RTA}_j = \tau \]
\[ \eta^\text{RTA} = \tau P \]
\[ \lambda^\text{RTA} = \frac{\tau}{12mT} \left(7Q - \frac{75P^2}{n}\right) \]

\[ Q \equiv \frac{1}{m^2} \int_p \delta p^4 f^{eq} \quad \delta p \equiv m(v - u) \]
Viscous relaxation time of stress tensor


Our exact expressions

\[ \tau_{\text{exact}}^\pi = \frac{1}{10T\eta} \langle \hat{\pi}^{ij}, L^{-2} \hat{\pi}^{ij} \rangle \]

RTA

\[ \tilde{\tau}_{\text{RTA}}^\pi = \frac{\eta_{\text{exact}}}{P} \]

RTA well reproduces the exact results!!, which may also imply that our microscopic formulae of the relaxation times are correct!
Viscous relaxation time of heat conductivity


**Temperature dependence**

**Exact expressions**

\[ \tau_{J}^{\text{exact}} = \frac{1}{3T^2\lambda} \langle \hat{f}^i, L^{-2} \hat{f}^i \rangle \]

**RTA**

\[ \tilde{\tau}_{J}^{\text{RTA}} = \frac{12mT\lambda^{\text{exact}}}{(7Q - 75P^2/n)} \]

Considerably violated in contrast to \( \tau_{\pi} \)

\[ \tau_{\pi}^{\text{RTA}} = \tau_{J}^{\text{RTA}} = \tau \text{ is clearly invalid.} \]
A geometrical formulation of the reduction of the dynamics is given on the basis of the renormalization-group/envelope method, which may give a partial and intermediate resolution of Hilbert’s 6th problem.\(\rightarrow\) variational principle (Hilbert)?

The microscopic expressions of the transport coefficients that coincide with those of Chapman-Enskog and \textbf{viscous relaxation times} are derived from the Boltzmann equation (quasi-particle approx.) by an adaptation of the RG method, and \textbf{numerical evaluations are performed without recourse to any approximation}.

\textbf{Quantum statistics} makes significant contributions to the shear viscosity (and the others as well).

We have numerically examined that the relation \(\tau_\pi = \eta/P\), which is derived in the RTA, \textbf{is satisfied quite well}.

The analoguous relation for \(\tau_J\) is \textbf{satisfied only approximately}.