Hadronic molecule resonances

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Constituent quark model systematically reproduces the spectra by $qqq$ and $q\bar{q}$.

Exotic hadrons: exceptions to the $qqq/q\bar{q}$ + more than four quarks

~ 130 baryons

~ 160 mesons
Introduction

Structure of hadron resonances

Example) baryon excited state

Excited states
= resonances in hadron scattering

Exotic structure near threshold?
c.f. \(^{12}\text{C}\) Hoyle state
Study of the internal structure

Number of quarks and antiquarks are not conserved.

\[ | B \rangle = N_3 | qqq \rangle + N_5 | qqq \ q\bar{q} \rangle + N_7 | qqq \ q\bar{q} \ q\bar{q} \rangle + \ldots \]

\[ \langle qqq | qqq \ q\bar{q} \rangle \neq 0 \]

How to investigate the internal structure?

- **Comparison of model calculation v.s. experiments**
  (mass, width, decay properties, etc.)
  : Any model can describe data with appropriate corrections
  : Model-dependent definition

- **Extrapolation** to the ideal world, change the environment
  (large N_c, symmetry restoration, etc.)
  : Structure may change during the extrapolation
  : Qualitative discussion only

--> **model-independent and quantitative distinction?**
Contents

Introduction

Weinberg’s study of the deuteron structure
  • Field renormalization constant Z
  • relation to couplings and observables

  S. Weinberg, Phys. Rev. 137 B672-B678 (1965)

Application to chiral unitary approach
  • natural renormalization scheme
  • Z as “compositeness”


Summary
Weinberg’s study of the deuteron structure

Main result: theorem

\[ |\text{deuteron}\rangle = \begin{array}{c} \text{N} \\ \text{N} \end{array} \text{ or } \begin{array}{c} \text{N} \\ \text{N} \end{array} \]

∉ NN model space

~ elementary particle

\begin{align*}
Z &= 0 \\
Z &= 1
\end{align*}

\( Z \): probability of finding deuteron in a bare elementary state

For a bound state with small binding energy, the following equation should be satisfied model independently:

\[
\begin{align*}
a_s &= \left[ \frac{2(1-Z)}{2-Z} \right] R + \mathcal{O}(m^{-1}_\pi), \\
r_e &= \left[ -\frac{Z}{1-Z} \right] R + \mathcal{O}(m^{-1}_\pi)
\end{align*}
\]

\( a_s \): scattering length

\( r_e \): effective range

\( R \): deuteron radius (binding energy)

\[
\begin{align*}
a_s &= +5.41 \text{ [fm]}, \\
r_e &= +1.75 \text{ [fm]}, \\
R &\equiv (2\mu B)^{-1/2} = 4.31 \text{ [fm]}
\end{align*}
\]

\[ \Rightarrow Z \lesssim 0.2 \quad \Rightarrow \text{deuteron is almost composite!} \]
The theorem is derived in two steps:

**Step 1 (Sec. II):** Z --> p-n-d coupling constant \( g \)

\[
g^2 = \frac{2\sqrt{B}(1 - Z)}{\pi \rho} \quad \rho = 4\pi \sqrt{2\mu^3}
\]

**Step 2 (Sec. III):** coupling constant \( g \) --> \( a_s, r_e \)

\[
a_s = 2R \left[ 1 + \frac{2\sqrt{B}}{\pi \rho g^2} \right] \quad r_e = R \left[ 1 - \frac{2\sqrt{B}}{\pi \rho g^2} \right]
\]

Assumption: \( B \) is sufficiently smaller than the typical energy scale of the NN interaction

\[ p \sim m_\pi, \quad B \ll m_\pi^2/2\mu \quad \Rightarrow \quad R^2 \gg m_\pi^2 \]

--> uncertainty for order \( R \) quantity: \( m_\pi^{-1} \)
Weinberg’s study of the deuteron structure

**Definition of the probability Z**

**Hamiltonian of NN system:** \( \mathcal{H} = \mathcal{H}_0 + V \)

**Complete set for free Hamiltonian:** bare \( |d_0\rangle \) + continuum

\[
1 = |d_0\rangle\langle d_0| + \int dk |k\rangle\langle k|
\]

\[
\mathcal{H}_0|d_0\rangle = E_0|d_0\rangle, \quad \mathcal{H}_0|k\rangle = E(k)|k\rangle
\]

(Original, \( d_0 \): sum of discrete states, \( k \): \( \alpha \))

**Physical deuteron \( |d\rangle \): eigenstate of full Hamiltonian**

\[
(\mathcal{H}_0 + V)|d\rangle = -B|d\rangle
\]

**Z: overlap of \( d \) and \( d_0 \)**

(wavefunction renormalization factor)

\[
Z \equiv |\langle d_0 | d \rangle|^2
\]

\[
|d\rangle = \sqrt{Z}|d_0\rangle + \sqrt{1-Z} \int dk |k\rangle
\]
Weinberg’s study of the deuteron structure

**p-n-d coupling constant**

After some algebra, we arrive at

\[
1 - Z = \int d\mathbf{k} \frac{|\langle \mathbf{k} | V | d \rangle|^2}{[E(\mathbf{k}) + B]^2}
\]

\[
\langle \mathbf{k} | V | d \rangle = g(\mathbf{k}) : d \rightarrow V \rightarrow \mathbf{k}
\]

**Typical energy scale** $E_0$: below $E_0$, coupling is constant

\[
\langle \mathbf{k} | V | d \rangle = g(\mathbf{k}) \sim g \quad \text{for} \quad |E(\mathbf{k})| \leq E_0 \quad (\text{NN scattering} : E_0 \approx \frac{m_\pi^2}{2\mu})
\]

**Assumption:** $B \ll E_0$

\[
\Rightarrow 1 - Z \sim g^2 \int \frac{d\mathbf{k}}{[E(\mathbf{k}) + B]^2} = g^2 \rho \int_0^\infty \frac{\sqrt{E} \, dE}{[E + B]^2}
\]

\[
\rho = 4\pi \sqrt{2\mu^3}
\]

Integrate analytically

\[
g^2 = \frac{2\sqrt{B}(1 - Z)}{\pi \rho}
\]
Scattering equations

The Lippmann-Schwinger equation

\[ T(W) = V + V \frac{1}{W - \mathcal{H}_0} T(W) \]

\[ \Rightarrow T(W) = V + V \frac{1}{W - \mathcal{H}} \] (Chew-Goldberger solution)

Complete set for full Hamiltonian (asymptotic completeness)

\[ 1 = |d\rangle\langle d| + \int dk |k, \text{in}\rangle\langle k, \text{in}| \]

\[ V |k, \text{in}\rangle = T |k\rangle \]

\[ T_{k'k}(W) = V_{k'k} + \frac{\langle k' |V|d\rangle\langle d|V|k\rangle}{W + B} + \int dk'' \frac{\langle k' |V|k''\rangle, \text{in}\rangle\langle k'', \text{in}|V|k\rangle}{W - E(k'')} \]

Setting \( W = E(k) + i\epsilon \), we obtain the Low equation

\[ T_{k'k} = V_{k'k} + \frac{\langle k' |V|d\rangle\langle d|V|k\rangle}{E(k) + B} + \int dk'' \frac{T_{k'k''}T_{k''k}}{E(k) - E(k'')} + i\epsilon \]

So far no approximations.
Weinberg’s study of the deuteron structure

Solution for the scattering equation

The same assumption: \( B << E_0 \), external energy \( E << E_0 \)

\[
\frac{\langle k' \mid V \mid d \rangle \langle d \mid V \mid k \rangle}{E(k) + B} \sim \frac{g^2}{E(k) + B} \propto \frac{1}{\sqrt{B}} \gg V_{k'k}
\]

We neglect the 1st term (information of \( V \) is lost!!).

\[
T_{k'k} = \frac{g^2}{E(k) + B} + \int dk'' \frac{T_{k'k''}T_{k''k}}{E(k) - E(k'') + i\epsilon}
\]

S-wave scattering (no angular dependence)

\[
T_{k'k} \to t[E(k)] \delta_{k'k}
\]

\[
t(E) = \frac{g^2}{E + B} + \rho \int_0^\infty dE'' \frac{\sqrt{E''} |t(E)|^2}{E - E'' + i\epsilon}
\]

The solution of the integral equation (well-known? We should solve \( t^{-1}(E) \) using optical theorem and analyticity)

\[
t(E) = \left[ \frac{E + B}{g^2} + \frac{\pi \rho (B - E)}{2\sqrt{B}} + i\pi \rho \sqrt{E} \right]^{-1}
\]
Amplitude, phase shift, and scattering length

The result of low-energy scattering amplitude

\[ t(E) = \left[ \frac{E+B}{g^2} + \frac{\pi \rho (B-E)}{2\sqrt{B}} + i\pi \rho \sqrt{E} \right]^{-1} \]

S-wave phase shift

\[ e^{2i\delta(E)} = 1 - 2i\pi \rho \sqrt{Et(E)} \]

\[ \cot \delta = -\frac{1}{\pi \rho \sqrt{E}} \left[ \frac{E+B}{g^2} + \frac{\pi \rho (B-E)}{2\sqrt{B}} \right] \]

Scattering length \( a_s \), effective range \( r_e \)

\[ k \cot \delta = -\frac{1}{a_s} + r_e \frac{k^2}{2}, \quad E = \frac{k^2}{2\mu}, \quad R = \frac{1}{\sqrt{2\mu B}} \]

We obtain the final result (no expansion needed)

\[ a_s = 2R \left[ 1 + \frac{2\sqrt{B}}{\pi \rho g^2} \right] \quad r_e = R \left[ 1 - \frac{2\sqrt{B}}{\pi \rho g^2} \right] \]
Weinberg’s study of the deuteron structure

Main result: theorem

\[ |\text{deuteron}\rangle = \begin{cases} N & \text{or} \\ N & \end{cases} \]

\[ Z = 0 \quad \text{or} \quad Z = 1 \]

\( Z \): probability of finding deuteron in a bare elementary state

For a bound state with small binding energy, the following equation should be satisfied model independently:

- coupling constant \( \leftrightarrow Z \)

\[ g^2 = \frac{2\sqrt{B}(1 - Z)}{\pi \rho} \]

- scattering length, effective range \( \leftrightarrow Z \)

\[ a_s = \left[ \frac{2(1 - Z)}{2 - Z} \right] R + \mathcal{O}(m^{-1}_\pi), \quad r_e = \left[ \frac{-Z}{1 - Z} \right] R + \mathcal{O}(m^{-1}_\pi) \]
Description of $S = -1$, $\Sigma N$ s-wave scattering: $\Lambda(1405)$ in $l=0$

- Interaction $\leftarrow$ chiral symmetry
  

- Amplitude $\leftarrow$ unitarity in coupled channels
  

\[
T = \frac{1}{1 - VG} V
\]

\[\text{chiral} + \text{cutoff}\]

- Many others:
  

It works successfully, also in $S=0$ sector, meson-meson scattering sectors, systems including heavy quarks, ...
Natural renormalization condition

Conditions for natural renormalization

- Loop function $G$ should be negative below threshold.
- $T$ matches with $V$ at low energy scale.

“$a$” is uniquely determined such that

$$G(\sqrt{s} = M_T) = 0 \iff T(M_T) = V(M_T)$$

Subtraction constant: $a_{\text{natural}}$

We regard this condition as the **exclusion of the CDD pole contribution from $G$**.

- $\Lambda(1405)$ is dominated by meson-baryon structure
- $N(1535)$ requires some additional component


We expect that $Z$ should be zero or small for $a_{\text{natural}}$.

How to check? --> calculate the **coupling constant $g$**
Single-channel problem: $M_T$ and $m$

\[ T = \frac{1}{1 - VG(a)} V \]

\[ V = -\frac{C}{2f^2}(\sqrt{s} - M_T) = \tilde{C}(\sqrt{s} - M_T) \]

2 parameters: $(\tilde{C}, a)$

For the system with a bound state

\[ 1 - VG|_{\sqrt{s}=M_B} = 1 - \tilde{C}(M_B - M_T)G(M_B; a) = 0 \]

relation among $\tilde{C}, a, M_B$

--> bound state can be characterized by $(\tilde{C}, a)$ or $(a, M_B)$

Check the Z factor in natural renormalization scheme from the residue of the pole

\[ g^2(M_B; a) = \lim_{\sqrt{s} \to M_B} (\sqrt{s} - M_B)T(\sqrt{s}) \]
Application to chiral unitary approach

Field renormalization constant

The residue can be calculated analytically:

\[
g^2(M_B; a) = -\frac{M_B - M_T}{G(M_B; a) + (M_B - M_T)G'(M_B)} \quad (a, M_B)
\]

\[
1 - Z = \sqrt{\frac{2mM_T}{(M_T + m)(M_T + m - M_B)}} \frac{M_T}{8\pi M_B} g^2(M_B; a)
\]

valid for small
\[B = (M_T + m) - M_B\]

1) \(a = a_{\text{natural}}, \) vary \(B\)

2) \(B = 10\) MeV, vary \(a\)

natural scheme --> \(Z \sim 0\)

large deviation --> \(Z \sim 1\)
Weinberg’s study of the deuteron structure

Field renormalization constant $Z$: quantitative measure of \textit{compositeness}

For \textbf{small $B$}, $Z$ is related to the coupling constant and scattering observables model independently.

S. Weinberg, Phys. Rev. 137 B672-B678 (1965)
Summary 2

Application to chiral unitary approach

Natural renormalization scheme
exclude CDD pole contribution from
the loop function to generate purely
circle molecule resonance


Residue of the pole --> coupling constant
natural scheme corresponds to $Z \sim 0$
--> composite particle is generated