Feshbach resonances with large background scattering length: Interplay with open-channel resonances


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Coupled channels $(P + Q)$

- Bound state in $Q$ channel $E_Q < 0$
- Above $P$ threshold $E_P > 0$
- Unstable by transition
- (CDD pole contribution)

1) Potential resonance

- Single channel ($P$)
- Bound by potential barrier
- Energy $E > 0$
- Unstable by tunneling
- (Dynamically generated state)

2) Feshbach resonance

- Coupled channels ($P + Q$)
- Bound state in $Q$ channel $E_Q < 0$
- Above $P$ threshold $E_P > 0$
- Unstable by transition
- (CDD pole contribution)
In alkali-metal atoms, (P, Q) are different spin configurations. The magnetic field B modifies the threshold energy difference.

In channel P, the energy of resonance is changed by B. The scattering amplitude in P channel is changed by the magnetic field. The scattering length of P is also changed by the magnetic field.
Introduction -- resonance

Scattering length

Scattering length of P as a function of magnetic field B

\[ a(B) = a_{bg} \left(1 - \frac{\Delta B}{B - B_0}\right) \]

P interaction

bound state in Q

\[ B = B_0 : \text{unitary limit} \]

Interaction strength is adjustable by the magnetic field

(Hidden) assumption: the lower energy P channel is weakly interacting \((a_{bg} \sim r_0)\).

This is not always the case, e.g., \(^{133}\text{Cs}, ^{85}\text{Rb}, ^{6}\text{Li},...\)

What happens if \(|a_{bg}| >> r_0\)?
Feshbach resonance theory: projection formalism

Reduction of two-channel problem into a single channel


P: open channel, lower energy.
Q: closed channel, higher energy.

\[ H |\Psi\rangle = E |\Psi\rangle, \quad H = \begin{pmatrix} H_{PP} & H_{PQ} \\ H_{QP} & H_{QQ} \end{pmatrix}, \quad |\Psi\rangle = \begin{pmatrix} |\Psi_P\rangle \\ |\Psi_Q\rangle \end{pmatrix} \]

(full) Green’s operator in Q channel $\rightarrow |\Psi_Q\rangle$

\[ |\Psi_Q\rangle = \frac{1}{E^+ - H_{QQ}} H_{QP} |\Psi_P\rangle, \quad E^+ = E + i\delta \]

Eliminating $|\Psi_Q\rangle$, we obtain effective Hamiltonian in P

\[ E |\Psi_P\rangle = \left( H_{PP} + H_{PQ} \frac{1}{E^+ - H_{QQ}} H_{QP} \right) |\Psi_P\rangle \]

\[ \equiv H_{\text{eff}} |\Psi_P\rangle \]

original P interaction + coupling effect to channel Q
Feshbach resonance

**T-matrix for effective single channel interaction**

**Structure of the effective interaction**

\[
H_{\text{eff}} = H_{PP}^{0} + V_{PP} + H_{PQ} \frac{1}{E^{+} - H_{QQ}} H_{QP} \equiv H_{PP}^{0} + V_{I} + V_{II}
\]

**t-matrix:** Two-potential theorem
(c.f. DWBA for nuclear reaction)

\[
t = t_{I} + \langle \Psi_{P}^{-} | V_{II} | \Psi_{P} \rangle
\]

\[
= \langle \chi_{P} | V_{PP} | \Psi_{P}^{+} \rangle + \langle \Psi_{P}^{-} | H_{PQ} \frac{1}{E^{+} - H_{QQ}} H_{QP} | \Psi_{P} \rangle
\]

- \( | \chi_{P} \rangle \): free P state
- \( | \Psi_{P}^{+} \rangle \): full P state with \( V_{PP} \)
- \( | \Psi_{P} \rangle \): full P state with \( V_{\text{eff}} \)

\[
| \Psi_{P}^{\pm} \rangle = | \chi_{P} \rangle + \frac{1}{E^{\pm} - H_{PP}} V_{PP} | \chi_{P} \rangle, \quad H_{PP} = H_{PP}^{0} + V_{PP}
\]

\[
| \Psi_{P} \rangle = | \Psi_{P}^{+} \rangle + \frac{1}{E^{+} - H_{PP}} H_{PQ} \frac{1}{E^{+} - H_{QQ}} H_{QP} | \Psi_{P} \rangle
\]
Feshbach resonance

**Single-pole dominance**

Pick up one bound state (relevant for P threshold) from the expansion of Green’s operator of Q channel

\[
\frac{1}{E^+ - H_{QQ}} = \sum_i \frac{\phi_i \langle \phi_i |}{E - \epsilon_i^Q} + \int \frac{\phi(\epsilon) \langle \phi(\epsilon) |}{E^+ - \epsilon} d\epsilon \rightarrow \frac{\phi_b \langle \phi_b |}{E - \epsilon_b^Q}
\]

**t-matrix for single-pole dominance**

\[
t = \langle \chi_P | V_{PP} | \Psi_P^+ \rangle + \frac{\langle \Psi_P^- | H_{PQ} | \phi_b \rangle \langle \phi_b | H_{QP} | \Psi_P \rangle}{E - \epsilon_b^Q}
\]

\[
= \langle \chi_P | V_{PP} | \Psi_P^+ \rangle + \frac{|\langle \phi_b | H_{QP} | \Psi_P^+ \rangle|^2}{E - \epsilon_b^Q - A(E)}
\]

**Coupling to P channel modifies the bare mass (self-energy)**

\[
A(E) = \langle \phi_b | H_{QP} \frac{1}{E^+ - H_{PP}} H_{PQ} | \phi_b \rangle = \Delta_{\text{res}}(E) - \frac{i}{2} \Gamma(E)
\]

\[
\uparrow \quad \text{mass shift} \quad \uparrow \quad \text{width}
\]
Feshbach resonance

**S-matrix (E=k^2)**

\[
S(E) = S_P(E) \left( 1 - 2\pi i \frac{| \langle \phi_b | H_{QP} | \Psi_P^+ \rangle |^2}{E - \epsilon_b^Q - A(E)} \right)
\]

**single P channel S-matrix**

\[
A(E) = \langle \phi_b | H_{QP} \frac{1}{E^+ - H_{PP}} H_{PQ} | \phi_b \rangle = \Delta_{\text{res}}(E) - \frac{i}{2} \Gamma(E)
\]

**Near threshold (E ~ 0)**

\[
\Delta_{\text{res}}(E) \sim \text{const.}, \quad \Gamma(E) = 2\pi | \langle \phi_b | H_{QP} | \Psi_P^+ \rangle |^2 \sim 2Ck,
\]

so the S-matrix is given by

\[
S(k) = \exp[-2ika_{bg}] \left( 1 - \frac{2iCk}{E - \epsilon_b^Q - \Delta_{\text{res}} + iCk} \right)
\]

**Result of the single-resonance approximation**

\[
e^Q_b \propto B \quad \Rightarrow \quad a(B) = a_{bg} \left( 1 - \frac{\Delta B}{B - B_0} \right)
\]
Open channel singularities

For a while, we consider single P channel.

Bound state/virtual state/resonances: pole of S-matrix

bound state (I)

virtual state (II)

pole close to the threshold
--> large scattering length
--> strong energy dependence
--> affect to the Feshbach resonance?
Significance of virtual state in open channel

Previous assumption: P channel has smooth amplitude --> scattering length governs the low energy behavior

\[ S_P(k) = \exp[-2ika_{bg}] \]

If a pole exists near threshold, then

\[ \rightarrow S_{bg}^P(k)S_{res}^P(k) = \exp[-2ika_{bg}^P]S_{res}^P(k) \]

potential resonance, not Feshbach

For a virtual state, it is explicitly written as

\[ S_P(k) = \exp[-2ika_{bg}^P] \frac{i\kappa_{vs} - k}{i\kappa_{vs} + k} \]

Example for \(^{85}\text{Rb} \) (P-channel only)

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Scattering length parameter only does not fully encapsulate the energy dependence of the scattering physics.```

" virtual state"
Open channel resonance

**Mittag-Leffle series**

**Pole in (full) Green’s operator: common with T-matrix**

Expansion of Green’s operator in P channel: Mittag-Leffle series

$$\frac{1}{E - H_{PP}} = \sum_{n=1}^{\infty} \frac{\Omega_n \langle \Omega_n^D \rangle}{2k_n (k - k_n)}$$

Gamow vector and its dual

$k_n$: pole position, arbitrary complex number

- **One virtual state**

$$\frac{1}{E - H_{PP}} \rightarrow \frac{\Omega_{vs} \langle \Omega_{vs}^D \rangle}{2k_{vs} (k - k_{vs})}, \quad k_{vs} = -i\kappa_{vs}$$

- **One virtual state + one bound state**

$$\frac{1}{E - H_{PP}} \rightarrow \frac{\Omega_{vs} \langle \Omega_{vs}^D \rangle}{2k_{vs} (k - k_{vs})} + \frac{\phi_{bs} \langle \phi_{bs} \rangle}{2k_{bs} (k - k_{bs})}, \quad k_{vs} = -i\kappa_{vs}, \quad k_{bs} = i\kappa_{bs}$$
Feshbach resonance with P-channel virtual state

Self-energy function with one virtual state:

\[
A(E) = \frac{\langle \phi_b | H_{QP} | \Omega_{vs} \rangle \langle \Omega_{vs}^D | H_{PQ} | \phi_b \rangle}{2 \kappa_{vs} (k - k_{vs})} = \Delta_{\text{res}}(E) - \frac{i}{2} \Gamma(E)
\]

Numerator: no energy dependence

\[
A(E) = \frac{-i A_{vs}}{2 \kappa_{vs} (k + i \kappa_{vs})}
\]

Mass modification in the presence of a virtual state

\[
\Delta_{\text{res}}(E) = \frac{-A_{vs}/2}{k^2 + \kappa_{vs}^2} = -\frac{A_{vs}}{2 \kappa_{vs}^2} \left(1 - \frac{k^2}{\kappa_{vs}^2} + \ldots\right)
\]

\[
\Gamma(E) = \frac{A_{vs} k}{\kappa_{vs} (k^2 + \kappa_{vs}^2)} = \frac{A_{vs} k}{\kappa_{vs}^3} \left(1 - \frac{k^2}{\kappa_{vs}^2} + \ldots\right)
\]

For \( k \ll \kappa_{vs} \), single-resonance approximation is recovered

a-B relation becomes more complicated

\[
a(B) \neq a_{bg} \left(1 - \frac{\Delta B}{B - B_0}\right)
\]
Mass of the Feshbach resonance of $^{85}$Rb

$$\epsilon_b - \Delta_{\text{res}}(E), \quad \epsilon_b \propto B$$

**Application to physical example**

**red**
- single-resonance app.

**Dotted**
- full coupled-channel result with realistic interaction

**Solid**
- model with 1 virtual and 1 bound state
We study the Feshbach resonance with near threshold singularity in open channel.

- Feshbach resonance: a bound state embedded in a continuum
- Projection method: effective single-channel interaction
- Large $a_{bg}$: open channel singularity
- Open-channel singularity: modifies the linear $B$ dep. of Feshbach resonance

In hadron physics: 

Λ(1405) in KN-πΣ amplitude?

Characteristic feature?

Response to open-channel resonance?