M2-branes, quantum curves and q-Painlevé equations

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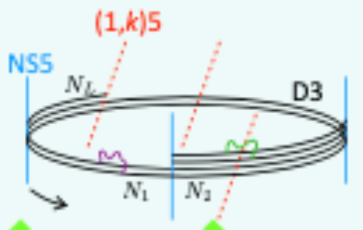
based on [Bonelli,Globlek,Kubo,**TN**,Tanzini,2202.10654], [Moriyama,**TN**,2305.03978]

M2-branes → quantum algebraic curves

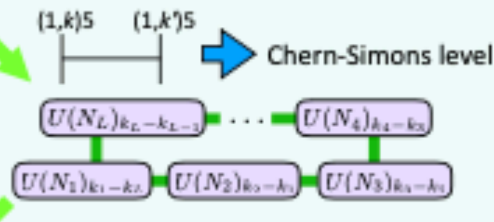


	x_{012}	x_3	x_{456}	x_{789}
D3	– – –	–		
NS5	– – –		– – –	
D5	– – –			– – –

(1,k)5 = bound state of NS5 and k D5

T-duality and M-uplift

$N=\min(N_1,\cdots,N_L)$ M2-branes on $(\mathbb{C}^2/\mathbb{Z}_{\#(NS5)}\times\mathbb{C}^2/\mathbb{Z}_{\#((1,k)5)})/\mathbb{Z}_k$



Chern-Simons level = $k'-k$

S^3 partition function:

$$Z(N)=\frac{1}{\prod_a N_a!}\int\prod_{a=1}^{L}\frac{d^{N_a}\lambda_i^a}{(2\pi)^{N_a}}e^{\frac{ik_a}{4\pi}\sum_i(\lambda_i^a)^2}\frac{\prod_{i<j}2\sinh\frac{\lambda_i^a-\lambda_j^a}{2}\prod_{i<j}2\sinh\frac{\lambda_i^{a+1}-\lambda_j^{a+1}}{2}}{\prod_{i,j}2\cosh\frac{\lambda_i^a-\lambda_j^{a+1}}{2}}$$

Fermi gas formalism

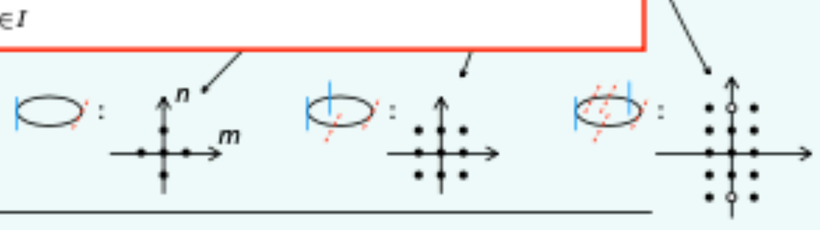
$$Z(N)=\frac{1}{N!}\int\frac{d^Nx}{(2\pi)^N}\det_{i,j}\langle x_i|\hat\rho|x_j\rangle$$

$$N_1=\cdots=N_L=N \;\Rightarrow\; \hat\rho=\prod_{a=1}^{L}\frac{1}{2\cosh\frac{\hat x\text{ or }\hat p}{2}}\qquad[\hat x,\hat p]=2\pi ik$$

(1,k)5 / NS5

General ranks $\Rightarrow$ $\hat O=\hat\rho^{-1}=\sum_{(m,n)\in I}c_{mn}(\Delta N_i)e^{m\hat x+n\hat p}$ $\quad I$: independent of ΔN_i

examples: I for



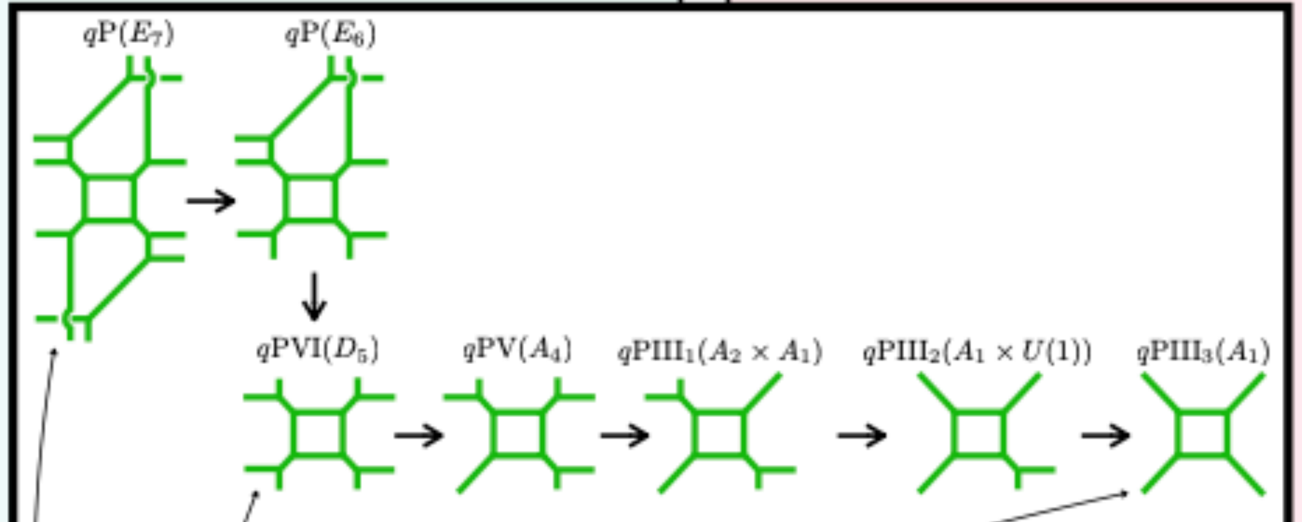
M2-branes ← quantum algebraic curves?

Fermi gas formalism manifestates $-\log Z(N)\sim-\log \mathrm{Ai}[\bigcirc(N-\square)]\sim N^{3/2}$ = M2 entropy

argued to be universal for $\mathrm{AdS}_4\times Y_7$

[Drukker,Dabholkar,Gomes,'14][Caputa,Hirano,'18][Hristov,'22] [Klebanov,Tseytlin,'96]

$Z(N)$ with any Newton polygon I might be regarded as a M2-brane setup



Conjecture:

$$\sum_{N=0}^{\infty}\kappa^N Z^{\rm M2}_{S^3}(N)=Z_{S^3}(0)\mathrm{Det}(1+\kappa\hat O^{-1})=\tau^{q\rm PX}$$

satisfies κ-independent bilinear relations called q-Painlevé

$\Rightarrow$ ∞-ly many identities for M2-partition functions with different N's!

q-Painlevé equations classified by algebraic curves

Painlevé eq: 2nd order non-linear ODE which is non-trivial and does not have movable branch points

example (PIII_3): $\ddot\lambda-\frac{\dot\lambda^2}{\lambda}+\frac{\dot\lambda}{t}-\frac{2\lambda^2}{t^2}+\frac{2}{t}=0$

PIII_3 is equivalent to 2-particle periodic Toda $\frac{d^2q_i}{d(\log t)^2}=t^{\frac12}(e^{q_{i+1}-q_i}-e^{q_i-q_{i-1}})$, $\quad q_1-q_2=-\log(-t^{-\frac12}\lambda)$

$\lambda=-t\left(\frac{d^2\log\tau_1}{d(\log t)^2}\right)^{-1}=-\frac{d^2\log\tau_2}{d(\log t)^2}$

bilinear form: $\tau_i\frac{d^2\tau_i}{d(\log t)^2}-\left(\frac{d\tau_i}{d(\log t)}\right)^2-4t^{\frac12}\tau_{i+1}\tau_{i-1}=0$

$t\to(q-1)^4t,\ q\to1$

q-uplift: $\tau_i(qt)\tau_i(q^{-1}t)-\tau_i(t)^2-4t^{\frac12}\tau_{i+1}\tau_{i-1}=0$: $q\mathrm{PIII}_3$ equation

More general q-Painlevé eqs are classified by genus 1 algebraic curves $\mathcal{O}(u,v)=0$ with E_n symmetry

Math (Sakai's classification): [Sakai,'00]

$\tau_i(t;\{\theta_a\})\to\tau_M$: defined on lattice sites

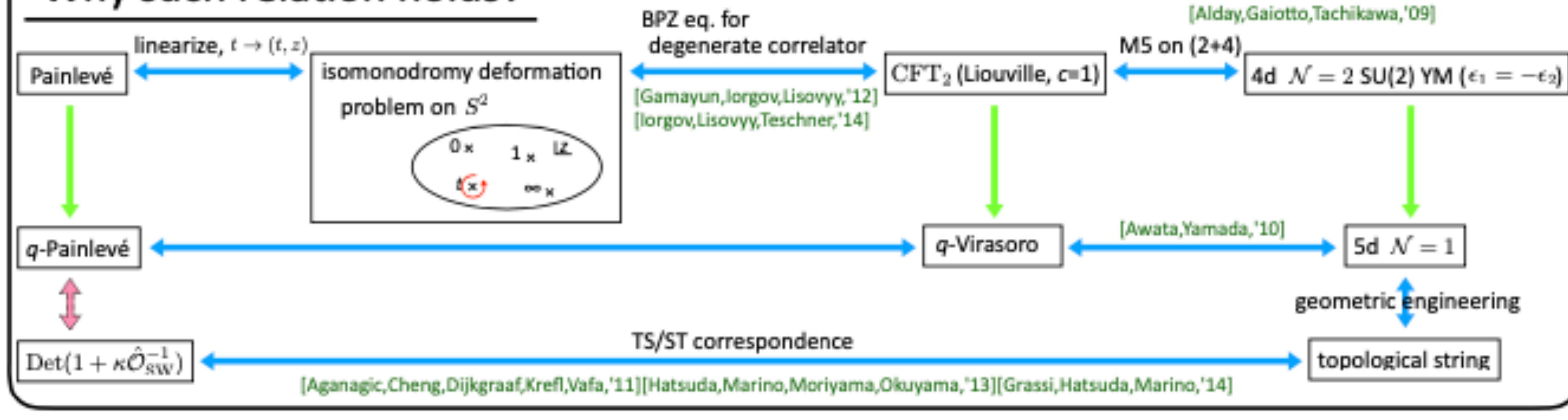
q-discrete bilinear relation ⇔ τ_M is a birational representation of affine Weyl group $\widehat W(E_n)$

Physics (Painlevé/gauge correspondence): [Bonelli,Lisovyy,Maruyoshi,Sciarappa,Tanzini,'16]

q-Painlevé eq. ⇔ bilinear relations (thought to be) satisfied by $Z^{\mathcal O}_{\rm NO}=\sum_{n\in\mathbb Z}s^nZ^{\mathcal O}_{\rm Nek}(\sigma+2\pi in)$ of 5d $\mathcal N=1$ SU(2) YM whose SW curve is $\mathcal O(u,v)$

due to blowup equations [Nakajima,Yoshioka,'05]

Why such relation holds?



Example 1: ABJ(M) theory ↔ $q\mathrm{PIII}_3$ [Bonelli,Grassi,Tanzini,'17]

$U(N)_k$ — $U(N+M)_{-k}$

Quantum curve

Apply Cauchy-Vandermonde identity to SUSY localization formula

$$\frac{\prod_{i<j}^N2\sinh\frac{x_i-x_j}{2k}\prod_{i<j}^{N+M}2\sinh\frac{y_i-y_j}{2k}}{\prod_{i=1}^N\prod_{j=1}^{N+M}2\cosh\frac{x_i-y_j}{2k}}=k^{N+\frac M2}\det\ldots=k^{N+\frac M2}\det\begin{pmatrix}\langle x_i|\frac{1}{2\cosh\frac{\hat p-i\pi M}{2}}|y_j\rangle\\ \langle(2\pi i(\frac{M+1}{2}-r))|y_j\rangle\end{pmatrix}\qquad[\hat x,\hat p]=2\pi ik$$

$$Z=\frac{1}{N!(N+M)!}\int\frac{d^N\lambda_i^{(1)}d^{N+M}\lambda_i^{(2)}}{(2\pi)^N(2\pi)^{N+M}}\det\left[\ldots\right]\times\det\left[\ldots\right]$$

$\hat U=\hat V=e^{\frac{i}{4\pi k}\hat x^2}$

Reorganize as

$$\frac{\prod_{i<j}^N2\sinh\frac{\lambda_i^{(1)}-\lambda_j^{(1)}}{2k}\prod_{i<j}^N2\sinh\frac{\lambda_i^{(2)}-\lambda_j^{(2)}}{2k}}{\prod_{i,j}^N2\cosh\frac{\lambda_i^{(1)}-\lambda_j^{(2)}}{2k}}\ldots=\det_{i,j}\left[\langle\lambda_i^{(1)}|\frac{1}{2\cosh\frac{\hat p}{2}}|\lambda_j^{(2)}\rangle\right]$$

$$\Rightarrow\ Z=\frac{Z(0)}{N!}\int\frac{d^N\lambda_i^{(1)}}{(2\pi)^N}\det_{i,j}\langle\lambda_i^{(1)}|\hat\rho|\lambda_j^{(1)}\rangle$$

$$Z(0)=\frac{1}{k^{\frac M2}}\prod_{i>j}^{N}2\sin\frac{\pi(i-j)}{k}$$

$$\hat\rho=\frac{1}{2\cosh\frac{\hat x'}{2}}\frac{\prod_{r=1}^M\tanh\frac{\hat x+2\pi i(\frac{M+1}{2}-r)}{2k}}{2\cosh\frac{\hat x+i\pi M}{2}}\qquad \hat x',\hat p'=\frac{\hat x\mp\hat p}{2}$$

$$\hat O=\hat\rho^{-1}=e^{\pi iM-\frac{\pi ik}{4}}e^{\hat x'}+e^{\frac{\pi ik}{4}}e^{\hat p'}+e^{\frac{\pi ik}{4}}e^{-\hat p'}+e^{\pi iM-\frac{\pi ik}{4}}e^{-\hat x'}$$

[Awata,Hirano,Shigemori,'12] [Kashaev,Marino,Zakany,'15]

q-difference relation

$(q)\mathrm{PIII}_3$	2d	4d/5d	ABJ(M)
t	cross ratio	gauge coupling = moduli of SW curve	M

$$\Xi_M(\kappa)=\sum_{N=0}^{\infty}\kappa^NZ_M(N)$$

precise dictionary: $t=e^{-2\pi i(1-\frac{2M}{k})}$, $q=e^{\frac{4\pi i}{k}}$ ⇒ $q^{\pm1}t\leftrightarrow M\pm1$; $q\mathrm{PIII}_3$: difference relation in relative rank

$$\Xi_M(\kappa)^2-e^{\frac{2\pi iM}{k}}\Xi_M(-\kappa)^2+ie^{\frac{\pi iM}{k}}\Xi_{M+1}(-i\kappa)\Xi_{M-1}(i\kappa)=0$$

[Grassi,Hatsuda,Marino,'14][Bonelli,Grassi,Tanzini,'17]

- checked against exact values of $Z_{k=1,2,\cdots,\ M=0,1,\cdots,k}(N=0,1,\cdots)$, but not proved yet
- $q\mathrm{PIII}_3$ is satisfied also by $Z(N{=}0)$ although it is not directly related to quantum curve!

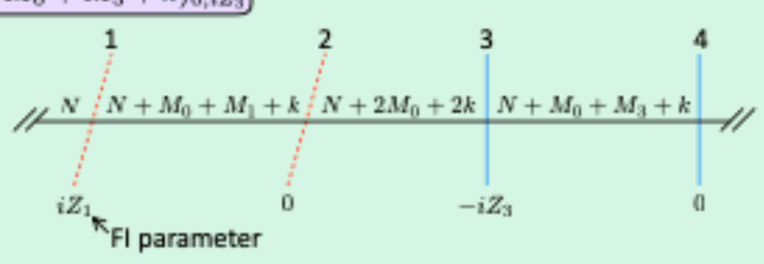
Example 2: (2,2) model ↔ $q\mathrm{PVI}$ [BGKNT,'20][MN,'22]

$U(N+M_0+M_1+k)_{0,-iZ_1}$, $U(N+2M_0+2k)_{-k,-iZ_2}$, $U(N)_{k,iZ_1}$, $U(N+M_0+M_3+k)_{0,iZ_3}$

Quantum curve

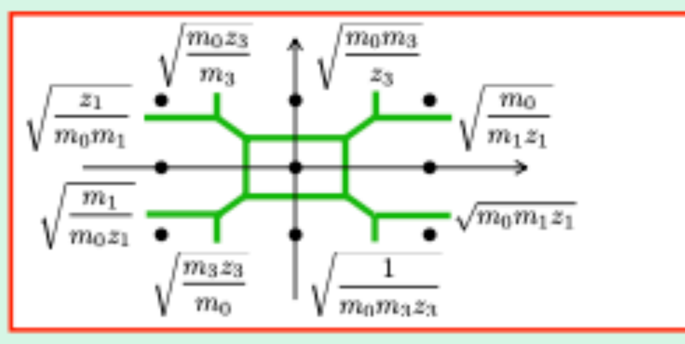
$$\hat\rho^{-1}=\sum_{m,n\in I}c_{mn}(M_0,M_1,M_3,Z_1,Z_3)e^{m\hat x+n\hat p}$$

- $c_{mn}=e^{\text{1st order pol. of }M_0,M_1,M_3,Z_1,Z_3}$
- Hanany-Witten trsf. = IR duality ⇒ symmetry of curve



moduli parameters of curve can be determined by extrapolation from

$(M_0,M_1,M_3)=(0,0,0)$, $(M_0,M_1,M_3)=(0,-k,0)$, $(M_0,M_1,M_3)=(-\frac k2,-\frac k2,-\frac k2)$, $(M_0,M_1,M_3)=(0,0,-k)$



$m_0=e^{2\pi iM_0},m_1=e^{2\pi iM_1},m_3=e^{2\pi iM_3},z_1=e^{2\pi iZ_1},z_3=e^{2\pi iZ_3}$

$W(D_5)$ symmetry of curve (= exchange of asymptotic loci) acts as $M_0\leftrightarrow M_1$, $(M_0,M_1)\to(-M_0,-M_1)$, etc.

[Furukawa,Matsumura,Moriyama,Nakanishi,'21] [BGKNT,'20]

q-difference relation

natural guess from $W(D_5)$ actions or translate bilinear eq of $Z^{\rm 5d\ SU(2)\ N_f=4}_{\rm NO}$ [Jimbo,Nagoya,Sakai,'17]

$$\bigcirc_1\prod_{\pm}\Xi_{M_0\pm\frac12,M_1\pm\frac12}(\gamma_1^{\pm1}\kappa)+\bigcirc_2\prod_{\pm}\Xi_{M_0\pm\frac12,M_1\mp\frac12}(\gamma_2^{\pm1}\kappa)+\bigcirc_3\prod_{\pm}\Xi_{M_3\pm\frac12,Z_1\pm\frac12,Z_3\pm\frac12}(\gamma_3^{\pm1}\kappa)=0$$

$\bigcirc_1,\bigcirc_2,\bigcirc_3$ can be guessed by using exact values of $Z(N{=}0)$

$\gamma_1,\gamma_2,\gamma_3$ can be guessed by using exact values of $Z(N{=}1)$

$$e^{-\frac{\pi i}{2k}(\sigma_cM_c+\sigma_dM_d+\sigma_eM_e)}S^{(1)}_M\prod_\pm\Xi_{M_a\pm\frac12,M_b\pm\frac12}(\kappa)+e^{\frac{\pi i}{2k}(\sigma_cM_c+\sigma_dM_d+\sigma_eM_e)}S^{(2)}_M\prod_\pm\Xi_{M_a\pm\frac12,M_b\mp\frac12}(-\kappa)+S^{(3)}_M\prod_\pm\Xi_{M_c\pm\frac{\sigma_c}{2},M_d\pm\frac{\sigma_d}{2},M_e\pm\frac{\sigma_e}{2}}(\mp i\kappa)=0$$

: 40 bilinear relations for the choices of (a,b): two directions of (M_0,M_1,M_3,Z_1,Z_3) and $(\sigma_c,\sigma_d,\sigma_e)=(+++),(+--),(-+-),(--+)$

$(a,b)=(M_1,Z_1)$: $S^{(1)}_M=2\sin\frac{\pi(M_1+Z_1)}{k}$, $S^{(2)}_M=2\sin\frac{\pi(M_1-Z_1)}{k}$, $S^{(3)}_M=2\sin\frac{\pi(M_3+\sigma_{M_3}Z_3)}{k}$

$(a,b)=(M_3,Z_3)$: $S^{(1)}_M=2\sin\frac{\pi(M_3+Z_3)}{k}$, $S^{(2)}_M=2\sin\frac{\pi(M_3-Z_3)}{k}$, $S^{(3)}_M=2\sin\frac{\pi(M_1+\sigma_{M_1}Z_1)}{k}$

$(a,b)=(M_0,M_1)$: $S^{(1)}_M=S^{(2)}_M=1$, $S^{(3)}_M=2\sin\frac{\pi(M_3+\sigma_{Z_3}Z_3)}{k}$

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- 40 relations are $W(D_5)$-covariant (non-covariance of $\bigcirc_1,\bigcirc_2,\bigcirc_3$ is canceled by non-covariance of $Z(N{=}0)$)

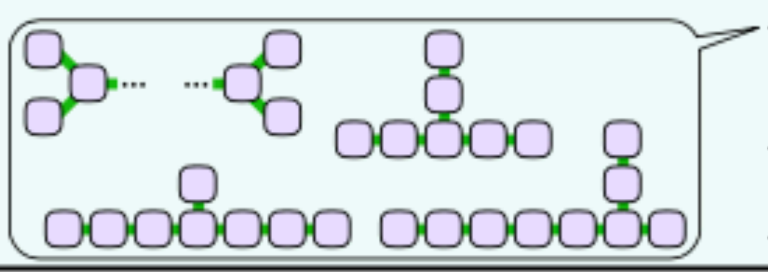
Future directions

① Proof of bilinear relations from matrix model/physics of M2-branes?

② q-difference equations for other theories of M2-branes?

- Other $\mathcal N=4$ circular quivers?
- additional matter fields?
- mass deformation?
- $\mathcal N=3$ circular quiver? [TN,'20]

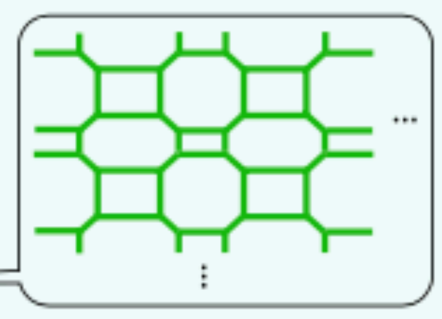
- non-unitary gauge groups?
- non-circular quiver?



- Other manifolds / other observables?
- Other 3d theories (not M2-type)?

③ Matrix models for other q-difference equations (or 5d super Yang-Mills)?

- Other q-painlevé?
- 5d $\mathcal N=1$ Yang-Mills with...
  - more matter fields?
  - SU(>2) gauge group?
  - quiver type?



- 5d $\mathcal N=2^{(*)}$ Yang-Mills? ⇒ Calogero-Moser
- non unitary gauge group (ABCDEFG)? ⇒ q-Toda on affine Dynkin diagram
- non self-dual Ω-backgroud? ⇒ *quantum* Painlevé/Toda