Universal physics of three-bosons with isospin

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Universal physics

*Universal*: different systems share the identical feature

Critical phenomena around phase transition

- large correlation length $\xi$
- scaling, critical exponent, ...
- liquid-gas transition $\sim$ ferromagnet

N. Goldenfeld, “*Lectures on phase transitions and the renormalization group*” (1992)

Universal physics in few-body system

- large two-body scattering length $|a|$
- “scaling”, Efimov effect, ...
- $^4$He atom ($\text{vdW}$) $\sim$ nucleon (strong)

Two-body system

We consider the **low-energy** phenomena \((1/p \gg r_0)\) of the system with **large scattering length** \(|a| \gg r_0\).

\[
f(\theta, p) = \sum_l (2l + 1) f_l(p) P_l(\cos \theta)
\]

\[
\rightarrow f_0(p)
\]

\[
= \frac{1}{p \cot \delta_0(p) - ip}
\]

\[
\rightarrow \frac{1}{-1/a - ip}
\]

**Consequence:** one shallow bound state exists for \(a > 0\)

\[
B_2 = \frac{1}{2 \mu a^2}, \quad \hbar = 1,
\]

- **determined only by** \(a\)

- **scale invariance**

\[
a \rightarrow \lambda a, \quad p \rightarrow \lambda^{-1} p \quad E \rightarrow \lambda^{-2} E
\]

<table>
<thead>
<tr>
<th>(B_2)</th>
<th>(N \text{ [MeV]})</th>
<th>(4\text{He} \text{ [mK]})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.22</td>
<td>2.22</td>
<td>1.31</td>
</tr>
<tr>
<td>(1/2 \mu a^2)</td>
<td>1.41</td>
<td>1.12</td>
</tr>
</tbody>
</table>

Introduction: universal few-body physics
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Three-body system: scaling and its violation

Three-body system in hyperspherical coordinates

\[(r_{12}, r_{3,12}) \leftrightarrow (R, \alpha_3, \hat{r}_{12}, \hat{r}_{3,12})\]

hyperradius hyperangular variables \(\Omega\)
(dimENSIONLESS)

For \(r_0/a \to 0\), system is scale invariant.

\[V(R, \Omega) \propto \frac{1}{R^2}\]

Efimov effect: attractive \(1/R^2\) for identical three bosons


- infinitely many bound states
- discrete scale invariance \(\to\) limit cycle

Experimental realization by ultracold cesium atoms


- tuning \( a \) by magnetic field (Feshbach resonance)

Universal theory \( \iff \) data (three-body recombination rate)
Hadrons with a large scattering length

Hadron systems ($r_0 \sim 1$ fm) with a large scattering length

- nucleon system


\[ 2.2 \text{ MeV} \updownarrow \quad p+n \quad a_{pn} \sim -22 \text{ fm} \; (^1S_0) \]
\[ \sim 5 \text{ fm} \; (^3S_1) \]

\[ \Rightarrow 3\text{H} \]

- charmed meson system ($D \sim c\bar{u}, \; c\bar{d}$)


\[ 0.1-0.5 \text{ MeV} \updownarrow \quad D^0 + \bar{D}^{0*} \quad a_{D^0\bar{D}^{0*}} \sim 6-14 \text{ fm} \]

\[ \Rightarrow \text{not bound} \]

These are the examples of accidental fine tuning.
Is there a “Feshbach resonance”?
Introduction to pion

Yukawa: pion mediates the nuclear force

- pseudoscalar particle
- isospin $I=1$
- lightest hadron (~140 MeV)

Nambu: spontaneous breaking of chiral symmetry

Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I*

Y. Nambu and G. Jona-Lasinio
The Enrico Fermi Institute for Nuclear Studies and the Department of Physics, The University of Chicago, Chicago, Illinois
(Received October 27, 1960)

It is suggested that the nucleon mass arises largely as a self-energy of some primary fermion field through the same mechanism as the appearance of energy gap in the theory of superconductivity. The idea can be put into a mathematical formulation utilizing a generalized Hartree-Fock approximation which regards real nucleons as quasi-particle excitations. We consider a simplified model of nonlinear four-fermion interaction which allows a $\gamma_5$-gauge group. An interesting consequence of the symmetry is that there arise automatically pseudoscalar zero-mass bound states of nucleon-antinucleon pair which may be regarded as an idealized pion. In addition, massive bound states of nucleon number zero and two are predicted in a simple approximation.

The theory contains two parameters which can be explicitly related to observed nucleon mass and the pion-nucleon coupling constant. Some paradoxical aspects of the theory in connection with the $\gamma_5$ transformation are discussed in detail.
Tuning pion interaction

**Pion interaction**

Interaction ← chiral low energy theorem

- **S-wave** $\pi\pi$ scattering length


\[
a^{I=0} \propto -\frac{7}{4} \frac{m_\pi}{f_\pi^2}, \quad a^{I=2} \propto \frac{1}{2} \frac{m_\pi}{f_\pi^2}
\]

attractive repulsive

The scattering lengths are proportional to

- $1/f_\pi^2$ ~ **spontaneous** breaking of chiral symmetry
- $m_\pi$ ~ **explicit** breaking of chiral symmetry

In nature, the scattering lengths are small:

- $a^{I=0} \sim -0.31$ fm, $a^{I=2} \sim 0.06$ fm / **QCD scale** ~ 1 fm

← explicit symmetry breaking is small.
Tuning pion interaction

If we can adjust $m_\pi$ or $f_\pi$, $|a|$ increases by $m_\pi \uparrow$ or $f_\pi \downarrow$

$$a_{I=0} \propto -\frac{7}{4} \frac{m_\pi}{f_\pi^2}, \quad a_{I=2} \propto \frac{1}{2} \frac{m_\pi}{f_\pi^2}$$

Can $|a|$ be extremely large?
- low energy theorem ~ Born approximation
- sufficient attraction --> bound state in $I=0$ --> diverging $|a|$

$\sigma$ meson: resonance in $\pi\pi$ scattering
- scalar particle
- isospin $I=0$
- experimentally established
- chiral partner of $\pi$
Tuning pion interaction

Increase pion mass

Lattice QCD and chiral effective field theory (EFT)

\[ 2m_{\pi} \]

\[ m_{\sigma} \]

\[ \sigma \text{ pole} \]

\[ \text{bound } \sigma \]

\[ \text{heavy pion} \]

\[ \text{large } |a| ! \]

\[ \text{physical} \]

\[ \Rightarrow \text{Numerical experiment (lattice QCD)!} \]
Decrease pion decay constant

Chiral symmetry restoration $\sim$ reduction of $f_\pi$

$m_\sigma = m_\pi < 2m_\pi$

$m_\sigma = 2m_\pi$

$m_\sigma > 2m_\pi$

restoration

large $|a|$!

physical


$\Rightarrow$ Real experiment (in-medium symmetry restoration)!
Isospin symmetric three pions

Large scattering length: zero range theory ($l=0$ interaction)

$$
\mathcal{L} = \sum_{i=1,2,3} \phi_i^\dagger \left( i \partial_t + \frac{\nabla^2}{2m} \right) \phi_i + \nu \left| \sum_{i=1,2,3} \phi_i \phi_i \right|^2 \quad I = 0 \quad \begin{bmatrix}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{bmatrix} \quad I = 0
$$

- two-body amplitude: $l=0$ and $l=2$

$$

it_0(p) = \frac{8\pi}{m} \frac{i}{\frac{1}{a} - \sqrt{\frac{E^2}{4} - mp_0 - i0^+}}, \quad it_2(p) = 0
$$

S-wave three-pion system in total $l=1$

$$

\begin{pmatrix}
| \pi \otimes [\pi \otimes \pi]_{I=0} \rangle_{I=1} \\
| \pi \otimes [\pi \otimes \pi]_{I=2} \rangle_{I=1}
\end{pmatrix} = \begin{pmatrix} 1/3 & \sqrt{5}/3 \\ \sqrt{5}/3 & 1/6 \end{pmatrix} \begin{pmatrix}
| \pi \otimes \pi \rangle_{I=0} \otimes \pi \rangle_{I=1} \\
| \pi \otimes \pi \rangle_{I=2} \otimes \pi \rangle_{I=1}
\end{pmatrix}
$$

\begin{align*}
I = 0 & \quad \begin{pmatrix}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{pmatrix} \\
I = 0 & \quad \begin{pmatrix}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{pmatrix}
\end{align*}

\text{coupled-channel effect}
Isospin symmetric three pions

Three-body scattering equation

\[ iT(E; k, p) = iG(P - k - p) - \int_T \frac{T(E; k, q)t_K(P - q)G(q)G(P - q - p)}{q} \]

Eigenstate: homogeneous equation with pole condition

\[ T_{\text{on}}(E; |k|, |p|) \rightarrow \frac{z^*(|k|)z(|p|)}{E + B_3} \]

Eigenvalue equation (eigenvalue \( B_3 \) for eigenfunction \( z(|p|) \))

\[ z(|p|) = \frac{2}{3\pi} \int_0^\infty d|q| \frac{|q|}{|p|} \ln \left( \frac{q^2 + p^2}{q^2 + p^2 - |q||p| + mB_3} \right) \frac{z(|q|)}{\sqrt{\frac{3}{4}q^2 + mB_3 - \frac{1}{a}}} \]

Factor 1/3 difference from the identical boson case
Result: one **universal** three-pion bound state

\[
B_3 = \frac{1.04391}{ma^2} \quad \text{for } 1/\alpha > 0
\]

c.f. \( B_2 = \frac{1}{ma^2} \)

**Resonances?**

- phase rotation of binding energy = phase rotation of \( a \)

\[
B_3 \rightarrow B_3 e^{i\theta} \iff \frac{1}{\alpha} \rightarrow \frac{1}{\alpha} e^{-i\theta/2}
\]

**Negative \( a \): virtual state**


\[<-- \text{rotation of } B_3 \text{ by } 2\pi = \text{sign flip of } a\]

No resonance for all \( a \)

\[<-- \text{interchange of Riemann sheet = sign flip of } a\]
With isospin breaking

In nature, \( m_{\pi^\pm} = m_{\pi^0} + \Delta \) with \( \Delta > 0 \)

- In the energy region \( E \ll \Delta \), heavy \( \pi^\pm \) can be neglected.

Identical three-boson system with a large scattering length

\[ z(|p|) = \frac{2}{\pi} \int_0^\infty d|q| \frac{|q|}{|p|} \ln \left( \frac{q^2 + p^2 + |q||p| + mB_3}{q^2 + p^2 - |q||p| + mB_3} \right) \]

\[ \times \frac{z(|q|)}{\sqrt{\frac{3}{4} q^2 + mB_3 - \frac{1}{a}}} f_\Lambda(|q|) \]

\( \text{cutoff} \sim 1/r_0 \)

Universal physics at \( E \ll (2m\Lambda)^{1/2} \)

\( \leftarrow \) Efimov parameter \( \kappa^* \)
Efimov resonances

Resonance solution is now possible.

- phase rotation of binding energy = phase rotation of $a$ and $\Lambda +$ proper treatment of singularity in $f_{\Lambda}(|q|)$

$$B_3 \rightarrow B_3 e^{i\theta} \iff \frac{1}{a} \rightarrow \frac{1}{a} e^{-i\theta/2} \quad \text{and} \quad \Lambda \rightarrow \Lambda e^{-i\theta/2}$$

Efimov bound state $\rightarrow$ resonance
Coupled-channel effect

Two universal phenomena: existence of the coupled channel

\[ z(|p|) = \frac{2}{\lambda \pi} \int_{0}^{\infty} d|q| \frac{|q|}{|p|} \ln \left( \frac{q^2 + p^2 + |q||p| + mB_3}{q^2 + p^2 - |q||p| + mB_3} \right) \frac{z(|q|)}{\sqrt{\frac{3}{4} q^2 + mB_3 - \frac{1}{a}}} \]

\[ \lambda < 2.41480 \quad 2.41480 < \lambda < 3.66811 \quad 3.66811 < \lambda \]

no universal bound state

discrete scale invariance

scale invariance

Both can be realized in three-pion systems.
Interpolation by model

A model with finite mass difference $\Delta = m_\pm - m_0$

$$\mathcal{L} = \sum_{i=0,\pm} \pi_i^\dagger \left( i \partial_t + \frac{\nabla^2}{2m_i} - m_i \right) \pi_i + \frac{g}{4} \frac{\pi_0^\dagger \pi_0^\dagger - 2\pi_+^\dagger \pi_-^\dagger}{\sqrt{3}} \frac{\pi_0 \pi_0 - 2\pi_- \pi_+}{\sqrt{3}}$$

- $E \ll \Delta$ : Efimov states, ($\Lambda \gg$) $E \gg \Delta$ : single bound state
- cutoff for the Efimov effect is introduced by $\Delta$.

Lowest Efimov level --> universal bound state
Implication in hadron physics

Two-body $\pi\pi$ bound state ($\sigma$) --> at least one bound state in three-body channel with $I=1$ and $J=0$ channel: $\pi^*$

Remnant of universal bound state: $\pi^*(1300)$

$M = 1300 \pm 100$ MeV, $\Gamma = 200-600$ MeV,

$\Gamma(\pi(\pi\pi)_{s\text{-wave}})/\Gamma(\pi\rho) \sim 2.2$

When the $\sigma$ softens, $\pi^*$ also softens simultaneously.

- caveats for the $\sigma$ softening in practice: final state interaction, mixing with quark number fluctuation, ...
Large $\pi\pi$ scattering length ($l=0$) can be obtained by $m_\pi \uparrow$ or $f_\pi \downarrow$.

Universal phenomena with large $a$:
- single bound state (isospin symmetry)
- Efimov states (isospin breaking)

Consequence in hadron physics:
- realization in lattice QCD
- simultaneous softening of $\sigma$ and $\pi^*$