Supersymmetric indices of M2-branes and M2/M5 giant graviton expansion

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Based mainly on
- Hayashi,TN,Okazaki,2409.13239 (JHEP12(2024)109)
- Hayashi,TN,Okazaki,2508.20663

@Supersymmetric Indices, Giant Gravitons, and Matrix Models (4-8 Dec 2025)

Plan of talk

1. Introduction

2. 3d U(N) ADHM theory

3. Coulomb limit and giant graviton expansion

4. Comparison with M5-indices

5. Higgs limit

Supersymmetric index $I(q_i)$

$I = \mathrm{Tr}(-1)^F e^{-\beta\{Q,Q^\dagger\}}$ (Q : a supercharge) is independent of β

$$\Rightarrow \boxed{I = I_{\beta=\infty} = \mathrm{Tr}_{Q=Q^\dagger=0}(-1)^F}$$

$\mathcal{H}_{Q=Q^\dagger=0}$ is still ∞-dimensional (and hence $I_{\beta=\infty}$ is divergent),

but we can make it finite by a refinement

$$I(q_i) = \mathrm{Tr}(-1)^F e^{-\beta\{Q,Q^\dagger\}} \prod_i q_i^{C_i}$$

: β-indep. if $[Q, C_i] = 0$

$$\overset{\beta=\infty}{=} \mathrm{Tr}_{Q=Q^\dagger=0}(-1)^F \prod_i q_i^{C_i}$$

$$= 1 + \bigcirc q_1 + \bigcirc q_1^2 q_2 + \cdots$$

$I(q_i)$ is independent of continuous parameters of the theory

$\Rightarrow$ we can study strongly coupled QFTs from weak coupling analysis

dualities, symmetry enhancement, ...; holography

A common structure of SI in holographic theories

Examples: *N* D3-branes $\longleftrightarrow \mathrm{AdS}_5 \times Y_5$

N M2-branes $\longleftrightarrow \mathrm{AdS}_4 \times Y_7$

N M5-branes $\longleftrightarrow \mathrm{AdS}_7 \times Y_4$

A "large *N* limit" exists:

$$\boxed{\begin{aligned} I_{N=1} &= 1 + a_1^{(\infty)} q + a_2^{(1)} q^2 + a_3^{(1)} q^3 + a_4^{(1)} q^4 + \cdots \\ I_{N=2} &= 1 + a_1^{(\infty)} q + a_2^{(\infty)} q^2 + a_3^{(2)} q^3 + a_4^{(2)} q^4 + \cdots \\ I_{N=3} &= 1 + a_1^{(\infty)} q + a_2^{(\infty)} q^2 + a_3^{(\infty)} q^3 + a_4^{(3)} q^4 + \cdots \\ &\vdots \end{aligned}} \Rightarrow I_\infty \equiv \sum_n a_n^{(\infty)} q^n$$

From the viewpoint of $N = \infty$, deviation at $\mathcal{O}(q^{N+1})$ is due to "overcounting" of gauge invariant operators

c.f. trace constraint on $\mathrm{tr}X^n$: $\mathrm{tr}X^{N+1} = f(\mathrm{tr}X, \mathrm{tr}X^2, \cdots \mathrm{tr}X^N)$

Gravitiy interpretation:

I_∞ = superconformal index of graviton multiplet in $\mathrm{AdS}_m \times Y_n$

[Kinney,Maldacena,Minwalla,Raju,'05][Bhattacharya,Bhattacharyya,Minwalla,Raju,'08]

Finite *N* corrections = branes wrapped on Y_n

Wrapped branes and "giant graviton expansion"

$\mathrm{AdS}_5 \times S^5 \longrightarrow$ D3 on $\mathbb{R} \times S^3$

$\mathrm{AdS}_4 \times S^7 \longrightarrow$ M5 on $\mathbb{R} \times S^5$

$\mathrm{AdS}_7 \times S^4 \longrightarrow$ M2 on $\mathbb{R} \times S^2$

- energy of wrapped *p*-brane $\sim (R_{S^n})^{p+1} \sim N$
- topologically trivial cycle ⇒ $^\exists$tachyonic mode

⇒ $\dfrac{q^N}{1-q^{-1}}(\cdots) = \boxed{-\dfrac{q^{N+1}}{1-q}(\cdots)}$

[Arai,Imamura,'19]

(c.f. finite *N* corrections can be understood as volume constraint on **"giant gravitons"**

- graviton with angular momentum L in $S^n \longrightarrow$ expands in S^n by Myers effect (size $\sim L^{\bigcirc}$)
- cannot be bigger than $S^n \longrightarrow L \leq N$)

[McGreevy,Susskind,Toumbas,'00]

"Giant graviton expansion" of supersymmetric index:

$$\boxed{I_N = I_\infty(1 + c_1(q)q^N + c_2(q)q^{2N} + \cdots)}$$

fluctuation modes of wrapped brane

fluctuation modes of 2 wrapped branes

Self-similarity structure

Fluctuation modes = supersymmetric index on wrapped branes

$$\boxed{\begin{aligned} \frac{I_{N\,\mathrm{M2}}(q)}{I_{\infty\,\mathrm{M2}}(q)} &= 1 + \sum_{m=1}^{\infty} q^{mN} I_{m\,\mathrm{M5}}(q') \\ \frac{I_{N\,\mathrm{M5}}(q)}{I_{\infty\,\mathrm{M5}}(q)} &= 1 + \sum_{m=1}^{\infty} q^{mN} I_{m\,\mathrm{M2}}(q') \end{aligned}}$$

change of variables (later)

This talk: check these relations explicitly

- Requires calculation of indices for higher *N* and to higher order in *q* (at least $\mathcal{O}(q^{N+1})$)

  ⇨ difficult in general

- Key: specialization (1/4 BPS limit) of full indices

  ⇨ finite *N* indices in closed form expression (∞-order in *q* !)

By-product: new method to calculate indices of multiple M5-branes

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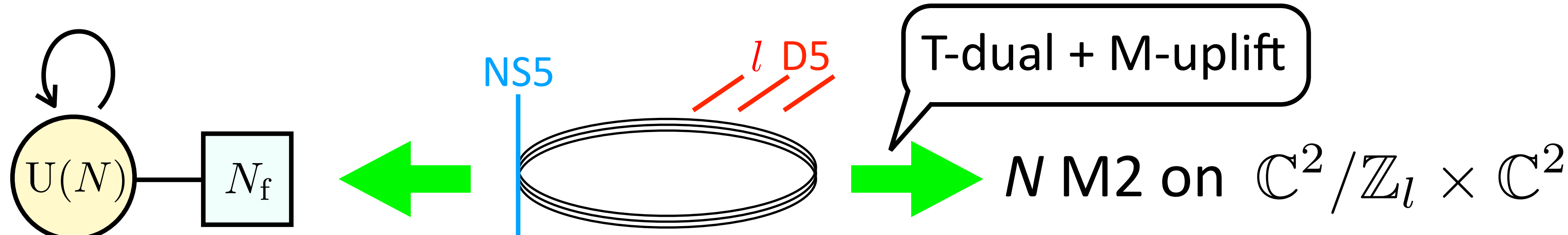
3d $\mathcal{N}=4$ ADHM theory



[de Boer,Hori,Ooguri,Oz,'96][Benini,Closset,Cremonesi,'09]

$\mathcal{N}=4$ supersymmetry:

- spacetime: $p_\mu,\ j_{\mu\nu}$
- SUSY: $\mathcal{Q}^I_\alpha$
- R-symmetry: $SO(4) = SU(2)_H \times SU(2)_C$

Supersymmetric index:

Choose one $\mathcal{Q}$ of $\{\mathcal{Q}^I_\alpha\}$ s.t.

	h	j_{12}	H	C
$\mathcal{Q}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	1

$\left(\{\mathcal{Q},\mathcal{Q}^\dagger\} = h - j_{12} - \frac{H+C}{2}\right)$

$$\Rightarrow\ I = \mathrm{Tr}[(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} q^{h-\frac{H+C}{4}} t^{H-C} z^{f_C} x^{f_H}]$$

$[\cdot,\mathcal{Q}]=0 \Rightarrow I$ is β-indep.

$$\boxed{I \overset{\beta=\infty}{=} \mathrm{Tr}_{\{\mathcal{Q},\mathcal{Q}^\dagger\}=0}(-1)^F q^{h-\frac{H+C}{4}} t^{H-C} z^{f_C} x^{f_H}}$$

$f_{C/H}$: global symmetry (topological U(1) / flavor sym.)

Supersymmetric index of ADHM theory

projection to gauge singlet

vector

adj. hyp

$$I^{\mathrm{U}(N)\text{-}[l]}=\frac{1}{N!}\sum_{m_i\in\mathbb{Z}}\int\frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\Big(1-q^{\frac{|m_i-m_j|}{2}}\frac{s_i}{s_j}\Big)\prod_{i,j}\frac{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}}t^2\frac{s_i}{s_j};q)_\infty}{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}}t^{-2}\frac{s_i}{s_j};q)_\infty}\prod_{i,j}\prod_{\pm}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_j|}{2}}t^{-1}x^{\pm1}\frac{s_i}{s_j};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_j|}{2}}tx^{\pm1}\frac{s_i}{s_j};q)_\infty}$$

$$\times\prod_{i}\prod_{\alpha=1}^{l}\prod_{\pm}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}t^{-1}(s_iy_\alpha)^{\pm1};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}t(s_iy_\alpha)^{\pm1};q)_\infty}z^{l\sum_i m_i}q^{\frac{l}{4}\sum_i|m_i|}t^{-l\sum_i|m_i|}$$

monopole flux on S^2

fund. hyp

[Kim,'09]

$$\Big(\frac{1}{(x;q)_\infty}=\prod_{n=0}^{\infty}\frac{1}{1-xq^n}=\mathrm{PE}[x+\underset{\substack{\uparrow\\ \partial}}{qx}+\underset{\substack{\uparrow\\ \partial^2}}{q^2x}+\cdots]\Big)$$

Examples:

$$I^{\mathrm{U}(1)\text{-}[1]}=1+[(x+x^{-1})t+(z+z^{-1})t^{-1}]q^{\frac{1}{4}}$$
$$+[(x^2+1+x^{-2})t^2+xz+xz^{-1}+x^{-1}z+x^{-1}z^{-1}+(z^2+1+z^{-2})t^{-2}]q^{\frac{1}{2}}+\cdots$$

$$I^{\mathrm{U}(2)\text{-}[1]}=1+[(x+x^{-1})t+(z+z^{-1})t^{-1}]q^{\frac{1}{4}}$$
$$+[(2x^2+2+2x^{-2})t^2+2xz+2xz^{-1}+2x^{-1}z+2x^{-1}z^{-1}+(2z^2+2+2z^{-2})t^{-2}]q^{\frac{1}{2}}+\cdots$$

$$I^{\mathrm{U}(3)\text{-}[1]}=1+[(x+x^{-1})t+(z+z^{-1})t^{-1}]q^{\frac{1}{4}}$$
$$+[(2x^2+2+2x^{-2})t^2+2xz+2xz^{-1}+2x^{-1}z+2x^{-1}z^{-1}+(2z^2+2+2z^{-2})t^{-2}]q^{\frac{1}{2}}+\cdots$$

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Coulomb limit ($q, t \to 0$ with $\mathfrak{t} = q^{\frac{1}{4}} t^{-1}$ kept fixed)

Rewrite SI as $\boxed{I = \mathrm{Tr}_{\{\mathcal{Q},\mathcal{Q}^\dagger\}=0}[(-1)^F (q^{\frac{1}{4}} t)^{2h-C} (q^{\frac{1}{4}} t^{-1})^{2h-H} z^{f_C} x^{f_H}]}$

$$\{\mathcal{Q}^I_\alpha, (\mathcal{Q}^I_\alpha)^\dagger\} = h \pm j_{12} \pm' \frac{C}{2} \pm'' \frac{H}{2} \quad \Rightarrow \quad 2h - C, 2h - H \geq 0$$

$$\mathcal{I}_C = \lim_{\substack{q,t\to 0 \\ (q^{\frac{1}{4}} t^{-1} = \mathfrak{t})}} I = \mathrm{Tr}_{\{\mathcal{Q},\mathcal{Q}^\dagger\}=0,2h-C=0}[\mathfrak{t}^{2h-H} z^{f_C} x^{f_H}]$$

$$0 = h + j_{12} + \frac{H-C}{2} = \{\mathcal{Q}', (\mathcal{Q}')^\dagger\}$$

$\Rightarrow$ counts only 1/4 BPS states with $\mathcal{Q}^{(\dagger)} = (\mathcal{Q}')^{(\dagger)} = 0$

SCI simplifies drastically in Coulomb limit

$$\boxed{\begin{aligned} \mathcal{I}_C^{\mathrm{U}(N)\text{-}[l]} = \frac{1}{N!} \sum_{m_i \in \mathbb{Z}} \int \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(1 - q^{\frac{|m_i - m_j|}{2}} \frac{s_i}{s_j}\right) \prod_{i,j} \frac{\cancel{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}} t^2 \frac{s_i}{s_j}; q)_\infty}}{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}} t^{-2} \frac{s_i}{s_j}; q)_\infty} \cancel{\prod_{i,j} \prod_\pm \frac{(q^{\frac{3}{4}+\frac{|m_i-m_j|}{2}} t^{-1} x^{\pm 1} \frac{s_i}{s_j}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_j|}{2}} t x^{\pm 1} \frac{s_i}{s_j}; q)_\infty}} \\ \times \cancel{\prod_i \prod_{\alpha=1}^{l} \prod_\pm \frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}} t^{-1} (s_i y_\alpha)^{\pm 1}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}} t (s_i y_\alpha)^{\pm 1}; q)_\infty}} z^{l \sum_i m_i} q^{\frac{l}{4} \sum_i |m_i|} t^{-l \sum_i |m_i|} \end{aligned}}$$

$$\to \begin{cases} 1 - (\mathfrak{t}^2) \frac{s_i}{s_j} & (m_i = m_j) \\ 1 & (\text{otherwise}) \end{cases} \quad \Rightarrow \quad \int d^N s \quad \text{factorize in pieces}$$

Coulomb limit as Fermi gas

It is useful to label monopole charges by $\nu_m = \#\{i \,|\, m_i = m\}$

example (N=5): $m_i = (1, 0, -3, 2, 1)$

$$\to \nu_m = \begin{cases} 1 & (m = -3, 0, 2) \\ 2 & (m = 1) \\ 0 & (otherwise) \end{cases}$$

$$\boxed{\mathcal{I}_C^{\mathrm{U}(N)\text{-}[l]} = \left\{ \sum_{\substack{\nu_m \geq 0 \\ (\sum_m \nu_m = N)}} \right\} \left(\prod_{m=-\infty}^{\infty} \frac{1}{\nu_m!} \int d^{\nu_m}\sigma (\cdots) \right)}$$

Constrained sum simplifies in grand canonical sum

$$\boxed{\Xi(u) = \sum_{N=0}^{\infty} \mathcal{I}_C^{\mathrm{U}(N)\text{-}[l]} = \prod_{m=-\infty}^{\infty} \tilde{\Xi}(\mathfrak{t}^{l|m|} z^{lm} u)}$$

$$\tilde{\Xi}(u) = \sum_{\nu=0}^{\infty} u^{\nu} \frac{1}{\nu!} \int \frac{d\sigma_i}{2\pi i \sigma_i} \frac{\prod_{i \neq j}^{\nu} (1 - \frac{\sigma_i}{\sigma_j})}{\prod_{i,j=1}^{\nu} (1 - \mathfrak{t}^2 \frac{\sigma_i}{\sigma_j})} = \sum_{\nu=0}^{\infty} u^{\nu} \frac{\mathfrak{t}^{-\nu(\nu-1)}}{\nu!} \int \frac{d\sigma_i}{2\pi i \sigma_i} \det_{i,j} \left[\frac{1}{\frac{\sigma_i}{\sigma_j} - \mathfrak{t}^2} \right]$$

$$= \langle \sigma_i | \rho(\hat{p}) | \sigma_j \rangle$$

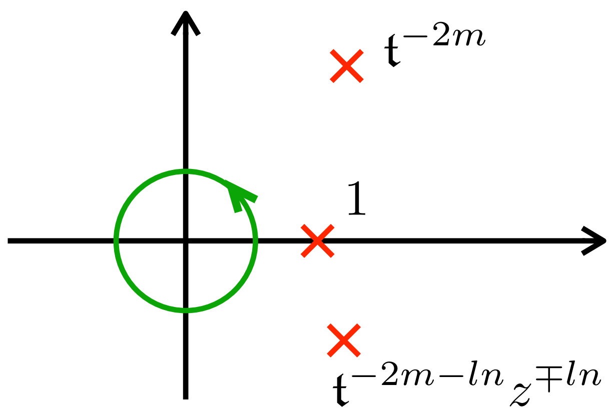
: ν-particle free Fermions on S^1

$\mathrm{tr}\hat{\rho}^n$ can be calculated in momentum basis as

$$\mathrm{tr}\hat{\rho}^n = \frac{1}{1 - \mathfrak{t}^{2n}} \Rightarrow \boxed{\Xi(u) = \prod_{m=0}^{\infty} \frac{1}{1 - \mathfrak{t}^{2m} u} \prod_{m=0}^{\infty} \prod_{n=1}^{\infty} \prod_{\pm} \frac{1}{1 - \mathfrak{t}^{2m+ln} z^{\pm ln} u}}$$

Large *N* limit and finite *N* corrections

$$\mathcal{I}_C^{\mathrm{U}(N)\text{-}[l]} = \oint_{|u|=\epsilon} \frac{du}{2\pi i u} u^{-N} \Xi(u) = -\sum_{w\neq 0} \mathrm{Res}\left[\frac{u^{-N}}{u}\Xi(u), u \to w\right]$$



$u = 1 \to \mathcal{I}_C^{\mathrm{U}(\infty)\text{-}[l]}$

$u = \mathfrak{t}^{-\bigcirc} \to \mathfrak{t}^{\bigcirc N}$: finite *N* corrections

$$\frac{\mathcal{I}_C^{\mathrm{U}(N)\text{-}[l]}}{\mathcal{I}_C^{\mathrm{U}(\infty)\text{-}[l]}} = 1 + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left[f_{m,n}^{(l)}(z^{lm}\mathfrak{t}^{2n+lm})^N + g_{m,n}^{(l)}(z^{-lm}\mathfrak{t}^{2n+lm})^N\right] + \sum_{m=1}^{\infty} h_m^{(l)} \mathfrak{t}^{2mN}$$

[Gaiotto,Lee,'21][Hayashi,TN,Okazaki,'22]

For, $l = 1$ we find

$$\frac{\mathcal{I}_C^{\mathrm{U}(N)\text{-}[1]}}{\mathcal{I}_C^{\mathrm{U}(\infty)\text{-}[1]}} = \sum_{m_1,m_2\geq 0} \hat{F}_{m_1,m_2}^{(l=1)}(x_1,x_2) x_1^{m_1 N} x_2^{m_2 N} \qquad (x_1 = z^{-1}\mathfrak{t},\ x_2 = z\mathfrak{t})$$

$$\hat{F}_{m_1,m_2}^{(l=1)}(x_1,x_2) = \prod_{a=1}^{m_1}\prod_{b=0}^{\infty}\frac{1}{1-x_1^{-a}x_2^{b}}\prod_{a=1}^{m_2}\prod_{b=0}^{\infty}\frac{1}{1-x_2^{-a}x_1^{b}}\prod_{a=1}^{m_1}\prod_{b=1}^{m_2}\frac{1}{1-x_1^{-a}x_2^{-b}}$$

Single-sum expansion

Originally we consider small $\mathfrak{t}$ expansion with $|z|=1$.

Now let us consider "single-sum expansion" defined by

- Treat $(x_1,x_2)=(z^{-1}\mathfrak{t},z\mathfrak{t})$ as independent variables
- Expand first in x_2 and truncate at finite (though arbitrary high) order

$$\left(\hat{F}^{(l=1)}_{m_1,m_2} \text{ with } m_2>0\right) \propto x_2^\infty \overset{\text{s.s.}}{=} 0 \quad \Rightarrow \quad \boxed{\frac{\mathcal{I}_C^{\mathrm{U}(N)\text{-}[1]}(x_1,x_2)}{\mathcal{I}_C^{\mathrm{U}(\infty)\text{-}[1]}(x_1,x_2)} \overset{\text{s.s.}}{=} \sum_{m=0}^{\infty} \hat{F}^{(l=1)}_{m,0}(x_1,x_2)x_1^{mN}}$$

Same simplification works for $l\geq 2$

$$\boxed{\frac{\mathcal{I}_C^{\mathrm{U}(N)\text{-}[l]}(x_1,x_2)}{\mathcal{I}_C^{\mathrm{U}(\infty)\text{-}[l]}(x_1,x_2)} \overset{\text{s.s.}}{=} \sum_{m=0}^{\infty} \hat{F}^{(l)}_{m,0}(x_1,x_2)x_1^{lmN} \qquad \hat{F}^{(l)}_{m}(x_1,x_2)=\prod_{a=0}^{l-1}\prod_{b=1}^{m}\prod_{c=0}^{\infty}\frac{1}{1-x_1^{-lb+a}x_2^{lc+a}}}$$

Inverse single-sum expansion

The finite *N* corrections $\hat{F}_m^{(l)}$ behaves like an index after change of variables

$$F_m^{(l)}(x_1, x_2) = \hat{F}_m^{(l)}(x^{-1}, x_1 x_2) = \prod_{a=0}^{l-1} \prod_{b=1}^{m} \prod_{c=0}^{\infty} \frac{1}{1 - x_1^{l(b+c)} x_2^{lc+a}} = 1 + (\text{positive powers of } x_1, x_2)$$

Moreover, finite *m* corrections to $F_m^{(l)}$ turn out to be given by the original index!

$$\frac{F_m^{(l)}}{F_\infty^{(l)}} \overset{\text{s.s.}}{=} \sum_{N=0}^{\infty} \hat{G}_N(x_1, x_2) x_1^{lmN} \Rightarrow \boxed{\hat{G}_N(x_1, x_2) = \mathcal{I}_C^{\mathrm{U}(N)\text{-}[l]}(x_1^{-1}, x_1 x_2)}$$

[Hayashi,TN,Okazaki, '24]

Expected interpretation $(l = 1)$:

$\hat{F}_m^{(l=1)}(x_1, x_2) \Rightarrow$ Contribution of *m* wrapped M5 in $\mathrm{AdS}_4 \times S^7$ to index of *N* M2

$F_m^{(l=1)}(x_1, x_2) \Rightarrow$ Index on *m* M5

$\hat{G}_N(x_1, x_2) = \mathcal{I}_C^{\mathrm{U}(N)\text{-}[1]}(x_1^{-1}, x_1 x_2) \Rightarrow$ Contribution of *N* wrapped M2 in $\mathrm{AdS}_7 \times S^4$ to index of *m* M5

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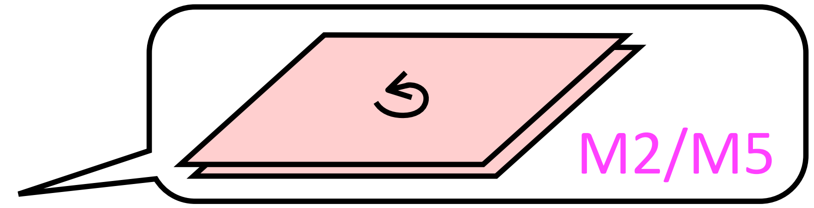
Comparison with M5 indices

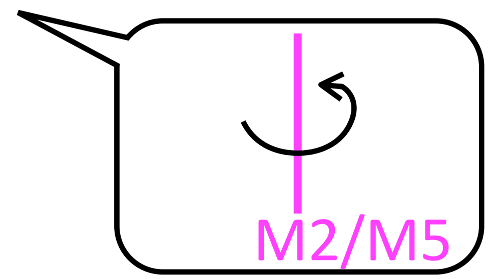
Gravity picture suggests $\hat{F}_m(x_1, x_2) = I^{\mathrm{6d}}_{m\mathrm{M5}}$ under some change of variables

Rough idea:

- spacetime rotation = rotation along M2/M5-branes



- R-symmetry = rotation around M2/M5-branes



Since worldvolume of M5-branes in $\mathrm{AdS}_4 \times S^7$ are wrapped on $\mathbb{R} \times \underset{\substack{\cap \\ S^7}}{S^5}$, we identify

R-symmetry in M2 $\sim$ $\begin{cases} \text{spacetime rotation in M5} \\ \text{R-symmetry in M5} \end{cases}$

spacetime rotation in M2 $\sim$ R-symmetry in M5

dilatation in M2 $\sim$ dilatation in M5

Convention for M2/M5 indices

M2 index

spacetime: $p_\mu,\ j_{\mu\nu},\ h,\ k_\mu$

SUSY: $\mathcal{Q}^I_\alpha$ $\longrightarrow$ Choose one s.t.

	h	j_{12}	r_{12}	r_{34}	r_{56}	r_{78}
$\mathcal{Q}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

R-symmetry: $SO(8)_r$

$$\left(\{\mathcal{Q},\mathcal{Q}^\dagger\} = h - j_{12} - \frac{r_{12}+r_{34}+r_{56}+r_{78}}{2}\right)$$

$$\boxed{I_{\text{M2}} = \mathrm{Tr}[(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} q^{j_{12}+\frac{r_{12}+r_{34}+r_{56}+r_{78}}{4}} u_1^{r_{12}} u_2^{r_{34}} u_3^{r_{56}} u_4^{r_{78}}]}$$

$$(u_1 = z^{-1}t^{-1},\ u_2 = x^{-1}t,\ u_3 = zt^{-1},\ u_4 = xt)$$

M5 index

spacetime: $P_\mu, J_{\mu\nu}, H, K_\mu$

SUSY: Q^I_α $\longrightarrow$ Choose one s.t.

	H	J_{12}	J_{34}	J_{56}	R_{12}	R_{34}
Q	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

R-symmetry: $SO(5)_R$

$$(\{Q,Q^\dagger\} = H - (J_{12}+J_{34}+J_{56}) - 2(R_{12}+R_{34}))$$

$$\boxed{I_{\text{M5}} = \mathrm{Tr}[(-1)^F e^{-\beta\{Q,Q^\dagger\}} p^{\frac{3(R_{12}+R_{34})}{2}+J_{12}+J_{34}+J_{56}} u^{R_{12}-R_{34}} y_1^{J_{12}} y_2^{J_{34}} y_3^{J_{56}}]}$$

$$(y_1 y_2 y_3 = 1)$$

To do: match 6 Cartans by comparing M2/M5 subalgebras preserved by wrapped M5/M2 brane

[Arai,Fujiwara,Imamura,Mori,Yokoyama,'20]

Parameter identification - step 1

$$\frac{I^{\mathrm{U}(N)\text{-}[1]}}{I^{\mathrm{U}(\infty)\text{-}[1]}} = 1 + \hat{F}_1^{(l=1)}(q^{\frac{1}{4}}t^{-1}z^{-1})^N + \cdots$$

ground state contribution of M5 wrapped on $S^5_{z_1=0} \subset S^7$ $\left(S^7 : \sum_{a=1}^{4} |z_a|^2 = 1\right)$

[Mikhailov,'00]

This M5 preserves $\mathcal{Q}^I_{\alpha\,(h-r_{12}=0)}$ ➡ subalgebra $\supset$ $\boxed{\underset{a,b=3,\cdots,8}{so(3)_{j_{ij}} \times so(6)_{r_{ab}}} \times u(1)_{h-\frac{1}{2}r_{12}} \times u(1)_{h-r_{12}}}$ ①

Taking into account inverse expansion, consider also the contribution in I_{M5}

from wrapped M2 on $S^2_{Z_1=0} \subset S^4$ $\left(S^4 : \sum_{a=1}^{2} |Z_a|^2 + X_5^2 = 1\right)$

: preserves $Q^I_{\alpha\,(H-R_{12}=0)}$ ➡ subalgebra $\supset$ $\boxed{\underset{a,b=3,4,5}{so(6)_{J_{ij}} \times so(3)_{R_{ab}}} \times u(1)_{H-2R_{12}} \times u(1)_{H-R_{12}}}$ ②

Match ① with ②

➡ $$\boxed{h = \frac{H}{2} - \frac{3R_{12}}{2},\ j_{12} = R_{34},\ r_{12} = -R_{12},\ r_{34} = J_{12},\ r_{56} = J_{34},\ r_{78} = J_{56}}$$

Parameter identification - step 2

Writing I_{M2} in terms of Cartans (H, J_{ij}, R_{ab}) of M5 SCFT, we find

$$I_{\mathrm{M2}} = \mathrm{Tr}[(-1)^F (q^{\frac{1}{4}} t^{\frac{1}{3}} z^{\frac{1}{3}})^{\frac{3(R_{12}+R_{34})}{2}+J_{12}+J_{34}+J_{56}} (q^{-\frac{5}{8}} t^{\frac{1}{2}} z^{\frac{1}{2}})^{R_{12}-R_{34}} (t^{\frac{2}{3}} z^{-\frac{1}{3}} x^{-1})^{J_{12}} (t^{-\frac{4}{3}} z^{\frac{2}{3}})^{J_{34}} (t^{\frac{2}{3}} z^{-\frac{1}{3}} x)^{J_{56}}]$$

Hence we conclude

$$\boxed{\frac{I^{\mathrm{U}(N)\text{-}[1]}}{I^{\mathrm{U}(\infty)\text{-}[1]}} = 1 + \sum_{m=1}^{\infty} (q^{\frac{1}{4}} t^{-1} z^{-1})^{mN} I_{m\mathrm{M5}}\Big|_{p=q^{\frac{1}{4}} t^{\frac{1}{3}} z^{\frac{1}{3}}, u=q^{-\frac{5}{8}} t^{\frac{1}{2}} z^{-\frac{1}{2}}, y_1=t^{\frac{2}{3}} z^{-\frac{1}{3}} x^{-1}, y_2=t^{-\frac{4}{3}} z^{\frac{2}{3}}, y_3=t^{\frac{2}{3}} z^{-\frac{1}{3}} x} + \cdots}$$

[Arai,Fujiwara,Imamura,Mori,Yokoyama,'20]

In Coulomb limit

$$\frac{\mathcal{I}_C^{\mathrm{U}(N)\text{-}[1]}}{\mathcal{I}_C^{\mathrm{U}(\infty)\text{-}[1]}} = 1 + \sum_{m=1}^{\infty} \hat{F}_m(x_1, x_2) x_1^{mN} + \cdots$$

$$\hat{F}_m(x_1, x_2) = \lim_{\substack{q,t\to 0 \\ (q^{\frac{1}{4}} t^{-1} = \mathfrak{t})}} I_{m\mathrm{M5}}|_{\ldots}$$

$$= I_{m\mathrm{M5}}(p^{\frac{3}{2}} u = x_1^{-1}, p y_2 = x_2; p^{\frac{3}{2}} u^{-1} = p y_1 = p y_3 = 0)$$

$x_1 = z^{-1}\mathfrak{t}$
$x_2 = z\mathfrak{t}$

Check for $m=1,\infty$

Recall

$$\hat{F}_m^{(l=1)}(x_1,x_2)=\prod_{b=1}^{m}\prod_{c=0}^{\infty}\frac{1}{1-x_1^{-b}x_2^{c}}\stackrel{?}{=}I_{m\mathrm{M5}}$$

$$I_{\mathrm{M5}}=\mathrm{Tr}[(-1)^F e^{-\beta\{Q,Q^\dagger\}}p^{\frac{3(R_{12}+R_{34})}{2}+J_{12}+J_{34}+J_{56}}u^{R_{12}-R_{34}}y_1^{J_{12}}y_2^{J_{34}}y_3^{J_{56}}]$$

Parameter identification with 3d Coulomb limit:

$$p,y_2^{-1},u^{-1}\to 0 \ \text{ with } \ p^{\frac{3}{2}}u=x_1^{-1},\ py_2=x_2,\ p^{\frac{3}{2}}u^{-1}=py_1=py_3=0)$$

① Single M5 ➡ free tensor multiplet

$$I_{\mathrm{singleM5}}=\mathrm{PE}\left[\frac{p^{\frac{3}{2}}(u+u^{-1})-p^2(y_1y_2+y_2y_3+y_3y_1)+p^3}{(1-py_1)(1-py_2)(1-py_3)}\right]$$

$$\stackrel{(C)}{\to}\mathrm{PE}\left[\frac{x_1^{-1}}{1-x_2}\right]=\hat{F}_1^{(l=1)}(x_1,x_2)$$

modes with $\{Q,Q^\dagger\}=H-(J_{12}+J_{34}+J_{56})-2(R_{12}+R_{34})=0$:

	H	$J_{12}+J_{34}+J_{56}$	R_{12}	R_{34}
$\Phi^{(1,0)}$	2	0	1	0
$\Phi^{(0,1)}$	2	0	0	1
$\Psi^{++}_{++-},\Psi^{++}_{+-+},\Psi^{++}_{-++}$	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\partial_1^+,\partial_2^+,\partial_3^+$	1	1	0	0
$(\partial\Psi)^{++}_{+++}=0$	$\frac{7}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

② ∞ M5 ➡ graviton index

$$I_{\infty\mathrm{M5}}=\mathrm{PE}\left[\frac{p^{\frac{3}{2}}(1-p^3)(u+u^{-1})-p^2(y_1y_2+y_2y_3+y_3y_1)+p^4(y_1+y_2+y_3)}{(1-p^{\frac{3}{2}}u)(1-p^{\frac{3}{2}}u^{-1})(1-py_1)(1-py_2)(1-py_3)}\right]$$

[Bhattacharya,Bhattacharyya,Minwalla,Raju,'08]

$$\stackrel{(C)}{\to}\mathrm{PE}\left[\frac{x_1^{-1}}{(1-x_1^{-1})(1-x_2)}\right]=\hat{F}_\infty^{(l=1)}(x_1,x_2)$$

Check for $2 \leq m < \infty$

Can be calculated by using the proposal m M5 on $S^1 =$ 5d U(m) $\mathcal{N}=2$ SYM

KK modes $\longleftrightarrow$ instanton particles

$$\boxed{I_{\mathrm{M5}} = Z^{\mathrm{M5}}_{S^5 \times_t S^1} \sim Z^{5\mathrm{d}\mathcal{N}=1^*\mathrm{SYM}}_{S^5_b}}$$

[Douglas,'10][Lambert,Papageorgakis,Schmidt-Sommerfeld,'10]

...but it is still difficult to obtain $I_{m\mathrm{M5}}$ in closed form ($\longleftrightarrow$ sum over all instantons)

Exception: "unrefined limit"

$$\boxed{I^{\mathrm{unref}}_{m\mathrm{M5}}(p) = I_{m\mathrm{M5}}(u = p^{-\frac{1}{2}}, y_1 = y_2 = y_3 = 1) = \prod_{b=1}^{m} \prod_{c=0}^{\infty} \frac{1}{1 - p^{b+c}}}$$

[Kim,Kim,Kim,'12][Kim,Kim,Kim,Lee,'13]

Remark: unrefined limit is equivalent to 3d Coulomb limit with $\mathfrak{t} = 1$

$$I_{\mathrm{M5}}|_{\mathrm{M2parameters}} = \mathrm{Tr}(-1)^F \mathfrak{t}^{-R_{12}+4R_{34}+J_{12}+J_{34}+J_{56}} x^{J_{56}-J_{12}} z^{R_{12}+J_{34}} t^{4R_{34}+2J_{12}+2J_{56}}$$

	H	J_{12}	J_{34}	J_{56}	R_{12}	R_{34}	$-R_{12}+4R_{34}+J_{12}+J_{34}+J_{56}$	$J_{56}-J_{12}$	$R_{12}+J_{34}$	$4R_{34}+2J_{12}+2J_{56}$
Q	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
Q'	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-2	0	0	0

$4R_{34} + 2J_{12} + 2J_{56} = \{Q', (Q')^\dagger\} - \{Q, Q^\dagger\}$ $\Rightarrow$ $I_{\mathrm{M5}}|_{\mathfrak{t}=1}$ is t-indep!

[Beem,Rastelli,van Rees,'14]

$$\boxed{I^{\mathrm{unref}}_{m\mathrm{M5}}(z) = I_{m\mathrm{M5}}(\mathfrak{t}=1, t = z^{\frac{1}{2}}, x=1) = I_{m\mathrm{M5}}(\mathfrak{t}=1, t=0, x=1) \overset{\checkmark}{=} \hat{F}_m(x_1 = z^{-1}, x_2 = z)}$$

Plan of talk

✓ 1. Introduction

✓ 2. 3d U(N) ADHM theory

✓ 3. Coulomb limit and giant graviton expansion

✓ 4. Comparison with M5-indices

5. Higgs limit

Higgs limit ($q, t^{-1} \to 0$ with $\mathfrak{t} = q^{\frac{1}{4}} t$ kept fixed)

$$\mathcal{I}_H^{\mathrm{U}(N)\text{-}[l]} = \lim_{\substack{q,t^{-1}\to 0 \\ (\mathfrak{t}=q^{\frac{1}{4}}t)}} I^{\mathrm{U}(N)\text{-}[l]}$$

$$= \frac{1}{N!} \frac{(1-\mathfrak{t}^2)^N}{\prod_\pm (1-x^{\pm 1}\mathfrak{t})^N} \oint \prod_{i=1}^N \frac{ds_i}{2\pi i s_i} \prod_\pm \frac{\prod_{i<j}(1-(\frac{s_i}{s_j})^{\pm 1})(1-\mathfrak{t}^2(\frac{s_i}{s_j})^{\pm 1})}{\prod_{i<j}(1-\mathfrak{t}x(\frac{s_i}{s_j})^{\pm 1})(1-\mathfrak{t}x^{-1}(\frac{s_i}{s_j})^{\pm 1})}$$

$$\times \prod_i \prod_{\alpha=1}^l \frac{1}{1-\mathfrak{t}(y_\alpha s_i)^{\pm 1}}$$

Integrations can be calculated by JK residue sum

$$\boxed{\mathcal{I}_H^{\mathrm{U}(N)\text{-}[l]} = \frac{1}{\mathfrak{t}^{2lN}} \sum_{\substack{\lambda^{(1)},\cdots,\lambda^{(l)} \\ (\sum_\alpha |\lambda^{(\alpha)}|=N)}} \prod_{\alpha,\beta}^{l} \frac{1}{\mathcal{N}_{\lambda^{(\alpha)},\lambda^{(\beta)}}(\frac{y_\beta}{y_\alpha})}}$$

(e.g. [Hwang,Kim,Kim,Park,'14])

choice of poles — Young diagrams

$$\frac{1}{1-\mathfrak{t}(y_\alpha s_i)^{-1}} \longleftrightarrow \lambda^{(\alpha)} = \boxed{i}$$

$$\frac{1}{1-\mathfrak{t}x\frac{s_i}{s_j}} \longleftrightarrow \boxed{i}\boxed{j}$$

$$\frac{1}{1-\mathfrak{t}x^{-1}\frac{s_i}{s_j}} \longleftrightarrow \begin{array}{c}\boxed{i}\\ \boxed{j}\end{array}$$

$$\mathcal{N}_{\lambda,\mu}(y) = \prod_{\Box\in\lambda}(1-yx_2^{\mathrm{arm}_\lambda(\Box)}x_1^{-\mathrm{leg}_\mu(\Box)-1}) \prod_{\Box'\in\mu}(1-yx_2^{-\mathrm{arm}_\mu(\Box')-1}x_1^{\mathrm{leg}_\lambda(\Box')})$$

$$(x_1 = x^{-1}\mathfrak{t},\ x_2 = x\mathfrak{t})$$

Observations (1/2)

① Inverse giant graviton expansion works!

$$\frac{\mathcal{I}_H^{\mathrm{U}(N)\text{-}[l]}}{\mathcal{I}_H^{\mathrm{U}(\infty)\text{-}[l]}} \overset{\text{s.s.}}{=} \sum_{m=0}^{\infty} x_1^{mN} \mathcal{F}_m^{(l)}(x_1^{-1}, x_1 x_2; y_\alpha)$$

$$\frac{\mathcal{F}_m(x_1, x_2; y_\alpha)}{\mathcal{F}_\infty(x_1, x_2; y_\alpha)} \overset{\text{s.s.}}{=} \sum_{N=0}^{\infty} x_1^{mN} \mathcal{G}_N^{(l)}(x_1^{-1}, x_1 x_2; y_\alpha) \Rightarrow \mathcal{G}_N^{(l)}(x_1, x_2; y_\alpha) = \mathcal{I}_H^{\mathrm{U}(N)\text{-}[l]}$$

② $\mathcal{F}_{m=1}^{(l)}$ are related to vacuum characters of affine Kac-Moody algebra with level 1

$$\mathcal{F}_{m=1}^{(2)} = \frac{1 + 3x_1x_2 + 4(x_1x_2)^2 + 7(x_1x_2)^3 + 13(x_1x_2)^4 + 19(x_1x_2)^5 + 29(x_1x_2)^6 + 43(x_1x_2)^7 + \cdots}{(x_1; x_1x_2)_\infty}$$

$$\mathcal{F}_{m=1}^{(3)} = \frac{1 + 8x_1x_2 + 17(x_1x_2)^2 + 46(x_1x_2)^3 + 98(x_1x_2)^4 + 198(x_1x_2)^5 + 371(x_1x_2)^6 + 692(x_1x_2)^7 + \cdots}{(x_1; x_1x_2)_\infty}$$

$$\mathcal{F}_{m=1}^{(4)} = \frac{1 + 15x_1x_2 + 51(x_1x_2)^2 + 172(x_1x_2)^3 + 453(x_1x_2)^4 + 1128(x_1x_2)^5 + 2539(x_1x_2)^6 + 5505(x_1x_2)^7 + \cdots}{(x_1; x_1x_2)_\infty}$$

$$\vdots$$

$$\Rightarrow \boxed{\mathcal{F}_1^{(l)}(x_1, x_2; y_\alpha) = \frac{1}{(x_1; x_1x_2)_\infty} \chi_{\widehat{\mathfrak{su}}(l)_1}(y_\alpha; x_1x_2)}$$

[Hayashi,TN,Okazaki,'24,'25]

Observations (2/2)

③ $\mathcal{F}_m^{(l)}$ contains vacuum characters of affine Kac-Moody algebra with level *m*

$$\mathcal{F}_{m=2}^{(2)} = \prod_{n=1}^{2} \frac{1}{(x_1^n; x_1 x_2)_\infty} \Big[1 + (3x_1 + 3x_1^2)x_2 + (9x_1^2 + 7x_1^3 + x_1^4)x_2^2 + (15x_1^3 + 19x_1^4 + 4x_1^5)x_2^3$$
$$+ (30x_1^4 + 40x_1^5 + 14x_1^6)x_2^4 + (54x_1^5 + 83x_1^6 + 32x_1^7 + 3x_1^8)x_2^5 + (94x_1^6 + 152x_1^7 + 72x_1^8 + 7x_1^9)x_2^6$$
$$+ (153x_1^7 + 275x_1^8 + 144x_1^9 + 22x_1^{10})x_2^7 + \cdots \Big]$$

$$\mathcal{F}_{m=3}^{(2)} = \prod_{n=1}^{3} \frac{1}{(x_1^n; x_1 x_2)_\infty} \Big[1 + (3x_1 + 3x_1^2 + 3x_1^3)x_2 + (9x_1^2 + 12x_1^3 + 13x_1^4 + 4x_1^5 + x_1^6)x_2^2$$
$$+ (22x_1^3 + 32x_1^4 + 39x_1^5 + 20x_1^6 + 7x_1^7)x_2^3 + (42x_1^4 + 77x_1^5 + 103x_1^6 + 65x_1^7 + 30x_1^8 + 4x_1^9)x_2^4$$
$$+ (81x_1^5 + 167x_1^6 + 241x_1^7 + 180x_1^8 + 97x_1^9 + 23x_1^{10} + 3x_1^{11})x_2^5$$
$$+ (151x_1^6 + 337x_1^7 + 517x_1^8 + 437x_1^9 + 264x_1^{10} + 84x_1^{11} + 17x_1^{12})x_2^6$$
$$+ (264x_1^7 + 643x_1^8 + 1042x_1^9 + 967x_1^{10} + 646x_1^{11} + 251x_1^{12} + 62x_1^{13} + 4x_1^{14})x_2^7 + \cdots \Big]$$

$\vdots$

$$\boxed{\lim_{\substack{x_1, x_2^{-1} \to 0 \\ (x_1 x_2 = \mathfrak{q})}} \mathcal{F}_m^{(l)} = \chi_{\hat{\mathfrak{su}}(l)_m}(y_\alpha; \mathfrak{q})}$$

[Hayashi,TN,Okazaki,'24,'25]

Summary

- M2-M5 giant graviton expansion works in 3d Coulomb/Higgs limit!

$$\frac{\mathcal{I}_{C/H}^{\mathrm{U}(N)\text{-}[1]}(x_1,x_2)}{\mathcal{I}_{C/H}^{\mathrm{U}(\infty)\text{-}[1]}(x_1,x_2)} \overset{\text{s.s.}}{=} \sum_{m=0}^{\infty} x_1^{mN} I_{m\mathrm{M5}}^{(\text{twist})}(x_1^{-1}, x_1 x_2), \qquad \frac{I_{m\mathrm{M5}}^{(\text{twist})}(x_1,x_2)}{I_{\infty\mathrm{M5}}^{(\text{twist})}(x_1,x_2)} \overset{\text{s.s.}}{=} \sum_{m=0}^{\infty} x_1^{mN} \mathcal{I}_{C/H}^{\mathrm{U}(N)\text{-}[1]}(x_1^{-1}, x_1 x_2)$$

- Propose 2-parameter refinement of multiple-M5 indices (with $\mathbb{Z}_l$-orbifold)

$$I_{\mathrm{M5}} = \mathrm{Tr}[(-1)^F e^{-\beta\{Q,Q^\dagger\}} p^{\frac{3(R_{12}+R_{34})}{2}+J_{12}+J_{34}+J_{56}} u^{R_{12}-R_{34}} y_1^{J_{12}} y_2^{J_{34}} y_3^{J_{56}}]$$

$$\Rightarrow \quad I_{\mathrm{M5}}^{(\text{twist})}(x_1,x_2) \equiv \lim_{\substack{p,y_1,y_3\to 0,\\ u,y_2\to\infty,\\ x_1=p^{\frac{3}{2}}:\ \text{fixed},\\ x_2=p^{-\frac{1}{2}}u^{-1}y_2:\ \text{fixed}}} I_{\mathrm{M5}}(p,u,y_1,y_2,y_3)$$

: "Twisted limit"

[Hayashi,TN,Okazaki,'24]

- In Higgs limit, M5 indices are related to affine KM characters (c.f. [Nishioka,Tachikawa,'11])

Future directions

- Derivation of finite *N* corrections from gravity side?

[Eleftherious,Murthy,Rossello,'23][Deddo,Liu,Pando-Zayas,Saskowski,'24]

- Derivation from direct M5 index calculation

  ⇒ understand simplification in "twisted limit"

- M5-interpretation of multi-sum expansion coefficients in Coulomb limit?

- Identify /derive full structure of M5 indices in Higgs limit?

  ⇒ we have to first understand $\mathbb{Z}_l$-action on M5 worldvolume/target space

- Multi-sum expansion in Higgs limit?

Quantum Algebras Meet Gauge Theory and String Theory

January 12 - 16, 2026

Auditorium on 18th Floor @ SIMIS

Key Topics:

- Quantum Algebras & BPS Counting, BPS/CFT Correspondence
- Integrable Structures in Non-Perturbative Gauge Theories
- Vertex Algebras in Gauge and String Theory
- Brane Constructions & Representation Theory

Confirmed Speakers:

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Ryo Fujita
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BKUP

Parameter identification - step 1 (detail)

In $I^{\rm 3d}_{\rm M2}$, wrapped M5 on $S^5_{z_1=0} \subset S^7$ preserves $\left(S^7 : \sum_{a=1}^{4} |z_a|^2 = 1\right)$

$\mathcal{Q}^I_{\alpha\,(h-r_{12}=0)}$ ➡ subalgebra $\supset$ $\boxed{\begin{array}{c} so(3)_{j_{ij}} \times so(6)_{r_{ab}} \times u(1)_{h-\frac{1}{2}r_{12}} \\ a,b=3,\cdots,8 \end{array}}$ ①

In $I^{\rm 6d}_{\rm M5}$, wrapped M2 on $S^2_{Z_1=0} \subset S^4$ preserves $\left(S^4 : \sum_{a=1}^{2} |Z_a|^2 + X_5^2 = 1\right)$

$Q^I_{\alpha\,(H-R_{12}=0)}$ ➡ subalgebra $\supset$ $\boxed{\begin{array}{c} so(6)_{J_{ij}} \times so(3)_{R_{ab}} \times u(1)_{H-2R_{12}} \\ a,b=3,4,5 \end{array}}$ ②

Match ① with ②

$$\begin{cases} h - \frac{1}{2} r_{12} = \lambda (H - 2R_{12}) \\ h - r_{12} = \rho (H - R_{12}) \end{cases} \Rightarrow \begin{cases} h = (2\lambda - \rho) H + (-4\lambda + \rho) R_{12} \\ r_{12} = (2\lambda - 2\rho) H + (-4\lambda + 2\rho) R_{12} \end{cases}$$

Impose $\{\mathcal{Q}, \mathcal{Q}^\dagger\} = (positive)\{Q, Q^\dagger\}$ and $r_{12} = \pm R_{12}$ ➡ $\boxed{\lambda = \rho = \frac{1}{2}}$

$$\Rightarrow \boxed{h = \frac{H}{2} - \frac{3R_{12}}{2}, \; j_{12} = R_{34}, \; r_{12} = -R_{12}, \; r_{34} = J_{12}, \; r_{56} = J_{34}, \; r_{78} = J_{56}}$$

Example: $l = 2, \mathcal{O}(x_2)$

N	$\left.\frac{\mathcal{I}^{\mathrm{U}(N)\text{-}[2]}}{\mathcal{I}^{\mathrm{U}(\infty)\text{-}[2]}}\right\|_{x_2}$
1	$-x_1 - 4x_1^2 - 3x_1^3 + x_1^4 + 5x_1^5 + 8x_1^6 + 8x_1^7 + 7x_1^8 + 3x_1^9 - x_1^{10} - 5x_1^{11} + \cdots$
2	$-x_1^2 - 4x_1^3 - 4x_1^4 - 3x_1^5 + x_1^6 + 5x_1^7 + 9x_1^8 + 12x_1^9 + 12x_1^{10} + 11x_1^{11} + 7x_1^{12} + \cdots$
3	$-x_1^3 - 4x_1^4 - 4x_1^5 - 4x_1^6 - 3x_1^7 + x_1^8 + 5x_1^9 + 9x_1^{10} + 13x_1^{11} + 16x_1^{12} + 16x_1^{13} + \cdots$
4	$-x_1^4 - 4x_1^5 - 4x_1^6 - 4x_1^7 - 4x_1^8 - 3x_1^9 + x_1^{10} + 5x_1^{11} + 9x_1^{12} + 13x_1^{13} + 17x_1^{14} + \cdots$
5	$-x_1^5 - 4x_1^6 - 4x_1^7 - 4x_1^8 - 4x_1^9 - 4x_1^{10} - 3x_1^{11} + x_1^{12} + 5x_1^{13} + 9x_1^{14} + 13x_1^{15} + \cdots$
6	$-x_1^6 - 4x_1^7 - 4x_1^8 - 4x_1^9 - 4x_1^{10} - 4x_1^{11} - 4x_1^{12} - 3x_1^{13} + x_1^{14} + 5x_1^{15} + 9x_1^{16} + \cdots$
7	$-x_1^7 - 4x_1^8 - 4x_1^9 - 4x_1^{10} - 4x_1^{11} - 4x_1^{12} - 4x_1^{13} - 4x_1^{14} - 3x_1^{15} + x_1^{16} + 5x_1^{17} + \cdots$
8	$-x_1^8 - 4x_1^9 - 4x_1^{10} - 4x_1^{11} - 4x_1^{12} - 4x_1^{13} - 4x_1^{14} - 4x_1^{15} - 4x_1^{16} - 3x_1^{17} + x_1^{18} + \cdots$
9	$-x_1^9 - 4x_1^{10} - 4x_1^{11} - 4x_1^{12} - 4x_1^{13} - 4x_1^{14} - 4x_1^{15} - 4x_1^{16} - 4x_1^{17} - 4x_1^{18} - 3x_1^{19} + \cdots$
10	$-x_1^{10} - 4x_1^{11} - 4x_1^{12} - 4x_1^{13} - 4x_1^{14} - 4x_1^{15} - 4x_1^{16} - 4x_1^{17} - 4x_1^{18} - 4x_1^{19} - 4x_1^{20} + \cdots$

$\Downarrow$

$$\left.\mathcal{F}_1^{(2)}\right|_{x_2} = -1 - 4x_1 - 4x_1^2 - 4x_1^3 - 4x_1^4 - 4x_1^5 - 4x_1^6 - 4x_1^7 - 4x_1^8 - 4x_1^9 - 4x_1^{10} + \cdots$$

$$= -\frac{1 + 3x_1}{1 - x_1}$$

Example: $l = 2, \mathcal{O}(x_2)$

N	$\left.\frac{\mathcal{I}^{\mathrm{U}(N)\text{-}[2]}}{\mathcal{I}^{\mathrm{U}(\infty)\text{-}[2]}}\right\rvert_{x_2} - x_1^N\left(-\frac{1+3x_1}{1-x_1}\right)$
1	$x_1^3 + 5x_1^4 + 9x_1^5 + 12x_1^6 + 12x_1^7 + 11x_1^8 + 7x_1^9 + 3x_1^{10} - x_1^{11} - 5x_1^{12} - 8x_1^{13} - 8x_1^{14} + \cdots$
2	$x_1^5 + 5x_1^6 + 9x_1^7 + 13x_1^8 + 16x_1^9 + 16x_1^{10} + 15x_1^{11} + 11x_1^{12} + 6x_1^{13} - x_1^{14} - 6x_1^{15} - 13x_1^{16} + \cdots$
3	$x_1^7 + 5x_1^8 + 9x_1^9 + 13x_1^{10} + 17x_1^{11} + 20x_1^{12} + 20x_1^{13} + 19x_1^{14} + 15x_1^{15} + 10x_1^{16} + 2x_1^{17} - 6x_1^{18} + \cdots$
4	$x_1^9 + 5x_1^{10} + 9x_1^{11} + 13x_1^{12} + 17x_1^{13} + 21x_1^{14} + 24x_1^{15} + 24x_1^{16} + 23x_1^{17} + 19x_1^{18} + 14x_1^{19} + 6x_1^{20} + \cdots$
5	$x_1^{11} + 5x_1^{12} + 9x_1^{13} + 13x_1^{14} + 17x_1^{15} + 21x_1^{16} + 25x_1^{17} + 28x_1^{18} + 28x_1^{19} + 27x_1^{20} + 23x_1^{21} + 18x_1^{22} + \cdots$
6	$x_1^{13} + 5x_1^{14} + 9x_1^{15} + 13x_1^{16} + 17x_1^{17} + 21x_1^{18} + 25x_1^{19} + 29x_1^{20} + 32x_1^{21} + 32x_1^{22} + 31x_1^{23} + 27x_1^{24} + \cdots$
7	$x_1^{15} + 5x_1^{16} + 9x_1^{17} + 13x_1^{18} + 17x_1^{19} + 21x_1^{20} + 25x_1^{21} + 29x_1^{22} + 33x_1^{23} + 36x_1^{24} + 36x_1^{25} + 35x_1^{26} + \cdots$
8	$x_1^{17} + 5x_1^{18} + 9x_1^{19} + 13x_1^{20} + 17x_1^{21} + 21x_1^{22} + 25x_1^{23} + 29x_1^{24} + 33x_1^{25} + 37x_1^{26} + 40x_1^{27} + 40x_1^{28} + \cdots$
9	$x_1^{19} + 5x_1^{20} + 9x_1^{21} + 13x_1^{22} + 17x_1^{23} + 21x_1^{24} + 25x_1^{25} + 29x_1^{26} + 33x_1^{27} + 37x_1^{28} + 41x_1^{29} + 44x_1^{30} + \cdots$
10	$x_1^{21} + 5x_1^{22} + 9x_1^{23} + 13x_1^{24} + 17x_1^{25} + 21x_1^{26} + 25x_1^{27} + 29x_1^{28} + 33x_1^{29} + 37x_1^{30} + 41x_1^{31} + 45x_1^{32} + \cdots$

$$\mathcal{F}_2^{(2)}\Big|_{x_2} = x_1 + 5x_1^2 + 9x_1^3 + 13x_1^4 + 17x_1^5 + 21x_1^6 + 25x_1^7 + 29x_1^8 + 33x_1^9 + 37x_1^{10} + 41x_1^{11} + 45x_1^{12} + \cdots$$

$$= \frac{x_1 + 3x_1^2}{(1-x_1)^2}$$

Example: $l = 2, \mathcal{O}(x_2)$

N	$\frac{\mathcal{I}^{\mathrm{U}(N)\text{-}[2]}}{\mathcal{I}^{\mathrm{U}(\infty)\text{-}[2]}}\Big\|_{x_2} - \{x_1^N(-\frac{1+3x_1}{1-x_1}) + x_1^{2N}(\frac{x_1+3x_1^2}{(1-x_1)^2})\}$
1	$-x_1^6 - 5x_1^7 - 10x_1^8 - 18x_1^9 - 26x_1^{10} - 34x_1^{11} - 42x_1^{12} - 49x_1^{13} - 53x_1^{14} - 57x_1^{15} - 60x_1^{16} - 60x_1^{17} + \cdots$
2	$-x_1^9 - 5x_1^{10} - 10x_1^{11} - 18x_1^{12} - 27x_1^{13} - 38x_1^{14} - 47x_1^{15} - 58x_1^{16} - 66x_1^{17} - 73x_1^{18} - 77x_1^{19} - 80x_1^{20} + \cdots$
3	$-x_1^{12} - 5x_1^{13} - 10x_1^{14} - 18x_1^{15} - 27x_1^{16} - 39x_1^{17} - 51x_1^{18} - 63x_1^{19} - 75x_1^{20} - 86x_1^{21} - 94x_1^{22} - 101x_1^{23} + \cdots$
4	$-x_1^{15} - 5x_1^{16} - 10x_1^{17} - 18x_1^{18} - 27x_1^{19} - 39x_1^{20} - 52x_1^{21} - 67x_1^{22} - 80x_1^{23} - 95x_1^{24} - 107x_1^{25} - 118x_1^{26} + \cdots$
5	$-x_1^{18} - 5x_1^{19} - 10x_1^{20} - 18x_1^{21} - 27x_1^{22} - 39x_1^{23} - 52x_1^{24} - 68x_1^{25} - 84x_1^{26} - 100x_1^{27} - 116x_1^{28} - 131x_1^{29} + \cdots$
6	$-x_1^{21} - 5x_1^{22} - 10x_1^{23} - 18x_1^{24} - 27x_1^{25} - 39x_1^{26} - 52x_1^{27} - 68x_1^{28} - 85x_1^{29} - 104x_1^{30} - 121x_1^{31} - 140x_1^{32} + \cdots$
7	$-x_1^{24} - 5x_1^{25} - 10x_1^{26} - 18x_1^{27} - 27x_1^{28} - 39x_1^{29} - 52x_1^{30} - 68x_1^{31} - 85x_1^{32} - 105x_1^{33} - 125x_1^{34} - 145x_1^{35} + \cdots$
8	$-x_1^{27} - 5x_1^{28} - 10x_1^{29} - 18x_1^{30} - 27x_1^{31} - 39x_1^{32} - 52x_1^{33} - 68x_1^{34} - 85x_1^{35} - 105x_1^{36} - 126x_1^{37} - 149x_1^{38} + \cdots$
9	$-x_1^{30} - 5x_1^{31} - 10x_1^{32} - 18x_1^{33} - 27x_1^{34} - 39x_1^{35} - 52x_1^{36} - 68x_1^{37} - 85x_1^{38} - 105x_1^{39} - 126x_1^{40} - 150x_1^{41} + \cdots$
10	$-x_1^{33} - 5x_1^{34} - 10x_1^{35} - 18x_1^{36} - 27x_1^{37} - 39x_1^{38} - 52x_1^{39} - 68x_1^{40} - 85x_1^{41} - 105x_1^{42} - 126x_1^{43} - 150x_1^{44} + \cdots$

$$\mathcal{F}_3^{(2)}\Big|_{x_2} = -x_1^3 - 5x_1^4 - 10x_1^5 - 18x_1^6 - 27x_1^7 - 39x_1^8 - 52x_1^9 - 68x_1^{10} - 85x_1^{11} - 105x_1^{12} - 126x_1^{13} - 150x_1^{14}$$
$$-175x_1^{15} + \cdots$$

$$= -\frac{x_1^3 + 3x_1^4}{(1-x_1)^2(1-x_1^2)}$$