Determination of the $a_0$-$a_2$ Pion Scattering Length from $K^+ \rightarrow \pi^+\pi^0\pi^0$ Decay

(see also N. Cabibbo and G. Isidori, JHEP 03, 021 (2005))

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supported by Global Center of Excellence Program “Nanoscience and Quantum Physics”

2010, Jul. 15th
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Scattering length

Scattering length: amplitude at threshold
- characterizes the low energy scattering
- changes the sign if there is a bound state

\[ a = -f(k)|_{k \to 0} \]

\[ \frac{d\sigma}{d\Omega} = |f(k)|^2, \quad \lim_{k \to 0} \sigma(k) = 4\pi a^2 \]

\[ f = \frac{1}{k \cot \delta_0 - i k}, \quad k \cot \delta_0 = -\frac{1}{a} + r_e \frac{k^2}{2} + \ldots \]

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J.J. Sakurai, Modern Quantum Mechanics, p. 415

(In hadron physics we usually adopt the opposite sign
--> positive for attraction, negative for repulsion)
**Introduction -- hadron scattering length**

**Hadron scattering length measurement**

**Extraction of hadron scattering length**
- **shift** and **width** of atomic state (coulomb bound state)
  - ex) Kaonic hydrogen

**Problem:** charged state only

- **extrapolation of phase shift**
  - ex) ππ scattering


**Problem:** uncertainty of the extrapolation

**- threshold effect in the decay spectrum --> today’s topic**
Cabibbo’s idea for $\pi\pi$ scattering length
- small isospin violation

\[ m_{\pi^\pm} \sim m_{\pi^0} + 5 \text{ MeV} \]

--> charged $\pi\pi$ state is heavier than the neutral channel

- threshold cusp effect in the $\pi^0\pi^0$ spectrum

--> cusp appears at a higher energy threshold

- transition amplitude is proportional to $a_0-a_2$
Goal: to show the cusp in the $K \rightarrow \pi^+\pi^0\pi^0$ spectrum

There are two (or more) processes:

Step 1) Threshold cusp effect

The rescattering amplitude has an imaginary part for

$$s_{\pi\pi} > 4m_{\pi^\pm}^2$$
$$s_{\pi\pi} = (q_2 + q_1)^2 = (k - q_3)^2$$

So we divide the amplitude into two pieces

$$\mathcal{M}(K^+ \rightarrow \pi^+\pi^0\pi^0) = \mathcal{M}_0 + \mathcal{M}_1$$

such that

$$\mathcal{M}_1 = \begin{cases} 
\text{pure imaginary} & s_{\pi\pi} > 4m_{\pi^\pm}^2 \\
0 & s_{\pi\pi} = 4m_{\pi^\pm}^2 \\
\text{real} & s_{\pi\pi} < 4m_{\pi^\pm}^2 
\end{cases}$$
Step 1) Threshold cusp effect

Imaginary part of $M_1$ amplitude

Imaginary part of the loop function

$$G(s_{\pi\pi}) \sim \int_{4m_{\pi^\pm}^2}^{\infty} ds' \frac{\rho(s')}{s_{\pi\pi} - s' + i\epsilon}$$

$$\text{Im} \ G(s_{\pi\pi}) = -\rho(s_{\pi\pi}) = -\frac{p}{8\pi \sqrt{s_{\pi\pi}}} \propto -v \quad \text{for} \quad s_{\pi\pi} > 4m_{\pi^\pm}^2$$

\(p\): phase space, \(p\): three-momentum, \(v\): velocity

$$p = \sqrt{s_{\pi\pi} - 4m_{\pi^\pm}^2} \quad \frac{2}{2} , \quad v = \frac{p}{E} = \sqrt{s_{\pi\pi} - 4m_{\pi^\pm}^2} / s_{\pi\pi} , \quad E = \sqrt{s_{\pi\pi}}$$

We can then write the amplitude as

$$M_1 \propto J = \begin{cases} J_+ = -i\pi v & s_{\pi\pi} > 4m_{\pi^\pm}^2 \\ 0 & s_{\pi\pi} = 4m_{\pi^\pm}^2 \\ J_- = \pi \tilde{v} & s_{\pi\pi} < 4m_{\pi^\pm}^2 \end{cases}$$

analytic continuation of $v$
Threshold cusp in the spectrum

Amplitude of the process

\[ M(K^+ \to \pi^+ \pi^0\pi^0) = M_0 + \boxed{M_1} \]

above threshold: imaginary
below threshold: real

real

The \( \pi^0\pi^0 \) invariant mass spectrum \((M_{\pi\pi} = \sqrt{s_{\pi\pi}})\)

\[ \frac{d\Gamma}{dM_{\pi\pi}} \propto |M|^2 = \begin{cases} (M_0)^2 + (iM_1)^2 & s_{\pi\pi} > 4m_{\pi^\pm}^2 \\ (M_0)^2 + (M_1)^2 + 2M_0M_1 & s_{\pi\pi} < 4m_{\pi^\pm}^2 \end{cases} \]

- Spectrum is continuous

\[ \lim_{s_{\pi\pi} \to 4m_{\pi^\pm}} M_1 = 0 \implies \lim_{+0} |M|^2 = \lim_{-0} |M|^2 \]

- Derivative of the spectrum is discontinuous

\[ \lim_{-0} \frac{d}{dM_{\pi\pi}} |M|^2 - \lim_{+0} \frac{d}{dM_{\pi\pi}} |M|^2 = 2M_0 \frac{d}{dM_{\pi\pi}} M_1 \]

--> threshold cusp

It is purely kinematical effect, independent of the interaction.
Step 2) Model for the one-loop amplitude

Amplitude for the subprocesses

Goal: to construct the $M_1$ amplitude from one-loop graph

- Imaginary part $\leftarrow$ loop function

$$M_1 \propto J$$

- Model for the $K$ decay (essentially ChPT)


$$K^+ \to \pi^- + \pi^+ \sim M_+ = A^+_{av} \left(1 + \frac{g^+(s_{\pi\pi} - s_0)}{2m_{\pi^\pm}^2}\right)$$

$$s_0 = (M_K^3 + 3m_{\pi}^3)/3$$

parameters

- Rescattering amplitude (around threshold)

$$\pi^- \sim 16\pi(a_0 - a_2)m_{\pi^\pm}$$

isospin relation

$$a_{\pi^+\pi^- \to \pi^0\pi^0} = \frac{a_0 - a_2}{3}$$
Step 2) Model for the one-loop amplitude

**M₁ amplitude**

**M₁ term from the one-loop amplitude**

\[
M_{1} = -\frac{2(a_{0} - a_{2})m_{\pi^{\pm}}}{3} \cdot J \cdot M_{\pi^{\pm}}
\]

Using a model for M₀ amplitude, we can calculate the mass spectrum.

If \( a_{0} - a_{2} = 0 \), no rescattering (dashed line)
For finite \( a_{0} - a_{2} \) (solid), cusp appears.
Step 3) Extraction from experimental data

**Experimental determination**

**Goal:** to analyze the experimental spectrum without the model for the $M_0$ amplitude.

**Region around threshold --> momentum expansion**

$$|M|^2 = a + b\delta + c\delta^2 + O(\delta^3) \equiv F(\delta^2) + O(\delta^3)$$

$$\delta = \frac{\sqrt{4m_{\pi\pm}^2 - s_{\pi\pi}}}{2m_{\pi}} = \frac{p}{m_{\pi\pm}}$$

Using this expansion and the $M_1$ amplitude, it is possible to extract the scattering length $a_0 - a_2$.

**Formulae:**

$$\frac{d\Gamma}{dM_{\pi\pi}} \propto |M|^2 = \begin{cases} (M_0)^2 + (iM_1)^2 & s_{\pi\pi} > 4m_{\pi\pm}^2 \\ (M_0)^2 + (M_1)^2 + 2M_0M_1 & s_{\pi\pi} < 4m_{\pi\pm}^2 \end{cases}$$

$$M_1 = -\frac{2(a_0 - a_2)m_{\pi\pm}}{3} \cdot J \cdot M_+ \quad M_+ = A_{av}^+ \left(1 + \frac{g^+(s_{\pi\pi} - s_0)}{2m_{\pi\pm}^2}\right)$$
Step 3) Extraction from experimental data

**Procedure for experimental analysis**

Four steps for the experimental determination of \(a_0-a_2\)

1) Determine \(M_+\) by \(K \rightarrow \pi^+\pi^+\pi^-\) decay spectrum

\[
M_+ = A^+_\text{av} \left( 1 + \frac{g^+ (s_{\pi\pi} - s_0)}{2m_{\pi^\pm}^2} \right) \quad \rightarrow \quad A^+_\text{av}, g^+ : \text{fixed}
\]

2) Fit the \(\pi^+\pi^0\pi^0\) spectrum above threshold by \(F(\delta^2)\)

\[
|M|^2_{\text{above}} = (M_0)^2 - (M_1)^2 = F(\delta^2) \quad \rightarrow \quad a, b, c : \text{fixed}
\]

3) Extract \(M_1\) from the \(\pi^+\pi^0\pi^0\) spectrum below threshold

\[
|M|^2_{\text{below}} = (M_0)^2 + (M_1)^2 + 2M_0M_1
\]

\[
= (M_0)^2 - (M_1)^2 + 2(M_1)^2 + 2[(M_0)^2 - (M_1)^2 + (M_1)^2]^{1/2}M_1
\]

\[
= F(\delta^2) + 2(M_1)^2 + 2[F(\delta^2) + (M_1)^2]^{1/2}M_1
\]

4) Calculate \(a_0-a_2\) from \(M_+\) and \(J\)

\[
M_1 = \frac{-2(a_0 - a_2)m_{\pi^\pm}}{3} \cdot J \cdot M_+
\]

This method does not require any model for \(M_0\)
Step 3) Extraction from experimental data

**Experimental feasibility**

Cusp is indeed seen in the experimental spectrum

A method to extract the $\pi\pi$ scattering length from the $K^+ \rightarrow \pi^+\pi^0\pi^0$ decay is discussed.

Isospin violation causes mass difference between charged and neutral pions.

Threshold cusp at $\pi^+\pi^-$ threshold is proportional to $a^0-a^2$ scattering length.

Experimental determination is possible with momentum expansion.

Higher order corrections:

N. Cabibbo and G. Isidori, JHEP 03, 021 (2005)
Summary + future plan

**Determination of πΣ scattering length**

**Similar approach for πΣ scattering length?**

T. Hyodo, M. Oka, work in progress

Σ⁺(~uus) < Σ⁰(~uds) < Σ⁻(~dds)

--> complicated spectrum

\[
\begin{align*}
\langle \pi^- Σ^+ | T | π^+ Σ^- \rangle |_{\text{threshold}} &= \frac{1}{3}a^0 - \frac{1}{2}a^1 + \frac{1}{6}a^2 \\
\langle π^0 Σ⁰ | T | π^+ Σ⁻ \rangle |_{\text{threshold}} &= \frac{1}{3}a^0 - \frac{1}{3}a^2 \\
\langle π^0 Σ⁺ | T | π^+ Σ⁰ \rangle |_{\text{threshold}} &= -\frac{1}{2}a^1 + \frac{1}{2}a^2
\end{align*}
\]
How to measure?
- \( \Lambda_c \rightarrow \pi \pi \Sigma \)

Branching fraction of the \( \Lambda_c \) decay (\( \Gamma_i/\Gamma \)) in PDG:
- \( \pi^+\Sigma^- \pi^+ \) (1.7%), \( \pi^-\Sigma^+ \pi^+ \) (3.6%), \( \pi^0\Sigma^0 \pi^+ \) (1.8%)

A lot of \( \Lambda_c \) in B decay (Belle, Babar) --> feasible?

Significance?
- Important constraint for \( K\pi-\pi\Sigma \) interaction at low energy
- Lower pole position of the \( \Lambda(1405) \) <-- sensitive to the \( \pi\Sigma \) scattering length

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