Hadron mass scaling near an s-wave threshold

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Introduction

Hadron mass scaling and threshold effect

Systematic expansion of hadron masses

- ChPT: light quark mass $m_q$
- HQET: heavy quark mass $m_Q$
- large $N_c$: number of colors $N_c$

What happens at two-body threshold?

Hadron mass scaling

$m_H(x); \quad x = \frac{m_q}{\Lambda}, \frac{\Lambda}{m_Q}, \frac{1}{N_c}$
Near-threshold behavior

Coupled-channel Hamiltonian (bare state + continuum)

\[
\begin{pmatrix}
M_0 & \hat{V} \\
\hat{V} & \frac{p^2}{2\mu}
\end{pmatrix}
\begin{pmatrix}
(c(E) | \psi_0 \rangle \\
\chi_E(p) | p \rangle
\end{pmatrix}
= E \begin{pmatrix}
(c(E) | \psi_0 \rangle \\
\chi_E(p) | p \rangle
\end{pmatrix}
\]

Equivalent single-channel scattering formulation

\[\hat{V}_{\text{eff}}(E) = \frac{\hat{V} | \psi_0 \rangle \langle \psi_0 | \hat{V}}{E - M_0} \sim \]

\[f(p, p', E) = -\frac{4\pi^2 \mu \langle p | \hat{V} | \psi_0 \rangle \langle \psi_0 | \hat{V} | p' \rangle}{E - M_0 - \Sigma(E)} \sim \]

Pole condition:

\[E_h - M_0 = \Sigma(E_h)\]

Question: How \(E_h\) behaves against \(M_0\) around \(E_h=0\)?
Near-threshold behavior

Near-threshold bound state

Bound state condition around $E_h=0$

$$E_h + \Sigma(0) - \delta M = \Sigma(E_h)$$

Leading contribution of the expansion:

$$E_h = \frac{1}{1 - \Sigma'(0)} \delta M = Z(0) \delta M, \quad \Sigma'(E) \equiv \frac{d\Sigma(E)}{dE}$$

Field renormalization constant

$$\left| \left\langle \Psi \left| \left( \begin{array}{c} \psi_0 \\ 0 \end{array} \right) \right\rangle \right|^2 + \int \left| \left\langle \Psi \left| \left( \begin{array}{c} 0 \\ q \end{array} \right) \right\rangle \right|^2 d^3q = 1$$

$Z(0)$ vanishes for $l=0$: compositeness theorem (shown later)

$$E_h \propto \begin{cases} \mathcal{O}(\delta M^2) & l = 0 \\ \delta M & l \neq 0 \end{cases}$$
Near-threshold bound state (general)

General argument by Jost function (Fredholm determinant)


\[ f_l(p) = \frac{J_l(-p) - J_l(p)}{2ipJ_l(p)} \]

- amplitude pole (eigenstate): Jost function zero

If there is a zero at \( p=0 \), then the expansion is

\[ J_l(p) = \begin{cases} i\gamma_0 p + O(p^2) & l = 0 \\ \beta_l p^2 + O(p^3) & l \neq 0 \end{cases} \]

- \( \gamma_0 \) and \( \beta_l \) are nonzero for a general local potential
  : simple (double) zero for \( l = 0 \) (\( l \neq 0 \))


Near-threshold scaling:

\[ E_h \propto \begin{cases} -\delta M^2 & l = 0 \\ \delta M & l \neq 0 \end{cases}, \quad \delta M < 0 \]
Near-threshold behavior

General threshold behavior

Near threshold scaling:

- $\delta M < 0$
  
  \[
  E_h \propto \begin{cases} 
  -\delta M^2 & l = 0 \\
  \delta M & l \neq 0 
  \end{cases}
  \]

- $\delta M > 0$
  
  \[
  E_h \propto -\delta M^2 \quad l = 0 \\
  \begin{cases}
  \text{Re } E_h \propto \delta M \\
  \text{Im } E_h \propto - (\delta M)^l + \frac{1}{2} 
  \end{cases} \quad l \neq 0
  \]

Numerical calculation

\[
\langle q | \hat{V} | \psi_0 \rangle = g_l |q|^l \Theta (\Lambda - |q|)
\]
Compositeness theorem

Theorem: $Z(0) = 0$ for s wave

If the s-wave scattering amplitude has a pole exactly at the threshold with a finite range interaction, then the field renormalization constant vanishes.


For bare state-continuum model ($c$: nonzero constant)

$$Z(B) = \frac{1}{1 - \Sigma'(-B)} \approx \frac{1}{1 - c \frac{g_0^2}{\sqrt{B}}}$$

$Z(0)$ vanishes for $g_0 \neq 0$. If $g_0 = 0$, no pole in the amplitude.

For a local potential: poles in the effective range expansion

$$p_1 = i \sqrt{2 \mu B}, \quad p_2 = -i \sqrt{2 \mu B} \frac{2 - Z(B)}{Z(B)}$$

If $Z(0) \neq 0$, then both $p_1$ and $p_2$ go to zero for $B \to 0$: contradiction with the simple pole at $p=0$. 

\[\operatorname{Im} \Sigma(p^2/2\mu) \propto p^{2l+1}\]
Interpretation of the compositeness theorem

\( Z(B): \) overlap of the bound state with bare state

\[
\left| \langle \Psi | \left( \begin{array}{c} \psi_0 \\ 0 \end{array} \right) \rangle \right|^2 + \int \left| \langle \Psi | \left( \begin{array}{c} 0 \\ q \end{array} \right) \rangle \right|^2 d^3q = 1
\]

- \( Z(B\neq0)=0 \rightarrow \) Bound state is completely composite.

Two-body wave function at \( E=0: \)

\( u_{l,E=0}(r) \xrightarrow{r \to \infty} r^{-l} \)

\( Z(0)=0: \) Bound state is completely composite.

Composite component is infinitely large so that the fraction of any finite admixture of bare state is zero.
We study the hadron mass scaling near threshold.

**General scaling laws:**

![Graph showing energy levels and states]

**Compositeness theorem:**

\[ Z(B = 0) = 0 \quad \text{for} \quad l = 0 \]

**Chiral extrapolation across the s-wave threshold should be carefully performed.**