Universal physics of three bosons with isospin

Yukawa Institute for Theoretical Physics, Kyoto Univ.

Tetsuo Hyodo

Yukawa Institute for Theoretical Physics, Kyoto Univ.

2014, Jun. 17th
Universal physics

Universal: different systems share the identical feature

Critical phenomena around phase transition

- large correlation length $\xi$
- scaling, critical exponent, ...
- liquid-gas transition $\sim$ ferromagnet

N. Goldenfeld, “Lectures on phase transitions and the renormalization group” (1992)

Universal physics in few-body system

- large two-body scattering length $|a|$
- shallow bound state $\leftrightarrow a \gg 0$

$$B_2 = \frac{1}{ma^2} \left[ 1 + O \left( \frac{r_s}{a} \right) \right]$$

<table>
<thead>
<tr>
<th></th>
<th>N [MeV]</th>
<th>$^4$He [mK]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_2$</td>
<td>2.22</td>
<td>1.31</td>
</tr>
<tr>
<td>$1/ma^2$</td>
<td>1.41</td>
<td>1.12</td>
</tr>
</tbody>
</table>

Two-body system

We consider the **low-energy phenomena** \(1/p \gg r_0\) of the system with **large scattering length** \(|a| \gg r_0\).

\[
f(\theta, p) = \sum_l (2l + 1) f_l(p) P_l(\cos \theta) \\
\rightarrow f_0(p) \\
= \frac{1}{p \cot \delta_0(p) - ip} \\
\rightarrow \frac{1}{-1/a - ip + r_s p^2/2 + \ldots}
\]

**Consequence:** one shallow bound state exists for \(a \gg 0\)

\[
B_2 = \frac{1}{ma^2} \left[ 1 + \mathcal{O} \left( \frac{r_s}{a} \right) \right]
\]

- determined only by \(a\)
- **scale invariance**

\[
a \rightarrow \lambda a, \quad p \rightarrow \lambda^{-1} p \quad E \rightarrow \lambda^{-2} E
\]
Introduction: universal few-body physics

Three-body system: scaling and its violation

Three-body system in hyperspherical coordinates

\[(r_{12}, r_{3,12}) \leftrightarrow (R, \alpha_3, \hat{r}_{12}, \hat{r}_{3,12})\]

hyperradius hyperangular variables \(\Omega\)
(dimensionless)

For \(|a| \rightarrow \infty\), system is scale invariant.

\[V(R, \Omega) \propto \frac{1}{R^2}\]

Efimov effect: attractive \(1/R^2\) for identical
three bosons


\[B_n^3 / B_{n+1}^3 \approx 22.7^2\]

- infinitely many bound states
- discrete scale invariance \(\rightarrow\) limit cycle

Introduction: universal few-body physics

Experimental realization by ultracold cesium atoms


- tuning $a$ by magnetic field (Feshbach resonance)

Universal theory $\iff$ data (three-body recombination rate)
Hadrons with a large scattering length

Hadron systems ($r_0 \sim 1$ fm) with a large scattering length

- nucleon system


\[ 2.2 \text{ MeV} \uparrow \quad p+n \quad a_{pn} \sim -22 \text{ fm (}^1S_0) \]
\[ \downarrow \quad d \quad \sim 5 \text{ fm (}^3S_1) \]

\[ \Rightarrow ^3\text{H} \]

- charmed meson system ($D^\sim c\bar{u}$, $c\bar{d}$)


\[ 0.1-0.5 \text{ MeV?} \uparrow \quad D^0 + \bar{D}^{0*} \quad a_{D^0\bar{D}^{0*}} \sim 6-14 \text{ fm} \]
\[ \downarrow \quad X(3872) \]

\[ \Rightarrow \text{not bound} \]

These are “accidental fine tuning” of $a$. Is there a tunable $a$ in hadron physics?
Tuning pion interaction

**Pion interaction**

$\pi\pi$ scattering length $\leftarrow$ chiral low energy theorem


$$a^{I=0} \propto -\frac{7}{4} \frac{m_\pi}{f_\pi^2}, \quad a^{I=2} \propto \frac{1}{2} \frac{m_\pi}{f_\pi^2}$$

- $1/f_\pi^2 \sim$ spontaneous breaking of chiral symmetry
- $m_\pi \sim$ explicit breaking of chiral symmetry

In nature, the scattering lengths are small $\leftarrow$ $m_\pi$ is small

- $a^{I=0} \sim -0.31$ fm, $a^{I=2} \sim 0.06$ fm / QCD scale $\sim 1$ fm

If we can adjust $m_\pi$ or $f_\pi$, $|a|$ may be increased by $m_\pi \uparrow$ or $f_\pi \downarrow$

- sufficient attraction
  —> bound state in $I=0$
  —> diverging $|a|$  
- sigma: $I=0$ resonance

$\Gamma = 2 \text{ Im}(\sqrt{s}_{\text{pole}})$.
Tuning pion interaction

Increase pion mass

Lattice QCD/chiral EFT can tune the pion mass


$2m_\pi$

$m_\sigma$

$\sigma$ pole

$\pi$ threshold

$\sigma$ pole

$\sigma$ bound

$\pi$ threshold

$\sigma$ resonance

heavy pion

large $|a|$ !

physical

$\rightarrow$ Numerical experiment (lattice QCD)!
The result in model C is also instructive in comparison with model A and model B. We have discussed the difference of the softening between model A and model B, but it should be noted that the pole moves toward the position of dynamically generated pole at \(\langle \sigma \rangle_0\). For large values of \(\langle \sigma \rangle\), the property of the pole changes from the bare pole origin to the dynamically generated one. This implies that the nature of the resonance becomes the same as those in model A. Comparing model C with model A, we conclude that the softening pattern of dynamically generated sigma meson for small \(\langle \sigma \rangle\) is changing from the CDD pole to the dynamically generated one, as the symmetry is gradually restored.

Actually, the change of the property of the pole near threshold can be further confirmed by tuning pion interaction. The result in model C is also instructive in comparison with model A and model B. We have discussed the difference of the softening between model A and model B, but it should be noted that the pole moves toward the position of dynamically generated pole at \(\langle \sigma \rangle_0\). For large values of \(\langle \sigma \rangle\), the property of the pole changes from the bare pole origin to the dynamically generated one. This implies that the nature of the resonance becomes the same as those in model A. Comparing model C with model A, we conclude that the softening pattern of dynamically generated sigma meson for small \(\langle \sigma \rangle\) is changing from the CDD pole to the dynamically generated one, as the symmetry is gradually restored.


\[ \langle \sigma \rangle = m_\pi < 2m_\pi \]

\[ m_\sigma = 2m_\pi \]

\[ m_\sigma > 2m_\pi \]

\[ f_\pi \]

**Chiral symmetry restoration ~ reduction of \( f_\pi \)**

**Decrease pion decay constant**

\( \langle \sigma \rangle = \langle \sigma \rangle_0 \)

**Tuning pion interaction**

\( \langle \sigma \rangle = \langle \sigma \rangle_0 \)

**softening**

\( f_\pi \)

\( f_\pi \)
Three pions with large scattering length

Large \( I = 0 \) scattering length

\[
f_{I=0} = \frac{1}{-1/a - ip}, \quad f_{I=2} = 0
\]

S-wave three-pion system in total \( |I = 1\)

\[
\left( | \pi \otimes [\pi \otimes \pi]_{I=0} \rangle I=1 \right) = \left( \begin{array}{cc} 1/3 & \sqrt{5}/3 \\ \sqrt{5}/3 & 1/6 \end{array} \right) \left( | [\pi \otimes \pi]_{I=0} \otimes \pi \rangle I=1 \right)
\]

Eigenvalue equation for 3-body system

\[
z(|p|) = \frac{2}{3\pi} \int_0^\infty dq \frac{|q|}{|p|} \ln \left( \frac{q^2 + p^2 + |q||p| + mB_3}{q^2 + p^2 - |q||p| + mB_3} \right)
\]

\[
\times \frac{z(|q|)}{\sqrt{\frac{3}{4}q^2 + mB_3 - \frac{1}{a}}}
\]

\[
B_3 = \frac{1.04391}{ma^2} \quad \text{for } 1/a > 0 \quad \text{c.f. } B_2 = \frac{1}{ma^2}
\]
Three pions with isospin breaking

Isospin breaking: \( m_{\pi^\pm} = m_{\pi^0} + \Delta \) with \( \Delta > 0 \)

- In the energy region \( E \ll \Delta \), heavy \( \pi^\pm \) can be neglected.

Identical three-boson system with a large scattering length → Efimov effect

\[
z(|p|) = \frac{2}{\pi} \int_0^\infty d|q| \frac{|q|}{|p|} \ln \left( \frac{q^2 + p^2 + |q||p| + mB_3}{q^2 + p^2 - |q||p| + mB_3} \right) \\
\times \frac{z(|q|)}{\sqrt{\frac{3}{4}q^2 + mB_3 - \frac{1}{a}}} f_\Lambda(|q|)
\]

**cutoff ~ \( 1/r_0 \)**

Universal physics at \( E \ll (2m\Lambda)^{1/2} \)

← Efimov parameter \( \kappa^* \)
Three pions with large scattering length

**Coupled-channel effect**

Two universal phenomena: existence of the coupled channel

\[
z(|p|) = \frac{2}{\lambda \pi} \int_0^\infty d|q| \frac{|q|}{|p|} \ln \left( \frac{q^2 + p^2 + |q||p| + mB_3}{q^2 + p^2 - |q||p| + mB_3} \right) \frac{z(|q|)}{\sqrt{\frac{3}{4}q^2 + mB_3 - \frac{1}{a}}}
\]

\[
\lambda < 2.41480 \\
2.41480 < \lambda < 3.66811 \\
3.66811 < \lambda
\]

- no universal bound state
- discrete scale invariance
- scale invariance

Both cases can be realized in three-pion systems.
Implication in hadron physics

Realization and consequences

Numerical experiment by lattice QCD: $m_\pi \uparrow$

- Find the quark mass for a shallow $\sigma$ ($\pi\pi$ bound states)
- Look for the three-$\pi$ bound state and measure the mass.

**single bound state**

$$B_3 = 1.04391 \ B_2$$

**Isospin symmetric**

**several bound states**

$$\frac{B_3^n}{B_3^{n+1}} = 515.03 \sim (22.7)^2$$

**Isospin breaking**

Note:

- $I=0 \ \pi\pi$ scattering is very difficult (disconnected graphs).
- Very high mass resolution is required.
- Shallow bound state $\rightarrow$ large volume?
Implication in hadron physics 2

Realization and consequences

In-medium restoration of chiral symmetry : $f_\pi \downarrow$

- $\sigma(I=J=0)$ softening in nuclear medium


- Existence of three-body bound state

$\rightarrow$ When $\sigma$ softens, $\pi^*(I=1, J=0)$ softens simultaneously.

Note:

- $\sigma$ softening is difficult to confirm (final state interaction,...)

T. Hatsuda, R.S. Hayano, Rev. Mod. Phys. 82, 2494 (2010)
Large $\pi\pi$ scattering length ($|l|=0$) can be obtained by $m_\pi \uparrow$ or $f_\pi \downarrow$.

Universal phenomena with large $a$:

- single bound state (isospin symmetric)
- Efimov states (isospin breaking)

Consequence in hadron physics:

- realization in lattice QCD
- simultaneous softening of $\sigma$ and $\pi^*$