Towards black hole interior by Magic of Chaos

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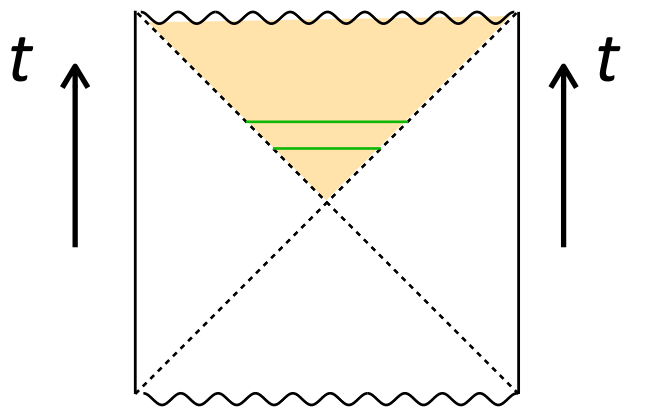
based on [Kanato Goto, TN, Masahiro Nozaki, 2112.14593]

Seminar at entanglement meeting, 28 July 2022

Motivation

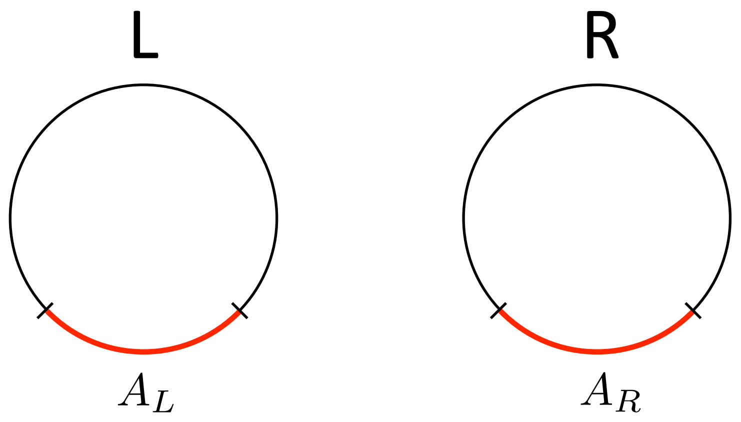
- How can we see (volume of) BH interior by AdS/CFT?

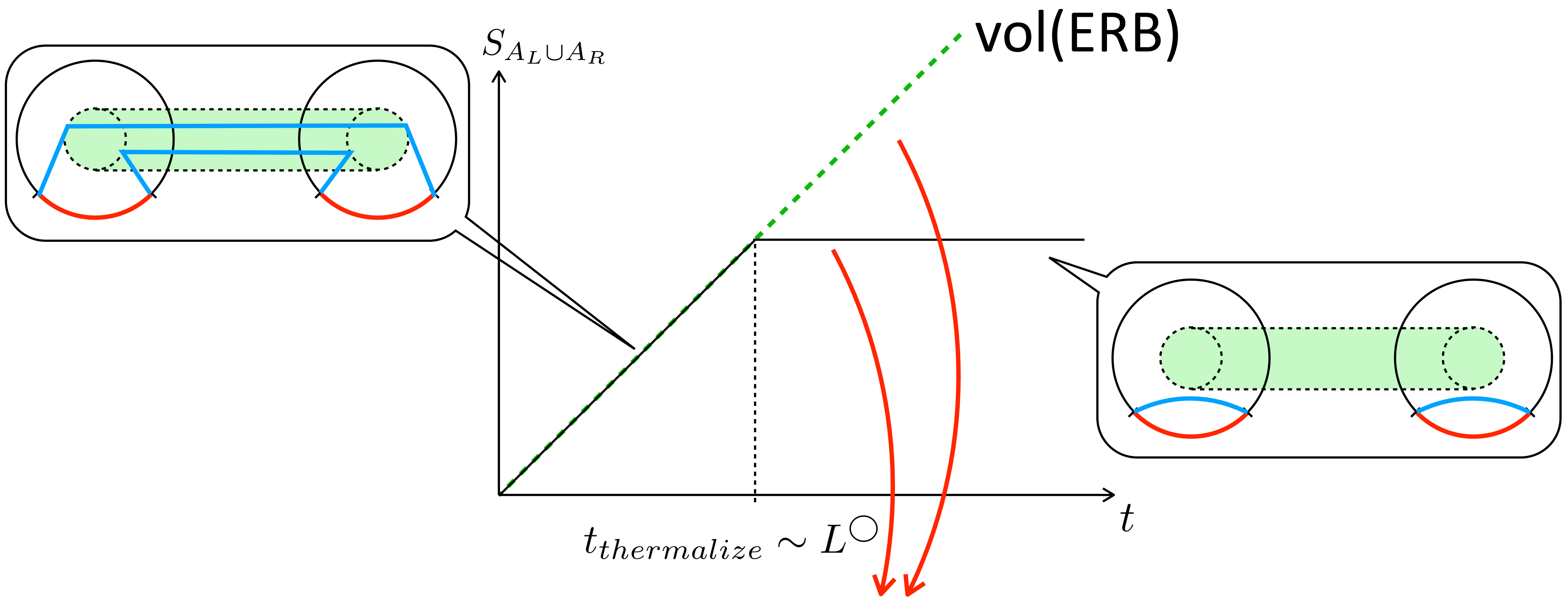


- Entanglement entropy? [Hartman,Maldacena,1303.1080]

$$|\Psi(t)\rangle = e^{-iHt}\sum_n e^{-\beta E_n}|n\rangle_L \otimes |n\rangle_R$$

⇒ $S_{A_L\cup A_R}(|\Psi(t)\rangle\langle\Psi(t)|)$





entanglement is not enough ⇒ Computational complexity?

[Susskind,1411.0690]

Complexity

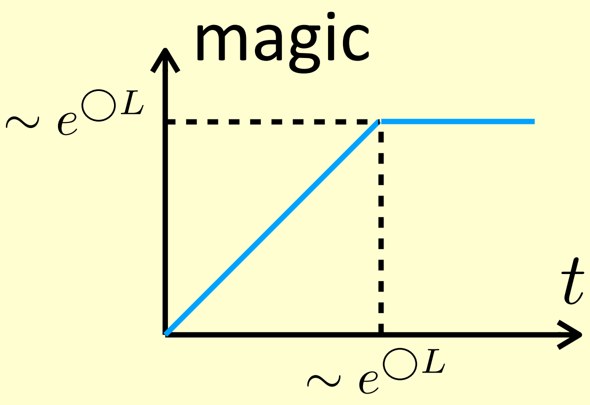
Computational complexity

- number of simple operations to reach $|\Psi(t)\rangle$
- For *L* qubits: 1- or 2-qubit unitary trsf. → max complexity: $\mathcal{O}(e^{\bigcirc L})$ $(\gg S_{EE} \sim L^{\bigcirc})$
- Complexity grows linearly in *t* until $t_{sat} \sim e^{\bigcirc L} \gg t_{therm.}$, same as vol(ERB)!

Question: Is there other quantity which shows the same long-time growth?

Magic

- A different "complexity" which counts only operations difficult to simulate in classical computer.
- We observed that magic in chaotic spin chain evolves as



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magic might also capture BH interior

Plan

√ 1. Introduction

2. Stabilizer formalism

3. Resource theory & magic monotones

4. Growth of magic in chaotic Ising model

5. Summary

Clifford group

Classical computer: states = discrete set $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

operators = discrete permutations $g \in S_{2^L}$

Quantum computer: $g \in U(2^L)$ ➡ need $2^L \times 2^L$ complex numbers to describe

If we restrict quantum operations to Clifford group Cl, *g* can be described much easier

$$Cl = \{g \in U(2^L) | gPg^\dagger = P\} \qquad P = \langle X_i, Z_i \rangle$$

: Pauli group

$$X_i = \begin{array}{c} \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_i \end{array}, \quad Z_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_i$$

- Cl is a finite group generated by $H_i, S_i, \mathrm{CNOT}_{ij}$ [Gottesman,9807006]

- $g \in Cl$ is permutation in $S_{|P|}$: characterized uniquely only by 2*N* images of X_i, Z_i

$H_i = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$X_i \to Z_i \quad Z_i \to X_i$
$S_i = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	$X_i \to Y_i$
$\mathrm{CNOT}_{ij} : \lvert s_i, s_j\rangle \to \lvert s_i, (s_j + s_i \,\mathrm{mod} 2)\rangle$	$X_i \to X_i X_j \quad Z_j \to Z_i Z_j$

➡ *g* can be identified with a classical operator

Stabilizer states

Define stabilizer pure states *St* as $St = \{g|0\cdots 0\rangle | g \in Cl\}$

$|\alpha\rangle \in St$: characterized as the simultaneous eigenstate of $gZ_ig^\dagger$ with $(gZ_ig^\dagger)|S\rangle = |S\rangle$

$|St| \sim 2^{\frac{L^2}{2}}$

Action of $g \in Cl$ on *St* can be regarded as a permutation.

↓

Gottesman-Knill's theorem

Quantum computer which consists only of Clifford gates, projection measurement with Pauli operators, state preparation with St can be efficiently simulated with classical computer (of $\mathcal{O}(L^2)$ bits)

Universal quantum computation and "magic"

If we add the *T*-gate to Clifford group, product of the elements do not close at finite order, and can approximate any element of $U(2^L)$ in infinitely high precision.

(=universal)

$$T=\begin{pmatrix}1 & 0\\ 0 & e^{\frac{\pi i}{4}}\end{pmatrix} \qquad \begin{aligned} X_i &\to \frac{1}{\sqrt{2}}(X_i+Y_i)\\ Z_i &\to Z_i \end{aligned} \qquad (T\notin Cl)$$

Let us consider a new notion of complexity of a state $|\psi\rangle\notin St$ which counts the smallest number of *T*-gates to obtain $|\psi\rangle$ from $|0\cdots0\rangle$, which is called "magic".

In order to evaluate magic concretely for any given state, we want to define it in a different way.

➡ use the idea of resource theory

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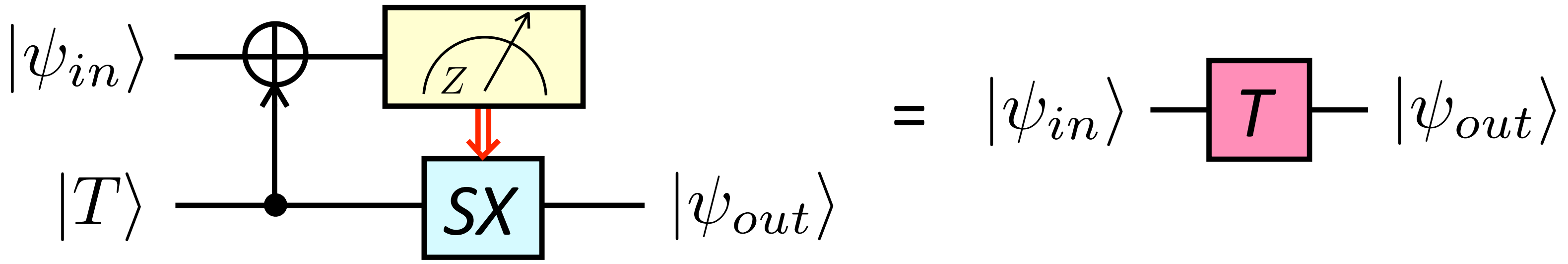
Non-stabilizer state is a "resource" of quantum computation

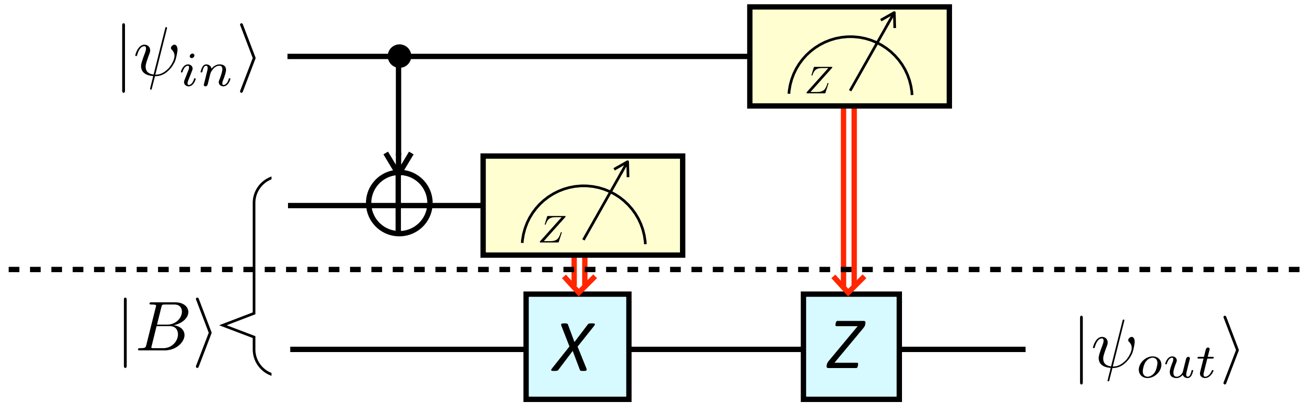
Suppose we are only allowed to act with Clifford gates and Pauli measurement.

If we have $|T\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{\frac{\pi i}{4}}|1\rangle)(\notin St)$, by consuming it we can run "*T*-circuit" which acts on $|\psi_{in}\rangle$ as a *T* gate

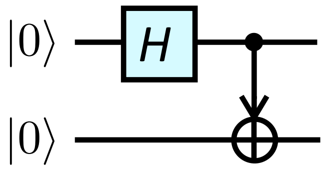


[Zhou,Leung,Chuang,0002039]

c.f. We can run teleportation circuit written only with LOCC by consuming a Bell pair



$$\left(|B\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \right)$$



Resource theory

Resource theory: a way to characterize quantum states which consists of

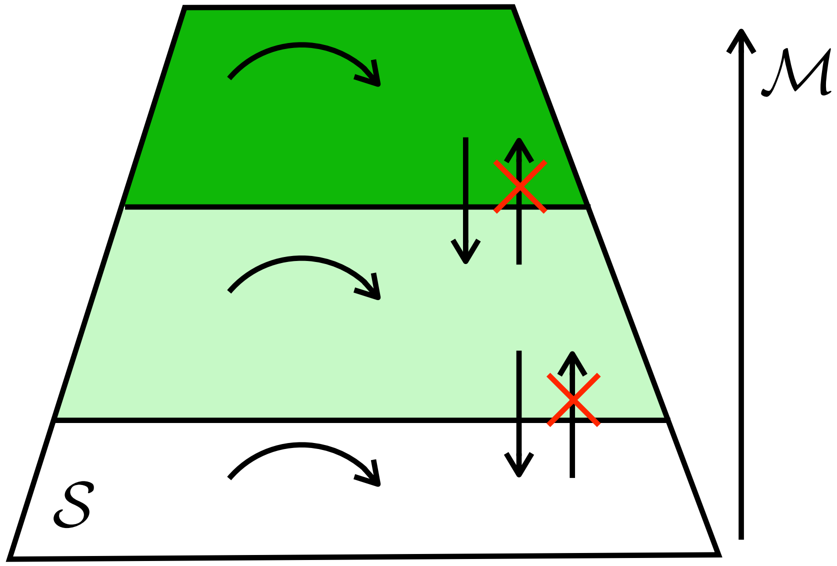
- $\mathcal{C}$ (free operations): subset of quantum operations ($U(2^L)$), measurement, preparation)
- $\mathcal{S}$ (free states): set of quantum states such that $\mathcal{C}|\psi\rangle = \mathcal{S}$ for any $|\psi\rangle \in \mathcal{S}$
- $\mathcal{M}$ (monotone): a quantity which is zero iff $|\psi\rangle \in \mathcal{S}$ and does not increase under $\mathcal{C}$

[Horodecki,Oppenheim,1209.2162]

One cannot obtain $|\psi_{target}\rangle$ by $\mathcal{C}$ from $|\psi_{resource}\rangle$ if $\mathcal{M}(|\psi_{target}\rangle) > \mathcal{M}(|\psi_{resource}\rangle)$

⬇

For those who only allowed to do $\mathcal{C}$, the states with high $\mathcal{M}$ are *resources to be consumed* to perform quantum computation.



Idea of resource theory

Motivated by an intuitive (but not so useful) notion of some property, define more useful quantity $\mathcal{M}$ to measure the same property.

Example

Resource theory of quantum communication (entanglement)

- free operations $\mathcal{C}$ = LOCC
- free states $\mathcal{S}$: tensor product states
- monotones $\mathcal{M}$: entanglement entropy, entanglement negativity, etc.

➡ We can use EE and negativity to measure the entanglement of a state instead of "counting Bell pairs"

Resource theory of quantum computation (magic)

$\mathcal{C}$ = Clifford gates + Pauli projection measurement + stabilizer state preparation

$\mathcal{S}$ = $\mathrm{STAB} = \left\{ \sum_{|\alpha_i\rangle \in St} a_i |\alpha_i\rangle\langle\alpha_i| \,\middle|\, a_i \geq 0 \right\}$: classical mixtures of stabilizer pure states

If there is a quantity which satisfies the requirements of *magic monotone*, we can use it to quantify the magic of a state instead of "counting minimum number of *T*-gates".

Examples of Magic Monotone

- Robustness of magic

$$\boxed{\mathrm{RoM}(\rho) = \inf_{a_i}\Big\{ \sum_{i\,(a_i<0)} |a_i| \,\Big|\, \rho = \sum_{|\alpha_i\rangle\in St} a_i |\alpha_i\rangle\langle\alpha_i| \Big\}}$$

[Howard,Campbell,1609.07488]

  - RoM quantifies how much *ρ* goes out of STAB $(a_i \geq 0)$
  - $\{|\alpha\rangle\langle\alpha\}_{|\alpha\rangle\in St}$ is overcomplete basis: $|St| \sim d^{\frac{L^2}{2}} \gg d^{2L}$ ⇒ linear optimization

- Relative entropy of magic

$$\boxed{r_M(\rho) = \min_{\sigma\in\mathrm{STAB}} (\mathrm{Tr}\rho\log\rho - \mathrm{Tr}\rho\log\sigma)}$$

[Veitch,Mousavian,Gottesman,Emerson,1307.7171]

*Optimization makes the computation very hard for a large system size.

- If we consider only pure states, we can define a faithful monotone without optimization.

Stabilizer Renyi entropy $M_2(|\psi\rangle)$

$$\boxed{M_2(|\psi\rangle) = -\log\left[\frac{1}{2^L}\sum_{p\in\widetilde{P}}|\langle\psi|p|\psi\rangle|^4\right]}$$

$$\widetilde{P} = \{1, X, Y, Z\}^{\otimes L}$$

[Leone,Oliviero,Hamma,2106.12587]

$M_2(|\psi\rangle)$ satisfies requirements for monotone:

- $M_2(|\psi\rangle) = M_2(U|\psi\rangle) \quad (U \in Cl)$
- Since $|\langle\psi|p|\psi\rangle| \leq 1$, $\sum_{p\in\widetilde{P}}|\langle\psi|p|\psi\rangle|^4 \leq \sum_{p\in\widetilde{P}}|\langle\psi|p|\psi\rangle|^2 = 2^L$ ➡ $M_2(|\psi\rangle) \geq 0$

"=" iff $|\langle\psi|p|\psi\rangle| = 1$ for some 2^L *p*'s

"=" iff $|\psi\rangle \in St$

Other properties of $M_2(|\psi\rangle)$:

- Bounded as $M_2(|\psi\rangle) \leq \log\left[\frac{2^L+1}{2}\right]$
- Haar random average: $M_{2,\text{Haar}} = -\log\left[\int_{U(2^L)} dU e^{-M_2(U|\psi\rangle)}\right] = \log\left[\frac{2^L+3}{4}\right]$

Mana $M(|\psi\rangle)$ (qu*d*it with *d* ≥ 3)

- Generalized Pauli group: $P = \langle Z_i, X_i \rangle$

$$Z = \sum_{s=0}^{d-1} \omega^s |s\rangle\langle s| \quad X = \sum_{s=0}^{d-1} |(s+1 \bmod d)\rangle\langle s| \quad \omega = e^{\frac{2\pi i}{d}}$$ (*d*: prime)

- Clifford group: $Cl = \langle H_i, S_i, Q_i, \text{C-SUM}_{ij} \rangle$

$H_i : \vert s\rangle \to \sum_{s'=0}^{d-1} \omega^{ss'} \vert s'\rangle$	$X_i \to Z_i,\ Z_i \to X_i^{-1}$
$S_i : \vert s\rangle \to \omega^{\frac{s(s-1)}{2}} \vert s\rangle$	$X_i \to X_i Z_i$
SUM_{ij} : $\vert s_i, s_j\rangle \to \vert s_i, (s_j + s_i \bmod d)\rangle$	$X_i \to X_i X_j,$ $Z_j \to Z_i^{-1} Z_j$
$Q_i : \vert s\rangle \to \vert 2s\rangle$	$X_i \to X_i^2,\ Z_i \to Z_i^{\frac{d+1}{2}}$

[Gottesman,quant-ph/9802007]

Define Mana as

$$\boxed{M(|\psi\rangle) = \log\left[\frac{1}{d^L} \sum_{\vec{a}} |\langle\psi|A_{\vec{a}}|\psi\rangle|\right]}$$

$$A_{\vec{a}} = d^{-L} T_{\vec{a}} \sum_{\vec{b}} T_{\vec{b}} T_{\vec{a}}^\dagger$$

$$T_{\vec{a}} = \otimes_{i=1}^L \omega^{-\frac{(d+1)a_{2i-1}a_{2i}}{2}} Z_i^{a_{2i-1}} X_i^{a_{2i}}$$

- $A_{\vec{a}}$ are orthonormal basis, hence $\sum_{\vec{a}} \langle\psi|A_{\vec{a}}|\psi\rangle = d^L$
- $\langle\psi|A_{\vec{a}}|\psi\rangle \geq 0$ for all $\vec{a}$ $\Longleftrightarrow$ $|\psi\rangle \in St$ ($\Rightarrow$ not true for *d* = 2)

⇨ $M(|\psi\rangle)$: monotone

[Veitch,et.al.,1307.7171]

Other properties:

- By Jensen's inequality for convex functions, $M(|\psi\rangle) \leq \dfrac{L \log d}{2}$ (not optimal)

optimal bound: $M_{d=3,L=1} \leq M(\rho) \leq \log(5/3), \quad M_{d=5,L=1} \leq \text{Arcsinh}(3+\sqrt{5}) - \log 5$

- Haar random average for *d* >> 1: $M_{\text{Haar}} \approx \dfrac{L \log(d\sqrt{\pi/2})}{2}$

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Higher spin generalized Ising model

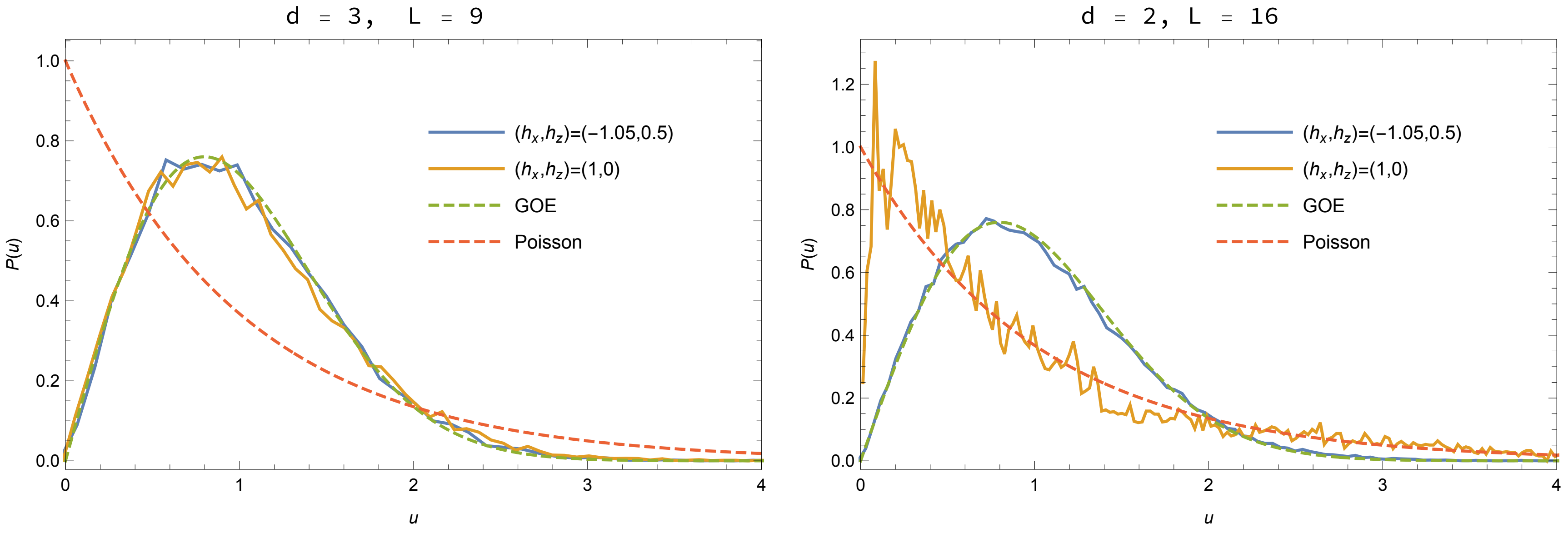
$$\boxed{H = -\sum_{i=1}^{L-1} G_z^{(i)} G_z^{(i+1)} - \sum_{i=1}^{L} (h_x H_x^{(i)} + h_z G_z^{(i)})}$$

$G_\mu^{(i)}$: SU(2) generators with $J = \frac{d-1}{2}$

- integrable for $(0, h_z)$ (if *d* = 2, also for $(h_x, 0)$)

- chaotic for $(h_x, h_z) \approx (-1.05, 0.5)$

level statistics (NNSD):



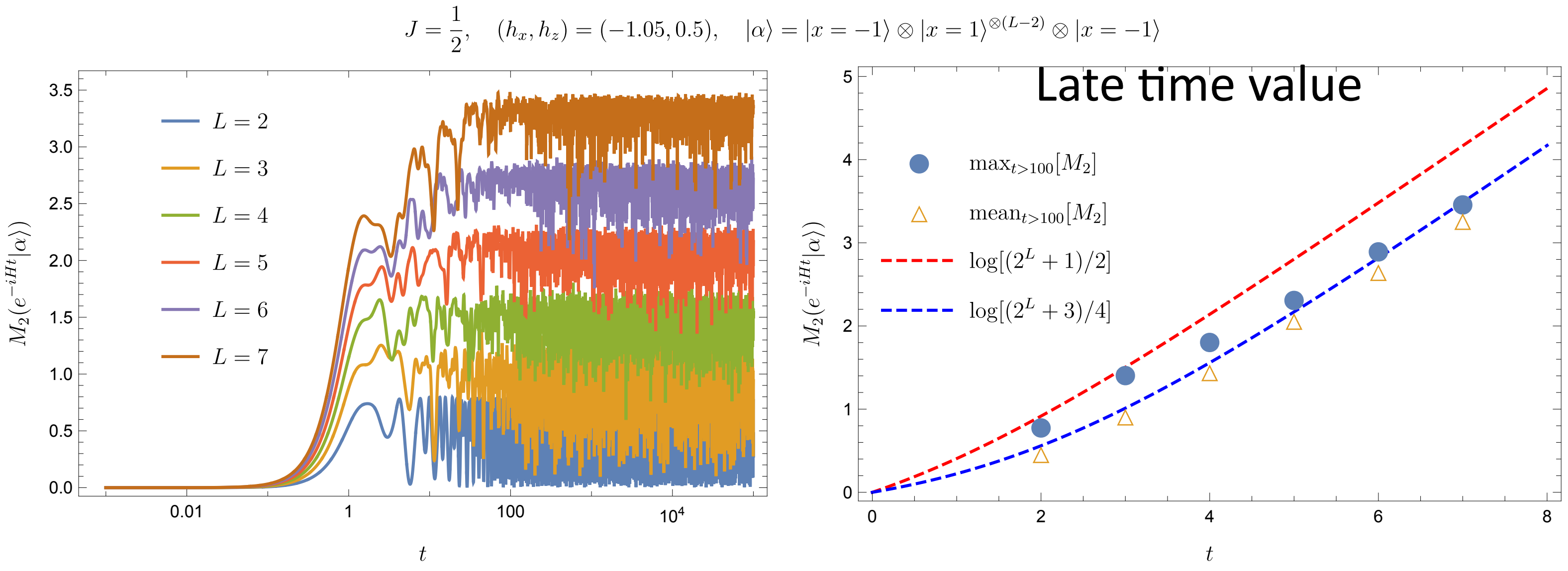
We compute Mana $M(|\psi\rangle)$ and stabilizer Renyi entropy $M_2(|\psi\rangle)$ with

$$|\psi\rangle = e^{-iHt}|\alpha\rangle \quad (|\alpha\rangle \in St)$$

Stabilizer Renyi entropy $M_2(|\psi\rangle)$ (*d*=2)

$$|\psi\rangle = e^{-iHt}|\alpha\rangle, \quad |\alpha\rangle = |x=-1\rangle \otimes |x=1\rangle^{\otimes(L-2)} \otimes |x=-1\rangle \in St$$

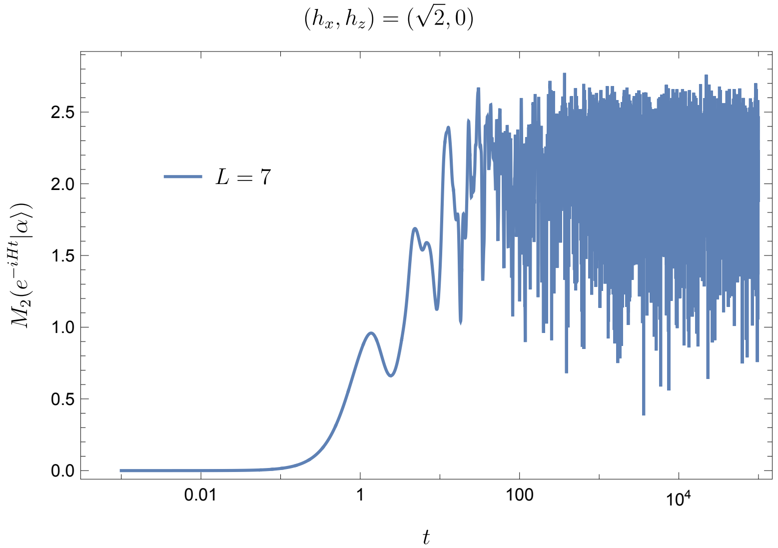
chaotic: $M_2(|\psi\rangle)$ increases monotonically at early time, and saturates at late time with

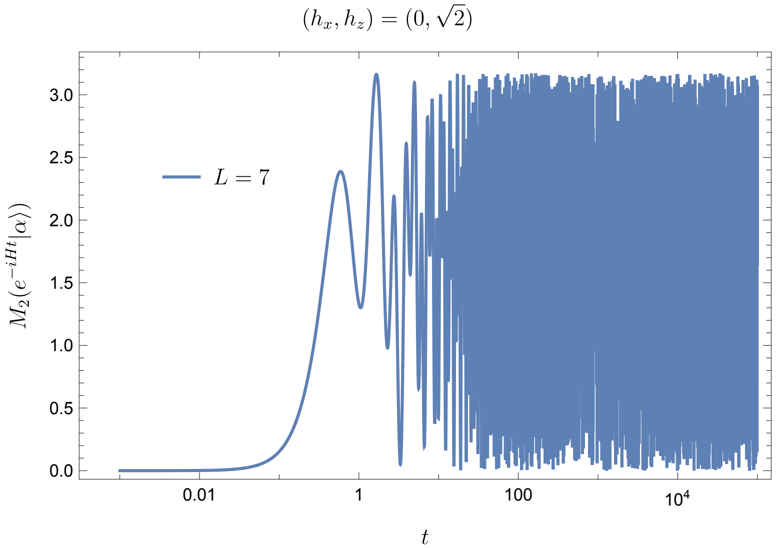


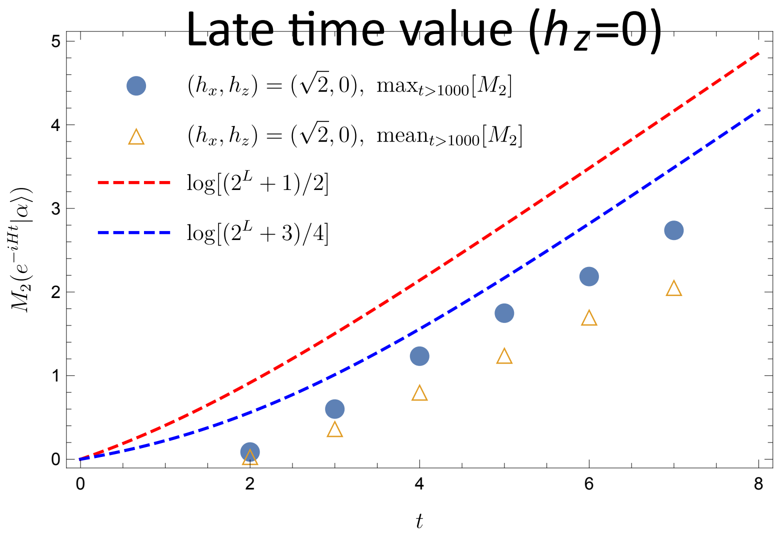
$$M_2(|\psi\rangle) \approx M_{\mathrm{Haar}} = \log\left[\frac{2^L+3}{4}\right]$$

integrable ($h_z=0$): larger oscillation, smaller late time value

integrable ($h_x=0$): $M_2(|\psi\rangle)$ comes back to $M_2 \approx 0$ even at late time.







Estimation of saturation time - 1

In order to capture BH interior better than entanglement, $M_2(e^{-iHt}|\alpha\rangle)$ should not saturate at $t_{thermalize} \sim L$.

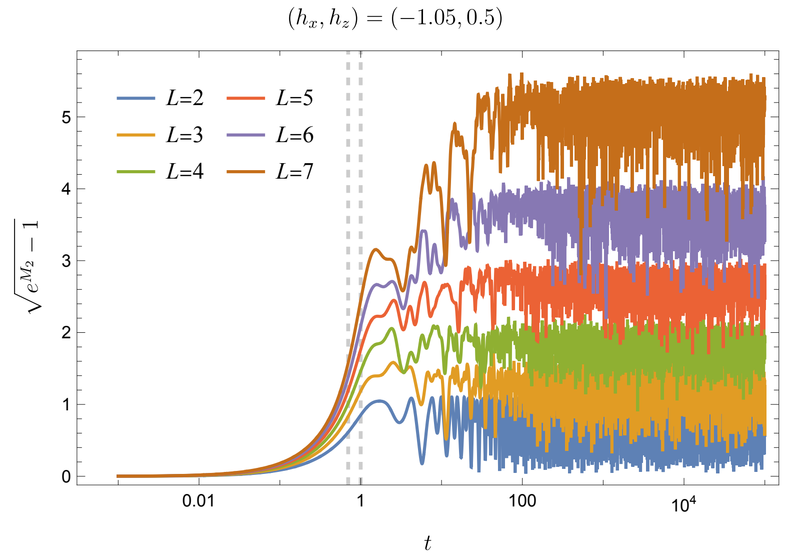
Observations:

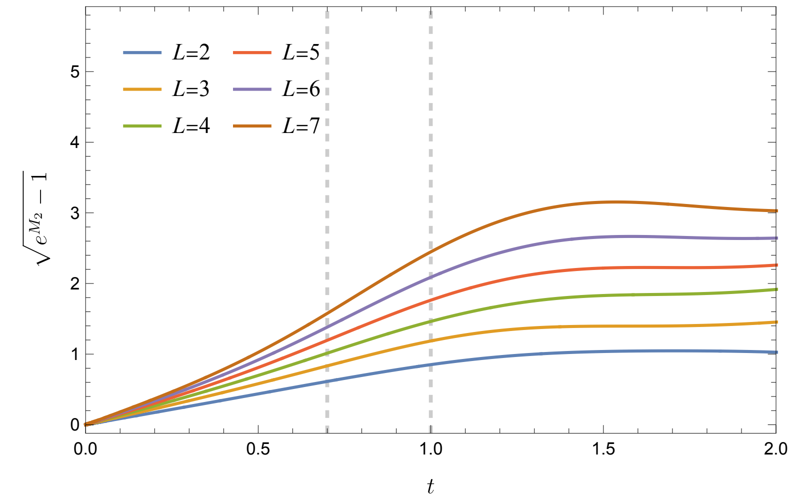
① M_2 saturates at late time with $M_2 \sim L$

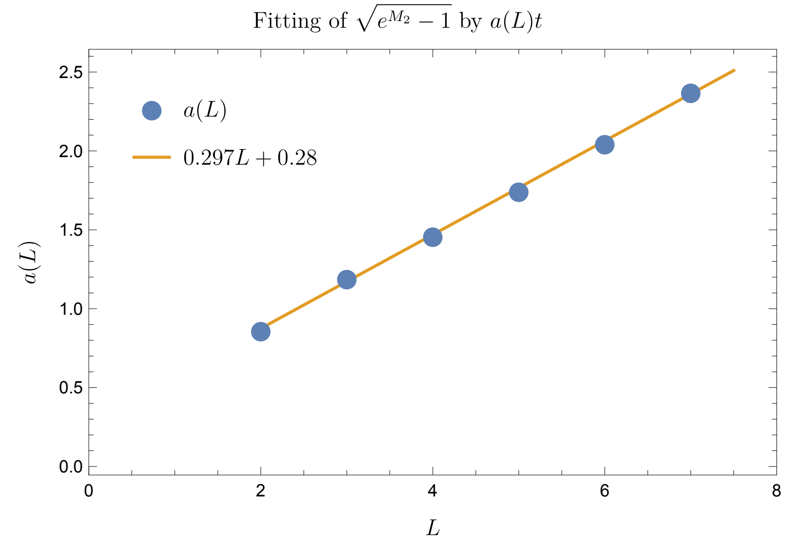
② $\sqrt{e^{M_2}-1}$ grows linearly in *t* at early time

③ growth rate $a(L)$ can be fit well with polynomial of *L*, rather than $e^{\bigcirc L}$

$$t_{saturation} \sim \frac{\sqrt{e^{M_2(\text{late time mean})}-1}}{a(L)} \sim \frac{2^{L/2}}{L} \gg L^{\bigcirc}$$ in large *L* limit



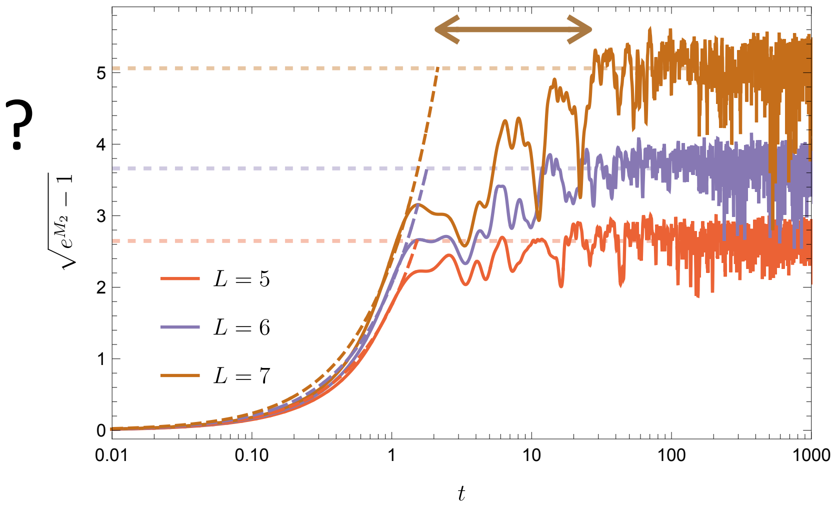




Estimation of saturation time - 2

- Time evolution of $\sqrt{e^{M_2(e^{-iHt}|\alpha\rangle)}-1}$ slow down before saturation

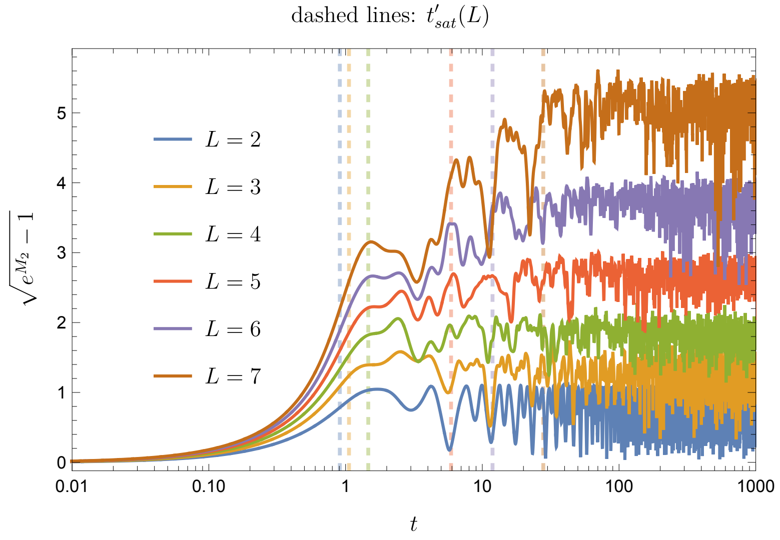
➡ $t_{saturation} \sim \frac{\sqrt{e^{M_2(\text{late time max})}-1}}{a(L)} <$ actual saturation time?

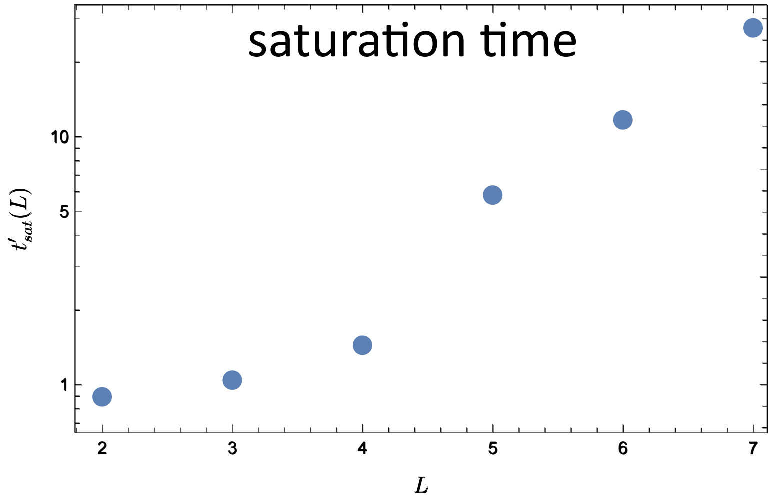


(dashed line: fitting with *a*(*L*)*t*)

- Defining t'_{sat} as the time when $\sqrt{e^{M_2}-1}$ reaches the late time mean value, we again observe $t'_{sat}(L) \sim e^{\bigcirc L}$.

- t'_{sat} is dominated by the "slow" regime (significant only for *L* ≥ 5).





Results for Mana $M(|\psi\rangle)$ (*d*=3)

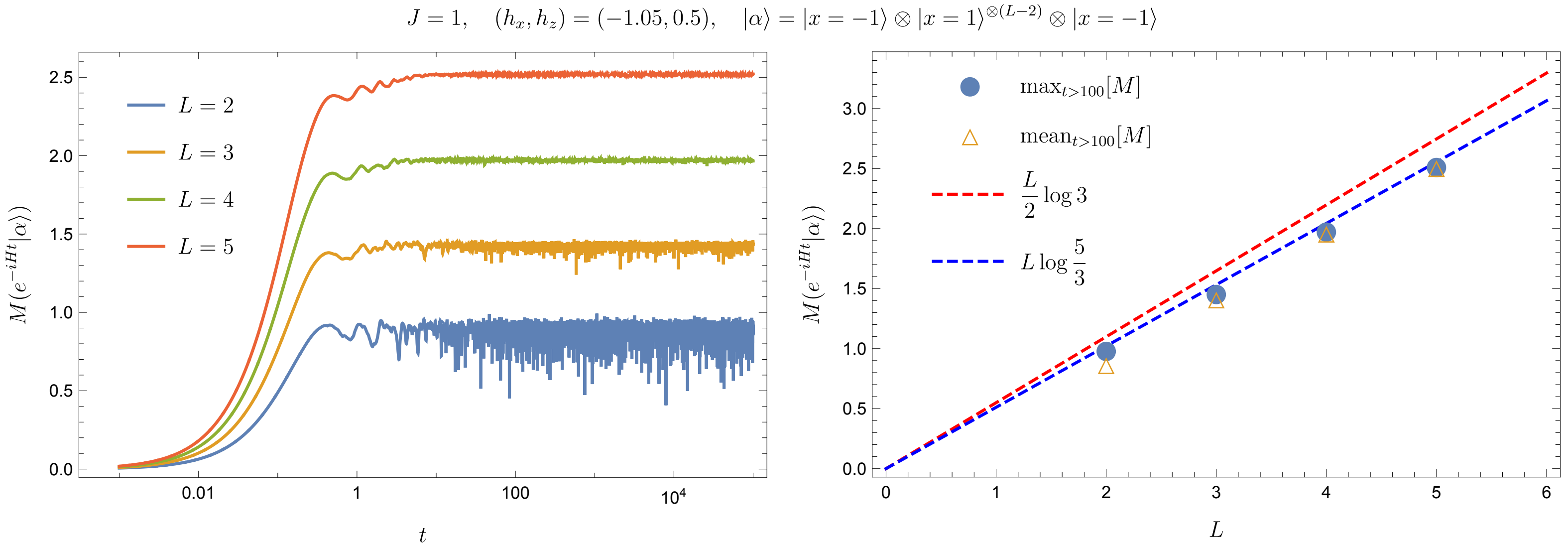
$$|\psi\rangle = e^{-iHt}|\alpha\rangle, \quad |\alpha\rangle = |x=-1\rangle \otimes |x=1\rangle^{\otimes(L-2)} \otimes |x=-1\rangle \in St$$

chaotic: $M(|\psi\rangle)$ increases monotonically at early time, and saturates at late time with

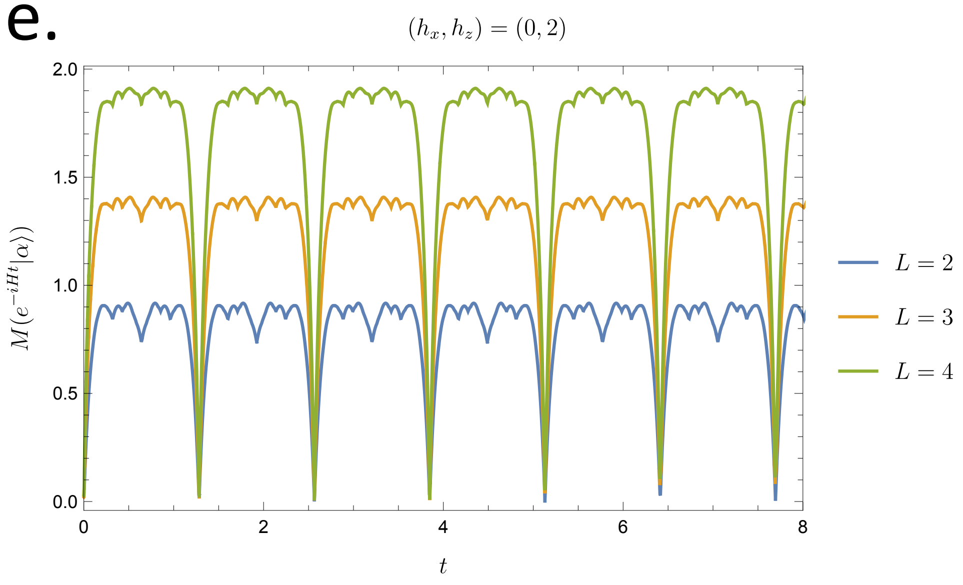
$$M(|\psi\rangle) \approx L \times \log\frac{5}{3}$$

upper bound for single site

[Goto,TN,Nozaki,2112.14593]

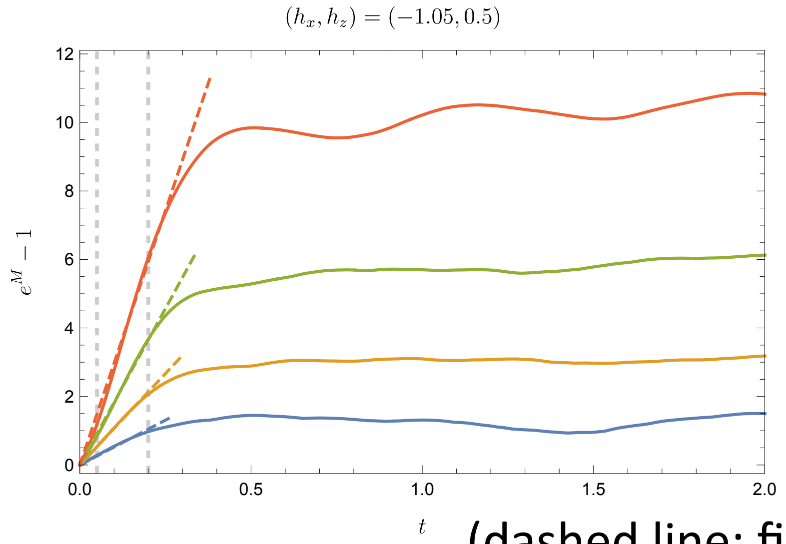


integrable: $M(|\psi\rangle)$ comes back to $M \approx 0$ even at late time.

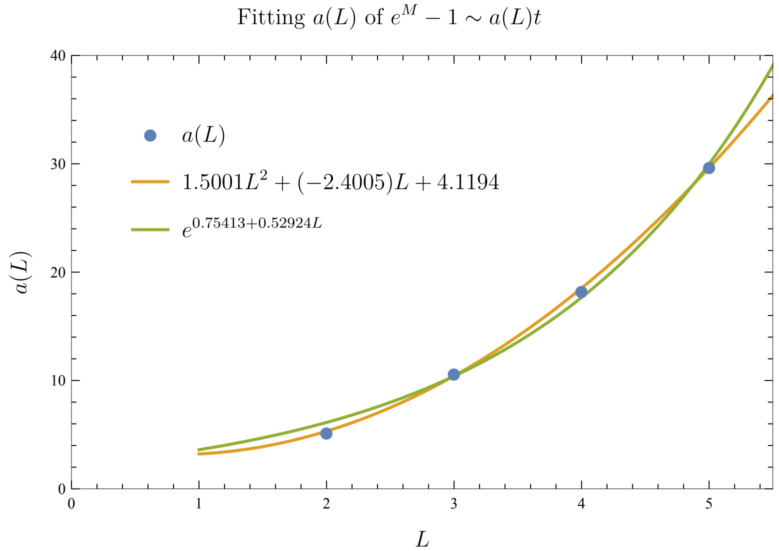


Estimations of saturation time (Mana)

Extrapolation of early time linear growth



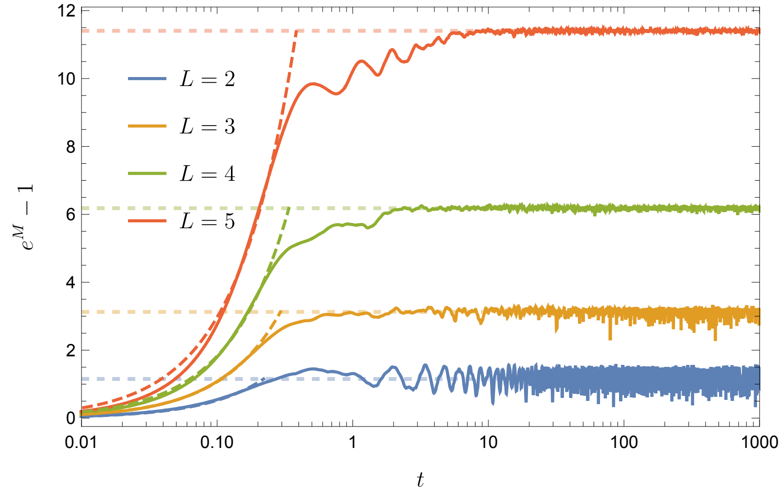
(dashed line: fitting with $a(L)t$)

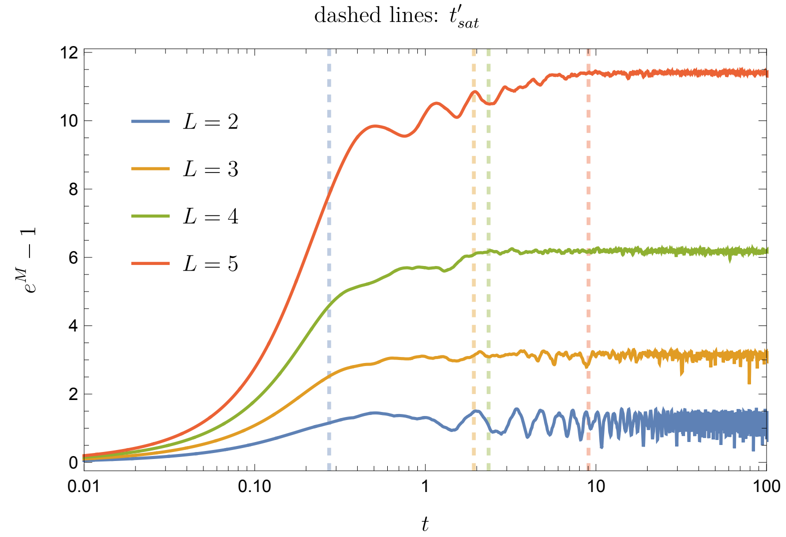


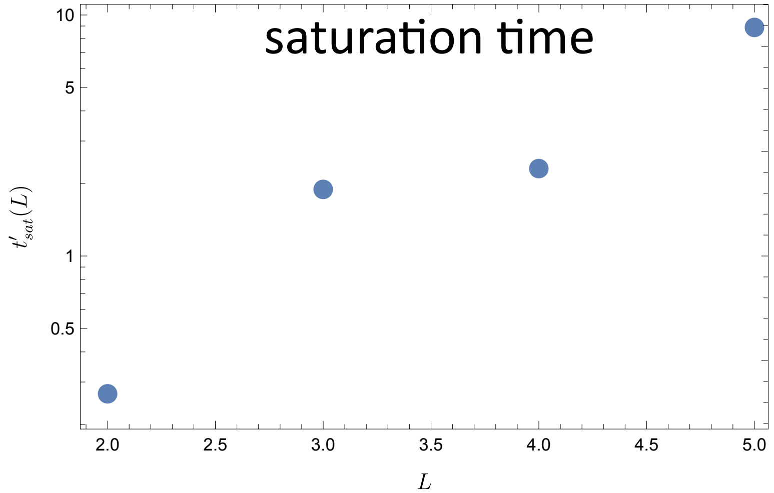
$$\boxed{t_{saturation} = \frac{e^{M(\text{late time})} - 1}{a(L)} \sim \frac{(5/3)^L}{L^2} \gg L^{\bigcirc}}$$

[Goto,TN,Nozaki,2112.14593]

t'_{sat} such that $M(e^{-it'_{sat}H}|\alpha\rangle) = \underset{t>100}{\text{mean}}[M]$







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Summary

To capture BH interior by AdS/CFT, entanglement is not enough since it saturates too fast. We need to see more refined property of states like computational complexity.

Magic: different kind of complexity which counts only "difficult" gates from viewpoint of classical simulation.

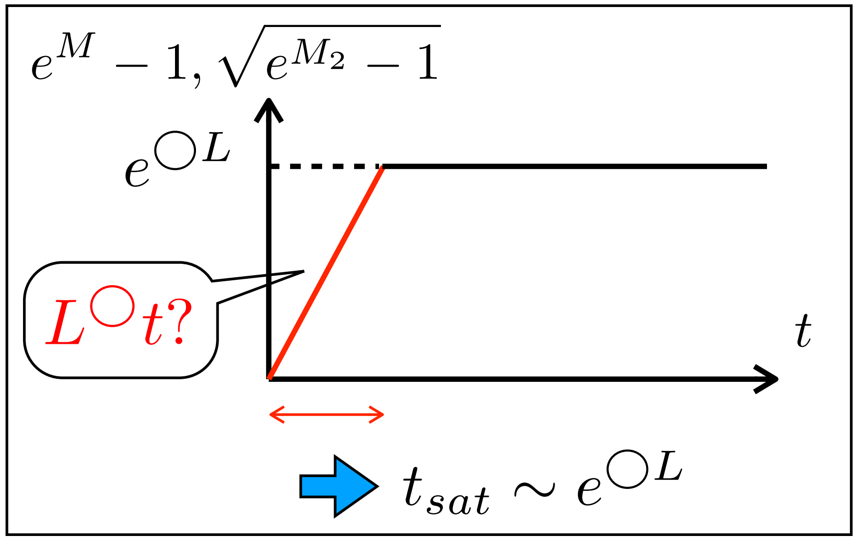
Numerical results for the chaotic spin chain (2 ≤ *L* ≤ 7 sites) suggest:

- Magic monotone $\mathcal{M}$ (stabilizer Renyi entropy M_2 /Mana M) approaches its Haar random value at late time only when *H* is chaotic.
- Saturation time of $\mathcal{M}(e^{-iHt}|\alpha\rangle)$ grows exponentially in *L*, which becomes larger than $t_{\text{thermalize}} \sim L$ in the large *L* limit.

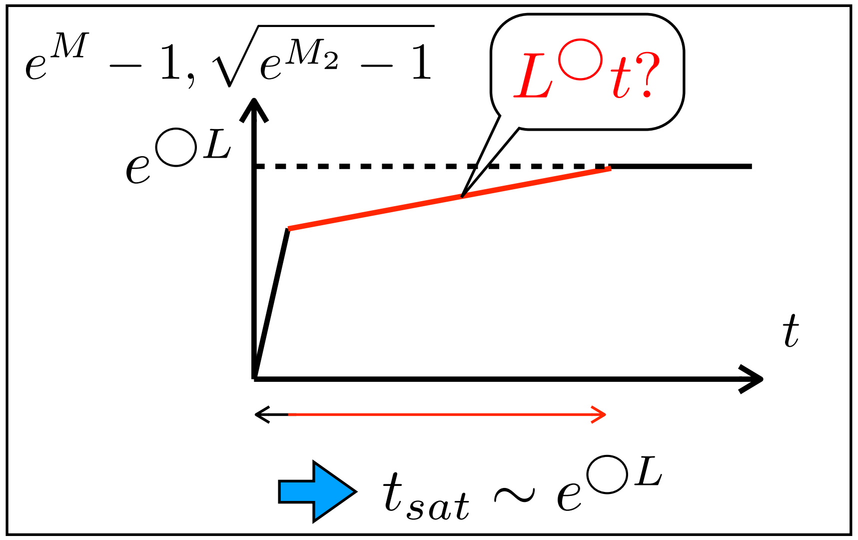
➡ Magic monotones might capture some information of BH interior which entanglement does not.

Further directions

- We need to increase system size *L* to confirm the structure of time dependence.



or



?

- Analytic approach for large *L* limit?
- Other magic monotones / other models (e.g. SYK model and its variants)
- Relation to other chaos/complexity measures
- What would be the gravity dual of magic?
- Field theory generalization of magic?