Weyl covariance of M2-brane matrix models and Painlevé equations

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Based on

[Bonelli,Globlek,Kubo,TN,Tanzini, LMP112(2022)109] [Moriyama,TN, arXiv:2305.03978]

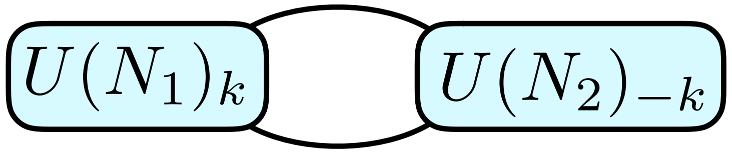
@Shing-Tung Yau Center of Southeast University (27 June 2023)

Introduction

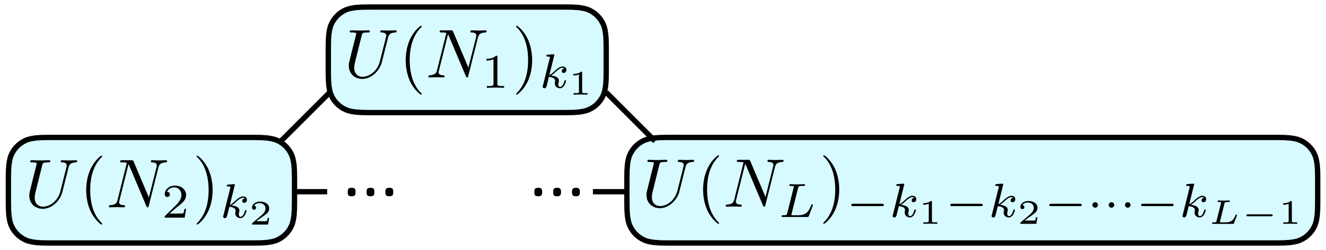
Motivation of research related to supersymmetric gauge theories:

discover hidden mathematical structures through exact (non-perturbative) analysis

We focus on 3d supersymmetric quiver Chern-Simons theories



: ABJM theory



⇨ M2-branes in M-theory

quantum gravity on $\mathrm{AdS}_4 \times Y_7$

Some observables are exactly calculable by SUSY localization+α

Exact large *N* expansion

$Z_{S^3}(k,N)$

- 't Hooft expansion ($k \to \infty, k^{-1}N$: fixed)
- Fermi gas formalism
  - ➡ semiclassical expansion ($k \to 0$)
  - ➡ TBA-like equation (k, N : finite)

All order in 1/*N* !

1/*N* perturbative = $N^{3/2} + N^{1/2} + \log N + \cdots = e^{A} C^{-\frac{1}{3}} \mathrm{Ai}[C^{-\frac{1}{3}}(N-B)]$

1/*N* non-pert. $(e^{-\sqrt{\frac{N}{k}}}, e^{-\sqrt{kN}})$ = refined topological string free energy

Partially reproduced from gravity side (not this talk)

New mathematical structure of M2-SCFT

Conceptually, Fermi gas formalism is

M2-branes ⟺ quantized algebraic curves

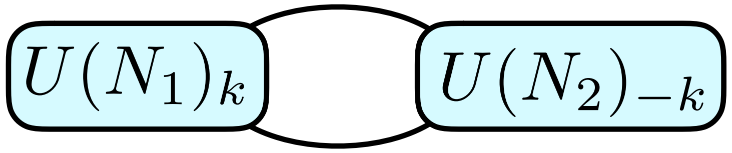
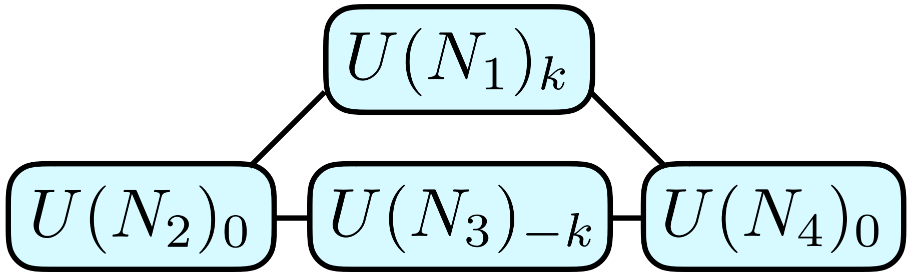
$Z^{\text{pert}}(N)=\text{Airy}$ follows automatically

encodes target CY of topological string (TS/ST correspondence)

[Grassi,Hatsuda,Marino,'14]

symmetry of curve ⇒ (enhanced) IR dualities

$Z_{S^3}(N)$ obey discrete integrable equations associated with symmetry

M2-SCFT	 $U(N_1)_k$ $U(N_2)_{-k}$	 $U(N_1)_k$ $U(N_2)_0$ $U(N_3)_{-k}$ $U(N_4)_0$
symmetry	$\mathbb{Z}_2(=W(A_1))$	$W(D_5)$
equation	q-Painlevé III$_3$ [Bonelli,Grassi,Tanzini,'17]	q-Painlevé VI [BGKNT][MN]

Plan of talk

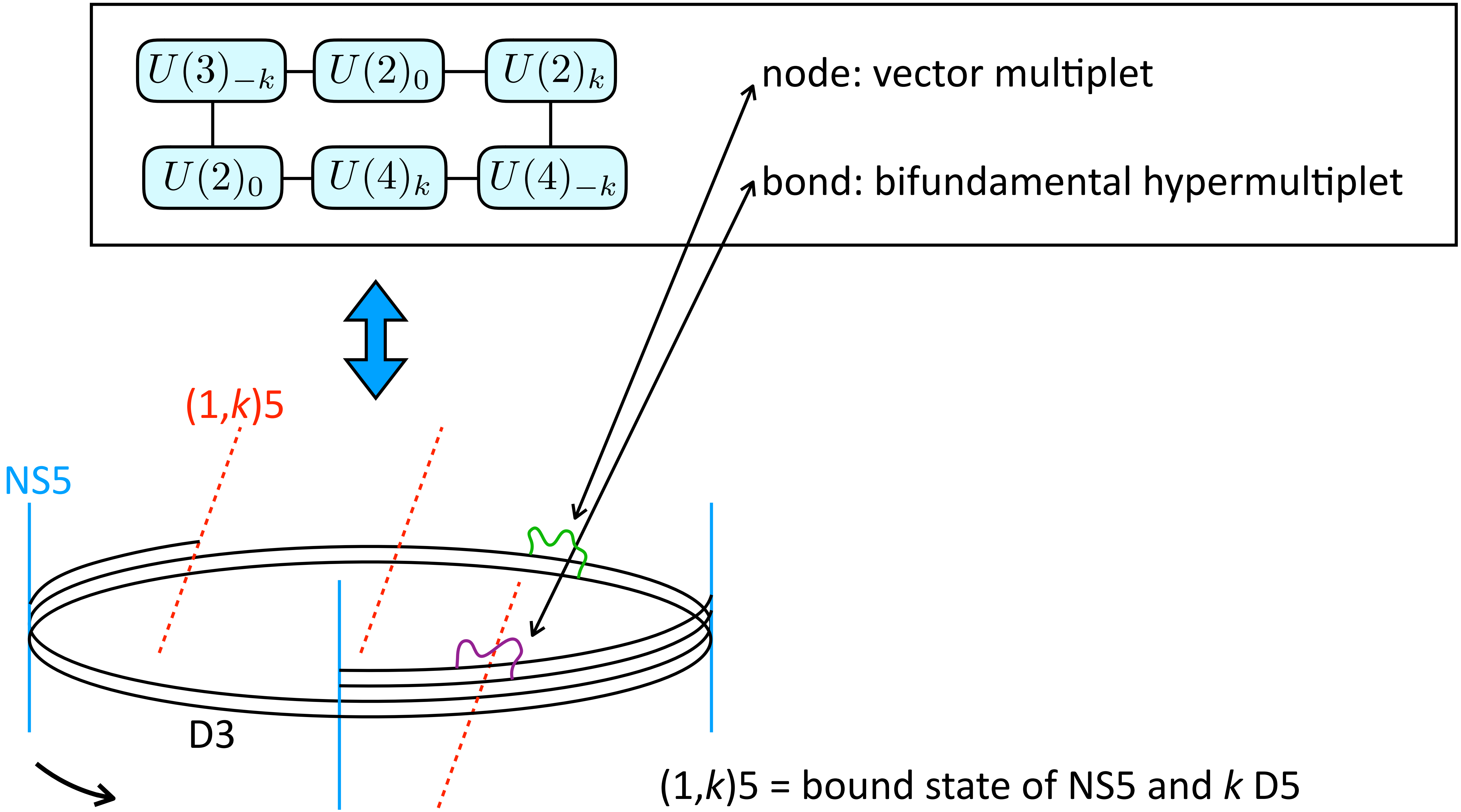
1. M2-branes and Fermi gas formalism

2. Quantum curve and symmetries

3. q-discrete Painlevé equations

4. Future problems

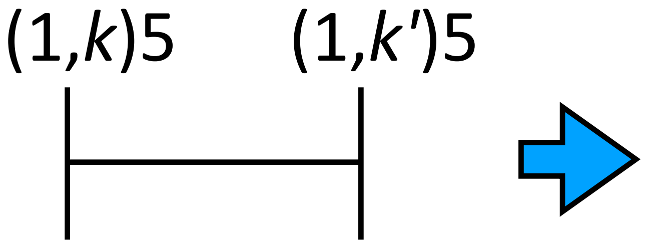
3d quiver Chern-Simons theories



node: vector multiplet

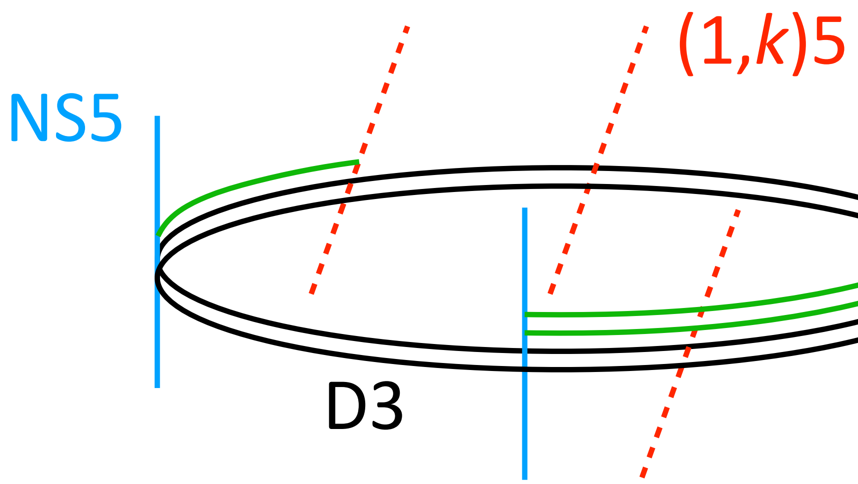
bond: bifundamental hypermultiplet

(1,*k*)5 = bound state of NS5 and *k* D5



Chern-Simons level = *k*′ - *k*

M-theory uplift



	x_{012}	x_3	x_{456}	x_{789}
D3	— — —	—		
NS5	— — —		— — —	
D5	— — —			— — —

IIB $\xrightarrow{\text{T-duality in } x_3}$ IIA $\xrightarrow{\text{uplift } x_{10}}$ M

wrapped D3 → D2 → M2

NS5 → geometry (KK monopole)

D5 → D6 → geometry

fractional D3 → discrete flux of C_3

$\min(N_1, N_2, \cdots)$ M2s on $(\mathbb{C}^2/\mathbb{Z}_{\#(\mathrm{NS5})} \times \mathbb{C}^2/\mathbb{Z}_{\#((1,k)5)})/\mathbb{Z}_k$

Note: orbifold is independent of ordering of 5-branes & fractional D3's

Supersymmetry localization

Suppose - action *S* is SUSY invariant

- $\mathcal{O}$: SUSY invariant operator $\delta\mathcal{O} = 0$

$$\int \mathcal{D}\Phi\mathcal{D}\Psi\mathcal{O}e^{-S-t\delta V}$$

is independent of *t*

$$V \sim (\Psi, \delta\Psi)$$

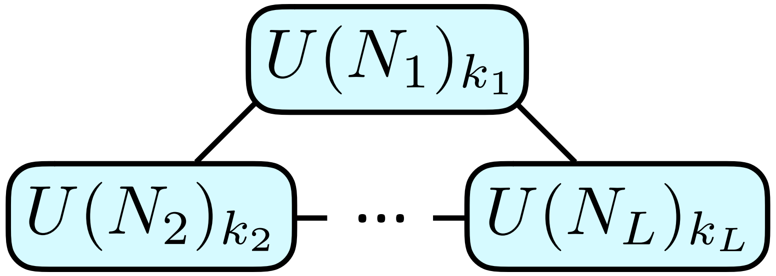
$t \to 0$ $t \to \infty$

$$\boxed{\langle\mathcal{O}\rangle = \int \mathcal{D}\Phi\mathcal{D}\Psi\mathcal{O}e^{-S} = \sum_{\Phi_{\rm saddle}} \left[\mathcal{O}e^{-S}\right]_{\Phi_{\rm saddle}} \mathrm{SDet}_{\Phi,\Psi}[\delta^2(\Phi_{\rm saddle})]}$$

$\Phi_{\rm saddle}$: solutions of $\delta V = 0$

Partition function of Super Chern-Simons theories

$$\sum_{\Phi_{\rm saddle}} \left[\mathcal{O}e^{-S}\right]_{\Phi_{\rm saddle}} {\rm SDet}_{\Phi,\Psi}[\delta^2(\Phi_{\rm saddle})]$$



$e^{-S_{\rm CS}}$

${\rm SDet}_{\rm vec}$

$$\boxed{Z_{S^3} = \frac{1}{N_a!}\int \frac{d^{N_a}\lambda_i^{(a)}}{(2\pi)^{N_a}} \prod_{a=1}^{L} e^{\frac{ik_a}{4\pi}\sum_{i=1}^{N_a}(\lambda_i^{(a)})^2} \prod_{a=1}^{L} \frac{\prod_{i<j}^{N_a}(2\sinh\frac{\lambda_i^{(a)}-\lambda_j^{(a)}}{2})^2}{\prod_{i=1}^{N_a}\prod_{j=1}^{N_{a+1}} 2\cosh\frac{\lambda_i^{(a)}-\lambda_j^{(a+1)}}{2}}}$$

${\rm SDet}_{\rm hyp}$

$\lambda_i^{(a)}$: vev of vectormultiplet

[Kapustin,Willett,Yaakov,'09]

Fermi gas formalism

$$\boxed{Z = \frac{1}{N!}\int \frac{d^N x}{(2\pi)^N} \det_{i,j}[\langle x_i|\rho(\hat{x},\hat{p})|x_j\rangle]}$$

[Marino,Putrov,'11]

N free fermions with $\hat{H}_{\text{one-particle}} = -\log\hat{\rho}$

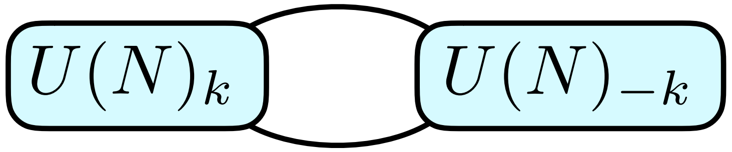
Key formula in deriving Fermi gas formalism:

$$\boxed{\frac{\prod_{i<j}^{N} 2\sinh\frac{x_i-x_j}{2}\prod_{i<j}^{N+M} 2\sinh\frac{y_i-y_j}{2}}{\prod_{i=1}^{N}\prod_{j=1}^{N+M} 2\cosh\frac{x_i-y_j}{2}} = \det_{(i\oplus r),j}\begin{pmatrix} \dfrac{e^{-\frac{M(x_i-y_j)}{2}}}{2\cosh\frac{x_i-y_j}{2}} \\ e^{(\frac{M+1}{2}-r)y_j} \end{pmatrix} = \det\begin{pmatrix} \langle x_i|\frac{1}{2\cosh\frac{\hat{p}-\pi i M}{2}}|y_j\rangle \\ {}_p\langle\langle 2\pi i(\frac{M+1}{2}-r)|y_j\rangle \end{pmatrix}}$$

(Cauchy-Vandermonde determinant)

$$[\hat{x},\hat{p}] = 2\pi i$$

Fermi gas formalism (ABJM)



$$[\hat{x},\hat{p}] = 2\pi i k$$

$$Z = \frac{1}{(N!)^2}\int \frac{d^N\lambda_i^{(1)}}{(2\pi)^N}\frac{d^N\lambda_i^{(2)}}{(2\pi)^N}\det\Big[\langle\lambda_i^{(1)}|\underset{\hat{U}^\dagger}{\underset{\uparrow}{}}\frac{1}{2\cosh\frac{\hat{p}}{2}}\underset{\hat{V}}{\underset{\uparrow}{}}|\lambda_j^{(2)}\rangle\Big]\underbrace{e^{-\frac{i}{4\pi k}\sum_i(\lambda^{(2)})^2}\det\Big[\langle\lambda_i^{(2)}|\underset{\hat{V}^\dagger}{\underset{\uparrow}{}}\frac{1}{2\cosh\frac{\hat{p}}{2}}\underset{\hat{U}}{\underset{\uparrow}{}}|\lambda_j^{(1)}\rangle\Big]e^{\frac{i}{4\pi k}\sum_i(\lambda^{(1)})^2}}$$

$$\hat{U} = \hat{V} = e^{\frac{i}{4\pi k}\hat{p}^2}$$

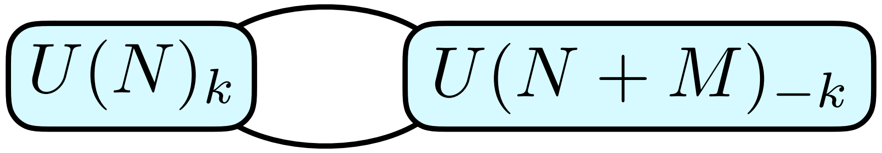
$$\Big\downarrow$$

$$\det[\langle\lambda_i^{(2)}|\frac{1}{2\cosh\frac{\hat{x}}{2}}|\lambda_j^{(1)}\rangle] = \det\left[\frac{2\pi\delta(\lambda_i^{(2)}-\lambda_j^{(1)})}{2\cosh\frac{\lambda_j^{(1)}}{2}}\right]$$

$$= N!\prod_{i=1}^{N}\frac{2\pi\delta(\lambda_i^{(2)}-\lambda_i^{(1)})}{2\cosh\frac{\lambda_i^{(1)}}{2}}$$

$$\boxed{Z = \frac{1}{N!}\int\frac{d^N\lambda_i^{(1)}}{(2\pi)^N}\det_{i,j}\Big[\langle\lambda_i^{(1)}|\frac{1}{2\cosh\frac{\hat{p}}{2}}\,\frac{1}{2\cosh\frac{\hat{x}}{2}}|\lambda_j^{(1)}\rangle\Big]}$$

Fermi gas formalism (ABJM with $N_1 \neq N_2$)



$$Z=\frac{1}{N!(N+M)!}\int\frac{d^N\lambda_i^{(1)}}{(2\pi)^N}\frac{d^{N+M}\lambda_i^{(2)}}{(2\pi)^{N+M}}\det\begin{bmatrix}\langle\lambda_i^{(1)}|\frac{1}{2\cosh\frac{\hat{p}-\pi iM}{2}}|\lambda_j^{(2)}\rangle\\ {}_p\langle\!\langle 2\pi i(\frac{M+1}{2}-r)|\lambda_j^{(2)}\rangle\end{bmatrix}$$

$$\underbrace{\times e^{-\frac{i}{4\pi k}\sum_i(\lambda^{(2)})^2}\det\left[\begin{array}{c:c}\langle\lambda_i^{(2)}|\frac{1}{2\cosh\frac{\hat{p}+\pi iM}{2}}|\lambda_j^{(1)}\rangle & \langle\lambda_i^{(2)}|-2\pi i(\frac{M+1}{2}-s)\rangle\!\rangle_p\end{array}\right]e^{\frac{i}{4\pi k}\sum_i(\lambda^{(1)})^2}}$$

$$\Downarrow$$

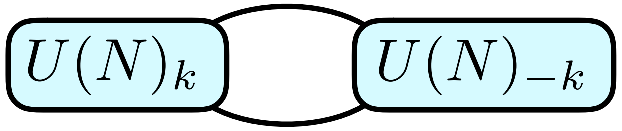
$$(N+M)!\prod_{i=1}^{N}\frac{2\pi\delta(\lambda_i^{(2)}-\lambda_i^{(1)})}{2\cosh\frac{\lambda_i^{(1)}+\pi iM}{2}}\prod_{r=1}^{M}2\pi\delta(\lambda_{N+r}^{(2)}-(-2\pi i(\frac{M+1}{2}-r)))$$

Reorganize ▨ as $\underbrace{\frac{\prod_{i<j}^N 2\sinh\frac{\lambda_i^{(1)}-\lambda_j^{(1)}}{2k}\prod_{i<j}^N 2\sinh\frac{\lambda_i^{(2)}-\lambda_j^{(2)}}{2k}}{\prod_{i,j}^N 2\cosh\frac{\lambda_i^{(1)}-\lambda_j^{(2)}}{2k}}}_{=\det\limits_{i,j}^{N}\left[\langle\lambda_i^{(1)}|\frac{1}{2\cosh\frac{\hat{p}}{2}}|\lambda_j^{(2)}\rangle\right]}\frac{\prod_{i=1}^N\prod_{r=1}^M 2\sinh\frac{\lambda_i^{(2)}-\lambda_{N+r}^{(2)}}{2k}}{\prod_{i=1}^N\prod_{r=1}^M 2\cosh\frac{\lambda_i^{(1)}-\lambda_{N+r}^{(2)}}{2k}}\prod_{r<s}^{M}2\sinh\frac{\lambda_{N+r}^{(2)}-\lambda_{N+s}^{(2)}}{2k}$

$$\boxed{Z=\frac{Z(0)}{N!}\int\frac{d^N\lambda_i^{(1)}}{(2\pi)^N}\det_{i,j}\left[\langle\lambda_i^{(1)}|\frac{1}{2\cosh\frac{\hat{p}}{2}}\frac{\prod_{r=1}^M\tanh\frac{\hat{x}+2\pi i(\frac{M+1}{2}-r)}{2k}}{2\cosh\frac{\hat{x}+\pi iM}{2}}|\lambda_j^{(1)}\rangle\right]}$$

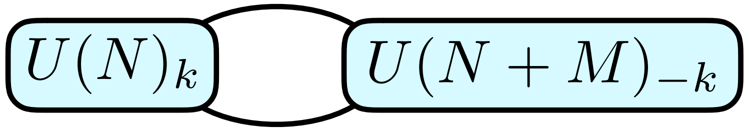
[Awata,Hirano,Shigemori,'12]

Quantum curve $\hat{\mathcal{O}} = \hat{\rho}^{-1}$



$$\hat{\mathcal{O}} = \left(2\cosh\frac{\hat{x}}{2}\right)\left(2\cosh\frac{\hat{p}}{2}\right) \qquad \hat{x}', \hat{p}' = \frac{\hat{x} \mp \hat{p}}{2}$$

$$= e^{\frac{\pi i k}{4}} e^{\hat{x}'} + e^{\frac{\pi i k}{4}} e^{\hat{p}'} + e^{\frac{\pi i k}{4}} e^{-\hat{p}'} + e^{\frac{\pi i k}{4}} e^{-\hat{x}'}$$



$$\hat{\mathcal{O}} = e^{\pi i M - \frac{\pi i k}{4}} e^{\hat{x}'} + e^{\frac{\pi i k}{4}} e^{\hat{p}'} + e^{\frac{\pi i k}{4}} e^{-\hat{p}'} + e^{\pi i M - \frac{\pi i k}{4}} e^{-\hat{x}'}$$

[Kashaev,Marino,Zakany,'15]

Obtained by continuing $\hat{\rho}$ to $M \in \mathbb{C}$ with quantum dilogarithm

$$\Phi_b(z) = \frac{(-e^{\pi i b^2 + 2\pi i b z}; e^{2\pi i b^2})_\infty}{(-e^{-\pi i b^{-2} + 2\pi b^{-1} z}; e^{-2\pi i b^{-2}})_\infty} \qquad (b = \sqrt{k})$$

Note $\hat{\mathcal{O}} \to \hat{\rho}$ is multivalued. Should not set $e^{\pi i M} = \pm 1$

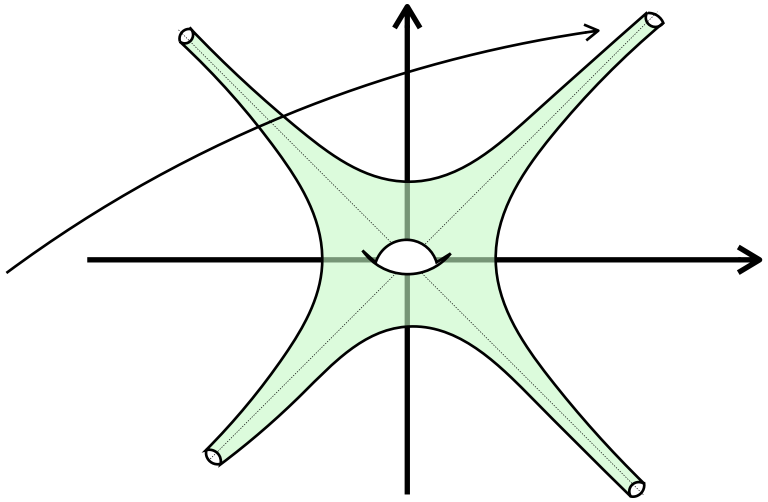
Quantum curve $\hat{\mathcal{O}} = \hat{\rho}^{-1}$

$$\hat{\mathcal{O}}_{\rm ABJM} = \bigcirc e^{\hat{x}'} + \bigcirc e^{\hat{p}'} + \bigcirc e^{-\hat{p}'} + \bigcirc e^{-\hat{x}'}$$

In classical limit, $\mathcal{O}_{\rm ABJM}(x', p') = const.$ is a complex genus-1 curve.

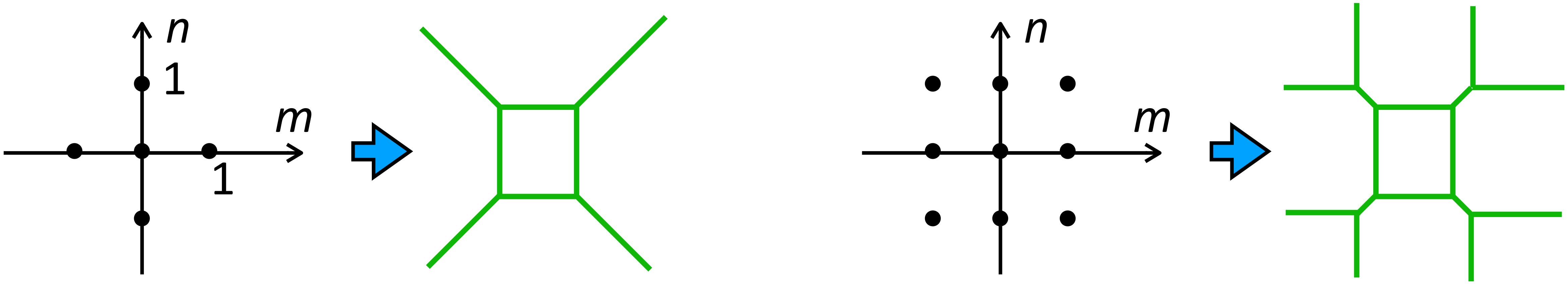
example: at $x', p' \to \infty$

$\rho^{-1} \sim \bigcirc e^{x'} + \bigcirc e^{p'} = const.$ ➡ $p' = x' + \square$

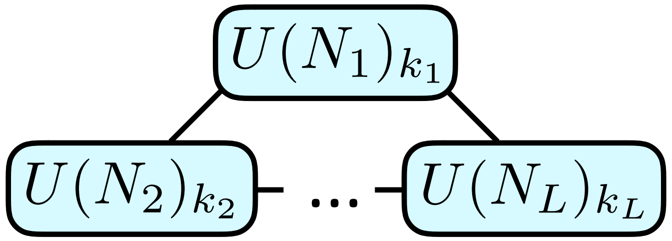


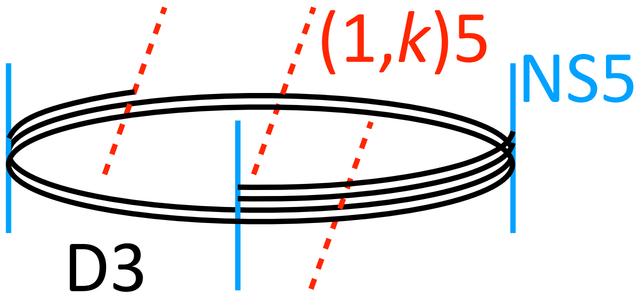
In general,

$\hat{\mathcal{O}} = \sum_{m,n} c_{mn} e^{m\hat{x} + n\hat{p}}$ ➡ $\mathcal{O} = const.$ is dual diagram of Newton polygon {(*m*,*n*)}



Summary so far





$$\boxed{Z = \frac{Z(N=0)}{N!}\int \frac{d^N x}{(2\pi)^N} \det_{i,j}\langle x_i|\hat{\rho}|x_j\rangle}$$

$$N = \min(N_1, \cdots, N_L)$$

(1,*k*)5

NS5

If $N_1 = \cdots = N_L$

$$\boxed{\hat{\rho} = \left(2\cosh\frac{\hat{x}\text{ or }\hat{p}}{2}\right)^{-1}\left(2\cosh\frac{\hat{x}\text{ or }\hat{p}}{2}\right)^{-1}\cdots\left(2\cosh\frac{\hat{x}\text{ or }\hat{p}}{2}\right)^{-1}}$$

$$\hat{\mathcal{O}} = \hat{\rho}^{-1} = \sum_{m,n} c_{mn} e^{m\hat{x}+n\hat{p}}$$

With fractional D3's it is difficult to obtain $\hat{\rho}$ and $\hat{\mathcal{O}}$, but we expect $\hat{\mathcal{O}}$ shares the same Newton polygon {(*m*,*n*)}, with c_{mn} depending on $N_i - N$

Plan of talk

1. M2-branes and Fermi gas formalism

2. Quantum curve and symmetries

3. q-discrete Painlevé equations

4. Future problems

Symmetry of quantum curve

$$\frac{Z}{Z(N=0)} = \frac{1}{N!}\int \frac{d^N x}{(2\pi)^N} \det_{i,j}\langle x_i|\hat{\rho}|x_j\rangle$$ is invariant under $\hat{\rho} \to \hat{U}\hat{\rho}\hat{U}^{-1}$

$$\hat{U}\rho(\hat{x},\hat{p})\hat{U}^{-1} = \rho(\hat{U}\hat{x}\hat{U}^{-1}, \hat{U}\hat{p}\hat{U}^{-1})$$

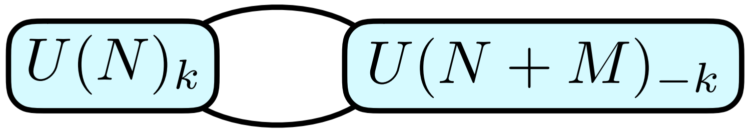
⇒ symmetry of $\frac{Z}{Z(N=0)}$ ⇔ coordinate trsf of curve $\mathcal{O}$

Two types of coordinate trsf.:

- continuous trsf $x \to x+a,\ p \to p+b$ ⇒ gauge degrees of freedom. Fix by hand

- discrete trsf ⇒ discrete symmetry on moduli space of curve

Symmetry of quantum curve $\hat{\mathcal{O}}_{\mathrm{ABJM}}$



$$\hat{\mathcal{O}} = e^{\pi i M - \frac{\pi i k}{4}} e^{\hat{x}} + e^{\frac{\pi i k}{4}} e^{\hat{p}} + e^{\frac{\pi i k}{4}} e^{-\hat{p}} + e^{\pi i M - \frac{\pi i k}{4}} e^{-\hat{x}}$$

By $x \to x + a,\ p \to p + b$ and overall rescaling (count separately),

we can set all but one coefficients to 1

$$\Rightarrow \boxed{\hat{\mathcal{O}} \sim e^{\hat{x}} + e^{\hat{p}} + e^{-\hat{p}} + e^{2\pi i M - \pi i k} e^{-\hat{x}}}$$

$\searrow$ moduli of curve

Discrete symmetry

$$(x, p) \to (p, -x) \quad \Rightarrow \quad \hat{\mathcal{O}} \sim e^{\hat{x}} + e^{\hat{p}} + e^{2\pi i M - \pi i k} e^{-\hat{p}} + e^{-\hat{x}}$$

gauge + overall to bring back to standard form

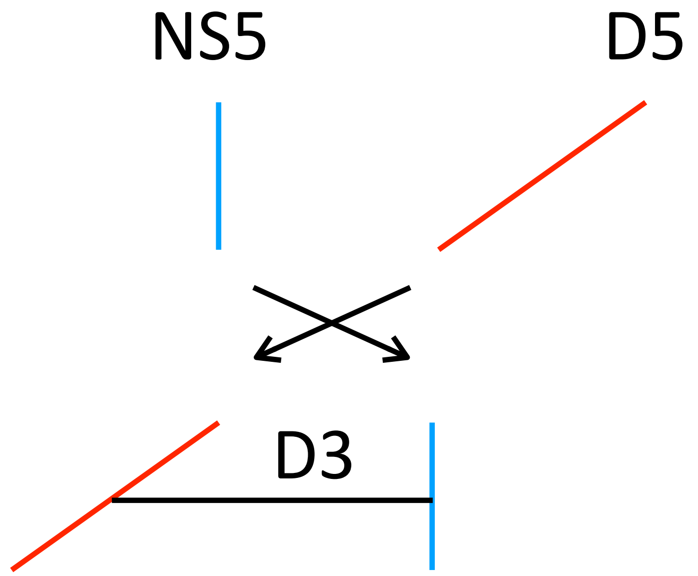
$$\Downarrow$$

$$\hat{\mathcal{O}} \sim e^{\hat{x}} + e^{\hat{p}} + e^{-\hat{p}} + e^{-2\pi i M + \pi i k} e^{-\hat{x}}$$

$\mathbb{Z}_2$-symmetry: $M \to k - M$

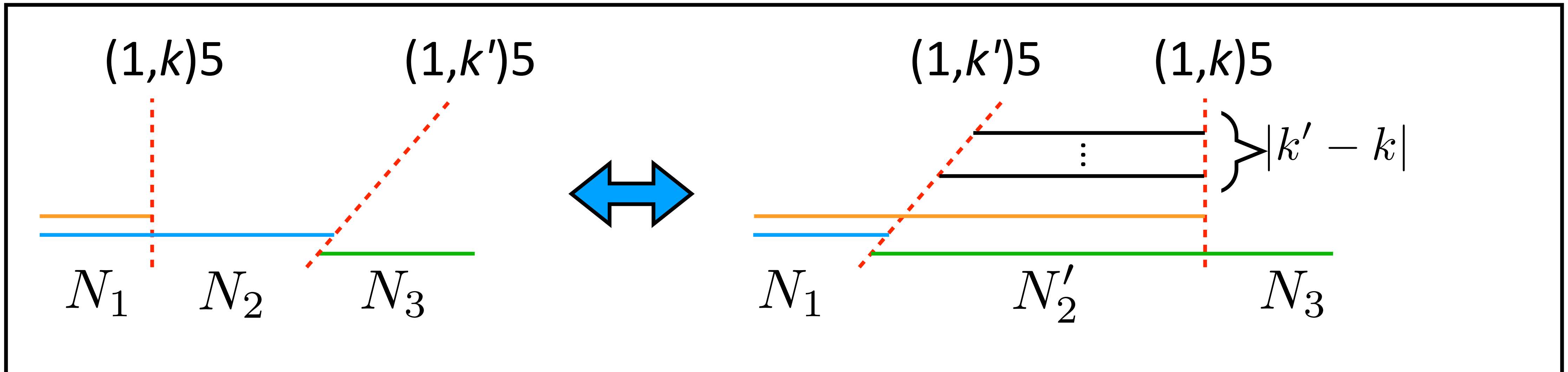
Physical interpretation

Hanany-Witten effect



	x_{012}	x_3	x_{456}	x_{789}
D3	$-\,-\,-$	$-$		
NS5	$-\,-\,-$		$-\,-\,-$	
D5	$-\,-\,-$			$-\,-\,-$

For (1,*k*)5-branes (=bound state of NS5 and *k* D5):

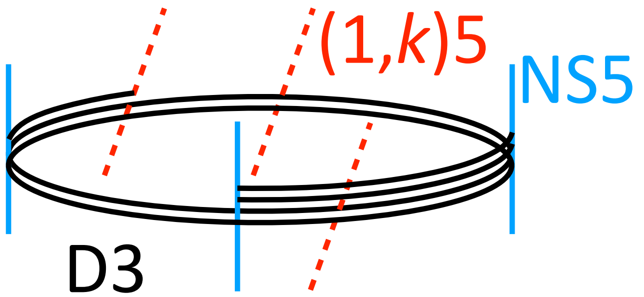


$$N_2' = N_1 + N_3 - N_2 + |k - k'|$$

For ABJM, we obtain

$$U(N)_k \times U(N+M)_{-k} \Leftrightarrow U(N)_k \times U(N+k-M)_{-k}$$

More general quantum curves?



$$\hat{\mathcal{O}} = \sum_{m,n} c_{mn} e^{m\hat{x}+n\hat{p}}$$

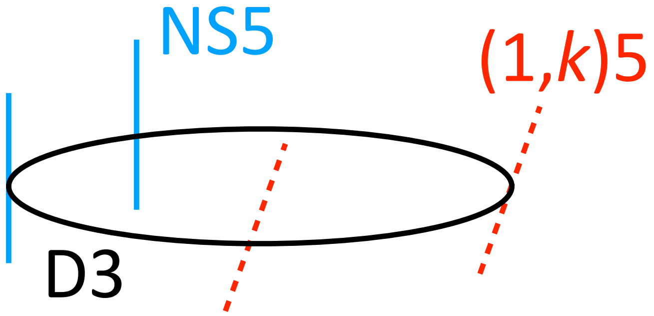
In ABJM we have found

- $c_{mn} = e^{(\text{1st order pol of } \{N_i - N\})}$
- Hanany-Witten trsf. is symmetry of curve

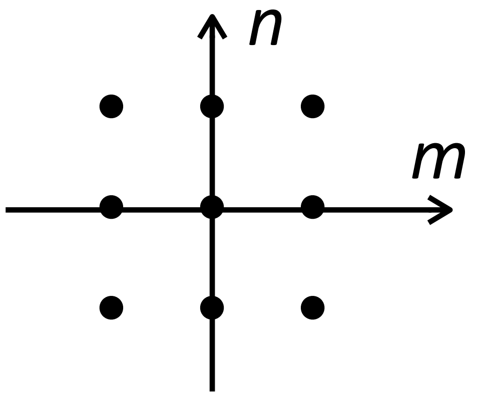
preserves moduli parameters up to discrete trsf.

We can identify $c_{mn}(N - N_i)$ for general theories by the extrapolation from special points which are HW dual to a "*ΔN*=0 configuration".

(2,2) model



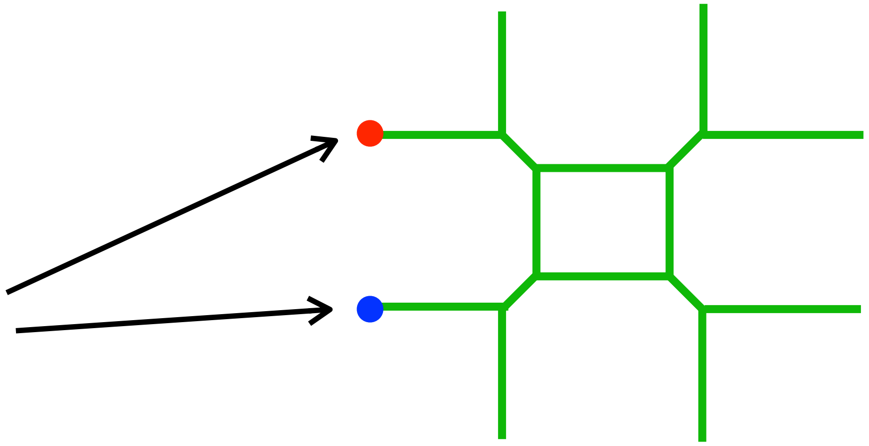
➡ $\hat{\mathcal{O}} = \sum_{m,n} c_{mn} e^{m\hat{x}+n\hat{p}}$



Parametrize moduli of curve by asymptotic loci

example: at $x \to -\infty$

$\mathcal{O} \sim e^{-x-p}(c_{-1,1}e^{2p} + c_{-1,0}e^{p} + c_{-1,-1})$ ➡ $e^{p} = \bigcirc, \square$

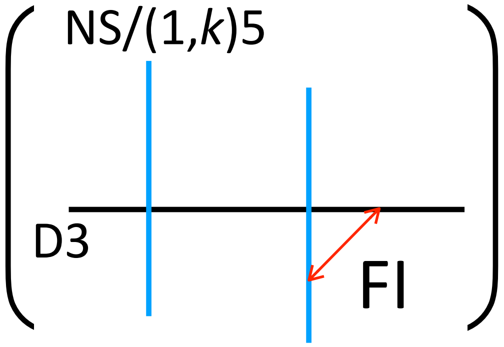


$\dim(\mathcal{M}_{\mathcal{O}})$ = 9(c_{mn}) - 2(gauge) - 1(overall) - 1($c_{0,0}$)

↘ ↙

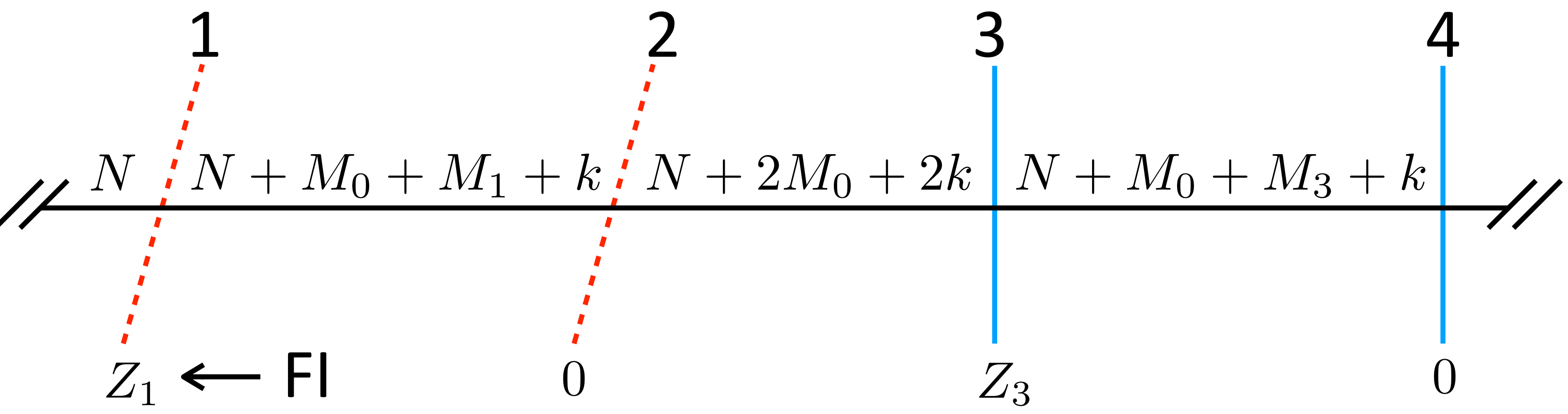
count separately

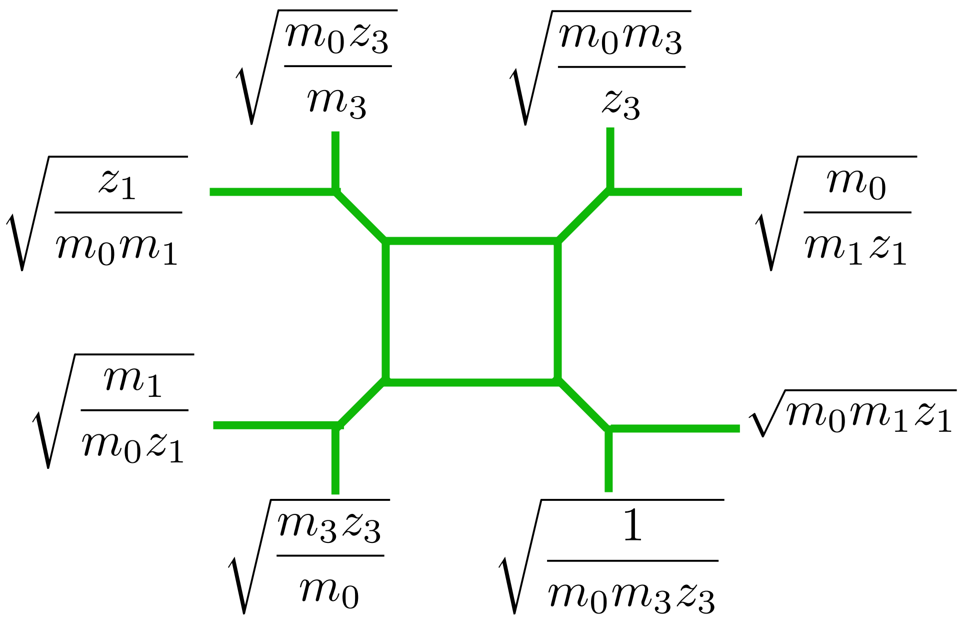
= 8(asymptotic loci) - 2(translation) - 1(vieta relation)



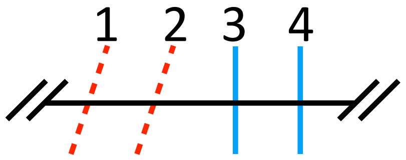
Agrees with $\dim(\mathcal{M}_{\text{SuperCS}})$ = 3(rank differences) + 2(FI parameters)

Parametrization of moduli

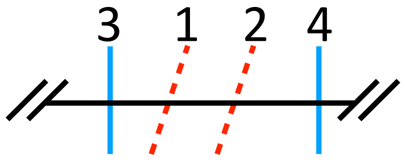




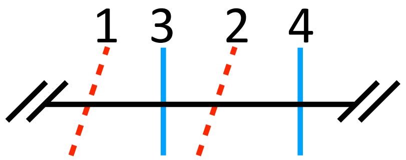
Special points:



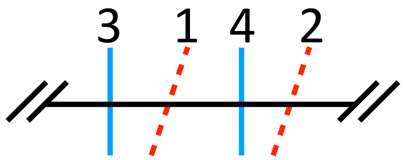
$(M_0, M_1, M_3) = (0, 0, 0)$



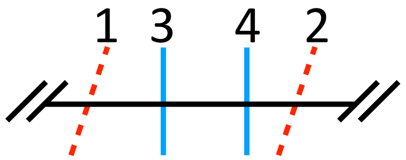
$(M_0, M_1, M_3) = (0, 0, -k)$



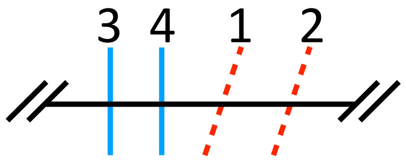
$(M_0, M_1, M_3) = (-\frac{k}{2}, -\frac{k}{2}, -\frac{k}{2})$



$(M_0, M_1, M_3) = (-\frac{k}{2}, \frac{k}{2}, \frac{k}{2})$



$(M_0, M_1, M_3) = (0, -k, 0)$



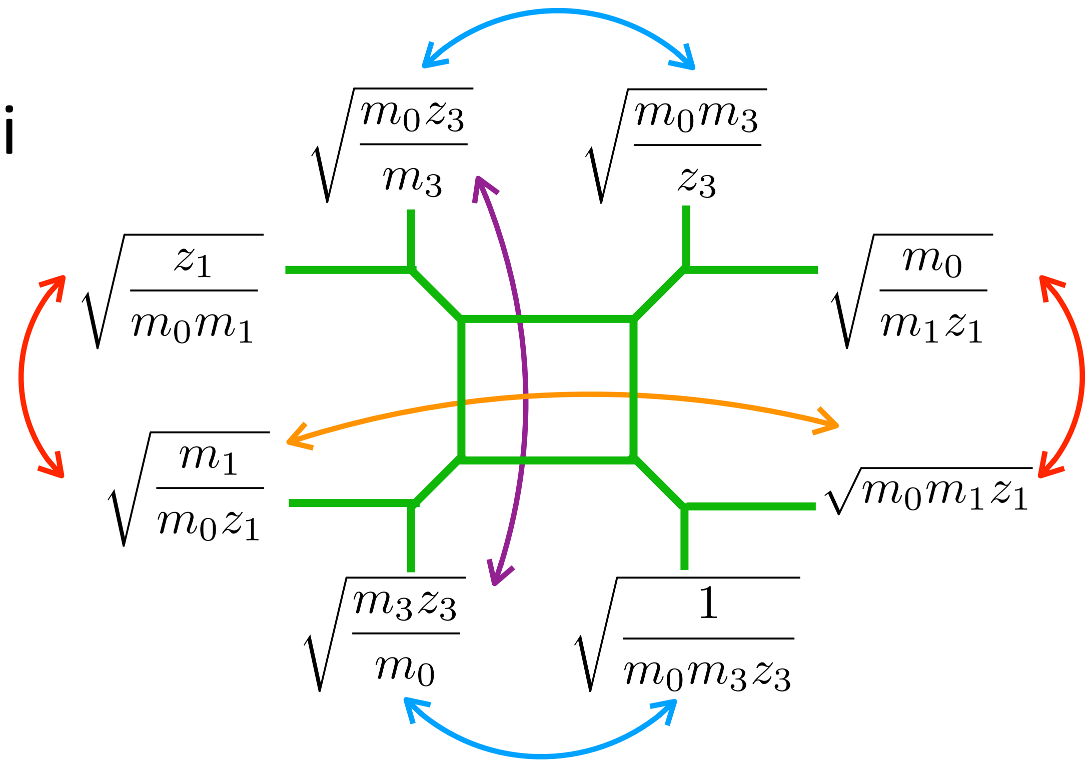
$(M_0, M_1, M_3) = (k, 0, 0)$

extrapolate!

$$\boxed{m_0 = e^{2\pi i M_0},\ m_1 = e^{2\pi i M_1},\ m_3 = e^{2\pi i M_3},\ z_1 = e^{2\pi i Z_1},\ z_3 = e^{2\pi i Z_3}}$$

Enhanced symmetry

Symmetry of curve = permutations of asymptotic loci



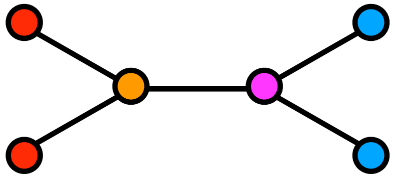
[Kubo,Moriyama,TN,'18]

generated by $(\vec{v} = (M_0, M_1, M_3, Z_1, Z_3))$

$$(v_a, v_b) \to (v_b, v_a)$$

$$(v_a, v_b) \to (-v_a, -v_b)$$

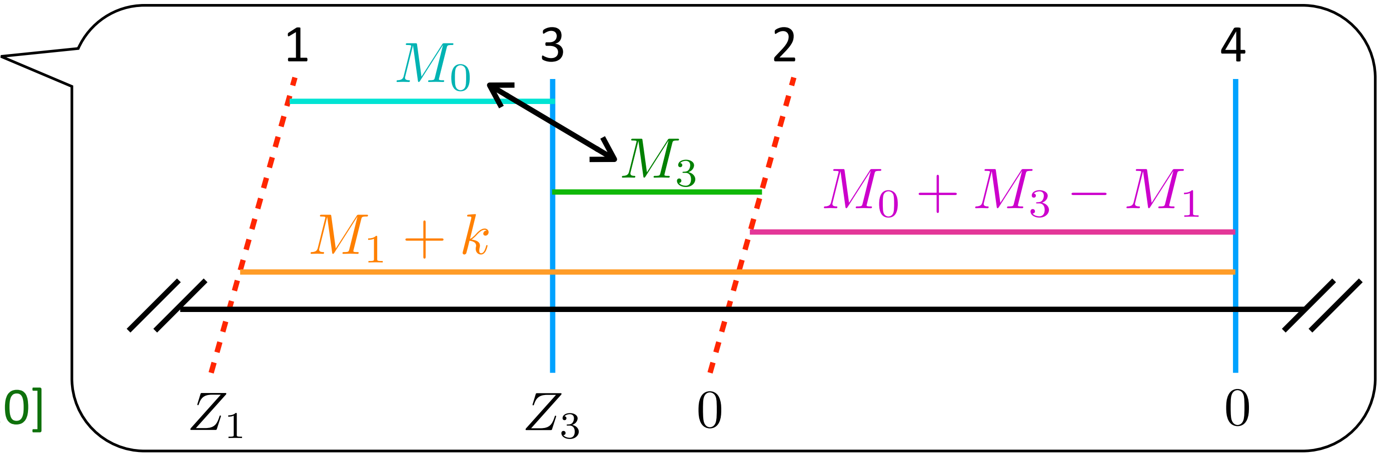
Weyl group of D_5



__Note__ This symmetry is larger than duality web of Hanany-Witten transitions.

examples not in HW: $(M_i, Z_j) \to (Z_i, M_j)$

$(M_0, M_3) \to (M_3, M_0)$



[Kubo,'20]

Implications of (enhanced) symmetry?

Group theoretical structures in large *N* expansion:

- $Z_{\mathrm{pert}}(N) = e^{A} C^{-\frac{1}{3}} \mathrm{Ai}[C^{-\frac{1}{3}}(N-B)]$

  C: independent of $\vec{v}$

  B: depends through $|\vec{v}|^2$ [Kubo,Moriyama,'19]

- 1/*N* non-pert. = topological string

  ➡ enumerative invariants are written as characters of *W*(*G*)

  [Moriyama,Nakayama,TN,17][Moriyama,TN,Yano,'17]

New:

- connection to q-Painlevé: discrete integrable systems classified by *W*(*G*)

Plan of talk

1. M2-branes and Fermi gas formalism

2. Quantum curve and symmetries

3. q-discrete Painlevé equations

4. Future problems

Painlevé equations

Original motivation: new special function from non-linear differential equations

Painlevé property: solution should not have an initial value-dependent branch point

example: $\dot{\lambda} = \lambda^{\alpha}$ ➡ $\lambda = (t - t_0)^{\frac{1}{1-\alpha}}$

$\alpha = 0, 1 \longrightarrow$ regular ○

$\alpha = 2 \longrightarrow$ pole ○

Non-trivial 2nd order ODE with Painlevé property were classified into six:

$$\begin{tikzcd}
\mathrm{PVI} \arrow[r] & \mathrm{PV} \arrow[r] \arrow[d] & \mathrm{PIII}_1 \arrow[r] \arrow[d] & \mathrm{PIII}_2 \arrow[r] \arrow[d] & \mathrm{PIII}_3 \\
 & \mathrm{PIV} \arrow[r] & \mathrm{PII} \arrow[r] & \mathrm{PI} &
\end{tikzcd}$$

Painlevé III_3 equation:

$$\boxed{\ddot{\lambda} = \frac{\dot{\lambda}^2}{\lambda} - \frac{\dot{\lambda}}{t} + \frac{2\lambda^2}{t^2} - \frac{2}{t}}$$

Hirota bilinear (τ-)form and q-uplift

Painlevé III$_3$ equation: $\ddot{\lambda} = \frac{\dot{\lambda}^2}{\lambda} - \frac{\dot{\lambda}}{t} + \frac{2\lambda^2}{t^2} - \frac{2}{t}$ (also known as SU(2) periodic Toda)

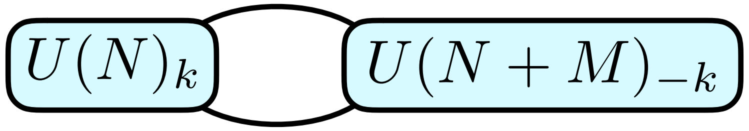
$$\partial^2_{\log t} \log \tau_1 = \lambda, \ \partial^2_{\log t} \log \tau_2 = t\lambda^{-1}$$

$$\boxed{\tau_i \partial^2_{\log t} \tau_i - (\partial_{\log t} \tau_i)^2 = t^{\frac{1}{2}} \tau_j^2 \qquad (i \neq j)}$$

q-uplift: $\partial_{\log t} f(t) \to \frac{f(\mathfrak{q}t) - f(t)}{\mathfrak{q} - 1}$

q-Painlevé III$_3$ equation: $\boxed{\tau_1(\mathfrak{q}t)\tau_1(\mathfrak{q}^{-1}t) = \tau_1(t)^2 + t^{\frac{1}{2}}\tau_2(t)^2}$

ABJM ⟷ q-Painlevé III$_3$



$$\Xi_M(\kappa) = \sum_{N=0}^{\infty} \kappa^N \frac{Z_{k,M}(N)}{Z_{k,M}(N=0)}$$

$k=2$

N	1	2	3
$M=0,2$	$\frac{1}{8}$	$\frac{1}{32\pi^2}$	$-\frac{1}{512}+\frac{5}{256\pi^2}$
$M=1$	$\frac{1}{4\pi}$	$\frac{1}{128}-\frac{1}{16\pi^2}$	$\frac{5}{4608\pi}-\frac{1}{96\pi^3}$

By using exact values of $\frac{Z_{k,M}(N)}{Z_{k,M}(0)}$, we find

$$\boxed{(1-e^{\frac{2\pi i M}{k}})\Xi_{M+1}(-i\kappa)\Xi_{M-1}(i\kappa) = \Xi_M(\kappa)^2 - e^{\frac{2\pi i M}{k}}\Xi_M(-\kappa)^2}$$

(still no rigorous proof)

[Grassi,Hatsuda,Marino,'14][Bonelli,Grassi,Tanzini,'17]

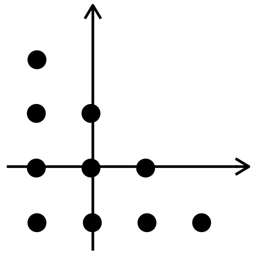
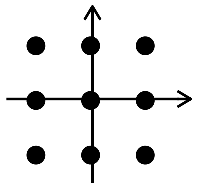
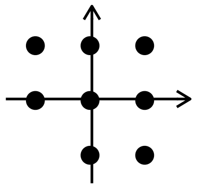
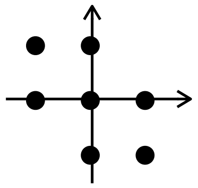
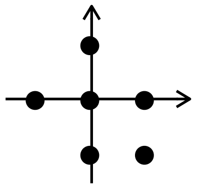
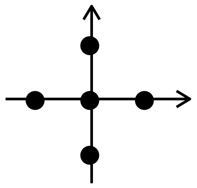
Identical to q-Painlevé III$_3$ equation with

$$\mathfrak{q} = e^{\frac{4\pi i}{k}} \qquad t = e^{-2\pi i(1-\frac{2M}{k})} \qquad \tau_1(t) \sim \Xi_M(e^{-\frac{\pi i M}{2}}\kappa) \qquad \tau_2(t) \sim \Xi_M(-e^{-\frac{\pi i M}{2}}\kappa)$$

Generalization of "O(2) matrix model ⟷ PIII$_3$" [Zamolodchikov,'94] to $q \neq 1$

q-Painleve equations and exceptional groups

Sakai's classification of q-Painlevé by exceptional groups (or curves): [Sakai,'01]

...	qP(E_6)	qPVI	qPV	qPIII$_1$	qPIII$_2$	qPIII$_3$
···	E_6	$E_5 = D_5$	$E_4 = A_4$	$E_3 = A_2 \times A_1$	$E_2 = A_1 \times A_1$	$E_1 = A_1$
...						

q-Painlevé equation

time evolution

$\tau_{\vec v}$ ($\vec v$: discrete lattice points)

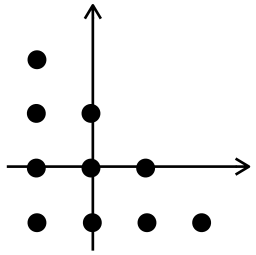
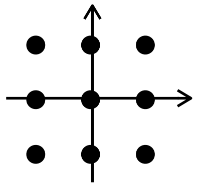
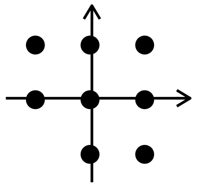
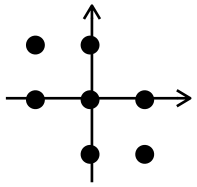
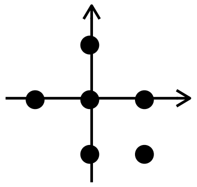
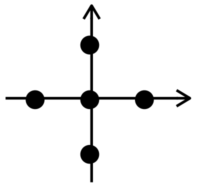
⟺ "affine Weyl group (= *W*(*G*) + translation) has a bi-rational representation"

$(w.\tau)_{\vec v} = \tau_{w(\vec v)}$ $(a.\tau)_{\vec v} = \frac{\tau\tau + \tau\tau}{\tau}$

bilinear eq

q-Painleve equations and exceptional groups

Sakai's classification of q-Painlevé by exceptional groups (or curves): [Sakai,'01]

…	qP(E_6)	qPVI	qPV	qPIII$_1$	qPIII$_2$	qPIII$_3$
…	E_6	$E_5 = D_5$	$E_4 = A_4$	$E_3 = A_2 \times A_1$	$E_2 = A_1 \times A_1$	$E_1 = A_1$
…						

time evolution

$\tau_{\vec{v}}$ ($\vec{v}$: discrete lattice points)

q-Painlevé equation

⟺ "affine Weyl group (= *W*(*G*) + translation) has a bi-rational representation"

$(w.\tau)_{\vec{v}} = \tau_{w(\vec{v})}$ $\quad$ $(a.\tau)_{\vec{v}} = \dfrac{\tau\tau + \tau\tau}{\tau}$

bilinear eq

From ABJM result, we expect

$$\boxed{\tau^{\mathrm{qPX}} \sim \sum_{N=0}^{\infty} \kappa^N \frac{Z^{\hat{\rho}=\hat{\mathcal{O}}^{-1}}(N)}{Z(N=0)}}$$

qPVI is the easiest to realize by M2-branes

bilinear relations of (2,2) model

Structure of q-Painlevé VI equations $(\mathfrak{q} = e^{\frac{2\pi i}{k}})$:

For each choice of two directions (*i,j*) in $\vec{v} = (M_0, M_1, M_3, Z_1, Z_3)$ and three signs $(\sigma_1, \sigma_2, \sigma_3) \in \{(+++), (+--), (-+-), (--+)\}$

$$\bigcirc \prod_{\pm} \tau_{\vec{v}\pm\text{shift1}} + \bigcirc \prod_{\pm} \tau_{\vec{v}\pm\text{shift2}} + \bigcirc \prod_{\pm} \tau_{\vec{v}\pm\text{shift3}} = 0$$

(read off from [Jimbo,Nagoya,Sakai,'17])

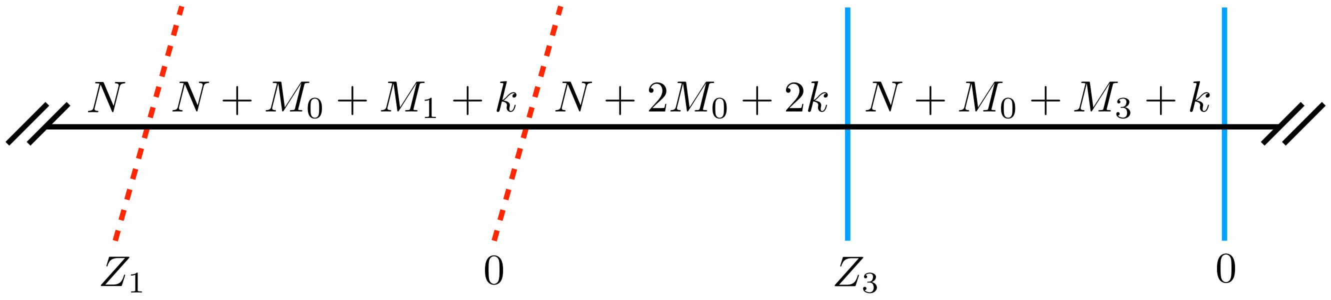
$(v_i, v_j) + (\frac{1}{2}, \frac{1}{2})$

$(v_i, v_j) + (\frac{1}{2}, -\frac{1}{2})$

$(v_k, v_l, v_m) + (\frac{\sigma_1}{2}, \frac{\sigma_2}{2}, \frac{\sigma_3}{2})$

Our conjecture:

$$\tau_{\vec{v}} \sim \Xi_{k,\vec{v}}(\kappa) = \sum_{N=0}^{\infty} \kappa^N \frac{Z_{k,\vec{v}}(N)}{Z_{k,\vec{v}}(N=0)}$$



satisfies 40 equations ☐ with some κ-indep. coefficients $\bigcirc's$

In [BGKTN,'22] we checked ☐ against exact values of $Z_{k,\vec{v}}(N)$, and partially fixed $\bigcirc's$

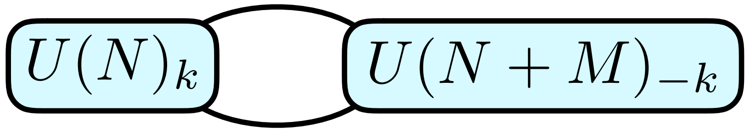
How about *Z*(*N*=0)?

$$\frac{Z}{Z(N=0)} = \frac{1}{N!}\int \frac{d^N x}{(2\pi)^N} \det_{i,j}\langle x_i|\hat{\rho}|x_j\rangle$$

So far we have omitted *Z*(*N*=0) since it is not related to quantum curve

However

$$Z^{\mathrm{ABJM}}_{k,M}(N=0) = Z_{\mathrm{CS}}(k,M) = \frac{1}{k^{\frac{M}{2}}}\prod_{j>j'} 2\sin\frac{\pi(j-j')}{k}$$



$$\Downarrow$$

$$\boxed{Z_{\mathrm{CS}}(k,M+1)Z_{\mathrm{CS}}(k,M-1) = \left(2\sin\frac{\pi M}{k}\right)Z_{\mathrm{CS}}(k,M)^2}$$

: q-Painlevé III_3 equation!

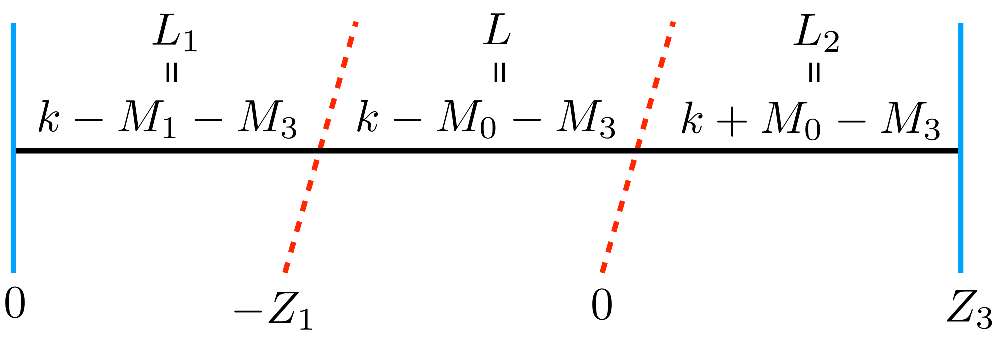
qPVI eqs for unnormalized grand partition function

The same relation holds for q-Painlevé VI and (2,2) model !

$$Z_{\vec{v}}(N=0)=Z_{\mathrm{CS}}(L_1)Z_{\mathrm{CS}}(L_2)$$

$$\times \det_{r,s}^{L}\left[\frac{1}{2\sin\pi(\frac{L_1+L_2}{2}-L+Z_1+r+s)}\sum_{\beta}\frac{e^{\frac{i(L+1-Z_1-r-s)\beta}{k}}}{\prod_{\beta'(\neq\beta)}2\sin\frac{\beta-\beta'}{2k}}\right]$$

$$\{\beta\}=\{2\pi(\tfrac{L_1+1}{2}-t)\}_{t=1}^{L_1}\cup\{2\pi(\tfrac{L_2+1}{2}-t-Z_3)\}_{t=1}^{L_2}$$



$$\boxed{\bigcirc\prod_{\pm}Z_{\vec{v}\pm\mathrm{shift}1}(N=0)+\bigcirc\prod_{\pm}Z_{\vec{v}\pm\mathrm{shift}2}(N=0)+\bigcirc\prod_{\pm}Z_{\vec{v}\pm\mathrm{shift}3}(N=0)=0}$$

[MN,'23]

Bilinear relations of $\Xi_{k,\vec{v}}(\kappa)$ are viewed as uplift of this to *κ* ≠ 0 :

$$\boxed{\bigcirc\prod_{\pm}\Xi^{\mathrm{un}}_{k,\vec{v}\pm\mathrm{shift}1}(\kappa)+\bigcirc\prod_{\pm}\Xi^{\mathrm{un}}_{k,\vec{v}\pm\mathrm{shift}2}(-\kappa)+\bigcirc\prod_{\pm}\Xi^{\mathrm{un}}_{k,\vec{v}\pm\mathrm{shift}3}(\mp i\kappa)=0}$$

$$\Xi^{\mathrm{un}}_{k,\vec{v}}(\kappa)=\sum_{N=0}^{\infty}\kappa^N Z_{k,\vec{v}}(N)$$

For (2,2) model, this approach is essential in finding bilinear relations with *κ* ≠ 0.

Plan of talk

1. M2-branes and Fermi gas formalism

2. Quantum curve and symmetries

3. q-discrete Painlevé equations

4. Future problems

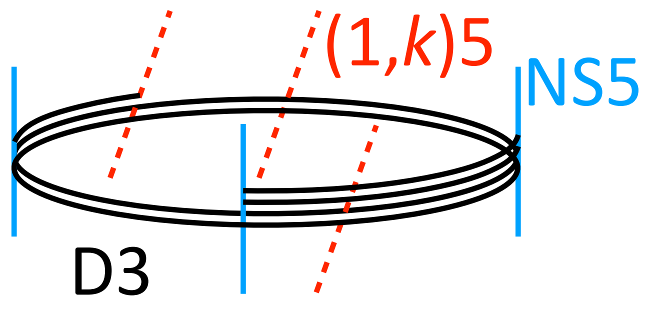
Summary

Fermi gas formalism:

(normalized) M2 partition functions ⟺ quantum curves $\hat{\mathcal{O}} = \sum_{m,n} c_{mn} e^{m\hat{x}+n\hat{p}}$

Symmetry of curve = Hanany-Witten duality (+ *α*)

⇨ determine $\hat{\mathcal{O}}$ for general rank/FI deformations



M2 partition functions solve q-Painlevé equations ($\mathfrak{q} \sim e^{\frac{\pi i}{k}}$, "time" $\sim \Delta N_i$)

prescription: identify symmetry *W*(*G*) and curve with those in Sakai classification

$$\tau^{\mathrm{qPX}} \sim \text{grand partition function } \Xi(\kappa) = \mathrm{Det}(1+\kappa\hat{\mathcal{O}})$$

Z(*N*=0) also solve q-Painlevé equations, although they are not related to quantum curves

Use of bilinear relations?

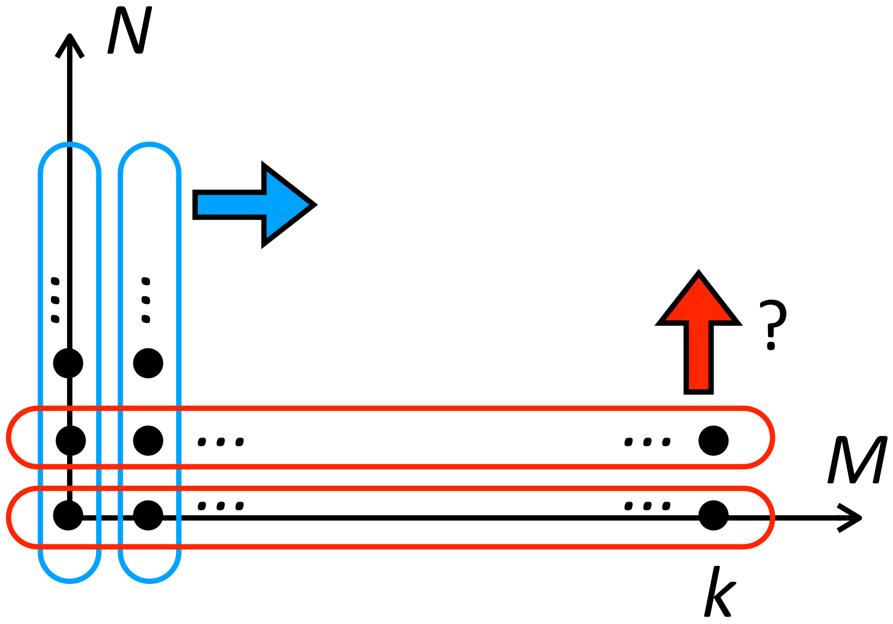
For ABJM theory

$$\Xi_M(\kappa)=\sum_{N=0}^{\infty}\kappa^N Z^{U(N)_k\times U(N+M)_{-k}}$$

$$\boxed{ie^{\frac{\pi iM}{k}}\Xi_{M+1}(-i\kappa)\Xi_{M-1}(i\kappa)+\Xi_M(\kappa)^2-e^{\frac{2\pi iM}{k}}\Xi_M(-\kappa)^2=0}$$

Can be solved recursively in *M* !

Ξ_0,Ξ_1 ⇨ Ξ_M for all *M* ≥ 2



$$\boxed{\sum_{\substack{\ell,m\geq 0\\(\ell+m=N)}}\left[i(-1)^{\ell}Z_{M+1}(\ell)Z_{M-1}(m)+\left(-e^{\frac{\pi iM}{k}}-(-1)^{n}e^{-\frac{\pi iM}{k}}\right)Z_M(\ell)Z_M(m)\right]=0}$$

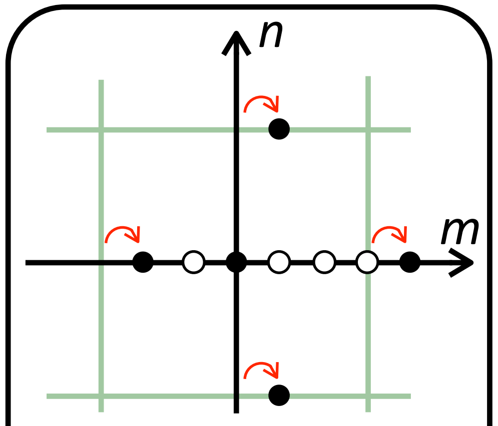
Can we solve recursively in *N* (possibly with additional constraints)?

[Moriyama,TN,*work in progress*]

Generalizations?

- Mass (R-charge) deformations?

$$\boxed{Z_{S^3} = \frac{1}{N_a!}\int \frac{d^{N_a}\lambda_i^{(a)}}{(2\pi)^{N_a}} \prod_{a=1}^{L} e^{\frac{ik_a}{4\pi}\sum_{i=1}^{N_a}(\lambda_i^{(a)})^2} \prod_{a=1}^{L} \frac{\prod_{i<j}^{N_a}(2\sinh\frac{\lambda_i^{(a)}-\lambda_j^{(a)}}{2})^2}{\prod_{i=1}^{N_a}\prod_{j=1}^{N_{a+1}} 2\cosh\frac{\lambda_i^{(a)}-\lambda_j^{(a+1)}-\xi_a}{2}}}$$



$$\nu = \frac{2}{1 - i\pi^{-1}\xi}$$

large *N* phase transition in ξ_a [Honda,TN,Shimizu,Terashima,'18]

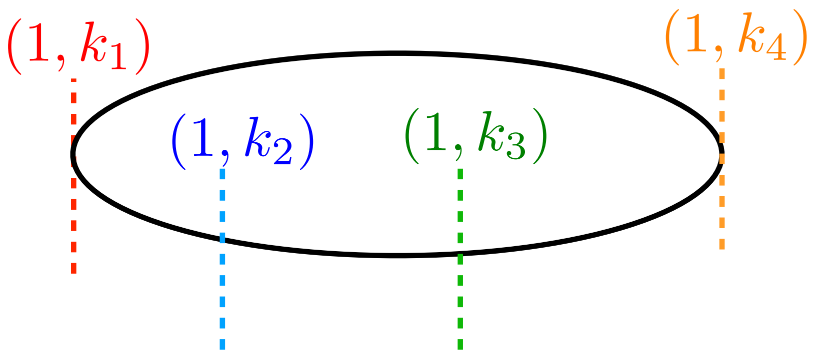
For ABJM with $\xi_1 = \xi_2$, q-PIII$_3$ eq → generalization of ν-site q-Toda eq

[TN,'20]

might be useful in understanding phase transition

- Other observables on S^3 ?

- Longer circular quivers / $\mathcal{N}=3$ circular quivers?



Backup slides

Precision holography (not this talk)

$$\boxed{-\log Z_{S^3} = \underbrace{N^{3/2} + N^{1/2} + \log N + \cdots}_{\text{Airy}} + e^{-\sqrt{\frac{N}{k}}} + \cdots + e^{-\sqrt{kN}} + \cdots}$$

- $\log N$: graviton 1-loop [Bhattacharyya,Grassi,Marino,Sen]

- $N^{1/2}$: higher derivative terms [Bobev,Charles,Hristov,Reys]

- Proposals for Airy : SUGRA localization [Dabholkar,Drukker,Gomes]

  partial gravitational path integral [Caputa,Hirano]

- 1/*N* non-perturbative : closed M2 wrapped on $\mathrm{AdS}_4 \times Y_7$

  [Cagnazzo,Sorokin,Wulff][Becker,Becker,Strominger][Drukker,Marino,Putrov]
  [Gautason,Puletti,vanMuiden]